# ANALYZING A MAXIMUM PRINCIPLE FOR FINITE HORIZON STATE CONSTRAINED PROBLEMS VIA PARAMETRIC EXAMPLES. PART 2: PROBLEMS WITH BILATERAL STATE CONSTRAINTS

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Abstract. In the present paper, a maximum principle for finite horizon state constrained problems is analyzed via parametric examples. These parametric examples resemble typical optimal growth problems in mathematical economics. Since the maximum principle is only a necessary condition for local optimal processes, a large amount of additional investigations is needed to obtain a comprehensive synthesis of finitely many processes suspected for being local minimizers. Our analysis not only helps to understand the principle in depth, but also serves as a sample of applying it to meaningful prototypes of optimal economic growth models. Problems with unilateral state constraints have been studied in Part 1 [J. Nonlinear Convex Anal. 21 (2020), 157–182] of the paper. Problems with bilateral state constraints are addressed in this Part 2.

## 1. Introduction

It is well known that optimal control problems with state constraints are models of importance, but one usually faces with a lot of difficulties in analyzing them. These models have been considered since the early days of the optimal control theory. For instance, the whole Chapter VI of the classical work [1, pp. 257–316] is devoted to problems with restricted phase coordinates. There are various forms of the maximum principle for optimal control problems with state constraints; see, e.g., [2], where the relations between several forms are shown and a series of numerical illustrative examples have been solved. To deal with state constraints, one has to use functions of bounded variation, Borel measurable functions, Lebesgue-Stieltjes integral, nonnegative measures on the $\sigma$-algebra of the Borel sets, the Riesz Representation Theorem for the space of continuous functions, and so on.

By using the maximum principle presented in [3, pp. 233–254], Phu proposed two ingenious methods called the *Method of Region Analysis* (MRA) [4, 5, 6, 7] and the *Method of Orienting Curves* (MOC) [8, 9] to solve several classes of optimal control problems with state constraints. The effectiveness of MRA and MOC in dealing with difficulties caused by state constraints in the just-mentioned papers has been

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shown [10, 11, 12, 13, 14] in the analysis of many important theoretical and practical problems. Afterwards, MRA and MOC were developed for more general models in [15, 16, 17, 18, 19] with concrete applications being given in [20, 21, 22, 23].

In the present paper, the maximum principle for finite horizon state constrained problems from the book by Vinter [24, Theorem 9.3.1] is analyzed via three parametric examples, which have the origin in Example 1 of the recent paper by Basco, Cannarsa, and Frankowska [25], where infinite horizon optimal control problems with state constraints are studied. Each of the considered three parametric examples is an optimal control problem of the Lagrange type with five parameters: the first one appears in the description of the objective function, the second one appears in the differential equation, the third one is the initial value, the fourth one is the initial time, and the fifth one is the terminal time. The difference among three problems is in the appearance of state constraints: The first one does not contain state constraints, the second one is a problem with an unilateral state constraint, and the third one is a problem with a bilateral state constraint. Due to the exponential term in the description of the objective function, the linearity w.r.t. the control variable in the differential equation, and the pure state constraints, these optimal control problems resemble optimal growth models in mathematical economics (see, e.g., [26, pp. 617–625]). Thus, our analysis not only helps to understand the principle in depth, but also serves as a sample of applying it to meaningful prototypes of optimal economic growth models. In fact, the similarity of the optimal control problems considered in our paper and the parametric optimal economic growth models in [27, 28, 29] can be easily observed. It is worthy to stress that the techniques of reasoning in this paper were successfully used in the just cited three papers.

The solution existence of optimal control problems considered herein is established by invoking Filippov's existence theorem [30, Theorem 9.2.i and Section 9.4]. Since the maximum principle is only a necessary condition for local optimal processes, a large amount of additional investigations is needed to obtain a comprehensive synthesis of finitely many processes suspected for being local minimizers. *In the vast literature on optimal control, we have not found any synthesis of optimal processes like the ones in our paper, where the dependence of the optimal processes on all the involved parameters can be explicitly seen.*

Note that the maximum principle for finite horizon state constrained problems in [24, Chapter 9] covers several known ones for smooth problems and allows us to deal with nonsmooth problems by using the concepts of *limitting normal cone* and *limitting subdifferential* of Mordukhovich [31, 32]. This principle is a necessary optimality condition which asserts the existence of a nontrivial multipliers set consisting of an *absolutely continuous function*, a *function of bounded variation*, a *Borel measurable function*, and a *real number*, such that the four conditions (i)–(iv) in Theorem 2.5 below are satisfied. The relationships among these conditions in concrete optimal control problems are worthy a detailed analysis. Just to have an idea about the importance of such analysis, observe that Section 22.1 of the book by Clarke [33] presents a maximum principle [33, Theorem 22.2] for an optimal control problem without state constraints denoted by $(OC)$. The whole Section 22.2 of [33] (see also [33, Exercise 26.1]) is devoted to applying this maximum principle to solve a very special example of $(OC)$ having just one parameter. The analysis contains a series of additional propositions on the properties of the unique global solution.

Problems without state constraints and problems with unilateral state constraints have been studied in Part 1 ([34]) of the paper. Problems with bilateral state constraints are addressed in this Part 2, which is organized as follows. Section 2 presents the above-mentioned maximum principle from [24, Chapter 9]. Control problems with bilateral state constraints are studied in Section 3. Section 4 is devoted to a discussion on the degeneracy phenomenon of the maximum principle, which sheds new lights on the synthesis of the optimal processes given in the previous section. Some concluding remarks are given in the last section.

In comparison with Part 1, to deal with bilateral state constraints, herein we have to prove a series of delicate lemmas and auxiliary propositions. Moreover, the synthesis of finitely many processes suspected for being local minimizers is rather sophisticated, and it requires a lot of refined arguments. Interestingly, the maximum principle plays a very important role in obtaining the synthesis of the optimal solutions of problems with bilateral state constraints. This is, firstly, due to the special structure of the problems in question. Secondly and more importantly, because we can fully exploit the relationships between the optimal solutions of problems with bilateral state constraints and the ones of problems with unilateral state constraints by means of the maximum principle.

We end this section by two remarks which are mainly due to one of the three anonymous referees of this paper. First, it would be interesting to apply the above-mentioned methods MRA and MOC to solve the optimal control problems considered herein and compare the obtained results with the ones of us. Second, one may try to find out what happens if the relationships of the parameters $a$ and $\lambda$ in the problem (3.1)–(3.2) below are described by the inequalities $\lambda > a > 0$.

## 2. Background Materials

In this section, we give some notations, definitions, and results that will be used repeatedly in the sequel.

2.1. **Notations.** The symbol $\mathbb{R}$ (resp., $\mathbb{N}$) denotes the set of real numbers (resp., the set of positive integers). The norm (resp., the inner product) in the $n$-dimensional Euclidean space $\mathbb{R}^n$ is denoted by $\|.\|$ (resp., $\langle \cdot, \cdot \rangle$). The closed unit ball in $\mathbb{R}^n$ is denoted by $\bar{B}$.

For a subset $C \subset \mathbb{R}^n$, we abbreviate its *convex hull* to $\operatorname{co} C$. For a given segment $[t_0, T]$ of the real line, we denote the $\sigma$-algebra of its Lebesgue measurable subsets (resp., the $\sigma$-algebra of its Borel sets) by $\mathcal{L}$ (resp., $\mathcal{B}$). The $\sigma$-algebra of the Borel sets in $\mathbb{R}^m$ is denoted by $\mathcal{B}^m$. The Sobolev space $W^{1,1}([t_0, T], \mathbb{R}^n)$ is the linear space of the absolutely continuous functions $x : [t_0, T] \to \mathbb{R}^n$ endowed with the norm $\|x\|_{W^{1,1}} = \|x(t_0)\| + \int_{t_0}^T \|\dot{x}(t)\|dt$ (see, e.g., [3, p. 21] for this and another equivalent norm).

2.2. **An Optimal Control Problem of the Mayer Type.** As in [24, p. 321], we consider the following *finite horizon optimal control problem of the Mayer type*, denoted by $\mathcal{M}$,

$$\text{Minimize } \ g(x(t_0), x(T)), \tag{2.1}$$

over $x \in W^{1,1}([t_0, T], \mathbb{R}^n)$ and measurable functions $u : [t_0, T] \to \mathbb{R}^m$ satisfying

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & \text{a.e. } t \in [t_0, T] \\ (x(t_0), x(T)) \in C \\ u(t) \in U(t), & \text{a.e. } t \in [t_0, T] \\ h(t, x(t)) \leq 0, & \forall t \in [t_0, T], \end{cases} \tag{2.2}$$

where $[t_0, T]$ is a given interval, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $h : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}$ are given functions, $C \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed set, and $U : [t_0, T] \rightrightarrows \mathbb{R}^m$ is a set-valued map.

A measurable function $u : [t_0, T] \to \mathbb{R}^m$ satisfying $u(t) \in U(t)$ for almost every $t \in [t_0, T]$ is called a *control function.* A *process* $(x, u)$ consists of a control function $u$ and an arc $x \in W^{1,1}([t_0, T]; \mathbb{R}^n)$ that is a solution to the differential equation in (2.2). A *state trajectory* $x$ is the first component of some process $(x, u)$. A process $(x, u)$ is called *feasible* if the state trajectory satisfies the *endpoint constraint* $(x(t_0), x(T)) \in C$ and the *state constraint* $h(t, x(t)) \leq 0$ for all $t \in [t_0, T]$.

Due to the appearance of the state constraint $h(t, x(t)) \leq 0, t \in [t_0, T]$, the problem $\mathcal{M}$ in (2.1)–(2.2) is said to be an *optimal control problem with state constraints.* But, if the state constraint is fulfilled for any state trajectory $x$ with $(x(t_0), x(T)) \in C$, i.e., the state constraint can be removed from (2.2), then one says that $\mathcal{M}$ an *optimal control problem without state constraints.*

**Definition 2.1.** A feasible process $(\bar{x}, \bar{u})$ is called a $W^{1,1}$ *local minimizer* for $\mathcal{M}$ if there exists $\delta > 0$ such that $g(\bar{x}(t_0), \bar{x}(T)) \leq g(x(t_0), x(T))$ for any feasible process $(x, u)$ satisfying $\|\bar{x} - x\|_{W^{1,1}} \leq \delta$.

**Definition 2.2.** A feasible process $(\bar{x}, \bar{u})$ is called a $W^{1,1}$ *global minimizer* for $\mathcal{M}$ if, for any feasible process $(x, u)$, one has $g(\bar{x}(t_0), \bar{x}(T)) \leq g(x(t_0), x(T))$.

2.3. **A Maximum Principle for State Constrained Problems.** The maximum principle for state constrained optimal control problems from [24, Theorem 9.3.1] was given in full in Part 1 of this paper (see [34, Theorem 2.4 ]). To simplify our presentation, here we will only present a version of that maximum principle for the problem $\mathcal{M}$ of which the input data have a certain level of smoothness and convexity. We first give some explanations and definitions.

Due to the appearance of the constraint $h(t, x(t)) \leq 0$ in $\mathcal{M}$, one has to introduce a multiplier that is an element in the topological dual $C^*([t_0, T]; \mathbb{R})$ of the space of continuous functions $C([t_0,T];\mathbb{R})$ with the supremum norm. By the Riesz Representation Theorem (see, e.g., [35, Theorem 6, p. 374] and [36, Theorem 1, pp. 113–115]), any bounded linear functional $f$ on $C([t_0,T];\mathbb{R})$ can be uniquely represented in the form $f(x)=\int_{[t_0,T]}x(t)dv(t)$ with $v$ being *a function of bounded variation* on $[t_0,T]$ vanishing at $t_0$ and continuous from the right at every point $\tau\in(t_0,T)$, and $\int_{[t_0,T]}x(t)dv(t)$ is the Riemann-Stieltjes integral of $x$ w.r.t. $v$ (see, e.g., [35, p. 364]). We denote by $C^{\oplus}(t_0,T)$ the set of the elements of $C^*([t_0,T];\mathbb{R})$ which are given by nondecreasing functions $v$. Every $v\in C^*([t_0,T];\mathbb{R})$ corresponds to *a finite regular measure*, which is denoted by $\mu_v$, on the $\sigma$-algebra $\mathcal{B}$ of the Borel subsets of $[t_0,T]$ by $\mu_v(A):=\int_{[t_0,T]}\chi_A(t)dv(t)$, where $\chi_A(t)=1$ for $t\in A$ and $\chi_A(t)=0$ if $t\notin A$. Due to the correspondence $v\mapsto\mu_v$, we call every element $v\in C^*([t_0,T];\mathbb{R})$ a "measure" and identify $v$ with $\mu_v$. Clearly, the measure corresponding to each $v\in C^{\oplus}(t_0,T)$ is nonnegative.

The integrals $\int_{[t_0,t)}\nu(s)d\mu(s)$ and $\int_{[t_0,T]}\nu(s)d\mu(s)$ of a Borel measurable function $\nu$ in Theorem 2.5 below are understood in the sense of the Lebesgue-Stieltjes integration [35, p. 364].

**Definition 2.3.** The *Hamiltonian* $\mathcal{H}:[t_0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ of the dynamic equation in (2.2) is defined by

$$\mathcal{H}(t,x,p,u):=\langle p,f(t,x,u)\rangle,\quad (t,x,p,u)\in[t_0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m. \tag{2.3}$$

**Definition 2.4** (See [24, p. 329])**.** The *partial hybrid subdifferential* $\partial_x^{>}h(t,x)$ of $h(t,x)$ w.r.t. $x$ is given by

$$\begin{aligned}\partial_x^{>}h(t,x):=\operatorname{co}\big\{\xi\ :\ &\text{there exists } (t_k,x_k)\xrightarrow{h}(t,x) \text{ such that}\\ &h(t_k,x_k)>0 \text{ for all } k \text{ and } \nabla_x h(t_k,x_k)\to\xi\big\},\end{aligned}\tag{2.4}$$

where $(t_k,x_k)\xrightarrow{h}(t,x)$ means $(t_k,x_k)\to(t,x)$ with $h(t_k,x_k)\to h(t,x)$.

**Theorem 2.5** (See [24, Theorem 9.3.1])**.** *Suppose that $C$ is a closed convex subset of $\mathbb{R}^n\times\mathbb{R}^n$ and $U(t)=U$ for all $t\in[t_0,T]$ with $U$ being a compact subset of $\mathbb{R}^m$. Let $(\bar{x},\bar{u})$ be a $W^{1,1}$ local minimizer for $\mathcal{M}$. Assume that, for some $\delta>0$,*

- (A1) *$f:[t_0,T]\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is continuous, $f(t,\cdot,u)$ is continuously differentiable on the ball $B(\bar{x}(t),\delta)$ for all $(t,u)\in[t_0,T]\times U$, and the derivative $f_x(t,x,u)$ of $f(t,\cdot,u)$ at $x$ is continuous on the set of vectors $(t,x,u)$ satisfying $(t,u)\in[t_0,T]\times U$ and $x\in B(\bar{x}(t),\delta)$;*
- (A2) *$g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is continuously differentiable;*
- (A3) *$h:[t_0,T]\times\mathbb{R}^n\to\mathbb{R}$ is upper semicontinuous and there exists $K>0$ such that*

$$\|h(t,x)-h(t,x')\|\le K\|x-x'\|,\ \forall x,x'\in\bar{x}(t)+\delta\bar{B},\ \forall t\in[t_0,T].$$

*Then there exist $p\in W^{1,1}([t_0,T];\mathbb{R}^n)$, $\gamma\ge 0$, $\mu\in C^{\oplus}(t_0,T)$, and a Borel measurable function $\nu:[t_0,T]\to\mathbb{R}^n$ with the property that $(p,\mu,\gamma)\ne(0,0,0)$, and for*

*$q(t) := p(t) + \eta(t)$ with*

$$\eta(t) := \int_{[t_0,t)} \nu(s)d\mu(s), \quad t \in [t_0, T)$$

*and*

$$\eta(T) := \int_{[t_0,T]} \nu(s)d\mu(s),$$

*the following holds true:*

(i) *$\nu(t) \in \partial_x^{>} h(t, \bar{x}(t))$ $\mu$-a.e.;*

(ii) *$-\dot{p}(t) = \nabla_x \mathcal{H}(t, \bar{x}(t), q(t), \bar{u}(t))$ a.e., where $\nabla_x \mathcal{H}(t, \bar{x}(t), q(t), \bar{u}(t))$ denotes the Fréchet derivative of the function $\mathcal{H}(t, ., q(t), \bar{u}(t))$ at $\bar{x}(t)$;*

(iii) *$(p(t_0), -q(T)) \in \gamma \nabla g(\bar{x}(t_0), \bar{x}(T)) + N_C(\bar{x}(t_0), \bar{x}(T))$, where $\nabla g(\bar{x}(t_0), \bar{x}(T))$ stands for the Fréchet derivative of $g$ at $(\bar{x}(t_0), \bar{x}(T))$ and $N_C(\bar{x}(t_0), \bar{x}(T))$ denotes the normal cone to the convex set $C$ at $(\bar{x}(t_0), \bar{x}(T))$ in the sense of convex analysis (see, e.g.,* [3, p. 205]*);*

(iv) *$\mathcal{H}(t, \bar{x}(t), q(t), \bar{u}(t)) = \max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), q(t), u)$ a.e.*

It remains to show that the assumptions made in Theorem 2.5 guarantee that the hypothesis (H1) in [24, Theorem 9.3.1] is fulfilled. To do so, take any $\delta' \in (0, \delta)$ and note that the set of all $(t, x, u)$ satisfying $(t, u) \in [t_0, T] \times U$ and $x \in \bar{B}(\bar{x}(t), \delta')$, denoted by $\mathcal{A}$, is compact. Hence, assumption (A1) in Theorem 2.5 implies that the number $\gamma = \max\{\|f_x(t, x, u)\| \,:\, (t, x, u) \in \mathcal{A}\}$ is well defined. By the mean value theorem for vector-valued functions (see, e.g., [3, p. 27]) we have

$$\|f(t, x, u) - f(t, x', u)\| \le \gamma \|x - x'\|, \;\; \forall t \in [t_0, T],\, x, x' \in \bar{B}(\bar{x}(t), \delta'),\, u \in U.$$

Thus, condition (H1) in [24, Theorem 9.3.1] is satisfied.

## 3. Optimal Control Problems with Bilateral State Constraints

By $(FP_3)$ we denote the following optimal control problem of the Lagrange type

$$\text{Minimize } \; J(x, u) = \int_{t_0}^{T} \big[ - e^{-\lambda t}(x(t) + u(t)) \big] dt \tag{3.1}$$

over $x \in W^{1,1}([t_0, T], \mathbb{R})$ and measurable functions $u : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}(t) = -au(t), & \text{a.e. } t \in [t_0, T] \\ x(t_0) = x_0 \\ u(t) \in [-1, 1], & \text{a.e. } t \in [t_0, T] \\ -1 \le x(t) \le 1, & \forall t \in [t_0, T] \end{cases} \tag{3.2}$$

with $a > \lambda > 0$, $T > t_0 \ge 0$, and $-1 \le x_0 \le 1$ being given.

In order to treat $(FP_3)$ in (3.1)–(3.2) as a problem of the Mayer type, we set $x(t) = (x_1(t), x_2(t))$, where $x_1(t)$ plays the role of $x(t)$ in $(FP_3)$ and

$$x_2(t) := \int_{t_0}^{t} \big[ - e^{-\lambda\tau}(x_1(\tau) + u(\tau))\big] d\tau \tag{3.3}$$

for all $t \in [0, T]$. Thus, $(FP_3)$ is equivalent to the problem

$$\text{Minimize } \ x_2(T) \tag{3.4}$$

over $x = (x_1, x_2) \in W^{1,1}([t_0, T], \mathbb{R}^2)$ and measurable functions $u : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}_1(t) = -au(t), & \text{a.e. } t \in [t_0, T] \\ \dot{x}_2(t) = -e^{-\lambda t}(x_1(t) + u(t)), & \text{a.e. } t \in [t_0, T] \\ (x(t_0), x(T)) \in \{(x_0, 0)\} \times \mathbb{R}^2 & \\ u(t) \in [-1, 1], & \text{a.e. } t \in [t_0, T] \\ -1 \le x_1(t) \le 1, & \forall t \in [t_0, T]. \end{cases} \tag{3.5}$$

The problem (3.4)–(3.5) is abbreviated to $(FP_{3a})$.

The solution existence of $(FP_3)$ is studied similarly as in Subsection 4.1 of [34]. Namely, we can apply the Filippov's Existence Theorem for Mayer problems (see [34, Theorem 2.6] and [30, Theorem 9.2.i and Section 9.4]) to conclude that $(FP_{3a})$ has a $W^{1,1}$ global minimizer. Thus, by the equivalence of $(FP_3)$ and $(FP_{3a})$, we can assert that $(FP_3)$ has a $W^{1,1}$ global minimizer.

To solve $(FP_{3a})$ by applying Theorem 2.5, we note that $(FP_{3a})$ is in the form of the Mayer problem $\mathcal{M}$ with $g(x, y) = y_2$, $f(t, x, u) = (-au, -e^{-\lambda t}(x_1 + u))$, $C = \{(x_0, 0)\} \times \mathbb{R}^2$, $U(t) = [-1, 1]$, and $h(t, x) = |x_1| - 1$ for all $t \in [t_0, T]$, $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$ and $u \in \mathbb{R}$. By (2.3), the Hamiltonian of $(FP_{3a})$ is

$$\mathcal{H}(t, x, p, u) = -aup_1 - e^{-\lambda t}(x_1 + u)p_2, \ (t, x, p, u) \in [t_0, T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}. \tag{3.6}$$

By (2.4), the partial hybrid subdifferential of $h$ at $(t, x) \in [t_0, T] \times \mathbb{R}^2$ is the set

$$\partial_x^{>} h(t, x) = \begin{cases} \{(-1, 0)\}, & \text{if } x_1 \le -1 \\ \emptyset, & \text{if } |x_1| < 1 \\ \{(1, 0)\}, & \text{if } x_1 \ge 1. \end{cases} \tag{3.7}$$

Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$ local minimizer for $(FP_{3a})$. Since all the assumptions (A1)–(A3) of Theorem 2.5 are satisfied for $(FP_{3a})$, by that theorem there exist $p \in W^{1,1}([t_0, T]; \mathbb{R}^2)$, $\gamma \ge 0$, $\mu \in C^{\oplus}(t_0, T)$, and a Borel measurable function $\nu : [t_0, T] \to \mathbb{R}^2$ such that $(p, \mu, \gamma) \neq (0, 0, 0)$, and for $q(t) := p(t) + \eta(t)$ with $\eta(t) := \int_{[t_0, t)} \nu(\tau) d\mu(\tau)$ for $t \in [t_0, T)$ and $\eta(T) := \int_{[t_0, T]} \nu(\tau) d\mu(\tau)$, the conditions (i)–(iv) in Theorem 2.5 hold true.

**Condition (i)**: As $-1 \le \bar{x}_1(t) \le 1$ for every $t$, using formula (3.7), one has

$$\begin{aligned}\mu\{t \in [t_0,T] : \nu(t) \notin \partial_x^{>} h(t,\bar{x}(t))\} =& \mu\{t \in [t_0,T] : -1 < \bar{x}_1(t) < 1\}\\ &+ \mu\{t \in [t_0,T] : \bar{x}_1(t) = 1, \nu(t) \neq (1,0)\}\\ &+ \mu\{t \in [t_0,T] : \bar{x}_1(t) = -1, \nu(t) \neq (-1,0)\}.\end{aligned}$$

So, from (i) it follows that

$$\mu\{t \in [t_0,T] \,:\, -1 < \bar{x}_1(t) < 1\} = 0, \tag{3.8}$$

$$\mu\big\{t \in [t_0,T] \,:\, \bar{x}_1(t) = 1,\ \nu(t) \neq (1,0)\big\} = 0, \tag{3.9}$$

and $\mu\big\{t \in [t_0,T] \,:\, \bar{x}_1(t) = -1,\ \nu(t) \neq (-1,0)\big\} = 0$.

**Condition (ii)**: By (3.6), $\mathcal{H}$ is differentiable in $x$ and

$$\nabla_x \mathcal{H}(t,x,p,u) = \{(-e^{-\lambda t}p_2, 0)\}, \quad (t,x,p,u) \in [t_0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}.$$

Thus, (ii) implies that $-\dot{p}(t) = (-e^{-\lambda t}q_2(t), 0)$ for a.e. $t \in [t_0,T]$. Therefore, $\dot{p}_1(t) = e^{-\lambda t}q_2(t)$ for a.e. $t \in [t_0,T]$ and $p_2(t)$ is a constant for all $t \in [t_0,T]$.

**Condition (iii)**: It is not hard to verify that $\nabla g(\bar{x}(t_0), \bar{x}(T)) = \{(0,0,0,1)\}$ and $N_C(\bar{x}(t_0), \bar{x}(T)) = \mathbb{R}^2 \times \{(0,0)\}$ from the formulas for $g$ and $C$. Thus, (iii) yields $(p(t_0), -q(T)) \in \{(0,0,0,\gamma)\} + \mathbb{R}^2 \times \{(0,0)\}$, which means that $q_1(T) = 0$ and $q_2(T) = -\gamma$.

**Condition (iv)**: By (3.6), from (iv) one gets

$$[aq_1(t) + e^{-\lambda t}q_2(t)]\bar{u}(t) = \min_{u \in [-1,1]} \big\{[aq_1(t) + e^{-\lambda t}q_2(t)]u\big\}, \quad \text{a.e. } t \in [t_0,T].$$

If the curve $\bar{x}_1(t)$ remains in the interior of $[-1,1]$ for all $t$ from an open interval $(\tau_1,\tau_2)$ of the time axis and touches the boundary of the segment at the moments $\tau_1$ and $\tau_2$, then it must have some special form. A formal formulation of this observation is as follows.

**Proposition 3.1.** *Suppose that $[\tau_1,\tau_2]$, $\tau_1 < \tau_2$, is a subsegment of $[t_0,T]$ with $\bar{x}_1(t) \in (-1,1)$ for all $t \in (\tau_1,\tau_2)$. Then, next statements hold true.*

S1) *If $\bar{x}_1(\tau_1) = -1$ and $\bar{x}_1(\tau_2) = 1$, then $\tau_2 - \tau_1 = 2a^{-1}$ and*

$$\bar{x}_1(t) = -1 + a(t-\tau_1), \quad t \in [\tau_1,\tau_2].$$

S2) *If $\bar{x}_1(\tau_1) = 1$ and $\bar{x}_1(\tau_2) = -1$, then $\tau_2 - \tau_1 = 2a^{-1}$ and*

$$\bar{x}_1(t) = 1 - a(t-\tau_1), \quad t \in [\tau_1,\tau_2].$$

S3) *If $\bar{x}_1(\tau_1) = \bar{x}_1(\tau_2) = -1$, then $\tau_2 - \tau_1 < 4a^{-1}$ and*

$$\bar{x}_1(t) = \begin{cases} -1 + a(t-\tau_1), & t \in [\tau_1, (\tau_1+\tau_2)/2] \\ -1 - a(t-\tau_2), & t \in ((\tau_1+\tau_2)/2, \tau_2]. \end{cases}$$

S4) *The situation where $\bar{x}_1(\tau_1) = \bar{x}_1(\tau_2) = 1$ cannot happen.*

*Proof.* Choose $\varepsilon_1 >$ and $\varepsilon_2 > 0$ small enough so as $[\tau_1+\varepsilon_1, \tau_2-\varepsilon_2] \subset [\tau_1, \tau_2]$. Then, $\bar{x}_1(t) \in (-1, 1)$ for all $t \in [\tau_1+\varepsilon_1, \tau_2-\varepsilon_2]$, i.e., $h(t, \bar{x}(t)) < 0$ for all $t \in [\tau_1+\varepsilon_1, \tau_2-\varepsilon_2]$. Thus, applying Proposition 4.3 in Part 1 ([34]) with $(FP_{3a})$ in the place of $(FP_{2a})$ in its formulation, one finds that the formula for $\bar{x}_1(.)$ on $[\tau_1+\varepsilon_1, \tau_2-\varepsilon_2]$ belongs to one of the following categories C1−C3:

$$\bar{x}_1(t) = \bar{x}_1(\tau_1+\varepsilon_1) + a(t-\tau_1-\varepsilon_1), \quad t \in [\tau_1+\varepsilon_1, \tau_2-\varepsilon_2],$$

$$\bar{x}_1(t) = \bar{x}_1(\tau_1+\varepsilon_1) - a(t-\tau_1-\varepsilon_1), \quad t \in [\tau_1+\varepsilon_1, \tau_2-\varepsilon_2],$$

and

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(\tau_1+\varepsilon_1) + a(t-\tau_1-\varepsilon_1), & t \in [\tau_1+\varepsilon_1, t_\zeta] \\ \bar{x}_1(t_\zeta) - a(t-t_\zeta), & t \in (t_\zeta, \tau_2-\varepsilon_2], \end{cases}$$

where $t_\zeta$ is some point in $(\tau_1+\varepsilon_1, \tau_2-\varepsilon_2)$.

To prove the statement S1, let $\varepsilon_2 = k^{-1}$ with $k$ being a positive integer, as large as $k^{-1} \in (\tau_1+\varepsilon_1, \tau_2)$. Since for each $k$ the formula for $\bar{x}_1(.)$ on the segment $[\tau_1+\varepsilon_1, \tau_2-k^{-1}]$ must be of the three types C1–C3, by the Dirichlet principle there must exist a subsequence $\{k'\}$ of $\{k\}$ such that the corresponding formulas belong to a fixed category. If the latter happens to be C2, then by the continuity of $\bar{x}_1(.)$ one has

$$\begin{aligned}\bar{x}_1(\tau_2) = \lim_{k'\to\infty} \bar{x}_1(\tau_2 - \frac{1}{k'}) &= \lim_{k'\to\infty} \Big[\bar{x}_1(\tau_1+\varepsilon_1) - a(\tau_2 - \frac{1}{k'} - \tau_1 - \varepsilon_1)\Big] \\ &= \bar{x}_1(\tau_1+\varepsilon_1) - a(\tau_2-\tau_1-\varepsilon_1).\end{aligned}$$

This is impossible, because $\bar{x}_1(\tau_2) = 1$. Similarly, the situation where the fixed category is C3 must also be excluded. In the case where the formulas for $\bar{x}_1(.)$ belong to the category C1, we have $\bar{x}_1(t) = \bar{x}_1(\tau_1+\varepsilon_1) + a(t-\tau_1-\varepsilon_1)$ for $t \in [\tau_1+\varepsilon_1, \tau_2]$. Now, letting $\varepsilon_1$ tend to zero and using the continuity of $\bar{x}_1(.)$, we obtain $\bar{x}_1(t) = \bar{x}_1(\tau_1) + a(t-\tau_1)$ for $t \in [\tau_1, \tau_2]$. As $\bar{x}_1(\tau_1) = -1$, the statement S1 is proved.

The statements S2 and S3 are proved similarly. To prove the assertion S4, it suffices to apply some arguments similar to the ones of Case 3 in Subsection 4.2 in Part 1 ([34]). □

The forthcoming technical lemma will be in use very frequently.

**Lemma 3.2.** *Given any $t_1, t_2 \in [t_0, T]$, $t_1 < t_2$, one puts*

$$J(x,u)|_{[t_1,t_2]} := \int_{t_1}^{t_2} \big[-e^{-\lambda t}\big(x_1(t)+u(t)\big)\big]dt \tag{3.10}$$

*for any feasible process $(x, u)$ of $(FP_{3a})$. If $(\widetilde{x}, \widetilde{u})$ and $(\check{x}, \check{u})$ are feasible processes for $(FP_{3a})$ with $\widetilde{x}_1(t) = 1$ for all $t \in [t_1, t_2]$ and*

$$\check{x}_1(t) = \begin{cases} 1 - a(t-t_1), & t \in [t_1, \check{t}] \\ 1 + a(t-t_2), & t \in (\check{t}, t_2], \end{cases}$$

*where* $\check{t} := 2^{-1}(t_1 + t_2)$*, then one has*

$$J(\check{x}, \check{u})|_{[t_1,t_2]} - J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]} = \frac{1}{\lambda}\big(\frac{a}{\lambda} - 1\big)\Delta(t_1, t_2) \tag{3.11}$$

*with*

$$\Delta(t_1, t_2) := e^{-\lambda t_1} - 2e^{-\frac{1}{2}\lambda(t_1+t_2)} + e^{-\lambda t_2}. \tag{3.12}$$

*Besides, it holds that* $\Delta(t_1, t_2) > 0$ *and* $J(\check{x}, \check{u})|_{[t_1,t_2]} > J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]}$.

*Proof.* Using the equation $\dot{x}_1(t) = -au(t)$ in (3.5), which is fulfilled for almost all $t \in [t_0, T]$, and the assumed properties of the processes $(\widetilde{x}, \widetilde{u})$ and $(\check{x}, \check{u})$, we have $\widetilde{u}(t) = 0$ for almost $t \in [t_1, t_2]$, $\check{u}(t) = 1$ for almost $t \in [t_1, \check{t}]$, and $\check{u}(t) = -1$ for almost $t \in (\check{t}, t_2]$. By the formulas for $\widetilde{x}_1$ and $\widetilde{u}$ on $[t_1, t_2]$,

$$J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]} = \int_{t_1}^{t_2} \big[ - e^{-\lambda t}\big(\widetilde{x}_1(t) + \widetilde{u}(t)\big)\big] dt = \frac{1}{\lambda}e^{-\lambda t_2} - \frac{1}{\lambda}e^{-\lambda t_1}.$$

Similarly, from the formulas for $\check{x}_1$ and $\check{u}$ on $[t_1, t_2]$ it follows that

$$\begin{aligned} J(\check{x}, \check{u})|_{[t_1,t_2]} &= \int_{t_1}^{t_2} \big[ - e^{-\lambda t}\big(\check{x}_1(t) + \check{u}(t)\big)\big] dt \\ &= \big(\frac{2}{\lambda} - \frac{2a}{\lambda^2}\big)e^{-\lambda \check{t}} + \big(\frac{a}{\lambda^2} - \frac{2}{\lambda}\big)e^{-\lambda t_1} + \frac{a}{\lambda^2}e^{-\lambda t_2}. \end{aligned}$$

Therefore,

$$J(\check{x}, \check{u})|_{[t_1,t_2]} - J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]} = \frac{1}{\lambda}\big(\frac{a}{\lambda} - 1\big)\big(e^{-\lambda t_1} - 2e^{-\lambda \check{t}} + e^{-\lambda t_2}\big). \tag{3.13}$$

So, formula (3.11) is proved. To obtain the second assertion of the lemma, put $\psi(t) = e^{-\lambda t}$ for all $t \in \mathbb{R}$. Since $\psi''(t) > 0$ for every $t$, the function $\psi$ is strictly convex. So, one has $\psi\big(\frac{1}{2}t_1 + \frac{1}{2}t_2\big) < \frac{1}{2}\psi(t_1) + \frac{1}{2}\psi(t_2)$. It follows that $\Delta(t_1, t_2) > 0$ for any $t_1 < t_2$. Combining this with (3.13) and the inequality $\dfrac{a}{\lambda} - 1 > 0$, we obtain $J(\check{x}, \check{u})|_{[t_1,t_2]} > J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]}$. □

The following analogue of Lemma 3.2 will be used latter on.

**Lemma 3.3.** *Let* $t_1, t_2$ *be as in Lemma 3.2. Let* $J(x, u)|_{[t_1,t_2]}$ *and* $\Delta(t_1, t_2)$ *be defined, respectively, by* (3.10) *and* (3.12)*. If* $(\widetilde{x}, \widetilde{u})$ *and* $(\hat{x}, \hat{u})$ *are feasible processes for* $(FP_{3a})$ *with* $\widetilde{x}_1(t) = -1$ *for all* $t \in [t_1, t_2]$ *and*

$$\hat{x}_1(t) = \begin{cases} -1 + a(t - t_1), & t \in [t_1, \hat{t}] \\ -1 - a(t - t_2), & t \in (\hat{t}, t_2], \end{cases}$$

*where* $\hat{t} := 2^{-1}(t_1 + t_2)$*, then one has*

$$J(\hat{x}, \hat{u})|_{[t_1,t_2]} - J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]} = -\frac{1}{\lambda}\big(\frac{a}{\lambda} - 1\big)\Delta(t_1, t_2).$$

*Therefore,* $J(\hat{x}, \hat{u})|_{[t_1,t_2]} < J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]}$.

*Proof.* By (3.5), by our assumptions, $\widetilde{u}(t) = 0$ for almost $t \in [t_1, t_2]$, $\hat{u}(t) = -1$ for almost $t \in [t_1, \hat{t}]$, and $\hat{u}(t) = 1$ for almost $t \in (\hat{t}, t_2]$. From formulas for $\widetilde{x}$, $\widetilde{u}$, $\hat{x}_1$, and $\hat{u}$ on $[t_1, t_2]$, we have $J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]} = -\frac{1}{\lambda}e^{-\lambda t_2} + \frac{1}{\lambda}e^{-\lambda t_1}$ and

$$J(\hat{x}, \hat{u})|_{[t_1,t_2]} = -(\frac{2}{\lambda} - \frac{2a}{\lambda^2})e^{-\lambda \check{t}} - (\frac{a}{\lambda^2} - \frac{2}{\lambda})e^{-\lambda t_1} - \frac{a}{\lambda^2}e^{-\lambda t_2}.$$

Thus, changing the sign of the expression $J(\hat{x}, \hat{u})|_{[t_1,t_2]} - J(\widetilde{x}, \widetilde{u})|_{[t_1,t_2]}$ we get the expression on the left-hand-side of (3.13). So, the desired results follow from Lemma 3.2. □

We will need two more lemmas.

**Lemma 3.4.** *Consider the function* $\Delta : \mathbb{R}^2 \to \mathbb{R}$ *defined by* (3.12). *For any* $t_1, t_2 \in \mathbb{R}$ *with* $t_1 < t_2$ *and for any* $\bar{\varepsilon} \in (0, t_2 - t_1)$*, one has*

$$\Delta(t_1 + \bar{\varepsilon}, t_2) < \Delta(t_1, t_2) \tag{3.14}$$

*and*

$$\Delta(t_1, t_2) > \Delta(t_1, t_1 + \bar{\varepsilon}) + \Delta(t_1 + \bar{\varepsilon}, t_2). \tag{3.15}$$

*Proof.* Fix a value $\bar{\varepsilon} \in (0, t_2 - t_1)$. To obtain the inequality (3.14), we consider the function $\psi_1(\varepsilon) := \Delta(t_1 + \varepsilon, t_2)$ of the variable $\varepsilon \in \mathbb{R}$. Since

$$\psi_1(\varepsilon) = e^{-\lambda(t_1+\varepsilon)} - 2e^{-\frac{1}{2}\lambda(t_1+\varepsilon+t_2)} + e^{-\lambda t_2},$$

one sees that $\psi_1(.)$ is continuously differentiable on $\mathbb{R}$ and

$$\psi_1'(\varepsilon) = \lambda\big(e^{-\frac{1}{2}\lambda(t_1+\varepsilon+t_2)} - e^{-\lambda(t_1+\varepsilon)}\big).$$

As the function $r(t) := e^{-\lambda t}$ is strictly decreasing on $\mathbb{R}$, the last equality implies that $\psi_1'(\varepsilon) < 0$ for every $\varepsilon \in [0, t_2 - t_1)$. Hence, the function $\psi_1(.)$ is strictly decreasing on $[0, t_2 - t_1)$. So, the inequality (3.14) is valid.

To obtain (3.15), we use (3.12) to calculate and get

$$\begin{aligned} &\Delta(t_1, t_2) - \Delta(t_1, t_1 + \bar{\varepsilon}) - \Delta(t_1 + \bar{\varepsilon}, t_2) \\ &= 2\big[e^{-\lambda(\frac{t_1+t_2}{2}+\frac{\bar{\varepsilon}}{2})} - e^{-\lambda\frac{t_1+t_2}{2}}\big] - 2\big[e^{-\lambda(t_1+\bar{\varepsilon})} - e^{-\lambda(t_1+\frac{\bar{\varepsilon}}{2})}\big]. \end{aligned}$$

Applying the classical mean value theorem to the function $r(t) = e^{-\lambda t}$, one can find $\tau_1 \in (t_1 + \frac{\bar{\varepsilon}}{2}, t_1 + \bar{\varepsilon})$ and $\tau_2 \in (\frac{t_1+t_2}{2}, \frac{t_1+t_2}{2} + \frac{\bar{\varepsilon}}{2})$ such that

$$e^{-\lambda(t_1+\bar{\varepsilon})} - e^{-\lambda(t_1+\frac{\bar{\varepsilon}}{2})} = \frac{-\bar{\varepsilon}\lambda}{2}e^{-\lambda\tau_1} \quad \text{and} \quad e^{-\lambda(\frac{t_1+t_2}{2}+\frac{\bar{\varepsilon}}{2})} - e^{-\lambda\frac{t_1+t_2}{2}} = \frac{-\bar{\varepsilon}\lambda}{2}e^{-\lambda\tau_2}.$$

Thus, $\Delta(t_1, t_2) - \Delta(t_1, t_1 + \bar{\varepsilon}) - \Delta(t_1 + \bar{\varepsilon}, t_2) = \bar{\varepsilon}\lambda\big[e^{-\lambda\tau_1} - e^{-\lambda\tau_2}\big]$. As the function $r(t)$ is strictly decreasing on $\mathbb{R}$ and $\tau_1 < \tau_2$, one gets $e^{-\lambda\tau_1} - e^{-\lambda\tau_2} > 0$; hence the inequality (3.15) is proved. □

**Lemma 3.5.** *Let there be given $t_1, t_2 \in [t_0, T]$, $t_1 < t_2$, and $\xi > 0$. Suppose that $(\widetilde{x}^\xi, \widetilde{u}^\xi)$ and $(\check{x}^\xi, \check{u}^\xi)$ are feasible processes for $(FP_{3a})$ with $\widetilde{x}_1^\xi(t) = \xi$ for all $t \in [t_1, t_2]$ and*

$$\check{x}_1^\xi(t) = \begin{cases} \xi - a(t - t_1), & t \in [t_1, \check{t}] \\ \xi + a(t - t_2), & t \in (\check{t}, t_2], \end{cases}$$

*where $\check{t} := 2^{-1}(t_1 + t_2)$. Then one has*

$$J(\check{x}^\xi, \check{u}^\xi)|_{[t_1,t_2]} - J(\widetilde{x}^\xi, \widetilde{u}^\xi)|_{[t_1,t_2]} = \frac{1}{\lambda}\Big(\frac{a}{\lambda} - 1\Big)\Delta(t_1, t_2),$$

*with $J(x, u)|_{[t_1,t_2]}$ and $\Delta(t_1, t_2)$ being defined respectively by (3.10) and (3.12). Besides, the strict inequality $J(\check{x}^\xi, \check{u}^\xi)|_{[t_1,t_2]} > J(\widetilde{x}^\xi, \widetilde{u}^\xi)|_{[t_1,t_2]}$ is valid.*

*Proof.* The proof is similar to that of Lemma 3.2. □

The following propositions are crucial for describing the behavior of the local solutions of $(FP_{3a})$.

**Proposition 3.6.** *The situation where $\bar{x}_1(t) = -1$ for all $t$ from a subsegment $[t_1, t_2]$ of $[t_0, T]$ with $t_1 < t_2$ cannot happen.*

*Proof.* Since $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer of $(FP_{3a})$, by Definition 2.1 there exists $\delta > 0$ such that $(\bar{x}, \bar{u})$ minimizes the quantity $g(x(t_0), x(T)) = x_2(T)$ over all feasible processes $(x, u)$ of $(FP_{3a})$ with $\|\bar{x} - x\|_{W^{1,1}([t_0,T];\mathbb{R}^2)} \leq \delta$.

To prove our assertion, suppose on the contrary that there are $t_1, t_2$ satisfying $t_0 \leq t_1 < t_2 \leq T$ such that $\bar{x}_1(t) = -1$ for all $t \in [t_1, t_2]$. Fixing a number $\varepsilon \in (0, t_2 - t_1)$, we consider the pair of functions $(\hat{x}^\varepsilon, \hat{u}^\varepsilon)$, where

$$\hat{x}_1^\varepsilon(t) := \begin{cases} \bar{x}_1(t), & t \in [t_0, t_1[ \cup (t_1 + \varepsilon, T] \\ -1 + a(t - t_1), & t \in [t_1, t_1 + 2^{-1}\varepsilon] \\ -1 - a(t - t_1 - \varepsilon), & t \in (t_1 + 2^{-1}\varepsilon, t_1 + \varepsilon] \end{cases}$$

and $\hat{u}^\varepsilon(t) := -a^{-1}\dfrac{d\hat{x}_1^\varepsilon(t)}{dt}$ for almost all $t \in [t_0, T]$. Clearly, $(\hat{x}^\varepsilon, \hat{u}^\varepsilon)$ is a feasible process of $(FP_{3a})$. By (3.3), (3.10), and the definition of $\hat{x}_1^\varepsilon(.)$, we have

$$\bar{x}_2(T) - \hat{x}_2^\varepsilon(T) = J(\bar{x}, \bar{u})_{|[t_1,t_1+\varepsilon]} - J(\hat{x}^\varepsilon, \hat{u}^\varepsilon)_{|[t_1,t_1+\varepsilon]}. \tag{3.16}$$

Besides, it follows from Lemma 3.3 and the constructions of $\bar{x}$ and $\hat{x}^\varepsilon$ on the interval $[t_1, t_1 + \varepsilon]$ that $J(\bar{x}, \bar{u})_{|[t_1,t_1+\varepsilon]} - J(\hat{x}^\varepsilon, \hat{u}^\varepsilon)_{|[t_1,t_1+\varepsilon]} > 0$. Combining this with (3.16) yields $\bar{x}_2(T) > \hat{x}_2^\varepsilon(T)$, which contradicts the $W^{1,1}$ local optimality of $(\bar{x}, \bar{u})$, because $\|\bar{x} - \hat{x}^\varepsilon\|_{W^{1,1}([t_0,T];\mathbb{R}^2)} \leq \delta$ for $\varepsilon > 0$ small enough. □

**Proposition 3.7.** *One must have $\bar{x}_1(t) > -1$ for all $t \in (t_0, T)$.*

*Proof.* By our standing assumption, $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer for $(FP_{3a})$. Let $\delta > 0$ be chosen as in the proof of Proposition 3.6. If the assertion is false, there would exist $\check{t} \in (t_0, T)$ with $\bar{x}_1(\check{t}) = -1$.

If there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\bar{x}_1(t) > -1$ for every $t \in (\check{t} - \varepsilon_1, \check{t}) \cup (\check{t}, \check{t} + \varepsilon_2)$. Then, thanks to the continuity of $\bar{x}_1(.)$, by shrinking $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ (if necessary) one may assume that $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (\check{t} - \varepsilon_1, \check{t}) \cup (\check{t}, \check{t} + \varepsilon_2)$. Then, since the curve $\bar{x}_1(.)$ cannot have more than one turning on the interval $(\check{t} - \varepsilon_1, \check{t})$ (resp., on the interval $(\check{t}, \check{t} + \varepsilon_2)$) by the observation given at the beginning of the proof of Proposition 3.1. So, replacing $\varepsilon_1$ (resp., $\varepsilon_2$) by a smaller positive number, one may assume that

$$\bar{x}_1(t) = \begin{cases} -1 - a(t - \check{t}), & t \in [\check{t} - \varepsilon_1, \check{t}] \\ -1 + a(t - \check{t}), & t \in (\check{t}, \check{t} + \varepsilon_2]. \end{cases}$$

To get a contradiction, we can apply the construction given in Lemma 3.5. Namely, choose $\varepsilon > 0$ as small as $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ and define a feasible process $(\widetilde{x}^\varepsilon, \widetilde{u}^\varepsilon)$ for $(FP_{3a})$ by setting

$$\widetilde{u}^\varepsilon(t) = \begin{cases} 0, & t \in [\check{t} - \varepsilon, \check{t} + \varepsilon] \\ \bar{u}(t), & t \in [t_0, \check{t} - \varepsilon) \cup (\check{t} + \varepsilon, T] \end{cases}$$

and

$$\widetilde{x}^\varepsilon(t) = \begin{cases} \bar{x}_1(\check{t} - \varepsilon), & t \in [\check{t} - \varepsilon, \check{t} + \varepsilon] \\ \bar{x}(t), & t \in [t_0, \check{t} - \varepsilon) \cup (\check{t} + \varepsilon, T]. \end{cases}$$

Then, by Lemma 3.5 one has $J(\bar{x}, \bar{u}) > J(\widetilde{x}^\varepsilon, \widetilde{u}^\varepsilon)$. This contradicts the $W^{1,1}$ local optimality of $(\bar{x}, \bar{u})$, as $\|\bar{x} - \hat{x}^\varepsilon\|_{W^{1,1}([t_0,T];\mathbb{R}^2)} \leq \delta$ for $\varepsilon > 0$ small enough.

Since one cannot find $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the strict inequality $\bar{x}_1(t) > -1$ holds for all $t \in (\check{t} - \varepsilon_1, \check{t}) \cup (\check{t}, \check{t} + \varepsilon_2)$, there must exist a sequence $\{t_k\}$ in $(t_0, T)$ converging to $\check{t}$ such that either $t_k < \check{t}$ for all $k$ or $t_k > \check{t}$ for all $k$, and $\bar{x}_1(t_k) = -1$ for each $k$. It suffices to consider the case $t_k < \check{t}$ for all $k$, as the other case can be treated similarly. By considering a subsequence (if necessary), we may assume that $t_k < t_{k+1}$ for all $k$. Choose $\bar{k}$ as large as

$$\check{t} - t_{\bar{k}} < \min\{2\delta a^{-1},\, 4a^{-1}\}. \tag{3.17}$$

This choice of $\bar{k}$ guarantees that $\bar{x}_1(t) < 1$ for every $t \in [t_{\bar{k}}, \check{t}]$. Indeed, otherwise there is some $\alpha \in (t_{\bar{k}}, \check{t})$ with $\bar{x}_1(\alpha) = 1$. Setting

$$\alpha_1 = \min\big\{t \in [t_{\bar{k}}, \alpha] \,:\, \bar{x}_1(t) = 1\big\},\ \ \alpha_2 = \max\big\{t \in [\alpha, \check{t}] \,:\, \bar{x}_1(t) = 1\big\},$$

one has $\alpha_1 \leq \alpha_2$, $[\alpha_1, \alpha_2] \subset [t_{\bar{k}}, \check{t}]$, and $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (t_{\bar{k}}, \alpha_1) \cup (\alpha_2, \check{t})$. Then, by assertion S1 of Proposition 3.1, one has $\alpha_1 - t_{\bar{k}} = 2a^{-1}$. Similarly, by assertion S2 in that proposition, one has $\check{t} - \alpha_2 = 2a^{-1}$. So, one gets $\check{t} - t_{\bar{k}} \geq 4a^{-1}$, which comes in conflict with (3.17).

By Proposition 3.6, one cannot have $\bar{x}_1(t) = -1$ for all $t \in [t_{\bar{k}}, t_{\bar{k}+1}]$. Thus, there is some $\tau \in (t_{\bar{k}}, t_{\bar{k}+1})$ with $\bar{x}_1(\tau) > -1$. Setting

$$\tau_1 = \max\big\{t \in [t_{\bar{k}}, \tau] \,:\, \bar{x}_1(t) = -1\big\},\ \ \tau_2 = \min\big\{t \in [\tau, t_{\bar{k}+1}] \,:\, \bar{x}_1(t) = -1\big\},$$

one has $\tau_1 < \tau_2$, $[\tau_1, \tau_2] \subset [t_{\bar{k}}, t_{\bar{k}+1}]$, and $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (\tau_1, \tau_2)$. Hence, replacing $t_{\bar{k}}$ (resp., $t_{\bar{k}+1}$) by $\tau_1$ (resp., $\tau_2$), one sees that all the above-described properties of the sequence $\{t_k\}$ remain and, in addition,

$$\bar{x}_1(t) \in (-1, 1), \quad t \in (t_{\bar{k}}, t_{\bar{k}+1}). \tag{3.18}$$

Let $F := \{t \in [t_{\bar{k}}, \check{t}] \,:\, \bar{x}_1(t) = -1\}$ and $E := [t_{\bar{k}}, \check{t}] \setminus F$. Since $F$ is a closed subset of $\mathbb{R}$ and $E = (t_{\bar{k}}, \check{t}) \setminus F$, $E$ is an open subset of $\mathbb{R}$. So, $E$ is the union of a countable family of disjoint open intervals (see [37, Proposition 9, p. 17]). Since $t_k \notin E$ for all $k$, we have a representation $E = \bigcup_{i=1}^{\infty} E_i$, where the intervals $E_i = (\tau_1^{(i)}, \tau_2^{(i)})$, $i \in \mathbb{N}$, are nonempty and disjoint. Thanks to (3.18), one may suppose that $E_1 = (\tau_1^{(1)}, \tau_2^{(1)}) = (t_{\bar{k}}, t_{\bar{k}+1})$. Note also that, for any $i \in \mathbb{N}$, $\bar{x}_1(t) \in (-1, 1)$ for all $t \in E_i$. Since $\bar{x}_1(\tau_1^{(i)}) = \bar{x}_1(\tau_2^{(i)}) = -1$, by assertion S3 of Proposition 3.1 one gets

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - \tau_1^{(i)}), & t \in [\tau_1^{(i)}, 2^{-1}(\tau_1^{(i)} + \tau_2^{(i)})] \\ -1 - a(t - \tau_2^{(i)}), & t \in (2^{-1}(\tau_1^{(i)} + \tau_2^{(i)}), \tau_2^{(i)}]. \end{cases}$$

If the set $F_1 := F \setminus \{t_{\bar{k}}\}$ has an isolated point in the induced topology of $[t_{\bar{k}}, \check{t}]$, says, $\bar{t}$. Then, one must have $\bar{t} \in [t_{\bar{k}+1}, \check{t})$. Thus, there exists $\varepsilon > 0$ such that $(\bar{t} - \varepsilon, \bar{t} + \varepsilon)$ is contained in $(t_{\bar{k}}, \check{t})$ and we have $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (\bar{t} - \varepsilon, \bar{t}) \cup (\bar{t}, \bar{t} + \varepsilon)$. Applying the construction given in the first part of this proof, we find a feasible process $(\widetilde{x}^\varepsilon, \widetilde{u}^\varepsilon)$ for $(FP_{3a})$ with the property that $J(\bar{x}, \bar{u}) > J(\widetilde{x}^\varepsilon, \widetilde{u}^\varepsilon)$. This contradicts the $W^{1,1}$ local optimality of $(\bar{x}, \bar{u})$, because (3.17) assures that $\|\bar{x} - \hat{x}^\varepsilon\|_{W^{1,1}([t_0,T];\mathbb{R}^2)} \leq \delta$.

Now, suppose that every point in the compact set $F_1$ is a limit point of this set in the induced topology of $[t_{\bar{k}}, \check{t}]$. Then, if the Lebesgue measure $\mu_L(F_1)$ of $F_1$ is null, then the structure of $F_1$ is similar to that of *the Cantor set*[1], constructed from the segment $[t_{\bar{k}+1}, \check{t}] \subset \mathbb{R}$. If $\mu_L(F_1) > 0$, the structure of $F_1$ is similar to that of a *fat Cantor set*, which is also called *a Smith-Volterra-Cantor set*[2]. Putting

$$\widetilde{u}(t) = \begin{cases} 0, & t \in [t_{\bar{k}}, \check{t}] \\ \bar{u}(t), & t \in [t_0, t_{\bar{k}}) \cup (\check{t}, T] \end{cases} \tag{3.19}$$

and

$$\widetilde{x}_1(t) = \begin{cases} -1, & t \in [t_{\bar{k}}, \check{t}] \\ \bar{x}_1(t), & t \in [t_0, t_{\bar{k}}) \cup (\check{t}, T], \end{cases} \tag{3.20}$$

we see that $(\widetilde{x}, \widetilde{u})$ is a feasible process for $(FP_{3a})$. Similarly, define

$$u(t) = \begin{cases} -1, & t \in [t_{\bar{k}+1}, 2^{-1}(t_{\bar{k}+1} + \check{t})] \\ 1, & t \in (2^{-1}(t_{\bar{k}+1} + \check{t}), \check{t}] \\ \bar{u}(t), & t \in [t_0, t_{\bar{k}+1}) \cup (\check{t}, T] \end{cases}$$

[1]https://en.wikipedia.org/wiki/Cantor_set.

[2]https://en.wikipedia.org/wiki/Smith-Volterra-Cantor_set.

and

$$x_1(t) = \begin{cases} -1 + a(t - t_{\bar{k}+1}), & t \in [t_{\bar{k}+1}, 2^{-1}(t_{\bar{k}} + \check{t})] \\ -1 - a(t - \check{t}), & t \in (2^{-1}(t_{\bar{k}+1} + \check{t}), \check{t}] \\ \bar{x}_1(t), & t \in [t_0, t_{\bar{k}+1}) \cup (\check{t}, T], \end{cases} \tag{3.21}$$

and observe that $(x, u)$ is a feasible process for $(FP_{3a})$. Using (3.17), it is easy to verify that $\|x - \bar{x}\|_{W^{1,1}([t_0,T];\mathbb{R}^2)} \le \delta$. Thus, if it can be shown that

$$J(x, u) < J(\bar{x}, \bar{u}), \tag{3.22}$$

then we get a contradiction to the $W^{1,1}$ local optimality of $(\bar{x}, \bar{u})$. Hence, the proof of the lemma will be completed.

By (3.19)–(3.21) and Lemma 3.3, $J(\widetilde{x}, \widetilde{u}) - J(x, u) = J(\widetilde{x}, \widetilde{u})|_{[t_{\bar{k}},\check{t}]} - J(x, u)|_{[t_{\bar{k}},\check{t}]}$. Therefore, we have

$$J(\widetilde{x}, \widetilde{u}) - J(x, u) = \frac{1}{\lambda}\Big(\frac{a}{\lambda} - 1\Big)\big[\Delta(t_{\bar{k}}, t_{\bar{k}+1}) + \Delta(t_{\bar{k}+1}, \check{t})\big], \tag{3.23}$$

where $\Delta(t_1, t_2)$, for any $t_1, t_2$ with $t_1 < t_2$, is given by (3.12). In addition, using (3.19), (3.20), the decomposition $[t_{\bar{k}+1}, \check{t}] = \big(\bigcup_{i=2}^{\infty} E_i\big) \cup F_1$, and the sum rule [35, Theorem 1', p. 297] and the decomposition formula [35, Theorem 4, p. 298] for the Lebesgue integrals, one gets

$$\begin{aligned} J(\bar{x}, \bar{u}) - J(\widetilde{x}, \widetilde{u}) &= J(\bar{x}, \bar{u})|_{[t_{\bar{k}},\check{t}]} - J(\widetilde{x}, \widetilde{u})|_{[t_{\bar{k}},\check{t}]} \\ &= \int_{[t_{\bar{k}},\check{t}]} \Big[ - e^{-\lambda t}\big(\big[\bar{x}_1(t) + \bar{u}(t)\big] - \big[\widetilde{x}_1(t) + \widetilde{u}(t)\big]\big)\Big] dt \\ &= \sum_{i=2}^{\infty} \int_{E_i} \Big[ - e^{-\lambda t}\big(\big[\bar{x}_1(t) + \bar{u}(t)\big] - \big[\widetilde{x}_1(t) + \widetilde{u}(t)\big]\big)\Big] dt \\ &\quad + \int_{F_1} \Big[ - e^{-\lambda t}\big(\big[\bar{x}_1(t) + \bar{u}(t)\big] - \big[\widetilde{x}_1(t) + \widetilde{u}(t)\big]\big)\Big] dt. \end{aligned}$$

Hence, it holds that

$$J(\bar{x}, \bar{u}) - J(\widetilde{x}, \widetilde{u}) = -\frac{1}{\lambda}\Big(\frac{a}{\lambda} - 1\Big) \sum_{i=2}^{\infty} \Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big) + I, \tag{3.24}$$

where $I := \int_{F_1} \Big[ - e^{-\lambda t}\big(\big[\bar{x}_1(t) + \bar{u}(t)\big] - \big[\widetilde{x}_1(t) + \widetilde{u}(t)\big]\big)\Big] dt$. Given any $t \in F_1$, we observe that $\bar{x}_1(t) = \widetilde{x}_1(t) = -1$ and $\widetilde{u}(t) = 0$. Since every point in $F_1$ is a limit point of this set in the induced topology of $[t_{\bar{k}}, \check{t}]$, we can find a sequence $\{\xi_j^t\}$ in $F_1$ satisfying $\lim\limits_{j\to\infty} \xi_j^t = t$. As the derivative $\dot{\bar{x}}_1(t)$ exists a.e. on $[t_0, T]$, it exists a.e. on $F_1$. In combination with the first differential equation in (3.5), this yields $\dot{\bar{x}}_1(t) = -a\bar{u}(t)$ a.e. $t \in F_1$. Since $\bar{x}_1(t) = -1$ for all $t \in F_1$, for a.e. $t \in F_1$ it holds that

$$\bar{u}(t) = -\frac{1}{a}\,\dot{\bar{x}}_1(t) = -\frac{1}{a}\lim_{j\to\infty}\frac{\bar{x}_1(\xi_j^t) - \bar{x}_1(t)}{\xi_j^t - t} = 0.$$

We have thus shown that $\big[\bar{x}_1(t) + \bar{u}(t)\big] - \big[\widetilde{x}_1(t) + \widetilde{u}(t)\big] = 0$ for a.e. $t \in F_1$. This implies that $I = 0$. Now, adding (3.23) and (3.24), we get

$$J(\bar{x},\bar{u}) - J(x,u) = \frac{1}{\lambda}\Big(\frac{a}{\lambda} - 1\Big)\Big[\Delta(t_{\bar{k}}, t_{\bar{k}+1}) + \Delta(t_{\bar{k}+1}, \check{t}) - \sum_{i=2}^{\infty}\Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big)\Big]. \tag{3.25}$$

We have

$$\sum_{i=2}^{\infty}\Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big) \le \Delta(t_{\bar{k}+1}, \check{t}). \tag{3.26}$$

To establish this inequality, we first show that

$$\sum_{i=2}^{m}\Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big) < \Delta(t_{\bar{k}+1}, \check{t}) \tag{3.27}$$

for any integer $m \ge 2$. Taking account of the fact that every point in $F_1$ is a limit point of this set in the induced topology of $[t_{\bar{k}}, \check{t}]$, by reordering the intervals $\big(\tau_1^{(i)}, \tau_2^{(i)}\big)$ for $i = 2, \dots, m$, we may assume that

$$t_{\bar{k}+1} < \tau_1^{(2)} < \tau_2^{(2)} < \tau_1^{(3)} < \tau_2^{(3)} < \cdots < \tau_1^{(m)} < \tau_2^{(m)} < \check{t}.$$

Then, by Lemma 3.4 and by induction, we have

$$\begin{aligned}
\sum_{i=2}^{m}\Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big) &< \Big[\Delta\big(t_{\bar{k}+1}, \tau_1^{(2)}\big) + \Delta\big(\tau_1^{(2)}, \tau_2^{(2)}\big)\Big] + \sum_{i=3}^{m}\Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big)\\
&< \Big[\Delta\big(t_{\bar{k}+1}, \tau_2^{(2)}\big) + \Delta\big(\tau_2^{(2)}, \tau_1^{(3)}\big)\Big] + \sum_{i=3}^{m}\Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big)\\
&\;\;\vdots\\
&< \Delta\big(t_{\bar{k}+1}, \tau_2^{(m)}\big) + \Delta\big(\tau_2^{(m)}, \check{t}\big)\\
&< \Delta(t_{\bar{k}+1}, \check{t}).
\end{aligned}$$

Thus, (3.27) is valid. Since $\Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big) > 0$ for all $i = 2, 3, \dots,$ the estimate (3.27) shows that the series $\sum_{i=2}^{\infty}\Delta\big(\tau_1^{(i)}, \tau_2^{(i)}\big)$ is convergent. Letting $m \to \infty$, from (3.27) one obtains (3.26). Since $\Delta(t_{\bar{k}}, t_{\bar{k}+1}) > 0$, the equality (3.25) and the inequality (3.26) imply (3.22).

The proof is complete. □

To continue, using the parameters tube $(\lambda, a, x_0, t_0, T)$ of the problem $(FP_{3a})$, we define $\rho = \dfrac{1}{\lambda}\ln\dfrac{a}{a-\lambda} > 0$, $\bar{t} = T - \rho$. Besides, for a given $x_0 \in [-1, 1]$, let

$$\rho_1 := a^{-1}(1 + x_0) \quad \text{and} \quad \rho_2 := a^{-1}(1 - x_0). \tag{3.28}$$

As $x_0 \in [-1, 1]$, one has $\rho_1 \in [0, 2a^{-1}]$ and $\rho_2 \in [0, 2a^{-1}]$. Moreover, since $\bar{x}_1(t)$ is a continuous function, $\mathcal{T}_1 := \{t \in [t_0, T] \,:\, \bar{x}_1(t) = 1\}$ is a compact set (which may be empty). If $\mathcal{T}_1$ is nonempty, then we consider the numbers $\alpha_1 := \min\{t \,:\, t \in \mathcal{T}_1\}$ and $\alpha_2 := \max\{t \,:\, t \in \mathcal{T}_1\}$.

By Proposition 3.7, one of next four cases must occur.

**Case 1:** $x_0 > -1$ *and* $\bar{x}_1(t) > -1$ *for all* $t \in (t_0, T]$. Then, condition (i) means that (3.8) and (3.9) are satisfied, while conditions (ii)–(iv) remain the same as those in Subsection 4.2 of Part 1 ([34]). Moreover, it is clear that $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer of the problem considered therein. So, the curve $\bar{x}_1(t)$ must be of one of the four types (a)–(d) depicted in Theorem 4.4 of Part 1 ([34]), where we let $\bar{x}_1(t)$ play the role of $\bar{x}(t)$. Of course, the condition $\bar{x}_1(t) > -1$ for all $t \in [t_0, T]$ must be satisfied. With respect to the just mentioned four types of $\bar{x}(t)$, we have the following four subcases.

<u>*Subcase 1a*</u>: $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = x_0 - a(t - t_0), \quad t \in [t_0, T]. \tag{3.29}$$

By statement (a) of Theorem 4.4 of Part 1 ([34]), this situation happens when $T - t_0 \le \rho$ (i.e., $\bar{t} \le t_0$). By (3.29), the condition $\bar{x}_1(t) > -1$ for all $t \in (t_0, T]$ is equivalent to $\bar{x}_1(T) > -1$ or, equivalently, $T - t_0 < \rho_1$. Therefore, if either $\rho < \rho_1$ and $T - t_0 \le \rho$, or $\rho \ge \rho_1$ and $T - t_0 < \rho_1$, then $\bar{x}_1(t)$ is given by (3.29).

<u>*Subcase 1b*</u>: $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ x_0 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T]. \end{cases} \tag{3.30}$$

According to statement (b) of Theorem 4.4 of Part 1 ([34]), this situation occurs when $\rho < T - t_0 < \rho + \rho_2$. By (3.30), the condition $\bar{x}_1(t) > -1$ for all $t \in (t_0, T]$ is equivalent to the requirement $\bar{x}_1(T) > -1$, which means that $T - t_0 > 2\rho - \rho_1$. Thus, if $\max\{\rho; 2\rho - \rho_1\} < T - t_0 < \rho + \rho_2$, then $\bar{x}_1(t)$ is given by (3.30).

<u>*Subcase 1c*</u>: $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T]. \end{cases} \tag{3.31}$$

In this situation, statement (c) of Theorem 4.4 of Part 1 ([34]) requires that $\rho < T - t_0 = \rho + \rho_2$. By (3.31), the condition $\bar{x}_1(t) > -1$ for all $t \in (t_0, T]$ is equivalent to the requirement $\bar{x}_1(T) > -1$, which means that $\rho < 2a^{-1}$. Thus, if $\rho < T - t_0 = \rho + \rho_2$ and $\rho < 2a^{-1}$, then $\bar{x}_1(t)$ is given by (3.31).

Subcase 1d: $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_0 + \rho_2) \\ 1, & t \in [t_0 + \rho_2, \bar{t}) \\ 1 - a(t - \bar{t}), & t \in [\bar{t}, T]. \end{cases} \tag{3.32}$$

By statement (d) of Theorem 4.4 of Part 1 ([34]), one need $\rho + \rho_2 < T - t_0$. Meanwhile, by (3.32), the condition $\bar{x}_1(t) > -1$ for all $t \in (t_0, T]$ is equivalent to the requirement $\bar{x}_1(T) > -1$, which means that $\rho < 2a^{-1}$. So, if $\rho + \rho_2 < T - t_0$ and $\rho < 2a^{-1}$, then $\bar{x}_1(t)$ is given by (3.32).

**Case 2:** $x_0 = -1$ *and* $\bar{x}_1(t) > -1$ *for all* $t \in (t_0, T]$. Let $\bar{\varepsilon} > 0$ be such that $t_0 + \bar{\varepsilon} < T$. For any $k \in \mathbb{N}$ with $k^{-1} \in (0, \bar{\varepsilon})$, based on the comments before Proposition 4.1 of Part 1 ([34]) and Proposition 4.2 therein, we can assert that the restriction of $(\bar{x}, \bar{u})$ on $[t_0 + k^{-1}, T]$ is a $W^{1,1}$ local minimizer for the Mayer problem obtained from $(FP_{3a})$ by replacing $t_0$ with $t_0 + k^{-1}$. Since $\bar{x}_1(t) > -1$ for all $t \in [t_0 + k^{-1}, T]$, repeating the arguments already used in Case 1 yields a formula for $\bar{x}_1(t)$ on $[t_0 + k^{-1}, T]$. With $\rho_1(k) := a^{-1}[1 + \bar{x}_1(t_0 + k^{-1})]$ and $\rho_2(k) := a^{-1}[1 - \bar{x}_1(t_0 + k^{-1})]$, for every $k \in \mathbb{N}$ we see that the function $\bar{x}_1(t)$ on $[t_0 + k^{-1}, T]$ must belong to one of the following four categories, which correspond to the four forms of the function $\bar{x}_1(t)$ in Case 1.

(C1) $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \bar{x}_1(t_0 + k^{-1}) - a(t - t_0 - k^{-1}), \quad t \in [t_0 + k^{-1}, T],$$

provided that $\rho < \rho_1(k)$ and $T - t_0 - k^{-1} \le \rho$, or $\rho \ge \rho_1(k)$ and $T - t_0 - k^{-1} < \rho_1(k)$.

(C2) $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(t_0 + k^{-1}) + a(t - t_0 - k^{-1}), & t \in [t_0 + k^{-1}, \bar{t}] \\ \bar{x}_1(t_0 + k^{-1}) - a(t + t_0 + k^{-1} - 2\bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $\max\{\rho; 2\rho - \rho_1(k)\} < T - t_0 - k^{-1} < \rho + \rho_2(k)$.

(C3) $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(t_0 + k^{-1}) + a(t - t_0 - k^{-1}), & t \in [t_0 + k^{-1}, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $\rho < T - t_0 - k^{-1} = \rho + \rho_2(k)$, and $\rho < 2a^{-1}$.

(C4) $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(t_0 + k^{-1}) + a(t - t_0 - k^{-1}), & t \in [t_0 + k^{-1}, t_0 + \rho_2(k)] \\ 1, & t \in (t_0 + \rho_2(k), \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $\rho + \rho_2(k) < T - t_0 - k^{-1}$ and $\rho < 2a^{-1}$.

By the Dirichlet principle, there exist an infinite number of indexes $k$ with $k^{-1} \in (0, \bar{\varepsilon})$ such that the formula for $\bar{x}_1(t)$ is given in the category C1 (resp., C2, C3, or C4). By considering a subsequence if necessary, we may assume that this happens for all $k$ with $k^{-1} \in (0, \bar{\varepsilon})$ .

If the first situation occurs, then $\bar{x}_1(t) = -1 - a(t - t_0)$ for all $t \in [t_0, T]$ by letting $k \to \infty$. This is impossible since the requirement $\bar{x}_1(t) > -1$ for all $t \in (t_0, T]$ is violated.

If the second situation occurs, then by letting $k \to \infty$ we have

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, \bar{t}] \\ -1 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $\max\{\rho; 2\rho - \rho_1\} \le T - t_0 \le \rho + \rho_2$. Since $\bar{x}_1(t) > -1$ for all $t \in (t_0, T]$, one must have $\bar{x}_1(T) > -1$; hence $2\rho < T - t_0$. Since $x_0 = -1$, by (3.28) one has $\rho_1 = 0$ and $\rho_2 = 2a^{-1}$. Thus, this situation happens when $2\rho < T - t_0 \le \rho + 2a^{-1}$.

If the third situation occurs, then $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $T - t_0 = \rho + 2a^{-1}$, and $\rho < 2a^{-1}$.

If the fourth situation occurs, then $\bar{x}_1(t)$ is given by

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(t_0) + a(t - t_0), & t \in [t_0, t_0 + 2a^{-1}] \\ 1, & t \in (t_0 + 2a^{-1}, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T], \end{cases}$$

provided that $\rho + 2a^{-1} < T - t_0$ and $\rho < 2a^{-1}$.

**Case 3:** $\bar{x}_1(T) = -1$ *and* $\bar{x}_1(t) > -1$ *for all* $t \in [t_0, T)$.

*Subcase 3a*: $\mathcal{T}_1 = \emptyset$. Then $\bar{x}_1(t) \in (-1, 1)$ for all $t \in [t_0, T)$ and $\bar{x}_1(T) = -1$. By some arguments similar to those of the proof of Proposition 3.1, one can show that formula for $\bar{x}_1(.)$ on $[t_0, T]$ is one of the following two types:

$$\bar{x}_1(t) = x_0 - a(t - t_0), \quad t \in [t_0, T], \tag{3.33}$$

and

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_\zeta] \\ -1 - a(t - T), & t \in (t_\zeta, T], \end{cases} \tag{3.34}$$

with $t_\zeta \in (t_0, T)$.

If $\bar{x}_1(.)$ is given by (3.33), then $\bar{x}_1(T) = -1$ if and only if $T - t_0 = \rho_1$. Since $x_0 \in (-1, 1]$, the latter yields $0 < T - t_0 = \rho_1 \le 2a^{-1}$.

If $\bar{x}_1(.)$ is of the form (3.34), then $t_\zeta = 2^{-1}[T + t_0 - \rho_1]$ as $\bar{x}_1(T) = -1$. Since $t_\zeta > t_0$, one must have $T - t_0 > \rho_1$. Meanwhile, by (3.34) and our standing assumption in this subcase, $\bar{x}_1(t_\zeta) < 1$. So, $T - t_0 < \rho_1 + 2\rho_2 = a^{-1}(3 - x_0)$.

Combining this and the inequality $T - t_0 > \rho_1$ yields $\rho_1 < T - t_0 < a^{-1}(3 - x_0)$. Our results in this subcase can be summarized as follows:

- $\bar{x}_1(.)$ is given by (3.33), provided that $T - t_0 = \rho_1$.
- $\bar{x}_1(.)$ is given by (3.34), provided that $\rho_1 < T - t_0 < a^{-1}(3 - x_0)$.

Subcase 3b: $\mathcal{T}_1 \neq \emptyset$. Then we have $t_0 \leq \alpha_1 \leq \alpha_2 < T$. It follows from assertion S2 of Proposition 3.1 that $T - \alpha_2 = 2a^{-1}$ and $\bar{x}_1(t) = 1 - a(t - \alpha_2)$ for all $t \in [\alpha_2, T]$. Thus, we have $\alpha_2 = T - 2a^{-1}$ and $\bar{x}_1(t) = 1 - a(t - T + 2a^{-1})$ for all $t \in [T - 2a^{-1}, T]$.

If $\alpha_1 < \alpha_2$, then $\bar{x}_1(t) = 1$ for all $t \in [\alpha_1, \alpha_2]$. Indeed, suppose on the contrary that there exists $\bar{t} \in (\alpha_1, \alpha_2)$ satisfying $\bar{x}_1(\bar{t}) < 1$. Set

$$\bar{\alpha}_1 = \max\{t \in [\alpha_1, \bar{t}] \ : \ \bar{x}_1(t) = 1\} \quad \text{and} \quad \bar{\alpha}_2 = \min\{t \in [\bar{t}, \alpha_2] \ : \ \bar{x}_1(t) = 1\}.$$

Clearly, $[\bar{\alpha}_1, \bar{\alpha}_2] \subset [\alpha_1, \alpha_2] \subset [t_0, T)$ and $\bar{x}_1(t) < 1$ for all $t \in (\bar{\alpha}_1, \bar{\alpha}_2)$. This and the condition $\bar{x}_1(t) > -1$ for all $t \in [t_0, T)$ imply that $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (\bar{\alpha}_1, \bar{\alpha}_2)$. So, by assertion S4 of Proposition 3.1, we obtain a contradiction. Our claim has been proved.

If $t_0 < \alpha_1$, then $\bar{x}_1(t) \in (-1, 1)$ for all $t \in [t_0, \alpha_1)$ and $\bar{x}_1(\alpha_1) = 1$. Thus, repeating the arguments in the proof of assertion S1 of Proposition 3.1, we find that $\bar{x}_1(t) = x_0 + a(t - t_0)$ for all $t \in [t_0, \alpha_1]$. As $\bar{x}_1(\alpha_1)=1$, we have $\alpha_1 = t_0 + \rho_2$. Consequently, the inequality $T - t_0 \geq (\alpha_1 - t_0) + (T - \alpha_2)$ implies that $T - t_0 \geq \rho_2 + 2a^{-1} = a^{-1}(3 - x_0)$. Our results in this subcase can be summarized as follows:

- $\bar{x}_1(.)$ is given by

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T], \end{cases}$$

provided that $T - t_0 = a^{-1}(3 - x_0)$.

- $\bar{x}_1(.)$ is given by

$$\bar{x}_1(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_0 + \rho_2] \\ 1, & t \in (t_0 + \rho_2, T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T], \end{cases}$$

provided that $T - t_0 > a^{-1}(3 - x_0)$.

**Case 4:** $\bar{x}_1(t_0) = \bar{x}_1(T) = -1$ *and* $\bar{x}_1(t) > -1$ *for all* $t \in (t_0, T)$.

Subcase 4a: $\mathcal{T}_1 = \emptyset$. Then, $\bar{x}_1(t) \in (-1, 1)$ for all $t \in (t_0, T)$. Thus, by assertion S3 of Proposition 3.1 one has $T - t_0 < 4a^{-1}$ and

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, 2^{-1}(t_0 + T)] \\ -1 - a(t - T), & t \in (2^{-1}(t_0 + T), T]. \end{cases}$$

Subcase 4b: $\mathcal{T}_1 \neq \emptyset$. Then, the numbers $\alpha_1$ and $\alpha_2$ exist and $t_0 < \alpha_1 \leq \alpha_2 < T$. It follows from statements S1 and S2 of Proposition 3.1 that $\alpha_1 - t_0 = T - \alpha_2 = 2a^{-1}$, $\bar{x}_1(t) = -1 + a(t - t_0)$ on the segment $[t_0, \alpha_1]$, and $\bar{x}_1(t) = 1 - a(t - \alpha_2)$ on the segment $[\alpha_2, T]$. Thus, $\alpha_1 = t_0 + 2a^{-1}$, $\alpha_2 = T - 2a^{-1}$, $\bar{x}_1(t) = -1 + a(t - t_0)$ for every $t \in [t_0, t_0+2a^{-1}]$, and $\bar{x}_1(t) = 1-a(t-T+2a^{-1})$ for all $t \in [T-2a^{-1}, T]$. Note that one must have $T - t_0 \geq 4a^{-1}$ in this subcase as $T - t_0 \geq (\alpha_1 - t_0) + (T - \alpha_2)$.

If $T - t_0 > 4a^{-1}$, i.e., $\alpha_1 < \alpha_2$, then by the result given in Subcase 3b we have $\bar{x}_1(t) = 1$ for all $t \in [t_0 + 2a^{-1}, T - 2a^{-1}]$.

Our results in this case can be summarized as follows:

• $\bar{x}_1(.)$ is given by

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, 2^{-1}(t_0 + T)] \\ -1 - a(t - T), & t \in (2^{-1}(t_0 + T), T], \end{cases}$$

provided that $T - t_0 \leq 4a^{-1}$.

• $\bar{x}_1(.)$ is given by

$$\bar{x}_1(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, t_0 + 2a^{-1}] \\ 1, & t \in (t_0 + 2a^{-1}, T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T], \end{cases}$$

provided that $T - t_0 > 4a^{-1}$.

Now we turn our attention back to the original problem $(FP_3)$. In the paragraph after formulating $(FP_{3a})$, it was observed that $(FP_3)$ has a global solution. So, the set of the local solutions of $(FP_3)$ is nonempty. Consequently, given the parameters tube $(\lambda, a, x_0, t_0, T)$, if we can show that for any $W^{1,1}$ local solution $(\bar{x}, \bar{u})$ of $(FP_3)$, $\bar{x}$ is described by a unique formula, then we can assert that the pair $(\bar{x}, \bar{u})$ is a unique local solution of $(FP_3)$, which is also a unique global solution of the problem.

Using the given constants $a, \lambda$ with $a > \lambda > 0$, we define

$$\rho = \frac{1}{\lambda} \ln \frac{a}{a - \lambda} > 0 \quad \text{and} \quad \bar{t} = T - \rho.$$

The number $\rho$ is a characteristic constant of $(FP_3)$. From the analysis given in the present section we can obtain a complete synthesis of optimal processes. Due to the complexity of the possible trajectories, we prefer to present our results in six separate theorems. The first one deals with the situation where $\rho \geq 2a^{-1}$, while the other five treat the situation where $\rho < 2a^{-1}$.

Based on the results obtained in Cases 1–4, we will provide a complete synthesis of the global solutions of $(FP_3)$. Recall that $\bar{x}_1(t)$ in $(FP_{3a})$ plays the role of $\bar{x}(t)$ in $(FP_3)$.

**Theorem 3.8.** *If $\rho \geq 2a^{-1}$, then problem $(FP_3)$ has a unique local solution $(\bar{x}, \bar{u})$, which is a unique global solution, where $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ for almost every $t \in [t_0, T]$ and $\bar{x}(t)$ can be described as follows:*

(a) *If $T - t_0 \leq a^{-1}(1 + x_0)$, then*

$$\bar{x}(t) = x_0 - a(t - t_0), \quad t \in [t_0, T].$$

(b) *If* $a^{-1}(1+x_0) < T - t_0 < a^{-1}(3 - x_0)$, *then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_\zeta] \\ -1 - a(t - T), & t \in (t_\zeta, T], \end{cases}$$

*with* $t_\zeta := 2^{-1}[T + t_0 - a^{-1}(1 + x_0)]$.

(c) *If* $T - t_0 \geq a^{-1}(3 - x_0)$, *then*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_0 + a^{-1}(1 - x_0)] \\ 1, & t \in (t_0 + a^{-1}(1 - x_0), T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T]. \end{cases}$$

*Proof.* Suppose that $\rho \geq 2a^{-1}$. Let $\rho_1, \rho_2$ be defined as in (3.28). Then, one has $\rho \geq \rho_1$, $2\rho - \rho_1 \geq \rho + \rho_2$, $\rho + \rho_2 \geq 2a^{-1} + \rho_2$, $2\rho \geq \rho + 2a^{-1}$, and $\rho + 2a^{-1} \geq 4a^{-1}$. Thus, the above Case 2 and the situations in Subcase 1b, Subcase 1c, Subcase 1d of Case 1 cannot happen. The situation in Subcase 1a happens when $T - t_0 < \rho_1$. Combining this with the results formulated in Case 3 and Case 4, we obtain the assertions of the theorem. □

If $\rho < 2a^{-1}$, then the locally optimal processes of $(FP_3)$ *depend greatly on the relative position of* $x_0$ in the segment $[-1, 1]$. In the forthcoming theorems, we distinguish five alternatives (one instance must occur, and any instance excludes others):

(i) $x_0 = -1$;
(ii) $x_0 > -1$ and $a^{-1}(1 + x_0) \leq \rho$;
(iii) $x_0 > -1$, $\rho < a^{-1}(1 + x_0)$, and $a^{-1}(1 + x_0) < \rho + a^{-1}(1 - x_0)$;
(iv) $x_0 > -1$, $\rho < a^{-1}(1 + x_0)$, and $a^{-1}(1 + x_0) = \rho + a^{-1}(1 - x_0)$;
(v) $x_0 > -1$, $\rho < a^{-1}(1 + x_0)$, and $\rho + a^{-1}(1 - x_0) < a^{-1}(1 + x_0)$.

It is worthy to stress that to describe the possibilities (i)–(v) we have used just the parameters $a, \lambda$, and $x_0$. In each one of the situations (i)–(v), the synthesis of the trajectories suspected for local minimizers of $(FP_3)$ is obtained by considering *the position of the number* $T - t_0 > 0$ *on the half-line* $[0, +\infty)$, which is divided into sections by the values $\rho$, $2\rho$, $\rho + 2a^{-1}$, $4a^{-1}$, and other constants appeared in (i)–(v).

**Theorem 3.9.** *If* $\rho < 2a^{-1}$ *and* $x_0 = -1$, *then any local solution of problem* $(FP_3)$ *must have the form* $(\bar{x}, \bar{u})$, *where* $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ *for a.e.* $t \in [t_0, T]$ *and* $\bar{x}(t)$ *is described as follows:*

(a) *If* $T - t_0 \leq 2\rho$, *then*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, 2^{-1}(t_0 + T)] \\ -1 - a(t - T), & t \in (2^{-1}(t_0 + T), T]. \end{cases} \tag{3.35}$$

*In this situation,* $(\bar{x}, \bar{u})$ *is a unique local solution of* $(FP_3)$, *which is also a unique global solution of the problem.*

(b) *If* $2\rho < T - t_0 < \rho + 2a^{-1}$*, then* $\bar{x}(t)$ *is given by either*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, \bar{t}] \\ -1 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T], \end{cases} \tag{3.36}$$

*or* (3.35)*.*

(c) *If* $T - t_0 = \rho + 2a^{-1}$*, then* $\bar{x}(t)$ *is given by either*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T], \end{cases} \tag{3.37}$$

*or* (3.35)*.*

(d) *If* $\rho + 2a^{-1} < T - t_0 \leq 4a^{-1}$*, then* $\bar{x}(t)$ *is given by either*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, t_0 + 2a^{-1}] \\ 1, & t \in (t_0 + 2a^{-1}, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T], \end{cases} \tag{3.38}$$

*or* (3.35)*.*

(e) *If* $T - t_0 > 4a^{-1}$*, then* $\bar{x}(t)$ *is given by either* (3.38) *or*

$$\bar{x}(t) = \begin{cases} -1 + a(t - t_0), & t \in [t_0, t_0 + 2a^{-1}] \\ 1, & t \in (t_0 + 2a^{-1}, T - 2a^{-1}] \\ -1 - a(t - T), & t \in (T - 2a^{-1}, T]. \end{cases}$$

*In situations* (b)–(e)*, the unique global solution of the problem* $(FP_3)$ *is given correspondingly by* (3.36)*,* (3.37)*,* (3.38)*, and* (3.38)*, where the last switching time of the optimal control function* $\bar{u}(\cdot)$ *is* $\bar{t}$*.*

*Proof.* Suppose that $\rho < 2a^{-1}$ and $x_0 = -1$. Since the above Case 1 and Case 3 are excluded, to obtain the assertions (a)–(e) we just need to combine the results formulated in Case 2 and Case 4. Here we have $2\rho < \rho + 2a^{-1} < 4a^{-1}$. In what follows, we will consider the position of the number $T - t_0$ on the half-line $[0, +\infty)$ marked by the values $2\rho$, $\rho + 2a^{-1}$, and $4a^{-1}$.

First, consider situation (a), where $T - t_0 \leq 2\rho$. Since all the three possibilities depicted in the conclusion part of Case 2 are excluded, Case 4 must occur. Since $T - t_0 \leq 2\rho < 4a^{-1}$, from the results in Case 4 it follows that $\bar{x}$ is described by (3.35). Thus, assertion (a) of the theorem is proved.

Now, consider situations (b)–(e). Here we have $T - t_0 > 2\rho$. So, our assertions follow from the results formulated in Case 2 and Case 4.

The fact that $(FP_3)$ has a global solution has been observed before. To prove the last assertion of the theorem about the uniqueness of the global solution of $(FP_3)$ in situations (b)–(e), we suppose on the contrary that the set of the global optimal processes of $(FP_3)$ is not a singleton. In each situation, by assertions(b)–(e) we know that the set of the local optimal processes contains no more than two elements. Hence, our supposition means that both distinctly feasible processes depicted therein in each situation are global ones. Observe that, in situations (b)–(c), we have $2^{-1}(t_0+T) < \bar{t}$ because $T - t_0 > 2\rho$ and $\bar{t} = T - \rho$. Moreover, the formulas given in each situation show that the state trajectories of the two global processes coincide on $[t_0, 2^{-1}(t_0+T)]$. Consider the problem denoted by $(FP_2)|_{[2^{-1}(t_0+T),T]}$, which is obtained from $(FP_3)$ by replacing the initial time $t_0$ and the bilateral state constraints $-1 \le x(t) \le 1$ respectively by $2^{-1}(t_0+T)$ and the unilateral state constraints $x(t) \le 1$. As $2^{-1}(t_0+T) < \bar{t}$, it follows from [34, Theorem 4.1] that $(FP_2)|_{[2^{-1}(t_0+T),T]}$ has a unique global process, in which the switching time of the optimal control is $\bar{t}$. Consequently, $(FP_3)$ has a unique global process, in which the switching time of the optimal control function is $\bar{t}$, a contradiction. Similarly, observing that $2^{-1}(t_0+T) \le t_0 + 2a^{-1} < \bar{t}$ in situation (d) (resp., $T - 2a^{-1} < \bar{t}$ in situation (e)) and using the previous arguments, we will arrive at a contradiction.

□

**Theorem 3.10.** *If $\rho < 2a^{-1}$, $x_0 > -1$, and $a^{-1}(1+x_0) \le \rho$, then any local solution of problem $(FP_3)$ must have the form $(\bar{x}, \bar{u})$, where $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ for almost everywhere $t \in [t_0, T]$ and $\bar{x}(t)$ can be described as follows:*

(a) *If $T - t_0 \le a^{-1}(1+x_0)$, then $\bar{x}(t)$ is given by*

$$\bar{x}(t) = x_0 - a(t - t_0), \quad t \in [t_0, T]. \tag{3.39}$$

*In this situation, $(\bar{x}, \bar{u})$ is a unique local solution of $(FP_3)$, which is also a unique global solution of the problem.*

(b) *If $a^{-1}(1+x_0) < T - t_0 \le 2\rho - a^{-1}(1+x_0)$, then $\bar{x}(t)$ is given by*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, t_\zeta] \\ -1 - a(t - T), & t \in (t_\zeta, T], \end{cases} \tag{3.40}$$

*with $t_\zeta := 2^{-1}[T + t_0 - a^{-1}(1+x_0)]$. In this situation, $(\bar{x}, \bar{u})$ is a unique local solution of $(FP_3)$, which is also a unique global solution of the problem.*

(c) *If $2\rho - a^{-1}(1+x_0) < T - t_0 < \rho + a^{-1}(1-x_0)$, then $\bar{x}(t)$ is given by either*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ x_0 - a(t + t_0 - 2\bar{t}), & t \in (\bar{t}, T], \end{cases} \tag{3.41}$$

*or (3.40).*

(d) *If $T - t_0 = \rho + a^{-1}(1-x_0)$, then $\bar{x}(t)$ is given by either*

$$\bar{x}(t) = \begin{cases} x_0 + a(t - t_0), & t \in [t_0, \bar{t}] \\ 1 - a(t - \bar{t}), & t \in (\bar{t}, T], \end{cases} \tag{3.42}$$

*or (3.40).*

(e) *If* $\rho + a^{-1}(1-x_0) < T - t_0 < a^{-1}(3-x_0)$, *then* $\bar{x}(t)$ *is given by either*

$$\bar{x}(t) = \begin{cases} x_0 + a(t-t_0), & t \in [t_0, t_0 + a^{-1}(1-x_0)) \\ 1, & t \in [t_0 + a^{-1}(1-x_0), \bar{t}) \\ 1 - a(t-\bar{t}), & t \in [\bar{t}, T], \end{cases} \tag{3.43}$$

*or* (3.40).

(f) *If* $T - t_0 = a^{-1}(3-x_0)$, *then* $\bar{x}(t)$ *is given by either* (3.43) *or*

$$\bar{x}(t) = \begin{cases} x_0 + a(t-t_0), & t \in [t_0, T - 2a^{-1}] \\ -1 - a(t-T), & t \in (T - 2a^{-1}, T]. \end{cases} \tag{3.44}$$

(g) *If* $T - t_0 > a^{-1}(3-x_0)$, *then* $\bar{x}(t)$ *is given by either* (3.43) *or*

$$\bar{x}(t) = \begin{cases} x_0 + a(t-t_0), & t \in [t_0, t_0 + a^{-1}(1-x_0)] \\ 1, & t \in (t_0 + a^{-1}(1-x_0), T - 2a^{-1}] \\ -1 - a(t-T), & t \in (T - 2a^{-1}, T]. \end{cases} \tag{3.45}$$

*In situations* (c)–(g), *the unique global solution of the problem* $(FP_3)$ *is the one in which the last switching time of the optimal control function* $\bar{u}(\cdot)$ *is* $\bar{t}$.

*Proof.* Suppose that $\rho < 2a^{-1}$, $x_0 > -1$, $a^{-1}(1+x_0) \le \rho$, and let $\rho_1, \rho_2$ be given by (3.28). Then, it is easy to verify that $\max\{\rho, 2\rho - \rho_1\} = 2\rho - \rho_1$ and

$$\rho_1 \le 2\rho - \rho_1 < \rho + \rho_2 < a^{-1}(3 - x_0).$$

The above Case 2 and Case 4 are excluded by the condition $x_0 > -1$. So, to obtain the desired assertions (a)–(g), it suffices to combine the results formulated in Case 1 and Case 3 with an observation on the position of the number $T - t_0$ on the half-line $[0, +\infty)$, which is marked by the values $\rho_1$, $2\rho - \rho_1$, $\rho + \rho_2$, and $a^{-1}(3 - x_0)$.

Recall that $(FP_3)$ has a global solution. To prove the last assertion of the theorem about the uniqueness of the global solution of $(FP_3)$ in situations (c)–(g), we suppose on the contrary that the set of the global optimal processes of $(FP_3)$ is not a singleton. In each situation, by assertions(c)–(g) we know that the set of the local optimal processes contains no more than two elements. Hence, our supposition means that both distinctly feasible processes depicted in each situation are global ones. In situations (c)–(g), as $T - t_0 > 2\rho - \rho_1 \ge \rho$ and $\bar{t} = T - \rho$, we have $t_0 < \bar{t}$. Thus, by [34, Theorem 4.1], the problem $(FP_2)$ obtained from $(FP_3)$ by replacing the constraint $-1 \le x(t) \le 1$ by $x(t) \le 1$ has a unique global solution, which is the one where the last switching time of the optimal control function $\bar{u}(\cdot)$ is $\bar{t}$. Consequently, $(FP_3)$ has a unique global solution, in which the switching time of the optimal control function is $\bar{t}$, a contradiction.

□

**Theorem 3.11.** *If* $\rho < 2a^{-1}$, $x_0 > -1$, $\rho < a^{-1}(1+x_0)$, *and*

$$a^{-1}(1+x_0) < \rho + a^{-1}(1-x_0), \tag{3.46}$$

*then any local solution of problem* $(FP_3)$ *must have the form* $(\bar{x}, \bar{u})$*, where* $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ *for a.e.* $t \in [t_0, T]$ *and* $\bar{x}(t)$ *is described as follows:*

(a) *If* $T - t_0 \le \rho$*, then* $\bar{x}(t)$ *is given by* (3.39)*. In this situation,* $(\bar{x}, \bar{u})$ *is a unique local solution of* $(FP_3)$*, which is also a unique global solution of the problem.*
(b) *If* $\rho < T - t_0 < a^{-1}(1 + x_0)$*, then* $\bar{x}(t)$ *is given by* (3.41)*. In this situation,* $(\bar{x}, \bar{u})$ *is a unique local solution of* $(FP_3)$*, which is also a unique global solution of the problem.*
(c) *If* $T - t_0 = a^{-1}(1 + x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.41)*, or* (3.39)*.*
(d) *If* $a^{-1}(1+x_0) < T - t_0 < \rho + a^{-1}(1 - x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.41)*, or* (3.40)*.*
(e) *If* $T - t_0 = \rho + a^{-1}(1 - x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.42)*, or* (3.40)*.*
(f) *If* $\rho + a^{-1}(1 - x_0) < T - t_0 < a^{-1}(3 - x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.43)*, or* (3.40)*.*
(g) *If* $T - t_0 = a^{-1}(3 - x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.43)*, or* (3.44)*.*
(h) *If* $T - t_0 > a^{-1}(3 - x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.43)*, or* (3.45)*.*

*In situations* (c)–(h)*, the unique global solution of* $(FP_3)$ *is the one in which the last switching time of the optimal control function* $\bar{u}(\cdot)$ *is* $\bar{t}$*.*

*Proof.* Suppose that $\rho < 2a^{-1}$, $x_0 > -1$, $\rho < a^{-1}(1+x_0)$, and the inequality (3.46) holds. Let $\rho_1, \rho_2$ be defined as in (3.28). By the assumptions made, $\max\{\rho, 2\rho - \rho_1\} = \rho$ and $\rho < \rho_1 < \rho + \rho_2 < a^{-1}(3 - x_0)$. Due to the condition $x_0 > -1$, the above Case 2 and Case 4 are excluded. So, to obtain the assertions in (a)–(h), it suffices to combine the results formulated in Case 1 and Case 3 and observe the position of the number $T - t_0$ on the half-line $[0, +\infty)$, which is marked by the values $\rho$, $\rho_1$, $\rho + \rho_2$, and $a^{-1}(3 - x_0)$.

The last assertion of the theorem about the uniqueness of the global solution of $(FP_3)$ in situations (c)–(h) is proved similarly as in the second part of the proof of Theorem 3.10. □

**Theorem 3.12.** *If* $\rho < 2a^{-1}$*,* $x_0 > -1$*,* $\rho < a^{-1}(1 + x_0)$*, and*

$$a^{-1}(1 + x_0) = \rho + a^{-1}(1 - x_0), \tag{3.47}$$

*then any local solution of problem* $(FP_3)$ *must have the form* $(\bar{x}, \bar{u})$*, where* $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ *for a.e.* $t \in [t_0, T]$ *and* $\bar{x}(t)$ *is described as follows:*

(a) *If* $T - t_0 \le \rho$*, then* $\bar{x}(t)$ *is given by* (3.39)*. In this situation,* $(\bar{x}, \bar{u})$ *is a unique local solution of* $(FP_3)$*, which is also a unique global solution of the problem.*
(b) *If* $\rho < T - t_0 < a^{-1}(1 + x_0)$*, then* $\bar{x}(t)$ *is given by* (3.41)*. In this situation,* $(\bar{x}, \bar{u})$ *is a unique local solution of* $(FP_3)$*, which is also a unique global solution of the problem.*
(c) *If* $T - t_0 = a^{-1}(1 + x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.42)*, or* (3.39)*.*
(d) *If* $a^{-1}(1 + x_0) < T - t_0 < a^{-1}(3 - x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.43)*, or* (3.40)*.*

(e) *If* $T - t_0 = a^{-1}(3 - x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.43) *or* (3.44)

(f) *If* $T - t_0 > a^{-1}(3 - x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.43) *or* (3.45).

*In situations* (c)–(f)*, the unique global solution of the problem* $(FP_3)$ *is the one in which the last switching time of the optimal control function* $\bar{u}(\cdot)$ *is* $\bar{t}$.

*Proof.* Suppose that $\rho < 2a^{-1}$, $x_0 > -1$, $\rho < a^{-1}(1 + x_0)$, and (3.47) holds. Let $\rho_1, \rho_2$ be as in (3.28). Then, we have $\max\{\rho, 2\rho - \rho_1\} = \rho$ and

$$\rho < \rho_1 = \rho + \rho_2 < a^{-1}(3 - x_0).$$

Since $x_0 > -1$, the above Case 2 and Case 4 are excluded. Hence, the desired assertions (a)–(g) follow from combining the results formulated in Case 1 and Case 3 with an observation on the position of the number $T - t_0$ on the half-line $[0, +\infty)$, which is marked by the values $\rho$, $\rho_1$, and $a^{-1}(3 - x_0)$.

The assertion on the uniqueness of the global solution of $(FP_3)$ in situations (c)–(g) is proved similarly as in the second part of the proof of Theorem 3.10. □

Finally, consider the situation where $\rho < 2a^{-1}$, $x_0 > -1$, $\rho < a^{-1}(1 + x_0)$, and $\rho + a^{-1}(1 - x_0) < a^{-1}(1 + x_0)$. As $x_0 \le 1$, we have $\rho \le \rho + a^{-1}(1 - x_0)$. Combining the latter with the inequality $\rho + a^{-1}(1 - x_0) < a^{-1}(1 + x_0)$ yields $\rho < a^{-1}(1 + x_0)$. So, the last inequality can be omitted in the formulation of the following theorem.

**Theorem 3.13.** *If* $\rho < 2a^{-1}$, $x_0 > -1$*, and* $\rho + a^{-1}(1 - x_0) < a^{-1}(1 + x_0)$*, then any local solution of problem* $(FP_3)$ *must have the form* $(\bar{x}, \bar{u})$*, where* $\bar{u}(t) = -a^{-1}\dot{\bar{x}}(t)$ *for a.e.* $t \in [t_0, T]$ *and* $\bar{x}(t)$ *is described as follows:*

(a) *If* $T - t_0 \le \rho$*, then* $\bar{x}(t)$ *is given by* (3.39).

(b) *If* $\rho < T - t_0 < \rho + a^{-1}(1 - x_0)$*, then* $\bar{x}(t)$ *is given by* (3.41).

(c) *If* $T - t_0 = \rho + a^{-1}(1 - x_0)$*, then* $\bar{x}(t)$ *is given by* (3.42).

(d) *If* $\rho + a^{-1}(1 - x_0) < T - t_0 < a^{-1}(1 + x_0)$*, then* $\bar{x}(t)$ *is given by* (3.43).

(e) *If* $T - t_0 = a^{-1}(1 + x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.43) *or* (3.39).

(f) *If* $a^{-1}(1 + x_0) < T - t_0 < a^{-1}(3 - x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.43) *or* (3.40).

(h) *If* $T - t_0 = a^{-1}(3 - x_0)$*, then* $\bar{x}(t)$ *is given by either* (3.43) *or* (3.44).

(g) *If* $T - t_0 > a^{-1}(3 - x_0)$*, then* $\bar{x}(t)$ *is given by* (3.43) *or* (3.45).

*In situations* (a)–(d)*,* $(\bar{x}, \bar{u})$ *is a unique local solution of* $(FP_3)$*, which is also a unique global solution of the problem. In situations* (e)–(g)*, the unique global solution of* $(FP_3)$ *is the one in which the last switching time of the optimal control function* $\bar{u}(\cdot)$ *is* $\bar{t}$.

*Proof.* Suppose that $\rho < 2a^{-1}$, $x_0 > -1$, and $\rho + a^{-1}(1 - x_0) < a^{-1}(1 + x_0)$. With $\rho_1, \rho_2$ being defined (3.28), one has $\max\{\rho, 2\rho - \rho_1\} = \rho$ and

$$\rho \le \rho + \rho_2 < \rho_1 \le a^{-1}(3 - x_0).$$

Since $x_0 > -1$, to obtain the desired assertions (a)–(g) we just need to combine the results formulated in Case 1 and Case 3 with an observation on the position of the number $T - t_0$ on the half-line $[0, +\infty)$, which is marked by the values $\rho$, $\rho + \rho_2$, $\rho_1$, and $a^{-1}(3 - x_0)$.

The uniqueness of the global solution of $(FP_3)$ in situations (a)–(d) is obvious. The claim on the uniqueness of the global solution of $(FP_3)$ in situations (e)–(g) is proved similarly as in the final part of the proof of Theorem 3.10.

□

## 4. On the Degeneracy Phenomenon of the Maximum Principle

Regarding maximum principles for optimal control problems with state constraints, there is a so-called *degeneracy phenomenon*, which has been widely discussed in the literature (see, e.g., the books [24, 38], the papers [39, 40, 41, 42, 43], and the references therein). Regarding a $W^{1,1}$ local minimizer $(\bar{x}, \bar{u})$ of the Mayer problem $\mathcal{M}$, as mentioned in [24, Remark (b), pp. 330–331], the maximum principle formulated in Theorem 2.5 is of interest only when the state constraint is *nondegenerate*, in the sense that $0 \notin \partial_x^> h(t, \bar{x}(t))$ *for every* $t$ *satisfying* $h(t, \bar{x}(t)) = 0$. Because when the state constraint is *degenerate*, i.e.,

$$0 \in \partial_x^> h(t', \bar{x}(t')) \quad \text{and} \quad h(t', \bar{x}(t')) = 0$$

*for some time* $t'$, the necessary conditions (i)–(iv) automatically hold with the choice of multipliers $\mu = \delta_{\{t'\}}$ (the unit measure concentrated on $\{t'\}$), $p(t) \equiv 0$, $\nu : [t_0, T] \to \mathbb{R}^n$ is a Borel measurable function with the property that $\nu(t') = 0$, and $\gamma = 0$. In other words, the maximum principle does not convey any useful information about the minimizer. In such situations, special treatments including extra conditions to eliminate degeneracy as in [24, Theorem 10.6.1 and Corollary 10.6.2] are needed. Luckily, though the state constraint is active, our problem $(FP_{3a})$ is nondegenerate. This is because formula (3.7) shows that, for any feasible process $(\bar{x}, \bar{u})$ of $(FP_{3a})$, one has $0 \notin \partial_x^> h(t, \bar{x}(t))$ for every $t$ satisfying $h(t, \bar{x}(t)) = 0$. One referee of this paper has observed that the degeneracy can also be avoided by applying the maximum principle in the book of Ioffe and Tihomirov [3] directly to the original optimal control problem of the Lagrange type $(FP_3)$.

## 5. Conclusions

We have analyzed a maximum principle for finite horizon optimal control problems with state constraints via one parametric example, which resembles the optimal economic growth problems in macroeconomics. This example is an optimal control problem with bilateral state constraints of the Lagrange type and has five parameters. We have proved that the optimal control problem can have at most two local optimal processes. Moreover, we have obtained explicit descriptions of the unique global optimal process with respect to all possible configurations of the five parameters.

The obtained results allow us to have a deep understanding of the maximum principle in question.

It seems to us that optimal economic growth models can be studied by advanced tools from functional analysis and optimal control theory via the approach adopted in this paper.

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## References

[1] Pontryagin, L.S., Boltyanskii, V.G., Gamkrelidze, R.V., Mishchenko, E.F. (1962). *The Mathematical Theory of Optimal Processes.* Interscience Publishers, New York–London.

[2] Hartl, R.F., Sethi, S.P., Vickson, R.G. (1995). A survey of the maximum principles for optimal control problems with state constraints. SIAM Rev. 37: 181–218.

[3] Ioffe, A.D., Tihomirov, V.M. (1979). *Theory of Extremal Problems.* North-Holland Publishing Co., Amsterdam-New York.

[4] Phu, H.X. (1985). Lineare Steuerungsprobleme mit engen Zustandsbereichen. (German) [Linear control problems with narrow state domains]. Optimization 16: 273–284.

[5] Phu, H.X. (1986). Einige notwendige Optimalitätsbedingungen für einfache reguläre Aufgaben der optimalen Steuerung. (German) [Some necessary optimality conditions for simple regular problems of optimal control]. Z. Anal. Anwend. 5: 465–475.

[6] Phu, H.X. (1987). Some necessary conditions for optimality for a class of optimal control problems which are linear in the control variable. Systems Control Lett. 8, 261–271.

[7] Phu, H.X. (1987). A method for solving a class of optimal control problems which are linear in the control variable. Systems Control Lett. 8: 273–280.

[8] Phu, H.X. (1987). Zur Lösung einer regulären Aufgabenklasse der optimalen Steuerung im Großen mittels Orientierungskurven. (German) [On the solution of regular problems of optimal control in the large by the means of orientor curves]. Optimization 18: 65–81.

[9] Phu, H.X. (1991). Method of orienting curves for solving optimal control problems with state constraints. Numer. Funct. Anal. Optim. 12: 173–211.

[10] Phu, H.X. (1987). On optimal control of a hydroelectric power plant. Systems Control Lett. 8: 281–288.

[11] Phu, H.X. (1987). Zur Lösung eines Zermeloschen Navigationsproblems. (German) [On the solution of a navigation problem of Zermelo]. Optimization 18: 225–236.

[12] Phu, H.X. (1987). Ein konstruktives Lösungsverfahren für das Problem des Inpolygons kleinsten Umfangs von J. Steiner. (German) [A constructive method of solution for J. Steiner's problem of the inpolygon with minimal circumference]. Optimization 18: 349–359.

[13] Phu, H.X. (1988). Optimal control of a hydroelectric power plant with unregulated spilling water. Systems Control Lett. 10: 131–139.

[14] Phu, H.X. (1988). Investigation of some inventory problems with linear replenishment cost by the method of region analysis. In Optimal Control Theory and Economic Analysis 3, Editor: G. Feichtinger, North–Holland, Amsterdam, 195–221.

[15] Phu, H.X. (1988). Solution of some high-dimensional linear optimal control problems by the method of region analysis. Internat. J. Control 47: 493–518.

[16] Phu, H.X. (1989). A solution method for regular optimal control problems with state constraints. J. Optim. Theory Appl. 62: 489–513.

[17] Dinh, N. and Phu, H.X. (1992). Solving a class of regular optimal control problems with state constraints by the method of orienting curves. Optimization 25: 231–247.

[18] Dinh, N. and Phu, H.X. (1992). Solving a class of optimal control problems which are linear in the control variable by the method of orienting curves. Acta Math. Vietnam. 17: 115–134.

[19] Phu, H.X., Dinh, N. (1995). Some remarks on the method of orienting curves. Numer. Funct. Anal. Optim. 16: 755–763.

[20] Phu, H.X. (1988). On a linear optimal control problem of a system with circuit-free graph structure. Internat. J. Control 48: 1867–1882.

[21] Phu, H.X. (1992). Investigation of a macroeconomic model by the method of region analysis. J. Optim. Theory Appl. 72: 319–332.

[22] Dinh, N. and Phu, H.X. (1992). The method of orienting curves and its application to an optimal control problem of hydroelectric power plants. Vietnam J. Math. 20: 40–53.

[23] Phu, H.X., Bock, H.G., and Schlöder, J. (1997). The method of orienting curves and its application for manipulator trajectory planning. Numer. Funct. Anal. Optim. 18: 213–225.

[24] Vinter, R (2000). *Optimal Control*. Birkhäuser, Boston.

[25] Basco, V., Cannarsa, P., Frankowska, H. (2018). Necessary conditions for infinite horizon optimal control problems with state constraints. Math. Control Relat. Fields 8: 535–555.

[26] Takayama, A. (1974). *Mathematical Economics*. The Dryden Press, Hinsdale, Illinois.

[27] Huong, V.T., Yao, J.-C., Yen, N.D. (2020). Optimal processes in a parametric optimal economic growth model. Taiwanese J. Math. 24: 1283–1306.

[28] Huong, V.T. (2020). Optimal economic growth problems with high values of total factor productivity. Appl. Anal., First Online, https://doi.org/10.1080/00036811.2020.1779231.

[29] Huong, V.T., Yao, J.-C., Yen, N.D. (2021). Optimal economic growth models with nonlinear utility functions. J. Optim. Theory Appl. 188: 571–596.

[30] Cesari, L. (1983). *Optimization Theory and Applications*. Springer-Verlag, New York.

[31] Mordukhovich, B.S. (2006). *Variational Analysis and Generalized Differentiation* (Vol. I. Basic Theory, Vol. II. Applications). Springer-Verlag, Berlin.

[32] Mordukhovich, B.S. (2018). *Variational Analysis and Applications*. Springer, Cham.

[33] Clarke, F. (2013). *Functional Analysis, Calculus of Variations and Optimal Control*. Springer, London.

[34] Huong, V.T., Yao, J.-C., Yen, N.D. (2020). Analyzing a maximum principle for finite horizon state constrained problems via parametric examples. Part 1: Problems with unilateral state constraints. J. Nonlinear Convex Anal. 21: 157–182.

[35] Kolmogorov, A.N., Fomin, S.V. (1970). *Introductory Real Analysis.* Translated from the Russian and edited by R. A. Silverman, Dovers Publications, Inc., New York.

[36] Luenberger, D.G. (1969). *Optimization by Vector Space Methods.* John Wiley & Sons, New York.

[37] Royden, H.L., Fitzpatrick, P.M. (2010). *Real Analysis.* China Machine Press.

[38] Arutyunov, A.V. (2000). *Optimality Conditions. Abnormal and Degenerate Problems.* Translated from the Russian by S. A. Vakhrameev, Kluwer Academic Publishers, Dordrecht.

[39] Ferreira, M.M.A., Vinter, R.B. (1994). When is the maximum principle for state constrained problems nondegenerate? J. Math. Anal. Appl. 187: 438–467.

[40] Arutyunov, A.V., Aseev, S.M. (1997). Investigation of the degeneracy phenomenon of the maximum principle for optimal control problems with state constraints. SIAM J. Control Optim. 35: 930–952.

[41] Frankowska, H. (2009). Normality of the maximum principle for absolutely continuous solutions to Bolza problems under state constraints. Control Cybernet. 38: 1327–1340.

[42] Fontes, F.A.C.C., Frankowska, H. (2015). Normality and nondegeneracy for optimal control problems with state constraints. J. Optim. Theory Appl. 166: 115–136.

[43] Karamzin, D.; Pereira, F.L. (2019). On a few questions regarding the study of state-constrained problems in optimal control. J. Optim. Theory Appl. 180: 235–255.

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