ALGEBRAIC INVARIANTS OF PROJECTIONS OF VARIETIES AND PARTIAL ELIMINATION IDEALS

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Abstract. In this paper, we are interested in the properties of inner and outer projections with a view toward the Eisenbud-Goto regularity conjecture or the characterization of varieties satisfying certain extremal conditions. For example, if X is a quadratic scheme, the depth and regularity X and those of its inner projection from a smooth point are equal. In general, the above equalities do not hold for non-quadratic schemes. Therefore it is natural to investigate the algebraic invariants (e.g., depth and regularity) of X and its projected image in general.

We develop a framework which provides partial answers and explains their relations using the partial elimination ideal theory. Our main theorems recover several preceding results in the literature. We also give some interesting examples and applications to illustrate our results.

Contents

2010 *Mathematics Subject Classification.* 14N05, 13D02, 13D05.

Sijong Kwak was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (Ministry of Science and ICT) (No. 2017R1E1A1A03070765). Hop D. Nguyen is partially supported by Project CT 0000.03/19-21 of Vietnam Academy of Science and Technology, and by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2019.313.

1. Introduction

Let k be a field, and $X \subseteq \mathbb{P}^N_k$ a non-degenerate closed subscheme of the projective N-space. Let $I_X \subseteq R = k[x_0, x_1, \ldots, x_N]$ be the saturated homogeneous ideal which defines X. Various geometric properties of X are reflected in the minimal graded free resolution of its homogeneous coordinate ring R/I_X. Especially interesting is the case where X is defined by quadratic equations. In this case, roughly speaking the linear syzygies, in particular the $N_{2,p}$ property of R/I_X measures how far X is from being a variety of minimal degree. See, for example, [4, 10, 11, 14, 15].

Several papers employ the inner and outer projections of X to study its algebraic and geometric properties. Recall that if $q \in \mathbb{P}^N$ is a closed point, then the projection of X from q corresponds to the elimination of its defining ideal I_X with respect to some linear form. Let X_q be the Zariski closure of the image of $X \setminus q$ under the projection map $\pi_q : X \to \mathbb{P}^{N-1}$. If we assume $q = (1 : 0 : \cdots : 0) \in \mathbb{P}^N$, then the saturated defining ideal of X_q is $I_X \cap k[x_1, \ldots, x_N]$. We define the Castelnuovo–Mumford regularity of X, denoted by $\operatorname{reg}(X)$, to be $\operatorname{reg}(I_X)$ (namely $1+\operatorname{reg}(R/I_X)$), and its arithmetic depth, denoted by $\operatorname{depth}(X)$, to be $\operatorname{depth}(R/I_X)$.

Several authors studied the properties of inner and outer projections with a view toward the Eisenbud-Goto conjecture or the classification of schemes satisfying certain extremal conditions; see, e.g., [2, 14, 15, 19]. If X is a quadratic scheme, there is an interesting relationship between the depth and regularity of X and those of its inner projection from a smooth point.

Theorem 1.1 (Han-Kwak [14, Theorem 4.1(b)],[15, Theorem 2.9(c)])**.** *Let X be a quadratic scheme of codimension at least 2. Let $q \in X$ be a smooth inner point. Then there are equalities* $\operatorname{reg}(X) = \operatorname{reg}(X_q)$ *and* $\operatorname{depth}(X) = \operatorname{depth}(X_q)$.

In general, the above equalities do not hold for non-quadratic schemes, and they may also fail to hold if X is quadratic but $q \in X$ is not a smooth point. Therefore it is natural to ask

Question 1.2. How closely are algebraic invariants (like depth and regularity) of X and its projected image X_q, related in general?

For example, the depths of inner projections of toric varieties are related to the conjectures due to Bøgvad and Oda on projective normality and quadratic generation of smooth projective toric varieties; see [21, Section 13]. Motivated by the above question, we develop a framework which provides partial answers. Our main tools are the partial elimination ideals introduced by M. Green [13], elimination mapping cone sequence [1], and standard "Depth Lemma" arguments.

Let I be any non-zero homogeneous proper ideal of $R = k[x_0, x_1, \ldots, x_N]$. The partial elimination ideals $K_i(I)$, introduced by Green (see Section 2), are ideals of $S = k[x_1, \ldots, x_N]$ satisfying

$$K_0(I) = I \cap S \subseteq K_1(I) \subseteq K_2(I) \subseteq \cdots \subseteq S.$$

Note that $J = K_0(I)$ is the elimination of I with respect to x_0. Since S is noetherian, the above chain eventually stabilizes, and we denote by s the first point where this stabilization occurs. When $I = I_X$, then $q = (1 : 0 : \cdots : 0)$ does not belong to X (outer projection case) if and only if $K_s(I_X) = S$. When $I = I_X$ and $q = (1 : 0 : \cdots : 0)$ belongs to X, then $K_s(I_X)$ defines scheme-theoretically the (projectivized) tangent cone of X at q. Our main results provide the following generalization of Theorem 1.1.

Theorem 1.3 (Theorem 6.1)**.** *With the above notation, assume that $s = 1$, $S/K_1(I)$ is a Cohen–Macaulay ring, and $I_{\le c} = 0$, where $c = \operatorname{reg} K_1(I)$. Then there are equalities*

$$\operatorname{depth}(R/I) = \operatorname{depth}(S/J),$$
$$\operatorname{reg} I = \operatorname{reg} J.$$

Theorem 1.1 is an immediate corollary of Theorem 1.3: Choose $I = I_X$, and assume that $q = (1 : 0 : \cdots : 0) \in X$ (by change of coordinates). The fact that q is a smooth point of X implies that $K_1(I)$ is generated by linear forms, and $\operatorname{reg} K_1(I) = 1$. Now the last hypothesis $I_{\le 1} = 0$ of Theorem 1.3 is obvious as X is defined by quadrics.

Theorem 1.3 is a consequence of general results on partial elimination ideals, which apply to *both* outer and inner projections. Our first main result in this direction is a statement about regularity of partial elimination ideals, generalizing previous work of Jones [17], and Ahn-Kwak [2]. We do not require I to be a radical ideal as in the cited papers.

Theorem 1.4 (Theorems 4.1, 4.6)**.** *With the above notation, there are inequalities*

$$\operatorname{reg}_R I \le \max_{1\le i\le s} \{\operatorname{reg}_S J, \operatorname{reg}_S K_i(I) + i\},$$
$$\operatorname{reg}_S J \le \max_{1\le i\le s-1} \{\operatorname{reg}_R I, \operatorname{reg}_S K_i(I) + i + 1, \operatorname{reg}_S K_s(I) + s - 1\}.$$

The two bounds provided above are sharp in general, and strict inequality may happen in both cases. For example, McCullough and Peeva [20] recently constructed several counterexamples to the Eisenbud-Goto conjecture. Applying Theorem 1.4 to their Example 4.7 in [20], we get

$$\operatorname{reg}_S J < \operatorname{reg}_R I < \max_{1\le i\le s} \{\operatorname{reg}_S J, \operatorname{reg}_S K_i(I) + i\};$$

see Example 4.5 for details.

Our second main result is a statement about depth of partial elimination ideals.

Theorem 1.5 (Theorems 3.1, 3.4)**.** *With the above notation, there is an inequality*

$$\operatorname{depth}_R \frac{R}{I} \ge \min_{1\le i\le s-1} \left\{\operatorname{depth}_S \frac{S}{J}, \operatorname{depth}_S \frac{S}{K_i(I)}, \operatorname{depth}_S \frac{S}{K_s(I)} + 1\right\}.$$

If moreover J is generated in degree at least $1 + \operatorname{reg}_S K_s(I)$, then there is another inequality

$$\operatorname{depth}_S \frac{S}{J} \ge \min_{1\le i\le s} \left\{\operatorname{depth}_R \frac{R}{I}, \operatorname{depth}_S \frac{S}{K_i(I)} + 1\right\}.$$

The hypothesis that J is generated in degree at least $1 + \operatorname{reg}_S K_s(I)$ in the last result looks rather strange, but it is indispensable (see Example 3.11). The hypothesis is automatically satisfied if $I = I_X$ and $q = (1 : 0 : \cdots : 0) \in X$ is a smooth point. The proofs of all four inequalities in Theorems 1.4 and 1.5 are similar, and are inspired by [14]. Among them, the proof of the second inequality in Theorem 1.5, concerning the lower bound for $\operatorname{depth}_S(S/J)$, is the most difficult one (see Theorem 3.4).

An interpretation of Theorems 1.4 and 1.5 is that the complexity of the minimal free resolution of I is governed by the complexity/singularity of its partial elimination ideals. This relationship is not surprising since the partial elimination ideals contain information about the fibers of the projection from $q = (1 : 0 : \cdots : 0)$.

For inner projections of irreducible varieties, we have even more precise statements. Let $X \subseteq \mathbb{P}^N$ be a non-degenerate irreducible closed subscheme with defining ideal $I = I_X$, and assume that $q = (1 : 0 : \cdots : 0) \in X$. Denote by s' the degree of the finite map $\pi_q : X \dashrightarrow X_q$. Interestingly, $s' \leq s$, and the partial elimination ideals satisfy the relations $K_0(I) = K_1(I) = \cdots = K_{s'-1}(I) \subsetneq K_{s'}(I)$ (see Lemma 2.9). Theorems 1.4 and 1.5 can be improved as follows.

Theorem 1.6 (Theorem 5.1)**.** *With notation as above, consider the inequalities*

$$\operatorname{depth}_R \frac{R}{I} \geq \min_{s' \leq i \leq s-1} \left\{ \operatorname{depth}_S \frac{S}{J}, \operatorname{depth}_S \frac{S}{K_i(I)}, \operatorname{depth}_S \frac{S}{K_s(I)} + 1 \right\}, \tag{1.1}$$

$$\operatorname{depth}_S \frac{S}{J} \geq \min_{s' \leq i \leq s} \left\{ \operatorname{depth}_R \frac{R}{I}, \operatorname{depth}_S \frac{S}{K_i(I)} + 1 \right\}, \tag{1.2}$$

$$\operatorname{reg}_R I \leq \max_{s' \leq i \leq s} \{ \operatorname{reg}_S J + s' - 1, \operatorname{reg}_S K_i(I) + i \}, \tag{1.3}$$

$$\operatorname{reg}_S J \leq \max_{s' \leq i \leq s-1} \{ \operatorname{reg}_R I, \operatorname{reg}_S K_i(I) + i + 1, \operatorname{reg}_S K_s(I) + s - 1 \} - s' + 1. \tag{1.4}$$

Then (1.1)*,* (1.3)*, and* (1.4) *hold unconditionally, while* (1.2) *holds if* $J_{\leq \operatorname{reg} K_s} = 0$.

Theorem 1.6 yields the following strengthening of Theorem 1.3 in the irreducible case. Below, we denote $\operatorname{vert}(X)$ the set of vertices of the scheme X, i.e. the closed points $q \in X$ such that X is a cone over q.

Theorem 1.7 (Theorem 6.2)**.** *Let $X \subseteq \mathbb{P}^N$ be a non-degenerate irreducible closed subscheme with homogeneous defining ideal $I = I_X$. Assume that $q = (1 : 0 : \cdots : 0) \in X \setminus \operatorname{vert}(X)$. Keep using the notation introduced above. Assume furthermore that the following conditions are satisfied:*

1. $\deg \pi_q = s$,
2. $S/K_s(I)$ *is Cohen-Macaulay,*
3. $I_{\leq c+\max\{0,s-2\}} = 0$ *where* $c = \operatorname{reg} K_s(I)$.

Then there are equalities

$$\operatorname{depth}_R(R/I) = \operatorname{depth}_S(S/J),$$
$$\operatorname{reg}_R I = \operatorname{reg}_S J + s - 1.$$

Our paper is organized as follows. In Section 2, we set up the notation and provide some background. In Sections 3 and 4, we prove our main results about depth and regularity of arbitrary projections, inner or outer. In Section 5, we prove Theorem 1.6, which is a strengthening of our main results 1.4 – 1.5 for inner projections of irreducible closed subschemes of $\mathbb{P}^N$. In Section 6, using our main results about depth and regularity of partial elimination ideals, we quickly recover many results in the literature. We also provide other interesting examples where our results can be applied. For completeness, we give a self-contained proof of the geometric interpretation of partial elimination ideals in the inner projection case in Appendix A. This appendix can be skipped by the experts.

2. Notation and preliminaries

2.1. **Depth and regularity.** Let R be a standard graded k-algebra, with the graded maximal ideal $\mathfrak{m}$. We usually write k also for the residue field $R/\mathfrak{m}$. For simplicity, for an R-module M, we will denote $\operatorname{Tor}_i^R(M, R/\mathfrak{m})_{i+j}$ and $\operatorname{Tor}_i^R(M, R/\mathfrak{m})$

by $T^R_{i,j}(M)$ and $T^R_i(M)$, respectively. Sometimes, we will also denote $T^R_{i,j}(M)$ and $T^R_i(M)$ by $T_{i,j}(M)$ and $T_i(M)$, respectively, if there is no danger of confusion.

Let $H^i_{\mathfrak{m}}(M)$ denote the i-th local cohomology of M with support at $\mathfrak{m}$. The depth of M and the (Castelnuovo-Mumford) regularity of M are defined respectively by

$$\operatorname{depth} M := \min\{i : H^i_{\mathfrak{m}}(M) \neq 0\},$$
$$\operatorname{reg} M := \sup\{i + j : H^i_{\mathfrak{m}}(M)_j \neq 0\}.$$

If $\operatorname{pd}_R M$, the projective dimension of M over R, is finite, by the Auslander-Buchsbaum formula, we also have $\operatorname{depth} M = \operatorname{depth} R - \operatorname{pd}_R M$.

When R is a standard graded polynomial ring over k, the regularity of M can also be computed from the minimal free resolution of M. Namely

$$\operatorname{reg} M = \max\{j : T^R_{i,j}(M) \neq 0 \quad \text{for some } i\}.$$

If X is a closed subscheme of the projective space $\mathbb{P}^N$, with saturated defining ideal $I_X \subseteq R = k[x_0, \ldots, x_N]$, we define the (arithmetic) depth and regularity of X as follows:

$$\operatorname{depth}(X) := \operatorname{depth}(R/I_X),$$
$$\operatorname{reg}(X) := \operatorname{reg}(R/I_X) + 1 = \operatorname{reg} I_X.$$

The reader may consult [9] for more information on depth, regularity, and other standard knowledge of commutative algebra.

2.2. **Partial elimination ideals.** Let X be a closed subscheme of the projective space $\mathbb{P}^N$, whose saturated defining ideal is $I_X \subseteq k[x_0, \ldots, x_N]$. As in [22, Example 1.27], let E be a t-dimension linear subspace of $\mathbb{P}^N$ defined by $N - t$ linearly independent linear equations $L_1 = \cdots = L_{N-t} = 0$. The *projection with center* E is the rational map

$$\pi_E : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-t-1}, x \mapsto (L_1(x) : \cdots : L_{N-t}(x)).$$

In the special case where E is the single point $q = (1 : 0 : \cdots : 0)$, denote the projection with center at q by π_q. It is a map from $\mathbb{P}^N \setminus q$ to $\mathbb{P}^{N-1} = \{x_0 = 0\}$. Let $X_q = \overline{\pi_q(X \setminus q)}$. Then the ideal defining X_q is $J = I_X \cap k[x_1, \ldots, x_N]$, the elimination ideal of I_X with respect to x_0.

We will use the following notation throughout.

Notation 2.1. Let $N \geq 0$ be an integer. Let $R = k[x_0, x_1, \ldots, x_N]$ be a standard graded polynomial ring and $I \subseteq R$ a non-zero proper homogeneous ideal. For each polynomial $f \in R$, denote by $d_0(f)$ the degree of f as a polynomial in x_0.

(1) Let $S = k[x_1, \ldots, x_N]$ and $J = I \cap S$.
(2) For any $f \in R$, we can write $f = x_0^{d_0(f)}\overline{f} + g$, where $\overline{f} \in S$ and $g \in R$ such that $d_0(g) < d_0(f)$. Such an expression is unique.
(3) For each $i \geq 0$, denote

$$\widetilde{K}_i(I) = \bigoplus_{m \geq 0} \{f \in I_m : d_0(f) \leq i\}.$$

Note that $\widetilde{K}_i(I)$ is an S-module but **not** an R-module. There is an injection of S-modules

$$\widetilde{K}_i(I) \hookrightarrow S \oplus Sx_0 \oplus \cdots \oplus Sx_0^i,$$

so $\widetilde{K}_i(I)$ is a noetherian S-module. Observe that $\widetilde{K}_0(I) = I \cap S = J$, and there is a chain of S-modules

$$J = \widetilde{K}_0(I) \subseteq \widetilde{K}_1(I) \subseteq \widetilde{K}_2(I) \subseteq \cdots .$$

This chain does **not** stabilize.

(4) For each $i \geq 0$, define the i-th partial elimination $K_i(I)$ of I as follows. For each $f \in R$, there is a unique expression

$$f = \sum_{j \geq 0} \overline{f}_j x_0^j,$$

where $\overline{f}_j \in S$ for all j, and almost all of $\overline{f}_j$ are zero.

Let $\theta_i : R \to S$ be the S-linear map $f \mapsto \overline{f}_i$. Then we define $K_i(I) := \theta_i(\widetilde{K}_i(I)) \subseteq S$. Observe that $K_i(I)$ is an ideal of S, $K_0(I) = J$, and there is a chain

$$J = K_0(I) \subseteq K_1(I) \subseteq K_2(I) \subseteq \cdots .$$

(5) Denote by s the minimal index from which the chain $J = K_0(I) \subseteq K_1(I) \subseteq K_2(I) \subseteq \cdots$ stabilizes, namely

$$s = \min\{t : K_i(I) = K_t(I) \text{ for all } i \geq t+1\}.$$

Since S is noetherian, s is a finite number.

(6) For simplicity, we will usually denote $\widetilde{K}_i(I)$ and $K_i(I)$ simply by $\widetilde{K}_i$ and K_i, respectively.

A standard result about the projectivized tangent cone is

Lemma 2.2 (Cf. [16, Lecture 20]). *Let X be a reduced closed subscheme of the projective space $\mathbb{P}^N$, whose saturated defining ideal is $I = I_X \subseteq k[x_0, \ldots, x_N]$. Assume that $q = (1 : 0 : \cdots : 0) \in X$. Using Notation 2.1, there is an equality* $\dim V(K_s) = \dim X - 1$.

A geometric interpretation of the partial elimination ideals is given by the following result, which subsumes Green's [13, Proposition 6.2] (strictly speaking, Green's result applies only to outer projections).

Lemma 2.3 (Han-Kwak [15, Proposition 2.5]). *For each $i \geq 0$, we have an equality of sets*

$$V(K_i) = \left\{z \in \mathbb{P}^{N-1} : \text{length}((X \setminus q) \cap \langle q, z \rangle) \geq i+1\right\} \bigcup V(K_\infty).$$

Here $\mathbb{P}^{N-1}$ is identified with $V(x_0) \subseteq \mathbb{P}^N$, and $\langle q, z \rangle$ denotes the line passing through $q = (1 : 0 : \cdots : 0)$ and z. The length of the finite scheme $(X \setminus q) \cap \langle q, z \rangle$ denotes its number of points counted with multiplicity.

Since we are not aware of any published proof for this result, we give one in Appendix A, following an idea of Kurmann [18].

The following lemma generalizes [14, Proposition 2.5(a)] and admits a similar proof. We give a detailed argument for completeness.

Lemma 2.4 (Bound for the stabilization index). *Use Notation 2.1. Let $f_1, \ldots, f_r$ be a system of homogeneous generators of I. Then the stabilization index s of the chain of partial elimination ideals satisfies the inequality $s \leq \max\{d_0(f_1), \ldots, d_0(f_r)\}$.*

Proof. Denote $s_0 = \max\{d_0(f_1), \dots, d_0(f_r)\}$. Any element $f \in R$ can be written uniquely as $f = x_0^{d_0(f)}\overline{f} + g$, where $\overline{f} \in S, g \in R$ and $d_0(g) < d_0(f)$. We prove by induction on $d_0(f)$ that $\overline{f} \in K_{s_0} = K_{s_0}(I)$.

If $d_0(f) \le s_0$ then we are done since $K_i \subseteq K_{i+1}$. Assume that $d_0(f) > s_0$. Since $f_1, \dots, f_r$ generate I, we can write

$$f = a_1 f_1 + \cdots + a_r f_r$$

where $a_1, \dots, a_r$ are homogeneous polynomials in R. Write $a_i = x_0 a_i' + b_i$, where $a_i' \in R, b_i \in S$. Denote $f^* = \sum_{i=1}^r a_i' f_i$, $h = \sum_{i=1}^r b_i f_i$. Then

$$f = x_0 \sum_{i=1}^r a_i' f_i + \sum_{i=1}^r b_i f_i = x_0 f^* + h.$$

Clearly $d_0(h) \le \max\{d_0(f_1), \dots, d_0(f_r)\} = s_0 < d_0(f)$. Hence $d_0(f) = d_0(x_0 f^*) = 1 + d_0(f^*)$. Furthermore, $\overline{f} = \overline{f^*}$.

By induction hypothesis, $\overline{f^*} \in K_{s_0}$, hence $\overline{f} \in K_{s_0}$, as claimed. The proof is completed. □

The following lemma provides a method to compute partial elimination ideals in practice.

Lemma 2.5 (Conca–Sidman [7, Proposition 3.4])**.** *Use Notation 2.1. Let τ be an elimination order for x_0 of R, and τ_0 the induced monomial order on S. Let G be a Gröbner basis of I with respect to τ. Then for every $i \ge 0$,*

$$G_i = \{\overline{f} : f \in G, d_0(f) \le i\}$$

is a Gröbner basis for $K_i(I)$ with respect to τ_0.

The next three results are our main tools in relating the algebraic invariants of I with that of its partial elimination ideals.

Proposition 2.6 (Elimination mapping cone sequence [14, Theorem 2.1])**.** *Let $S = k[x_1, \dots, x_N]$ be a standard graded polynomial ring and $R = S[x_0]$ a polynomial extension of S. Let M be a (not necessarily finitely generated) graded R-module. Then there is an exact sequence of k-modules*

$$\cdots \to T^S_{i,j}(M) \to T^R_{i,j}(M) \to T^S_{i-1,j}(M) \xrightarrow{\mu} T^S_{i-1,j+1}(M) \to T^R_{i-1,j+1}(M) \to \cdots$$

Here the connecting map μ is induced by the (S-linear) multiplication map $M \xrightarrow{\cdot x_0} M$.

Lemma 2.7. *Let $I \subseteq R$ be any homogeneous ideal. Then for all $i \ge 1$, there is an exact sequence of graded S-modules*

$$0 \to \widetilde{K}_{i-1} \to \widetilde{K}_i \to K_i(-i) \to 0.$$

This lemma follows directly from the definition of $K_i(I)$ via the map $\theta_i : R \to S$ in Notation 2.1(4).

The first assertion in the next lemma follows from [15, Proposition 2.3].

Lemma 2.8 (Approximation of syzygies)**.** *Let $I \subseteq R$ be any homogeneous ideal. Then for all $i \in \mathbb{Z}$, $d \ge j$, there is an isomorphism*

$$T^S_{i,j}(I) \cong T^S_{i,j}(\widetilde{K}_d).$$

In particular, $\operatorname{pd}_S I \le \liminf_{d\to\infty} \operatorname{pd}_S \widetilde{K}_d \le \dim S$.

Proof. Let $\ell = \liminf_{d\to\infty} \operatorname{pd}_S \widetilde{K}_d$ and $d_1 < d_2 < \cdots$ be a sequence of whole numbers such that $\operatorname{pd}_S \widetilde{K}_{d_i} = \ell$ for each $i \geq 1$.

Take $i > \ell$, then $T^S_{i,j}(\widetilde{K}_{d_m}) = 0$ for all $m \geq 1$. Assume that $T^S_i(I) \neq 0$ then for some j, $T^S_{i,j}(I) \neq 0$. For $m \gg 0$, we then have

$$T^S_{i,j}(\widetilde{K}_{d_m}) \cong T^S_{i,j}(I) \neq 0,$$

a contradiction. Hence $T^S_i(I) = 0$, and $\operatorname{pd}_S I \leq \ell$. □

2.3. **The case of irreducible projective varieties.** Assume that $X \subseteq \mathbb{P}^N$ is a non-degenerate *irreducible* closed subscheme of dimension n with defining ideal $I = I_X$, and $q = (1:0:\cdots:0) \in X \setminus \operatorname{vert}(X)$, where $\operatorname{vert}(X)$ is the set of vertices of X (i.e. $\operatorname{vert}(X)$ consists of the closed points $q \in X$ such that X is a cone over q). Keep using Notation 2.1. We note that

$$X_q = V(K_0) \supseteq \cdots \supseteq V(K_s) = V(K_\infty)$$

and $\dim V(K_s) = n-1$ by Lemma 2.2. Denote

$$s' = \min\{i \geq 0 : \dim V(K_i) = n-1\}.$$

Since $q \notin \operatorname{vert}(X)$, X is not a cone over q. Thus we have $\dim X_q = n$, so $1 \leq s' \leq s$. The next lemma provides a geometric interpretation of s'.

Lemma 2.9. *The following equalities hold:*

(i) $K_0 = K_1 = \cdots = K_{s'-1} \subsetneq K_{s'}$.
(ii) $s' = \deg(\pi_q : X \dashrightarrow X_q)$.

Proof. (i) The definition of s' yields $\dim V(K_{s'-1}) = n$, namely $\dim(S/K_{s'-1}) = n+1$. Since $\dim(X_q) = \dim X = n$, it follows that $\dim(S/K_0) = n+1$. Note that $I = I_X$ is a prime ideal, hence so is K_0. Combining this with the facts that $K_0 \subseteq K_{s'-1}$ and $\dim(S/K_0) = \dim(S/K_{s'-1})$, we infer that $K_0 = K_{s'-1}$.

(ii) By Lemma 2.3 and part (i), we have

$$X_q = V(K_{s'-1}) = V(K_\infty) \bigcup \left\{z \in \mathbb{P}^{N-1} : \operatorname{length}((X \setminus q) \cap \langle q, z\rangle) \geq s'\right\},$$
$$V(K_{s'}) = V(K_\infty) \bigcup \left\{z \in \mathbb{P}^{N-1} : \operatorname{length}((X \setminus q) \cap \langle q, z\rangle) \geq s'+1\right\}.$$

Note that $\dim V(K_{s'}) = n-1 < \dim V(K_{s'-1}) = n$. Hence any general point $z \in X_q \setminus V(K_{s'})$ is a general point of X_q. For such a point z, $\operatorname{length} \pi_q^{-1}(z) = \operatorname{length}((X \setminus q) \cap \langle q, z\rangle) = s'$. Hence $s' = \deg \pi_q$. □

3. Arithmetic depth of arbitrary projections

3.1. **Depth of R/I.** The following result and Theorem 3.4 compare the depth of I with that of the partial elimination ideals $K_i(I)$.

Theorem 3.1. *Keep using Notation 2.1. Then there is an inequality*

$$\operatorname{depth}_R \frac{R}{I} \geq \min_{1\leq i\leq s-1} \left\{\operatorname{depth}_S \frac{S}{J}, \operatorname{depth}_S \frac{S}{K_i(I)}, \operatorname{depth}_S \frac{S}{K_s(I)} + 1\right\}.$$

Below, recall that the notation $\widetilde{K}_i$ and K_i stand for $\widetilde{K}_i(I)$ and $K_i(I)$ respectively. The proof of Theorem 3.1 uses Lemma 3.2 and Proposition 3.3.

Lemma 3.2. *Denote* $t = \max\left\{\operatorname{pd}_S \widetilde{K}_{s-1} + 1, \operatorname{pd}_S \widetilde{K}_s\right\}$. *Then for all* $d \geq s-1$, *the map*

$$T_i^S(\widetilde{K}_d(-1)) \to T_i^S(\widetilde{K}_{d+1})$$

induced by the multiplication map $\widetilde{K}_d(-1) \xrightarrow{\cdot x_0} \widetilde{K}_{d+1}$ *is an injection if* $i = t$ *and an isomorphism if* $i > t$.

Proof. Induct on $d \geq s-1$. First let $d = s-1$. For $i > t$, we have

$$T_i^S(\widetilde{K}_{s-1}(-1)) = T_t^S(\widetilde{K}_{s-1}(-1)) = T_i^S(\widetilde{K}_s) = 0.$$

Hence we have the desired injection

$$0 = T_t^S(\widetilde{K}_{s-1}(-1)) \hookrightarrow T_t^S(\widetilde{K}_s).$$

and isomorphism

$$0 = T_i^S(\widetilde{K}_{s-1}(-1)) \cong T_i^S(\widetilde{K}_s).$$

Assume that the conclusion is true for $d-1 \geq s-1$. We prove it for d.

There is a commutative diagram with exact rows, where ι is the natural inclusion map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{K}_{d-1}(-1) & \longrightarrow & \widetilde{K}_d(-1) & \longrightarrow & K_d(-d-1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cdot x_0} & & \downarrow{\scriptstyle \cdot x_0} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \widetilde{K}_d & \longrightarrow & \widetilde{K}_{d+1} & \longrightarrow & K_{d+1}(-d-1) & \longrightarrow & 0
\end{array}$$

This induces a diagram with exact rows, where for simplicity, we omit the superscript S in the notation $T_i^S(M)$, and write $*$ for the shift $-d-1$:

$$\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \downarrow \\
T_{i+1}(K_d)(*) & \longrightarrow & T_i(\widetilde{K}_{d-1}(-1)) & \longrightarrow & T_i(\widetilde{K}_d(-1)) & \longrightarrow & T_i(K_d)(*) & \longrightarrow & T_{i-1}(\widetilde{K}_{d-1}(-1)) \\
\cong\downarrow{\scriptstyle \alpha} & & \cong\downarrow{\scriptstyle \beta_{d-1}} & & \downarrow{\scriptstyle \beta_d} & & \cong\downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \theta} \\
T_{i+1}(K_{d+1})(*) & \longrightarrow & T_i(\widetilde{K}_d) & \longrightarrow & T_i(\widetilde{K}_{d+1}) & \longrightarrow & T_i(K_{d+1})(*) & \longrightarrow & T_{i-1}(\widetilde{K}_d(-1)) \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

Take $i > t$. By the induction hypothesis, β_{d-1} is isomorphism. Since $d \geq s$, $K_d = K_{d+1} = K_s$, hence $K_d \xrightarrow{\iota} K_{d+1}$ is the identity map. Consequently, α and γ are isomorphisms. The map θ is an isomorphism if $i > t+1$ and an injection if $i = t+1$. By diagram chasing (and the Five Lemma in particular), β_d is an isomorphism.

For the case $i = t$, we consider the diagram:

$$\begin{array}{ccccccc}
 & & 0 & & & & 0 \\
 & & \downarrow & & & & \downarrow \\
T_{t+1}(K_d)(*) & \longrightarrow & T_t(\widetilde{K}_{d-1}(-1)) & \longrightarrow & T_t(\widetilde{K}_d(-1)) & \longrightarrow & T_t(K_d)(*) \\
\downarrow \alpha & & \downarrow \beta_{d-1} & & \downarrow \beta_d & & \downarrow \gamma \\
T_{t+1}(K_{d+1})(*) & \longrightarrow & T_t(\widetilde{K}_d) & \longrightarrow & T_t(\widetilde{K}_{d+1}) & \longrightarrow & T_t(K_{d+1})(*) \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

By the induction hypothesis, β_{d-1} is injective. Again α and γ are isomorphisms. By diagram chasing (and the Four Lemma in particular), we get that β_d is injective. This concludes the induction and the proof. □

The key step in the proof of Theorem 3.1 is accomplished by

Proposition 3.3. *There are inequalities*

$$\operatorname{pd}_R I \le \max\left\{\operatorname{pd}_S \widetilde{K}_{s-1} + 1, \operatorname{pd}_S \widetilde{K}_s\right\} \le \max\left\{\operatorname{pd}_S \widetilde{K}_{s-1} + 1, \operatorname{pd}_S K_s\right\}.$$

Proof. Denote $t = \left\{\operatorname{pd}_S \widetilde{K}_{s-1} + 1, \operatorname{pd}_S \widetilde{K}_s\right\}$. Taking any $i > t$, we show that $T^R_{i,j}(I) = 0$ for all j.

By Proposition 2.6 and Lemma 2.8, for all $d \ge \max\{j, s-1\}$, there is a diagram with vertical arrows being isomorphisms:

$$\begin{array}{ccccccccc}
T^S_{i,j-1}(I) & \xrightarrow{\mu_i} & T^S_{i,j}(I) & \longrightarrow & T^R_{i,j}(I) & \longrightarrow & T^S_{i-1,j}(I) & \xrightarrow{\mu_{i-1}} & T^S_{i-1,j+1}(I) \\
\downarrow \cong & & \downarrow \cong & & & & & & \\
T^S_{i,j-1}(\widetilde{K}_d) & \xrightarrow{\beta} & T^S_{i,j}(\widetilde{K}_{d+1}) & & & & & &
\end{array}$$

Here β is induced by the multiplication map $\widetilde{K}_d(-1) \xrightarrow{\cdot x_0} \widetilde{K}_{d+1}$.

Since $i > t$ and $d \ge s - 1$, by Lemma 3.2, β is an isomorphism, and thus so is μ_i. By similar arguments, μ_{i-1} is an isomorphism for $i > t+1$ and an injection for $i = t + 1$.

The first line of the diagram induces an exact sequence

$$0 = \operatorname{Coker} \mu_i \to T^R_{i,j}(I) \to \operatorname{Ker} \mu_{i-1} = 0.$$

Hence $T^R_{i,j}(I) = 0$, as desired.

The remaining inequality is a consequence of the fact that

$$\operatorname{pd}_S \widetilde{K}_s \le \max\{\operatorname{pd}_S \widetilde{K}_{s-1}, \operatorname{pd}_S K_s\},$$

which can be seen from the exact sequence $0 \to \widetilde{K}_{s-1} \to \widetilde{K}_s \to K_s(-s) \to 0$. □

Now we are ready to present the proof of our first main result.

Proof of Theorem 3.1. By Proposition 3.3,

$$\operatorname{pd}_R I \le \max\left\{\operatorname{pd}_S \widetilde{K}_{s-1} + 1, \operatorname{pd}_S K_s\right\}.$$

From Lemma 2.7, $\operatorname{pd}_S \widetilde{K}_i \le \max\{\operatorname{pd}_S \widetilde{K}_{i-1}, \operatorname{pd}_S K_i\}$. Using repeatedly this inequality and the fact that $\widetilde{K}_0 = K_0$, we have

$$\operatorname{pd}_S \widetilde{K}_{s-1} \le \max\{\operatorname{pd}_S K_0, \dots, \operatorname{pd}_S K_{s-1}\}.$$

Hence

$$\operatorname{pd}_R I \le \max_{0\le i\le s-1} \{\operatorname{pd}_S K_i + 1, \operatorname{pd}_S K_s\}.$$

Using the Auslander-Buchsbaum formula $\operatorname{pd}_R I = \dim R - \operatorname{depth}_R(R/I) - 1$, the last inequality becomes

$$\operatorname{depth}_R \frac{R}{I} \ge \min_{0\le i\le s-1} \left\{\operatorname{depth}_S \frac{S}{K_i}, \operatorname{depth}_S \frac{S}{K_s} + 1\right\},$$

as desired. □

3.2. **Depth of S/J.** The following theorem strengthens the second part of Theorem 1.5 in the Introduction.

Theorem 3.4. *Keep using Notation 2.1. Denote $p = \operatorname{pd}_S J, c = \operatorname{reg} K_s$. Assume in addition that $T^S_{p,j}(J) \neq 0$ for some $j \ge c+1$. Then there is an inequality*

$$\operatorname{depth}_S \frac{S}{J} \ge \min_{1\le i\le s} \left\{\operatorname{depth}_R \frac{R}{I}, \operatorname{depth}_S \frac{S}{K_i} + 1\right\}.$$

In particular, the inequality holds if $J_{\le c} = 0$.

Remark 3.5. (1) The condition $J_{\le \operatorname{reg} K_s} = 0$ in Theorem 3.4 is always satisfied if $q = (1 : 0 : \cdots : 0)$ is a smooth point of $X = V(I)$ and $I = I_X$ contains no linear form. Indeed, in that case K_s defines the tangent space of X at q, so K_s is generated by linear forms and $c = \operatorname{reg} K_s = 1$.

Example 3.11 below shows that the condition $T^S_{p,j}(J) \neq 0$ for some $j \ge c+1$ is crucial for the conclusion of Theorem 3.4.

(2) If $q = (1 : 0 : \cdots : 0) \notin V(I)$, namely $I \not\subseteq (x_1, \dots, x_N)R$ (the outer projection case), then $K_s = (1)$. In this case the condition $J_{\le \operatorname{reg} K_s} = 0$ of Theorem 3.4 is automatic, hence this result is always applicable to outer projections.

Before going to the proof of Theorem 3.4, we need a series of auxiliary results, Lemmas 3.6–3.9. Lemmas 3.6 and 3.7 focus on properties of the S-modules $\widetilde{K}_d/\widetilde{K}_{s-1}$ and $I/\widetilde{K}_{s-1}$.

Lemma 3.6. *Let $t + 1 = \operatorname{pd}_S K_s$. The following statements hold true.*

(i) *For all $d \ge s$, there is an equality*

$$\operatorname{pd}_S \frac{\widetilde{K}_d}{\widetilde{K}_{s-1}} = \operatorname{pd}_S K_s.$$

(ii) *For all j and all $d \ge s$, the inclusion $\widetilde{K}_d \to \widetilde{K}_{d+1}$ induces an injective map*

$$T^S_{t+1,j}(\widetilde{K}_d/\widetilde{K}_{s-1}) \to T^S_{t+1,j}(\widetilde{K}_{d+1}/\widetilde{K}_{s-1}).$$

(iii) *There is an equality*

$$\operatorname{pd}_S \frac{I}{\widetilde{K}_{s-1}} = \operatorname{pd}_S K_s.$$

Proof. (i) We prove by induction on $d \geq s$ that

$$\operatorname{pd}_S \frac{\widetilde{K}_d}{\widetilde{K}_{s-1}} = \operatorname{pd}_S K_s = t+1.$$

For $d = s$, it suffices to observe that $\widetilde{K}_s/\widetilde{K}_{s-1} \cong K_s(-s)$.

Taking $d \geq s+1$, we have an exact sequence

$$0 \to \frac{\widetilde{K}_{d-1}}{\widetilde{K}_{s-1}} \to \frac{\widetilde{K}_d}{\widetilde{K}_{s-1}} \to K_d(-d) \to 0.$$

Since $K_d = K_s$, $\operatorname{pd}_S \widetilde{K}_{d-1}/\widetilde{K}_{s-1} = \operatorname{pd}_S K_d = \operatorname{pd}_S K_s$ by the induction hypothesis. By standard "Depth Lemma" arguments, we deduce that

$$\operatorname{pd}_S \frac{\widetilde{K}_d}{\widetilde{K}_{s-1}} = \operatorname{pd}_S K_s.$$

(ii) From the sequence $0 \to \widetilde{K}_d/\widetilde{K}_{s-1} \to \widetilde{K}_{d+1}/\widetilde{K}_{s-1} \to K_{d+1}(-d-1) \to 0$, we get the exactness of

$$0 = T^S_{t+2,j-1}(K_{d+1}(-d-1)) \to T^S_{t+1,j}(\widetilde{K}_d/\widetilde{K}_{s-1}) \to T^S_{t+1,j}(\widetilde{K}_{d+1}/\widetilde{K}_{s-1}).$$

The equality holds since $\operatorname{pd}_S K_{d+1} = t+1$. The desired conclusion follows.

(iii) Arguing as in [15, Proposition 2.3], for all i and all $d \geq \max\{j, s-1\}$, there is an isomorphism

$$T^S_{i,j}\left(\frac{I}{\widetilde{K}_{s-1}}\right) \cong T^S_{i,j}\left(\frac{\widetilde{K}_d}{\widetilde{K}_{s-1}}\right).$$

This together with part (i) yield

$$\operatorname{pd}_S(I/\widetilde{K}_{s-1}) \leq \sup_{d \geq s-1} \operatorname{pd}_S(\widetilde{K}_d/\widetilde{K}_{s-1}) = \operatorname{pd}_S K_s = t+1.$$

As $\operatorname{pd}_S \widetilde{K}_s/\widetilde{K}_{s-1} = t+1$, for some j, $T^S_{t+1,j}(\widetilde{K}_s/\widetilde{K}_{s-1}) \neq 0$. Using part (ii),

$$T^S_{t+1,j}(\widetilde{K}_d/\widetilde{K}_{s-1}) \neq 0 \quad \text{for all } d \geq s.$$

Choosing $d \geq \max\{j, s\}$, we have $T^S_{t+1,j}(I/\widetilde{K}_{s-1}) \neq 0$. Thus $\operatorname{pd}_S(I/\widetilde{K}_{s-1}) \geq t+1$, and we are done. □

Lemma 3.7. *Denote $c = \operatorname{reg} K_s$ and $t+1 = \operatorname{pd}_S K_s$. Then for all $d \geq s$, the following statements hold:*

(i) *There is an equality* $\operatorname{reg}(\widetilde{K}_d/\widetilde{K}_{s-1}) = d + c$.
(ii) *The natural surjection $\widetilde{K}_d/\widetilde{K}_{s-1} \to \widetilde{K}_d/\widetilde{K}_{d-1} = K_d(-d)$ induces the following isomorphism*

$$T^S_{t+1,d+c}\left(\frac{\widetilde{K}_d}{\widetilde{K}_{s-1}}\right) \cong T^S_{t+1,d+c}(K_d(-d)).$$

Proof. (i) We use induction on $d \geq s$. For $d = s$, $\widetilde{K}_s/\widetilde{K}_{s-1} \cong K_s(-s)$, hence the claim is true. Assume that $d \geq s+1$.

Consider the exact sequence

$$0 \to \frac{\widetilde{K}_{d-1}}{\widetilde{K}_{s-1}} \to \frac{\widetilde{K}_d}{\widetilde{K}_{s-1}} \to K_d(-d) \to 0. \tag{3.1}$$

By the induction hypothesis and the fact that $K_d = K_s$,

$$\operatorname{reg} \frac{\widetilde{K}_{d-1}}{\widetilde{K}_{s-1}} = d + c - 1 < \operatorname{reg} K_d(-d) = d + c.$$

The standard short exact sequence argument ([9, Corollary 20.19]) yields the equality $\operatorname{reg}(\widetilde{K}_d/\widetilde{K}_{s-1}) = d + c$.

(ii) If $d = s$, we are done by the isomorphism $\widetilde{K}_s/\widetilde{K}_{s-1} \cong K_s(-s)$. Assume that $d \geq s + 1$, and denote $j = d + c$. From the sequence (3.1), we get a long exact sequence

$$T_{t+1,j}\left(\frac{\widetilde{K}_{d-1}}{\widetilde{K}_{s-1}}\right) \to T_{t+1,j}\left(\frac{\widetilde{K}_{d}}{\widetilde{K}_{s-1}}\right) \to T_{t+1,j}(K_d(-d)) \to T_{t,j+1}\left(\frac{\widetilde{K}_{d-1}}{\widetilde{K}_{s-1}}\right).$$

By (i), $\operatorname{reg}(\widetilde{K}_{d-1}/\widetilde{K}_{s-1}) = d + c - 1 < j$, hence the two terms from outside of the above sequence vanish. This yields the desired conclusion. □

Lemma 3.8. *There are inequalities*

$$\operatorname{pd}_S \widetilde{K}_{s-1} \leq \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_s - 1\} \leq \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_s\}. \tag{3.2}$$

Moreover, for all $0 \leq i \leq s - 1$, there are inequalities

$$\operatorname{pd}_S \widetilde{K}_i \leq \max\{\operatorname{pd}_R \widetilde{K}_{s-1}, \operatorname{pd}_S K_{i+1} - 1, \dots, \operatorname{pd}_S K_{s-1} - 1\} \tag{3.3}$$

$$\leq \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_{i+1} - 1, \dots, \operatorname{pd}_S K_s - 1\}. \tag{3.4}$$

Proof. Consider the exact sequence of S-modules

$$0 \to \widetilde{K}_{s-1} \to I \to \frac{I}{\widetilde{K}_{s-1}} \to 0.$$

The equality in the next chain follows from Lemma 3.6(iii)

$$\operatorname{pd}_S \widetilde{K}_{s-1} \leq \max\{\operatorname{pd}_S I, \operatorname{pd}_S(I/\widetilde{K}_{s-1}) - 1\} = \max\{\operatorname{pd}_S I, \operatorname{pd}_S K_s - 1\}.$$

Since $S \to R$ is a flat extension, $\operatorname{pd}_S I \leq \operatorname{pd}_R I$. Hence

$$\operatorname{pd}_S \widetilde{K}_{s-1} \leq \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_s - 1\} \leq \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_s\},$$

which is (3.2).

For $0 \leq i \leq s - 2$, from the exact sequence

$$0 \to \widetilde{K}_i \to \widetilde{K}_{i+1} \to K_{i+1}(-i-1) \to 0,$$

we get

$$\operatorname{pd}_S \widetilde{K}_i \leq \max\{\operatorname{pd}_S \widetilde{K}_{i+1}, \operatorname{pd}_S K_{i+1} - 1\}.$$

Repeating the argument, we get (3.3):

$$\operatorname{pd}_S \widetilde{K}_i \leq \max\{\operatorname{pd}_S \widetilde{K}_{s-1}, \operatorname{pd}_S K_{i+1} - 1, \dots, \operatorname{pd}_S K_{s-1} - 1\}.$$

Finally, using (3.2), we obtain (3.4). □

Lemma 3.9. *For an integer $0 \leq i \leq s - 1$, let $t + 1 = \max_{i+1 \leq j \leq s}\{\operatorname{pd}_R I, \operatorname{pd}_S K_j\}$. Then for all j and all $d \geq i$, the map*

$$T^S_{t+1,j}(\widetilde{K}_d) \to T^S_{t+1,j+1}(\widetilde{K}_{d+1})$$

induced by the multiplication $\widetilde{K}_d(-1) \xrightarrow{\cdot x_0} \widetilde{K}_{d+1}$, is injective.

Proof. Using the elimination mapping cone in Proposition 2.6, we get an exact sequence

$$T^R_{t+2,j}(I) = 0 \to T^S_{t+1,j}(I) \xrightarrow{\mu} T^S_{t+1,j+1}(I). \tag{3.5}$$

The equality holds since $\operatorname{pd}_R I \le t+1$.

We prove the lemma by reverse induction on d. If $d \ge \max\{i, j\}$, then by Lemma 2.8,

$$T^S_{t+1,j}(\widetilde{K}_d) \cong T^S_{t+1,j}(I),$$
$$T^S_{t+1,j+1}(\widetilde{K}_{d+1}) \cong T^S_{t+1,j+1}(I).$$

Hence the claim follows from the sequence (3.5).

Now assume that $d \ge i$ and the claim was established for $d+1$. From the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{K}_d(-1) & \longrightarrow & \widetilde{K}_{d+1}(-1) & \longrightarrow & K_{d+1}(-d-2) & \longrightarrow & 0 \\
 & & \downarrow{\cdot x_0} & & \downarrow{\cdot x_0} & & \downarrow{\iota} & & \\
0 & \longrightarrow & \widetilde{K}_{d+1} & \longrightarrow & \widetilde{K}_{d+2} & \longrightarrow & K_{d+2}(-d-2) & \longrightarrow & 0
\end{array}$$

we get the following diagram on homology with exact second row:

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
T_{t+2,j}(K_{d+1}(-d-2)) = 0 & \longrightarrow & T_{t+1,j}(\widetilde{K}_d) & \longrightarrow & T_{t+1,j}(\widetilde{K}_{d+1}) \\
 & & \downarrow{\beta_d} & & \downarrow{\beta_{d+1}} \\
 & & T_{t+1,j+1}(\widetilde{K}_{d+1}) & \longrightarrow & T_{t+1,j+1}(\widetilde{K}_{d+2})
\end{array}$$

The inequalities

$$\max\{\operatorname{pd}_S K_{d+1}, \operatorname{pd}_S K_{d+2}\} \le \max_{i+1 \le j \le s}\{\operatorname{pd}_S K_j\} \le t+1$$

implies the equality in the second row.

By the induction hypothesis, β_{d+1} is injective. This implies that β_d is injective as well. This finishes the proof of the lemma. □

The main work in the proof of Theorem 3.4 is accomplished by

Proposition 3.10. *Denote* $c = \operatorname{reg} K_s$ *and* $t+1 = \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_s\}$. *If the equality* $\operatorname{pd}_S \widetilde{K}_{s-1} = t+1$ *holds then*

$$T^S_{t+1,j}(\widetilde{K}_{s-1}) = 0 \quad \textit{for all } j \ge s+c.$$

Proof. We proceed through several steps.

Step 1: Recall that by (3.2) in Lemma 3.8,

$$\operatorname{pd}_S \widetilde{K}_{s-1} \le \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_s - 1\} \le \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_s\} = t+1.$$

If $\operatorname{pd}_S \widetilde{K}_{s-1} = t+1$, we deduce that $t+1 = \operatorname{pd}_R I \ge \operatorname{pd}_S K_s$.

Step 2: For each $j \ge s+c$, denote $d = j-c$. We show that there is an exact sequence with natural maps

$$0 \to T_{t+1,d+c}(\widetilde{K}_{s-1}) \to T_{t+1,d+c}(\widetilde{K}_d) \to T_{t+1,d+c}(K_d(-d)). \tag{3.6}$$

Indeed, consider the exact sequences

$$0 \to \widetilde{K}_{s-1} \to \widetilde{K}_d \to \frac{\widetilde{K}_d}{\widetilde{K}_{s-1}} \to 0,$$
$$\frac{\widetilde{K}_d}{\widetilde{K}_{s-1}} \to \frac{\widetilde{K}_d}{\widetilde{K}_{d-1}} \cong K_d(-d) \to 0.$$

We obtain exact sequences of homology, where the second one follows from Lemma 3.7 and the fact that $j = d + c$:

$$T_{t+2,j-1}\left(\frac{\widetilde{K}_d}{\widetilde{K}_{s-1}}\right) = 0 \to T_{t+1,j}(\widetilde{K}_{s-1}) \to T_{t+1,j}(\widetilde{K}_d) \to T_{t+1,j}\left(\frac{\widetilde{K}_d}{\widetilde{K}_{s-1}}\right),$$
$$T_{t+1,j}\left(\frac{\widetilde{K}_d}{\widetilde{K}_{s-1}}\right) \cong T_{t+1,j}\left(K_d(-d)\right).$$

The equality in the first one holds since by Lemma 3.6, $\operatorname{pd}_S(\widetilde{K}_d/\widetilde{K}_{s-1}) = \operatorname{pd}_S K_s < t+2$. Hence the sequence (3.6) exists and consists of natural maps.

Step 3: Now assuming that $T^S_{t+1,j_0}(\widetilde{K}_{s-1}) \neq 0$ for some $j_0 \geq s + c$, we will derive a contradiction.

Denote $d = j_0 - c \geq s$, and take $q \geq d$ arbitrary. Consider the following diagram of graded S-modules and homogeneous morphisms:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widetilde{K}_{s-1} & \longrightarrow & \widetilde{K}_d & \longrightarrow & \widetilde{K}_d/\widetilde{K}_{d-1} \cong K_d(-d) \\
 & & & & \downarrow{\scriptstyle \cdot x_0^{q-d}} & & \downarrow{\scriptstyle \iota} \\
0 & \longrightarrow & \widetilde{K}_{s-1}(q-d) & \longrightarrow & \widetilde{K}_q(q-d) & \longrightarrow & \left(\widetilde{K}_q/\widetilde{K}_{q-1}\right)(q-d) \cong K_q(-d)
\end{array}$$

Some healthy warnings and remarks:

(1) the rows need not be exact;
(2) there is no map from $\widetilde{K}_{s-1}$ to $\widetilde{K}_{s-1}(q-d)$ in the diagram(!);
(3) ι is the natural identity map (as $K_s = K_d = K_q$);
(4) the rectangle is commutative.

Taking homology, by Step 2, we get the following diagram with exact rows and a commutative rectangle:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
0 & \longrightarrow & T_{t+1,d+c}(\widetilde{K}_{s-1}) & \longrightarrow & T_{t+1,d+c}(\widetilde{K}_d) & \xrightarrow{\alpha_d} & T_{t+1,d+c}(K_d(-d)) \\
 & & & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
0 & \longrightarrow & T_{t+1,q+c}(\widetilde{K}_{s-1}) & \longrightarrow & T_{t+1,q+c}(\widetilde{K}_q) & \xrightarrow{\alpha_q} & T_{t+1,q+c}(K_q(-q))
\end{array}$$

Since β is induced by the multiplication map $\widetilde{K}_d(-(q-d)) \xrightarrow{\cdot x_0^{q-d}} \widetilde{K}_q$, and $t+1 = \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_s\}$, by Lemma 3.9 for $i = s-1$, it is injective. (Recall that $d \geq s$.)

Since $T_{t+1}(\widetilde{K}_{s-1})$ is a finite k-vector space, we can choose $q \gg 0$, such that $T_{t+1,q+c}(\widetilde{K}_{s-1}) = 0$. For such a q, the map α_q is injective. But then $\gamma \circ \alpha_d = \alpha_q \circ \beta$ is injective as well. This contradicts the fact that $0 \neq T_{t+1,d+c}(\widetilde{K}_{s-1}) \subseteq \operatorname{Ker} \alpha_d$.

The proof is concluded. □

Now we present the proof of the main result of this subsection.

Proof of Theorem 3.4. We proceed through several steps.

Step 1: By the Auslander-Buchsbaum formula, we need to show that

$$\operatorname{pd}_S J \leq \max_{1 \leq i \leq s} \{\operatorname{pd}_R I, \operatorname{pd}_S K_i\} - 1.$$

Denote the right-hand side by t.

Step 2: Assume on the contrary that $p = \operatorname{pd}_S J \geq t + 1$.

Using (3.3) and (3.4) in Lemma 3.8 for $i = 0$,

$$\begin{aligned}
\operatorname{pd}_S J = \operatorname{pd}_S \widetilde{K}_0 &\leq \max\{\operatorname{pd}_S \widetilde{K}_{s-1}, \operatorname{pd}_S K_1 - 1, \dots, \operatorname{pd}_S K_{s-1} - 1\} && (3.7)\\
&\leq \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_1 - 1, \dots, \operatorname{pd}_S K_{s-1} - 1, \operatorname{pd}_S K_s - 1\} && (3.8)\\
&\leq \max_{1 \leq i \leq s} \{\operatorname{pd}_R I, \operatorname{pd}_S K_i\} = t + 1. && (3.9)
\end{aligned}$$

The last equality is due to the definition of t. As $\operatorname{pd}_S J \geq t + 1$, all the inequalities in the last display are actually equalities. In particular (3.8) and (3.9) yield $\operatorname{pd}_R I \geq \max\{\operatorname{pd}_S K_1, \dots, \operatorname{pd}_S K_s\}$. Therefore

$$t + 1 = \operatorname{pd}_R I = \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_s\} = \max_{1 \leq i \leq s} \{\operatorname{pd}_R I, \operatorname{pd}_S K_i\}.$$

Thus $\operatorname{pd}_S J = t+1 > \max\limits_{1 \leq i \leq s-1} \{\operatorname{pd}_S K_i - 1\}$. Combining this with (3.7), we conclude that $\operatorname{pd}_S \widetilde{K}_{s-1} = t + 1$.

Step 3: Employing Lemma 3.9 for the case $i = 0$, we see that for all $d \geq 0$ and all j, the multiplication $\widetilde{K}_d(-1) \xrightarrow{\cdot x_0} \widetilde{K}_{d+1}$ induces an injective map

$$T^S_{t+1,j}(\widetilde{K}_d) \to T^S_{t+1,j+1}(\widetilde{K}_{d+1}).$$

In particular, we have injective maps

$$T^S_{t+1,j}(J) = T^S_{t+1,j}(\widetilde{K}_0) \hookrightarrow T^S_{t+1,j+1}(\widetilde{K}_1) \hookrightarrow \cdots \hookrightarrow T^S_{t+1,j+s-1}(\widetilde{K}_{s-1}). \tag{3.10}$$

Step 4: Take any $j \geq c + 1$. Now from Step 2, $t + 1 = \max\{\operatorname{pd}_R I, \operatorname{pd}_S K_s\}$, $\operatorname{pd}_S \widetilde{K}_{s-1} = t + 1$, and $j + s - 1 \geq s + c$, so Proposition 3.10 yields that

$$T^S_{t+1,j+s-1}(\widetilde{K}_{s-1}) = 0.$$

Together with the chain (3.10), we see that $T^S_{t+1,j}(J) = 0$ for all $j \geq c + 1$.

Since $\operatorname{pd}_S J = p = t + 1$, this contradicts the hypothesis that $T^S_{p,j}(J) \neq 0$ for some $j \geq c + 1$. In particular, the assumption $\operatorname{pd}_S J \geq t + 1$ in Step 2 is false. We thus have that $\operatorname{pd}_R J \leq t$, thereby establishing the first assertion of Theorem 3.4.

Step 5: For the remaining assertion, assume that $J_{\leq c} = 0$. If $J = (0)$, then as I is non-zero, $\operatorname{depth}(S/J) = \dim S \geq \dim(R/I) \geq \operatorname{depth}(R/I)$. Hence the desired inequality holds true.

Assume that $J \neq (0)$. Then J is generated in degree at least $c + 1$. This implies that $T^S_{p,j}(J) \neq 0$ for some $j \geq c + 1$. Hence the desired inequality holds by the first assertion.

The proof is completed. □

Example 3.11. The following example shows that the condition "$T^S_{p,j}(J) \neq 0$ for $p = \operatorname{pd}_S J$ and for some $j \geq 1 + \operatorname{reg} K_s$" in Theorem 3.4 is indispensable. Let $R = k[x, y, z, t]$ and I the ideal of 2-minors of the following matrix

$$X = \begin{pmatrix} x & y & t \\ y & z & 0 \end{pmatrix}.$$

Concretely, $I = (xz - y^2, yt, zt)$. Note that $I = (y, z) \cap (xz - y^2, t)$, hence using the exact sequence

$$0 \to \frac{R}{I} \to \frac{R}{(y,z)} \oplus \frac{R}{(xz - y^2, t)} \to \frac{R}{(y,z,t)} \to 0,$$

we deduce that $\operatorname{depth}(R/I) = 2$. Consider the partial elimination ideals of I with respect to x. In this case $J = (yt, zt) \subseteq k[y, z, t]$, $K_1 = (z, yt)$ and the stabilization index of the sequence of partial elimination ideals is $s = 1$. We have

$$\operatorname{depth}(S/J) = \operatorname{depth}(S/K_1) = 1.$$

Hence the conclusion $\operatorname{depth}(S/J) \geq \min\{\operatorname{depth}(R/I), \operatorname{depth}(S/K_1)+1\}$ of Theorem 3.4 is not satisfied. A reason here is that $T^S_{i,j}(J) = 0$ for all i and all $j \geq 1+\operatorname{reg} K_s = 3$, as J has a 2-linear resolution.

4. Regularity of arbitrary projections

The following result generalizes previous work of M. Jones [17, Theorem 2.1] and Ahn–Kwak [2, Corollary 3.6] on regularity of outer projections.

Theorem 4.1. *Keep using Notation 2.1. There is an inequality*

$$\operatorname{reg}_R I \leq \max_{1 \leq i \leq s} \{\operatorname{reg}_S J, \operatorname{reg}_S K_i + i\}.$$

In some sense, this result shows that the regularity of I is controlled by the complexity/singularity of its partial elimination ideals. For the proof of Theorem 4.1, we need an auxiliary statement, analogous to Lemma 3.2.

Lemma 4.2. *Denote* $\nu = \max\left\{\operatorname{reg}_S \widetilde{K}_{s-1}, \operatorname{reg}_S \widetilde{K}_s\right\}$. *Then for all i and all $d \geq s - 1$, the map*

$$T^S_{i,j}(\widetilde{K}_d) \longrightarrow T^S_{i,j+1}(\widetilde{K}_{d+1})$$

induced by multiplication $\widetilde{K}_d(-1) \xrightarrow{\cdot x_0} \widetilde{K}_{d+1}$ *is a surjection if $j = \nu$ and an isomorphism if $j > \nu$.*

Proof. Induct on $d \geq s - 1$.

Step 1: First let $d = s - 1$. We need to show that there is an isomorphism

$$T^S_{i,j}(\widetilde{K}_{s-1}) \xrightarrow{\cong} T^S_{i,j+1}(\widetilde{K}_s)$$

for $j > \nu$, and a surjection

$$T^S_{i,\nu}(\widetilde{K}_{s-1}) \twoheadrightarrow T^S_{i,\nu+1}(\widetilde{K}_s).$$

Since $j > \nu = \max\{\operatorname{reg}_S \widetilde{K}_{s-1}, \operatorname{reg}_S \widetilde{K}_s\}$, we see that $T^S_{i,j}(\widetilde{K}_{s-1}) = T^S_{i,j+1}(\widetilde{K}_s) = T^S_{i,\nu+1}(\widetilde{K}_s) = 0$ for all i. Therefore the desired isomorphism and surjection are established.

Step 2: Assume that the statement holds true for $d - 1 \geq s - 1$. We prove it for d.

From the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{K}_{d-1}(-1) & \longrightarrow & \widetilde{K}_d(-1) & \longrightarrow & K_d(-d-1) & \longrightarrow & 0\\ & & \Big\downarrow{\scriptstyle \cdot x_0} & & \Big\downarrow{\scriptstyle \cdot x_0} & & \Big\downarrow{\scriptstyle \iota} & & \\ 0 & \longrightarrow & \widetilde{K}_d & \longrightarrow & \widetilde{K}_{d+1} & \longrightarrow & K_{d+1}(-d-1) & \longrightarrow & 0 \end{array}$$

we get a diagram with exact rows:

$$\begin{array}{ccccccccc} & & & & & & & & 0\\ & & & & & & & & \Big\downarrow\\ T_{i+1,j-d-1}(K_d) & \to & T_{i,j}(\widetilde{K}_{d-1}) & \to & T_{i,j}(\widetilde{K}_d) & \to & T_{i,j-d}(K_d) & \to & T_{i-1,j+1}(\widetilde{K}_{d-1})\\ \cong\Big\downarrow{\scriptstyle \alpha_{i+1}} & & \cong\Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \pi} & & \cong\Big\downarrow{\scriptstyle \alpha_i} & & \Big\downarrow{\scriptstyle \gamma}\\ T_{i+1,j-d-1}(K_{d+1}) & \to & T_{i,j+1}(\widetilde{K}_d) & \to & T_{i,j+1}(\widetilde{K}_{d+1}) & \to & T_{i,j-d}(K_{d+1}) & \to & T_{i-1,j+2}(\widetilde{K}_d)\\ \Big\downarrow & & & & & & & & \\ 0 & & & & & & & & \end{array}$$

For $j > \nu$, by the induction hypothesis, β and γ are isomorphisms. Both α_i and α_{i+1} are isomorphisms, since $K_d = K_{d+1} = K_s$. Hence by a diagram chasing (in particular, the Five Lemma), π is also an isomorphism.

For $j = \nu$, we consider the diagram

$$\begin{array}{ccccccc} & & & & & & 0\\ & & & & & & \Big\downarrow\\ T_{i,\nu}(\widetilde{K}_{d-1}) & \to & T_{i,\nu}(\widetilde{K}_d) & \to & T_{i,\nu-d}(K_d) & \to & T_{i-1,\nu+1}(\widetilde{K}_{d-1})\\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \gamma}\\ T_{i,\nu+1}(\widetilde{K}_d) & \to & T_{i,\nu+1}(\widetilde{K}_{d+1}) & \to & T_{i,\nu-d}(K_{d+1}) & \to & T_{i-1,\nu+2}(\widetilde{K}_d)\\ \Big\downarrow & & & & \Big\downarrow & & \\ 0 & & & & 0 & & \end{array}$$

By the induction hypothesis, β is a surjection, γ is an isomorphism. Moreover, α is an isomorphism. By diagram chasing (in particular the Four Lemma), we get π is surjective. This finishes the induction on d and the proof. $\square$

The main step in the proof of Theorem 4.1 is accomplished by

Proposition 4.3. *There are inequalities*

$$\operatorname{reg}_R I \le \max\{\operatorname{reg}_S \widetilde{K}_{s-1}, \operatorname{reg}_S \widetilde{K}_s\} \le \max\{\operatorname{reg}_S \widetilde{K}_{s-1}, \operatorname{reg}_S K_s + s\}.$$

Proof. Denote $\nu = \max\{\operatorname{reg}_S \widetilde{K}_{s-1}, \operatorname{reg}_S \widetilde{K}_s\}$. We have to show that $T^R_{i,j}(I) = 0$ for all $j > \nu$.

Choose $d \geq \max\{j, s-1\}$ arbitrary. Consider the following diagram induced by the elimination mapping cone sequence:

$$\begin{array}{ccccccccc} T^S_{i,j-1}(I) & \xrightarrow{\mu_i} & T^S_{i,j}(I) & \longrightarrow & T^R_{i,j}(I) & \longrightarrow & T^S_{i-1,j}(I) & \xrightarrow{\mu_{i-1}} & T^S_{i-1,j+1}(I) \\ & & & & & & \downarrow \cong & & \downarrow \cong \\ & & & & & & T^S_{i-1,j}(\widetilde{K}_d) & \xrightarrow{\beta} & T^S_{i-1,j+1}(\widetilde{K}_{d+1}) \end{array}$$

The vertical isomorphisms are given by the approximation of syzygies in Lemma 2.8. Since $j > \nu$, by Lemma 4.2, β is an isomorphism, and thus so is μ_{i-1}. By similar arguments, μ_i is an isomorphism for $j > \nu+1$ and a surjection for $j = \nu+1$.

The first line of the diagram induces an exact sequence

$$0 = \operatorname{Coker} \mu_i \to T^R_{i,j}(I) \to \operatorname{Ker} \mu_{i-1} = 0.$$

Hence $T^R_{i,j}(I) = 0$, as desired.

The remaining inequality is a consequence of the fact that

$$\operatorname{reg}_S \widetilde{K}_s \leq \max\{\operatorname{reg}_S \widetilde{K}_{s-1}, \operatorname{reg}_S K_s + s\},$$

which can be seen from the exact sequence $0 \to \widetilde{K}_{s-1} \to \widetilde{K}_s \to K_s(-s) \to 0$. □

Now we present the proof of the first main result of this section.

Proof of Theorem 4.1. By Proposition 4.3, we get

$$\operatorname{reg}_R I \leq \max\{\operatorname{reg}_S \widetilde{K}_{s-1}, \operatorname{reg}_S K_s + s\}.$$

Using the exact sequences $0 \to \widetilde{K}_{i-1} \to \widetilde{K}_i \to K_i(-i) \to 0$, and induction, we get

$$\operatorname{reg}_S \widetilde{K}_{s-1} \leq \max_{1 \leq i \leq s-1} \{\operatorname{reg}_S \widetilde{K}_0, \operatorname{reg}_S K_i + i\} = \max_{1 \leq i \leq s-1} \{\operatorname{reg}_S J, \operatorname{reg}_S K_i + i\}.$$

Hence

$$\operatorname{reg}_R I \leq \max_{1 \leq i \leq s-1} \{\operatorname{reg}_S J, \operatorname{reg}_S K_i + i, \operatorname{reg}_S K_s + s\},$$

as desired. □

Example 4.4. The bound in Theorem 4.1 is sharp. For example, let X be the image of the map $\mathbb{P}^1 \to \mathbb{P}^3$ given by

$$(a : b) \mapsto (a^4 : a^2b^2 : ab^3 : b^4).$$

The defining ideal $I \subseteq k[x, y, z, t]$ of X is $I = (y^2 - xt, z^2 - yt)$. Projecting from the singular point $(1 : 0 : 0 : 0) \in X$ corresponds to eliminating x. We get

$$J = (z^2 - yt), K_1 = (z^2, t),$$

and the stabilization index $s = 1$. Hence $\operatorname{reg} J = 2 < \operatorname{reg} I = 3$. Furthermore, $\operatorname{reg} K_1 + 1 = 3$, so

$$\operatorname{reg} I = 3 = \max\{\operatorname{reg} J, \operatorname{reg} K_1 + 1\}.$$

Example 4.5 (McCullough-Peeva's counterexample [20, Example 4.7]). Denote $k = \mathbb{Z}/32003\mathbb{Z}$. Let $T = k[u, v, t, x, y, z]$ and $L = (u^6, v^6, u^2t^4 + v^2x^4 + uvty^3 + uvxz^3)$. Let $\operatorname{Rees}(L) = T[LX] \subseteq T[X]$ be the Rees algebra of L. Let $\phi : R = k[a_1, \ldots, a_9] \to \operatorname{Rees}(L)$ be the surjection map given by

$$(a_1, a_2, \ldots, a_9) \mapsto (u, v, t, x, y, z, u^6X, v^6X, (u^2t^4 + v^2x^4 + uvty^3 + uvxz^3)X).$$

Let $I = \operatorname{Ker}\phi$. Then I is a homogeneous prime ideal with respect to the standard grading of R. It is a counterexample to the Eisenbud-Goto conjecture: $\operatorname{reg} I = 38 > \deg(R/I) = 31$.

Eliminating a_4 corresponds to an inner projection. Thanks to computations with Macaulay2 [12], J is a principle ideal generated by a form of degree 30, the stabilization index $s = 6$ and $K_6 = (a_1, a_2)$. We also have

$$\operatorname{reg} K_1 = 35,\ \operatorname{reg} K_2 = 33,\ \operatorname{reg} K_3 = 37,$$
$$\operatorname{reg} K_4 = 35,\ \operatorname{reg} K_5 = 33,\ \operatorname{reg} K_6 = 1.$$

Hence $\operatorname{reg} I = 38 > \operatorname{reg} J = 30$ but

$$\operatorname{reg} I < \max_{0\le i\le 6}\{\operatorname{reg} K_i + i\} = \operatorname{reg} K_3 + 3 = 40.$$

We prove a weak reverse inequality of the one in Theorem 4.1 as follows.

Theorem 4.6. *Keep using Notation 2.1. Then there is an inequality*

$$\operatorname{reg}_S J \le \max_{1\le i\le s-1}\left\{\operatorname{reg}_R I, \operatorname{reg} K_i + i + 1, \operatorname{reg} K_s + s - 1\right\}.$$

The main technical step in the proof is given by the following slight improvement of Lemma 4.2. That Lemma 4.7 improves Lemma 4.2 can be seen from the inequality $\operatorname{reg}_R I \le \max\{\operatorname{reg}_S \widetilde{K}_{s-1}, \operatorname{reg}_S \widetilde{K}_s\}$ in Proposition 4.3. But somewhat unexpectedly, the proof of Lemma 4.7 is not applicable to deducing Lemma 4.2.

Lemma 4.7. *For all i and all $d \ge s-1$, the map*

$$T^S_{i,j}(\widetilde{K}_d) \longrightarrow T^S_{i,j+1}(\widetilde{K}_{d+1})$$

induced by the multiplication $\widetilde{K}_d(-1) \xrightarrow{\cdot x_0} \widetilde{K}_{d+1}$ is a surjection if $j = \operatorname{reg}_R I$ and an isomorphism if $j > \operatorname{reg}_R I$.

Proof. Denote $\rho = \operatorname{reg}_R I$. For each $j \ge \rho, d \ge s-1$, denote by $S(j,d)$ the following statement: For all i, the map

$$T^S_{i,j}(\widetilde{K}_d) \longrightarrow T^S_{i,j+1}(\widetilde{K}_{d+1})$$

induced by the multiplication $\widetilde{K}_d(-1) \xrightarrow{\cdot x_0} \widetilde{K}_{d+1}$ is a surjection if $j = \rho$ and an isomorphism if $j > \rho$.

We will prove the following claims:

(1) For all $j \ge \rho, d \ge s-1$, the statement $S(j,d)$ is true if $j \le d$.
(2) For all $d \ge s$, if $S(\rho,d)$ is true then so is $S(\rho, d-1)$.
(3) For $j \ge \rho+1$, $d \ge s$, if $S(j,d)$ and $S(j-1,d)$ are true, then so is $S(j,d-1)$.

Assuming these claims, we prove by induction on $j-d$ that for all $j \ge \rho$ and all $d \ge s-1$, $S(j,d)$ is true. This is the desired conclusion.

Indeed, if $j - d \le 0$, then by (1), $S(j,d)$ is true.

Assume that $S(j,d)$ is true for $j \ge \rho, d \ge s-1$ and $j - d \le g$, where $g \ge 0$. We prove that $S(j,d)$ is true when $j - d = g+1$.

If $j = \rho$, then by (1), $S(\rho, M)$ is true where $M = \max\{\rho, d\} \ge s-1$. Since $M \ge d$, using (2) repeatedly, we conclude that $S(\rho,d)$ is true.

Consider the case $j \ge \rho+1$. Now $j-1 \ge \rho$ and $d+1 \ge s$. Note that $j-1-(d+1) = g-1$ and $j-1-d = g$, so by induction hypothesis, $S(j-1,d+1)$ and $S(j,d+1)$ are true. By (3), we conclude that $S(j,d)$ is also true. This finishes the induction.

It remains to prove the claims (1)–(3).

For (1): Let us deal with the case $j > \rho$ first. Since $j > \rho \geq \operatorname{reg}_R I$, we obtain $T^R_{i,j}(I) = 0$ for all i. A part of the elimination mapping cone sequence looks as follows

$$T^S_{i,j-1}(I) \xrightarrow{\mu} T^S_{i,j}(I) \to T^R_{i,j}(I) = 0 \to T^S_{i-1,j}(I) \xrightarrow{\mu} T^S_{i-1,j+1}(I) \to T^R_{i-1,j+1}(I) = 0.$$

Hence $T^S_{i-1,j}(I) \xrightarrow{\mu} T^S_{i-1,j+1}(I)$ is an isomorphism, and choosing $j = \rho + 1$, we conclude that the map $T^S_{i,\rho}(I) \xrightarrow{\mu} T^S_{i,\rho+1}(I)$ is surjective.

By Lemma 2.8, $T^S_{i,j}(I) \cong T^S_{i,j}(\widetilde{K}_d)$ for all $d \geq j$. Hence for all i, all $j > \rho$, and $d \geq j$, the map $T^S_{i,j}(\widetilde{K}_d) \longrightarrow T^S_{i,j+1}(\widetilde{K}_{d+1})$ is an an isomorphism. In other words, $S(j,d)$ is true if $j > \rho$ and $j \leq d$.

If $j = \rho$, using the surjectivity of the map $T^S_{i,\rho}(I) \xrightarrow{\mu} T^S_{i,\rho+1}(I)$ and Lemma 2.8 once again, we see that $T^S_{i,\rho}(\widetilde{K}_d) \longrightarrow T^S_{i,\rho+1}(\widetilde{K}_{d+1})$ is a surjection for $d \geq \rho$. Hence $S(\rho,d)$ is true for $\rho \leq d$.

For (2): From the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{K}_{d-1}(-1) & \longrightarrow & \widetilde{K}_d(-1) & \longrightarrow & K_d(-d-1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cdot x_0} & & \downarrow{\scriptstyle \cdot x_0} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \widetilde{K}_d & \longrightarrow & \widetilde{K}_{d+1} & \longrightarrow & K_{d+1}(-d-1) & \longrightarrow & 0
\end{array}$$

we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
T_{i+1,\rho-d-1}(K_d) & \to & T_{i,\rho}(\widetilde{K}_{d-1}) & \to & T_{i,\rho}(\widetilde{K}_d) & \to & T_{i,\rho-d}(K_d) \\
\downarrow{\scriptstyle \alpha_{i+1}} & & \downarrow{\scriptstyle \beta_d} & & \downarrow{\scriptstyle \beta_{d+1}} & & \downarrow{\scriptstyle \alpha_i} \\
T_{i+1,\rho-d-1}(K_{d+1}) & \to & T_{i,\rho+1}(\widetilde{K}_d) & \to & T_{i,\rho+1}(\widetilde{K}_{d+1}) & \to & T_{i,\rho-d}(K_{d+1}) \\
\downarrow & & & & \downarrow & & \\
0 & & & & 0 & &
\end{array}$$

Since $d \geq s$, $K_d = K_{d+1} = K_s$, so α_i and α_{i+1} are isomorphisms. Since $S(\rho,d)$ is true, β_{d+1} is a surjection. By diagram chasing (the Four Lemma in particular), we get β_d is surjective. Namely $S(\rho, d-1)$ is true.

For (3): Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \downarrow \\
T_{i+1,j-1}(\widetilde{K}_d) & \to & T_{i+1,j-d-1}(K_d) & \to & T_{i,j}(\widetilde{K}_{d-1}) & \to & T_{i,j}(\widetilde{K}_d) & \to & T_{i,j-d}(K_d) \\
\downarrow{\scriptstyle \gamma} & & \cong\downarrow{\scriptstyle \alpha_{i+1}} & & \downarrow{\scriptstyle \beta_d} & & \cong\downarrow{\scriptstyle \beta_{d+1}} & & \downarrow{\scriptstyle \alpha_i} \\
T_{i+1,j}(\widetilde{K}_{d+1}) & \to & T_{i+1,j-d-1}(K_{d+1}) & \to & T_{i,j+1}(\widetilde{K}_d) & \to & T_{i,j+1}(\widetilde{K}_{d+1}) & \to & T_{i,j-d}(K_{d+1}) \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

Since $S(j,d)$ is true and $j>\rho$, β_{d+1} is an isomorphism. Since $S(j-1,d)$ is true, γ is a surjection for $j=\rho+1$ and is an isomorphism when $j>\rho+1$. Since $d\ge s$, we have $K_d=K_{d+1}=K_s$, thus both α_i and α_{i+1} are isomorphisms. Hence by a diagram chasing (in particular, the Five Lemma), β_d is also an isomorphism. In other words, $S(j,d-1)$ is true. □

The proof of Theorem 4.6 follows readily from

Proposition 4.8. *There is an inequality*

$$\operatorname{reg}_S \widetilde{K}_{s-1} \le \max\{\operatorname{reg}_R I, \operatorname{reg}_S K_s + s - 1\}.$$

Proof. Assume that $\operatorname{reg}_S \widetilde{K}_{s-1} > \max\{\operatorname{reg}_R I, \operatorname{reg}_S K_s + s-1\}$, we will derive a contradiction.

Indeed, let $j_0=\operatorname{reg}_S \widetilde{K}_{s-1}$, then $T^S_{i,j_0}(\widetilde{K}_{s-1})\ne 0$ for some i. Since $j_0>\operatorname{reg}_R I$, by Lemma 4.7, there is an isomorphism

$$T^S_{i,j_0}(\widetilde{K}_{s-1}) \cong T^S_{i,j_0+1}(\widetilde{K}_s).$$

In particular, $\operatorname{reg}_S \widetilde{K}_s \ge j_0+1 = \operatorname{reg} \widetilde{K}_{s-1}+1$. From the short exact sequence

$$0\to \widetilde{K}_{s-1} \to \widetilde{K}_s \to K_s(-s)\to 0,$$

standard "Depth Lemma" arguments yield $\operatorname{reg}_S \widetilde{K}_s = \operatorname{reg}_S K_s(-s) = \operatorname{reg}_S K_s + s$. But then

$$j_0+1\le \operatorname{reg}_S \widetilde{K}_s = \operatorname{reg}_S K_s + s \le \max\{\operatorname{reg}_R I, \operatorname{reg}_S K_s + s-1\}+1 \le j_0,$$

a contradiction. The proof is concluded. □

Now we are ready for the proof of the second main result of this section.

Proof of Theorem 4.6. For each $i\ge 1$, from the exact sequence

$$0\to \widetilde{K}_{i-1}\to \widetilde{K}_i \to K_i(-i)\to 0,$$

we have $\operatorname{reg}_S \widetilde{K}_{i-1} \le \max\{\operatorname{reg}_S \widetilde{K}_i, \operatorname{reg}_S K_i + i + 1\}$. Hence

$$\begin{aligned}
\operatorname{reg}_S J = \operatorname{reg}_S \widetilde{K}_0 &\le \max\{\operatorname{reg}_S \widetilde{K}_1, \operatorname{reg}_S K_1 + 2\}\\
&\le \cdots\\
&\le \max_{1\le i\le s-1}\{\operatorname{reg}_S \widetilde{K}_{s-1}, \operatorname{reg}_S K_i + i+1\}\\
&\le \max_{1\le i\le s-1}\{\operatorname{reg}_R I, \operatorname{reg}_S K_i + i + 1, \operatorname{reg}_S K_s + s-1\}.
\end{aligned}$$

The last inequality is due to Proposition 4.8. The proof is concluded. □

Example 4.9. The bound of Theorem 4.6 is sharp. Indeed, choose the polynomial rings $R=k[x,y,z,t,u]$, $U=k[a,b]$ and $\phi: R\to U$ given as follows

$$(x,y,z,t,u)\mapsto (a^{11}, a^{10}b, a^9b^2, a^7b^4, b^{11}).$$

Denote $I=\operatorname{Ker}\phi$. By Macaulay2, we have

$$I=(z^2-xt, y^2-xz, t^3-xyu, yzt^2-x^3u).$$

Eliminating x, we get the stabilization index $s=3$. Concretely,

$$\begin{gathered}
J=(z^3-y^2t, t^4-yz^2u, zt^3-y^3u),\\
K_1=K_2=(z,t,yu), K_3=(z,t,u).
\end{gathered}$$

Moreover, $\operatorname{reg}_R I = 4, \operatorname{reg}_S J = 5, \operatorname{reg}_S K_1 = \operatorname{reg}_S K_2 = 2$ and $\operatorname{reg}_S K_3 = 1$. Hence

$$\operatorname{reg}_S J = \operatorname{reg}_S K_2 + 3 = \max\{\operatorname{reg}_R I, \operatorname{reg}_S K_1 + 2, \operatorname{reg}_S K_2 + 3, \operatorname{reg}_S K_3 + 2\},$$

the upper bound given by Theorem 4.6.

5. Inner projections of irreducible projective varieties

We note that the results on depth and regularity of projections can be further refined for inner projections of irreducible varieties. The inequalities in the next result strengthen the corresponding inequalities in Theorems 3.1, 3.4, 4.1, and 4.6.

Theorem 5.1. *Let $X \subseteq \mathbb{P}^N$ be a non-degenerate irreducible closed subscheme with defining ideal $I = I_X$, and assume that $q = (1 : 0 : \cdots : 0) \in X \setminus \operatorname{vert}(X)$. Denote by s' the degree of the map $\pi_q : X \dashrightarrow X_q$. Keep using Notation 2.1.*

Consider the inequalities

$$\operatorname{depth}_R \frac{R}{I} \geq \min_{s' \leq i \leq s-1} \left\{\operatorname{depth}_S \frac{S}{J}, \operatorname{depth}_S \frac{S}{K_i}, \operatorname{depth}_S \frac{S}{K_s} + 1\right\}, \tag{5.1}$$

$$\operatorname{depth}_S \frac{S}{J} \geq \min_{s' \leq i \leq s} \left\{\operatorname{depth}_R \frac{R}{I}, \operatorname{depth}_S \frac{S}{K_i} + 1\right\}, \tag{5.2}$$

$$\operatorname{reg}_R I \leq \max_{s' \leq i \leq s} \{\operatorname{reg}_S J + s' - 1, \operatorname{reg}_S K_i + i\}, \tag{5.3}$$

$$\operatorname{reg}_S J \leq \max_{s' \leq i \leq s-1} \{\operatorname{reg}_R I, \operatorname{reg}_S K_i + i + 1, \operatorname{reg}_S K_s + s - 1\} - s' + 1. \tag{5.4}$$

Then (5.1)*,* (5.3)*, and* (5.4) *hold unconditionally, while* (5.2) *holds if* $J_{\leq \operatorname{reg} K_s} = 0$.

Proof. The proof combines minor variations of the arguments in Sections 3 and 4, and Lemma 2.9.

Step 1: We prove (5.1), and note that (5.3) can be established in the same manner.

By Theorem 3.1

$$\operatorname{depth}_R \frac{R}{I} \geq \min_{0 \leq i \leq s-1} \left\{\operatorname{depth}_S \frac{S}{K_i}, \operatorname{depth}_S \frac{S}{K_s} + 1\right\}.$$

By Lemma 2.9, $J = K_0 = K_1 = \cdots = K_{s'-1}$, hence

$$\operatorname{depth}_R \frac{R}{I} \geq \min_{s' \leq i \leq s-1} \left\{\operatorname{depth}_S \frac{S}{J}, \operatorname{depth}_S \frac{S}{K_i}, \operatorname{depth}_S \frac{S}{K_s} + 1\right\},$$

as desired.

Step 2: We prove (5.2) given that $J_{\leq \operatorname{reg} K_s} = 0$. By Theorem 3.4,

$$\operatorname{depth}_S \frac{S}{J} \geq \min_{1 \leq i \leq s} \left\{\operatorname{depth}_R \frac{R}{I}, \operatorname{depth}_S \frac{S}{K_i} + 1\right\}.$$

By Lemma 2.9, $J = K_0 = K_1 = \cdots = K_{s'-1}$, $\operatorname{depth}(S/J) < \operatorname{depth}(S/K_i) + 1$ for $1 \leq i \leq s' - 1$. This implies

$$\operatorname{depth}_S \frac{S}{J} \geq \min_{s' \leq i \leq s} \left\{\operatorname{depth}_R \frac{R}{I}, \operatorname{depth}_S \frac{S}{K_i} + 1\right\},$$

as desired.

Step 3: We prove (5.4). For each $0 \leq i \leq s' - 2$, we have

$$\operatorname{reg}_S \widetilde{K}_i \leq \max\{\operatorname{reg}_S \widetilde{K}_{i-1}, \operatorname{reg}_S K_i + i\} = \max\{\operatorname{reg}_S \widetilde{K}_{i-1}, \operatorname{reg}_S J + i\}.$$

The equality holds because of Lemma 2.9.

Hence

$$\begin{aligned}\operatorname{reg}_S \widetilde{K}_{s'-2} &\le \max\{\operatorname{reg}_S \widetilde{K}_{s'-3}, \operatorname{reg}_S J + s' - 2\}\\ &\le \max\{\operatorname{reg}_S \widetilde{K}_{s'-4}, \operatorname{reg}_S J + s' - 2\}\\ &\le \cdots\\ &\le \max\{\operatorname{reg}_S \widetilde{K}_{0}, \operatorname{reg}_S J + s' - 2\} = \max\{\operatorname{reg}_S J, \operatorname{reg}_S J + s' - 2\}.\end{aligned}$$

Consider the exact sequence

$$0 \to \widetilde{K}_{s'-2} \to \widetilde{K}_{s'-1} \to K_{s'-1}(-(s'-1)) = J(-(s'-1)) \to 0.$$

The last chain yields $\operatorname{reg}_S \widetilde{K}_{s'-2} < \operatorname{reg}_S J(-(s'-1)) + 1 = \operatorname{reg}_S J + s'$. Hence by the standard "Depth Lemma" argument,

$$\operatorname{reg}_S J(-(s'-1)) = \operatorname{reg}_S \widetilde{K}_{s'-1},$$

namely $\operatorname{reg}_S J = \operatorname{reg}_S \widetilde{K}_{s'-1} - (s'-1)$.

For each $i \ge s'$, from the exact sequence

$$0 \to \widetilde{K}_{i-1} \to \widetilde{K}_{i} \to K_i(-i) \to 0,$$

we have $\operatorname{reg}_S \widetilde{K}_{i-1} \le \max\{\operatorname{reg}_S \widetilde{K}_{i}, \operatorname{reg}_S K_i + i + 1\}$. Hence

$$\begin{aligned}\operatorname{reg}_S \widetilde{K}_{s'-1} &\le \max\{\operatorname{reg}_S \widetilde{K}_{s'}, \operatorname{reg}_S K_{s'} + s' + 1\}\\ &\le \cdots\\ &\le \max_{s' \le i \le s-1}\{\operatorname{reg}_S \widetilde{K}_{s-1}, \operatorname{reg}_S K_i + i + 1\}\\ &\le \max_{s' \le i \le s-1}\{\operatorname{reg}_R I, \operatorname{reg}_S K_i + i + 1, \operatorname{reg}_S K_s + s - 1\}.\end{aligned}$$

The last inequality is due to Proposition 4.8. Therefore

$$\operatorname{reg}_S J \le \max_{s' \le i \le s-1}\{\operatorname{reg}_R I, \operatorname{reg}_S K_i + i + 1, \operatorname{reg}_S K_s + s - 1\} - (s'-1).$$

The proof is completed. □

6. Applications

We summarize the results obtained thus far on the depth and regularity of arbitrary projections below.

- Theorem 3.1 (Depth of R/I):
$$\operatorname{depth}_R \frac{R}{I} \ge \min_{1 \le i \le s-1}\left\{\operatorname{depth}_S \frac{S}{J}, \operatorname{depth}_S \frac{S}{K_i}, \operatorname{depth}_S \frac{S}{K_s} + 1\right\}.$$
- Theorem 3.4 (Depth of S/J): If J is generated in degree at least $1 + \operatorname{reg} K_s$, then
$$\operatorname{depth}_S \frac{S}{J} \ge \min_{1 \le i \le s}\left\{\operatorname{depth}_R \frac{R}{I}, \operatorname{depth}_S \frac{S}{K_i} + 1\right\}.$$
- Theorem 4.1 (Regularity of R/I):
$$\operatorname{reg}_R I \le \max_{1 \le i \le s}\{\operatorname{reg}_S J, \operatorname{reg}_S K_i + i\}.$$
- Theorem 4.6 (Regularity of S/J):
$$\operatorname{reg}_S J \le \max_{1 \le i \le s-1}\{\operatorname{reg}_R I, \operatorname{reg}_S K_i + i + 1, \operatorname{reg}_S K_s + s - 1\}.$$

As a consequence to these theorems, we obtain the following a generalization of [14, Theorem 4.1], [15, Theorems 2.9(c) and 2.10]. Note that the last results in [14, 15] require the stronger condition that $s = 1$ and K_1 is generated by linear forms.

Theorem 6.1. *Keep using Notation 2.1. Assume that $s = 1$, K_1 is Cohen-Macaulay, and $I_{\le c} = 0$ where $c = \operatorname{reg} K_1$. Then there are equalities*

$$\operatorname{depth}_R(R/I) = \operatorname{depth}_S(S/J),$$
$$\operatorname{reg}_R I = \operatorname{reg}_S J.$$

Proof. Since $s = 1$, by Theorem 3.1, we get

$$\operatorname{depth}_R(R/I) \ge \min\{\operatorname{depth}_S(S/J), \operatorname{depth}_S(S/K_1) + 1\}.$$

Since K_1 is Cohen-Macaulay, $\operatorname{depth}_S(S/K_1) + 1 = \dim(S/K_1) + 1 = \dim(R/I)$. This implies

$$\operatorname{depth}_R(R/I) \ge \min\{\operatorname{depth}_S(S/J), \dim(R/I)\} = \operatorname{depth}_S(S/J).$$

Since $I_{\le c} = 0$, so is $J_{\le c}$, and the hypotheses of Theorem 3.4 are satisfied. In particular,

$$\begin{aligned}\operatorname{depth}_S(S/J) &\ge \min\{\operatorname{depth}_R(R/I), \operatorname{depth}_S(S/K_1) + 1\}\\ &= \min\{\operatorname{depth}_R(R/I), \dim(R/I)\} = \operatorname{depth}_R(R/I).\end{aligned}$$

Therefore, $\operatorname{depth}_R(R/I) = \operatorname{depth}_S(S/J)$. This yields the first assertion.

For the second assertion: By the hypothesis, $I_{\le c} = J_{\le c} = 0$, hence

$$\min\{\operatorname{reg}_R I, \operatorname{reg}_S J\} \ge c + 1.$$

Using Theorem 4.1,

$$\operatorname{reg}_R I \le \max\{\operatorname{reg}_S J, \operatorname{reg}_S K_1 + 1\} = \max\{\operatorname{reg}_S J, c + 1\} = \operatorname{reg}_S J.$$

Using Theorem 4.6,

$$\operatorname{reg}_S J \le \max\{\operatorname{reg}_R I, \operatorname{reg}_S K_1 + 1 - 1\} = \max\{\operatorname{reg}_R I, c\} = \operatorname{reg}_R I.$$

Therefore $\operatorname{reg}_R I = \operatorname{reg}_S J$. The proof is completed. □

For irreducible closed subschemes of the projective space, we have the following result, which gives a more precise conclusion than Theorem 6.1.

Theorem 6.2. *Let $X \subseteq \mathbb{P}^N$ be a non-degenerate irreducible closed subscheme with homogeneous defining ideal $I = I_X$. Assume that $q = (1 : 0 : \cdots : 0) \in X \setminus \operatorname{vert}(X)$. Keep using Notation 2.1. Assume furthermore that the following conditions are satisfied:*

1. $\deg \pi_q = s$,
2. *K_s is Cohen-Macaulay,*
3. $I_{\le c+\max\{0,s-2\}} = 0$ *where* $c = \operatorname{reg} K_s$.

Then there are equalities

$$\operatorname{depth}_R(R/I) = \operatorname{depth}_S(S/J),$$
$$\operatorname{reg}_R I = \operatorname{reg}_S J + s - 1.$$

Proof. The proof is similar to that of Theorem 6.1, but we invoke Theorem 5.1 instead. With the notation of the last result, we have $s' = s$, hence Theorem 5.1 yields the following inequalities

$$\operatorname{depth}(R/I) \geq \min\{\operatorname{depth}(S/J), \operatorname{depth}(S/K_s) + 1\},$$
$$\operatorname{depth}(S/J) \geq \min\{\operatorname{depth}(R/I), \operatorname{depth}(S/K_s) + 1\}.$$

Since S/K_s is Cohen-Macaulay, arguing as in the proof of Theorem 6.1, we get $\operatorname{depth}(R/I) = \operatorname{depth}(S/J)$.

For the equality of regularities, note that $J_{\leq c+\max\{0,s-2\}} = I_{\leq c+\max\{0,s-2\}} = 0$. Hence

$$\min\{\operatorname{reg} I, \operatorname{reg} J\} \geq c + \max\{1, s-1\}.$$

As $s' = s$, again by Theorem 5.1,

$$\operatorname{reg} I \leq \max\{\operatorname{reg} J + s - 1, c + s\} = \operatorname{reg} J + s - 1,$$
$$\operatorname{reg} J \leq \max\{\operatorname{reg} I - s + 1, c\} = \operatorname{reg} I - s + 1.$$

Therefore $\operatorname{reg} I = \operatorname{reg} J + s - 1$, as desired. □

Below, we need some terminology of Gröbner basis theory. For a polynomial $f \in R = k[x_0, \ldots, x_n]$ and a monomial order τ, denote by $\operatorname{lt}_\tau f$ the leading term of f with respect to the order τ. We usually write simply $\operatorname{lt} f$ if τ is clear from the context. For more details, see, for example, [8, Chapter 2].

The following example is perhaps known to experts; see, for example, [7, Lemma 3.11], [5, 6] for related results.

Example 6.3. Let $s \geq 1$ be any integer. Let $I = I(n)$ be the ideal of 2-minors of the matrix

$$M(n) = \begin{pmatrix} x_0^s & x_1^s & \cdots & x_{n-1}^s \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}.$$

We claim that $I(n)$ is a Cohen-Macaulay prime ideal of $R = k[x_0, x_1, \ldots, x_n]$ and $\operatorname{reg} I(n) = (n-1)(s-1) + 2$ for any $n \geq 2$.

Step 1: Note that I is generated by the 2-minors

$$f_{i,j} = \begin{vmatrix} x_i^s & x_j^s \\ x_{i+1} & x_{j+1} \end{vmatrix} = x_i^s x_{j+1} - x_j^s x_{i+1},$$

where $0 \leq i < j \leq n-1$. Let τ be any monomial order on R such that $\operatorname{lt}_\tau f_{i,j} = x_i^s x_{j+1}$ for all $0 \leq i < j \leq n-1$ (a *diagonal order*). The elements $f_{i,j}$ form a Gröbner basis for I with respect to any diagonal order. This can be proved using the Buchberger criterion and standard arguments on S-polynomials of minors.

Consider the elimination order τ for x_0 refining the graded lex order on S induced by $x_1 > \cdots > x_n$. One can check that $\operatorname{lt}_\tau f_{i,j} = x_i^s x_{j+1}$ for every $0 \leq i < j \leq n-1$. Hence with respect to τ, the 2-minors of $M(n)$ form a Gröbner basis for $I(n)$. The same thing happens if we take the graded lex order on R induced by $x_0 > x_1 > \cdots > x_n$ instead (which is also a diagonal order).

Similarly, we can show that:

(1) In the afore-mentioned graded lex order in S, the ideal I' of 2-minors of the matrix

$$\begin{pmatrix} 1 & x_1^s & \cdots & x_{n-1}^s \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

admits the collection of 2-minor generators as a Gröbner basis.

Note that I' is the defining ideal of the affine curve in k^n with the parametrization $b \mapsto (b, b^{s+1}, b^{s^2+s+1}, \ldots, b^{s^{n-1}+s^{n-2}+\cdots+s+1})$. Moreover, I' is a prime ideal since $S/I' \cong k[x_1]$.

Step 2: We have $I(n)$ is the homogenization of I'. Indeed, this follows from [8, Chapter 8, §4, Theorem 4], using the Gröbner basis statement (1) in Step 1. In particular, as I' is a prime ideal, so is $I(n)$.

Step 3: Consider the elimination of x_0 and the elimination order for x_0, called τ, in Step 1. By Lemma 2.4, the stabilization index of this elimination is at most s. With the order τ, by Step 1 and Lemma 2.5, we see that

$$K_0 = K_1 = \cdots = K_{s-1} = J,$$

where $J \cong I(n-1)$ is the ideal of 2-minors of

$$M' = \begin{pmatrix} x_1^s & \cdots & x_{n-1}^s \\ x_2 & \cdots & x_n \end{pmatrix}.$$

Moreover, $K_s = J + (x_2, \ldots, x_n) = (x_2, \ldots, x_n) \neq K_{s-1}$. Hence the stabilization index is exactly s.

Step 4: Let $q = (1 : 0 : \cdots : 0) \in \mathbb{P}^n$. In view of Lemma 2.9, we see that $\deg \pi_q = s$. So hypotheses (1) and (2) of Theorem 6.2 are satisfied. As $c = \operatorname{reg} K_s = 1$, I is generated in degree $s+1$, $I_{c+\max\{0,s-2\}} = I_{\max\{1,s-1\}} = 0$, and hypothesis (3) of *ibid.* is also satisfied. Applying this result, there are equalities

$$\begin{aligned} \operatorname{depth}(R/I(n)) &= \operatorname{depth}(S/I(n-1)), \\ \operatorname{reg} I(n) &= \operatorname{reg} I(n-1) + s - 1. \end{aligned}$$

Hence by induction on $n \geq 2$, we get that $\operatorname{depth} R/I(n) = 2$, i.e. $I(n)$ is Cohen-Macaulay, and $\operatorname{reg} I(n) = (n-1)(s-1) + 2$.

Example 6.4. Let $k = \mathbb{Z}/32003$. Let $C \subseteq \mathbb{P}^6$ be the rational normal curve, which is defined by the 2-minors of the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{pmatrix}.$$

Let $X = \operatorname{Sec}^2(C)$ be the second secant variety of C, which is defined by the 3-minors of the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_4 & a_5 & a_6 & a_7 \end{pmatrix}.$$

The ideal $I = I_X$ has the following Betti table.

	0	1	2
total:	10	15	6
3:	10	15	6

Take a closed point $q \in X$. Let J be the defining ideal of $X_q = \overline{\pi_q(X \setminus q)}$.

Case 1: If q is a general point in C, computations with Macaulay2 [12] show that the stabilization index of the partial elimination ideals is $s = 1$ and K_1 has the following Betti table.

	0	1	2
total:	6	8	3
2:	6	8	3

Hence K_1 is Cohen-Macaulay and has a 2-linear resolution. Note that $I_{\le 2} = 0$. Hence by Theorem 6.1, $\operatorname{depth}(R/I) = \operatorname{depth}(S/J)$ and $\operatorname{reg} I = \operatorname{reg} J$.

Case 2: If q is a general point of X (and hence $q \notin C$), computations with Macaulay2 [12] show that the stabilization index of the partial elimination ideals is $s = 2$, and K_2 is generated by linear forms. Moreover, K_1 has the following Betti table.

	0	1	2
total:	5	5	1
2:	5	5	.
3:	.	.	1

On the other hand, J has the following Betti table.

	0	1
total:	2	1
3:	2	.
4:	.	.
5:	.	1

In this case, K_1 is Cohen-Macaulay, $\operatorname{reg} K_1 = 3$. Since $\operatorname{pd}_S J = 1$ and $T^S_{1,5}(J) \neq 0$, the hypotheses of Theorem 3.4 are satisfied. Using Theorems 3.1 and 3.4, we can confirm that $\operatorname{depth}(R/I) = \operatorname{depth}(S/J)$.

In our last example, we apply Theorem 6.1 to study the ideal of maximal minors of a generic matrix.

Example 6.5. Let $X = (x_{i,j})_{m\times n}$ be a generic $m \times n$ matrix of indeterminates, where $m \le n$. Let $R = k[x_{i,j} : 1 \le i \le m, 1 \le j \le n]$, and $I = I_m(X)$, the ideal of maximal minors of X. Using Theorems 3.1 and 4.1, as well as a classical result about Gröbner bases of determinantal ideals, we give new proofs of the well-known facts that $I_m(X)$ is Cohen-Macaulay and has a linear resolution.

Induct on $m \ge 1$ and $n - m \ge 0$. If $m = 1$ then $R/I \cong k$, I is generated by linear forms, and we are done. If $n - m = 0$, then $I_m(X)$ is a principal ideal, hence again we are done. Assume that $m \ge 2$ and $n - m \ge 1$.

Let X' be the $(m-1) \times (n-1)$ matrix obtained by removing the first row and first column from X. Let Y be the $m \times (n-1)$ matrix obtained by removing the first column from X. We claim that when eliminating $x_{1,1}$ from $I = I_m(X)$, we get $J = I_m(Y)$ and $K_1 = I_{m-1}(X')$.

Indeed, let $S = k[x_{i,j} : 1 \le i \le m, 1 \le j \le n, (i,j) \neq (1,1)]$ and denote $x_0 = x_{1,1}$. Consider the graded lex order on S induced by the ordering of variables in which $x_{i,j} > x_{i',j'}$ if either $i > i'$ or $i = i'$ and $j > j'$. Let τ be the elimination order for x_0 on R refining the graded lex order on S. Then τ is a diagonal order, and the generators of I form a Gröbner basis in this order; see [3, Theorem 5.3] for more details. So J is the generated by the elements of I which do not involve $x_{1,1}$. This means $J = I_m(Y)$.

As in Notation 2.1, write any polynomial $f \in R$ in the form $f = x_0^{d_0(f)}\overline{f} + g$, where $\overline{f} \in S, g \in R$ and $d_0(g) < d_0(f)$. By Lemma 2.5, K_1 has a Gröbner basis $\overline{f}$, where f runs through the minimal generators of I. This implies that $K_1 = I_{m-1}(X') + I_m(Y)$. By Laplace's expansion, $I_m(Y) \subseteq I_{m-1}(X')$, so $K_1 = I_{m-1}(X')$. By Lemma 2.4, the stabilization index satisfies $s \le 1$. Clearly $J \subsetneq I_{m-1}(X')$, so $s = 1$.

By the induction hypothesis, K_1 is Cohen-Macaulay with a $(m-1)$-linear resolution. Applying Theorem 6.1 and the induction hypothesis,

$$\operatorname{reg} I_m(X) = \operatorname{reg} J = \operatorname{reg} I_m(Y) = m,$$
$$\operatorname{depth}(R/I) = \operatorname{depth}(S/J) = \dim(S/J) = \dim(R/I).$$

The proof is concluded.

Appendix A. Proof of Lemma 2.3

We give an algebraic proof for the geometric interpretation of partial elimination ideals in the inner projection case. The main technical result of this section, inspired by [18, Lemma 3.2], is as follows.

Proposition A.1. *Let $S = k[x_1, \dots, x_N]$, and $R = S[x_0]$ be polynomial rings. Denote $\mathfrak{q} = (x_1, x_2, \dots, x_N) \subseteq R$ and $L = (x_2, \dots, x_N) \subseteq S$. Let I be a homogeneous ideal of R contained in $\mathfrak{q}$. Consider the elimination of x_0. Assume that $K_\infty(I) \not\subseteq L$. Denote $i = \max\{j \ge 0 : K_j(I) \subseteq L\}$ if $K_0(I) \subseteq L$, and -1 otherwise.*

Identify $\overline{R} = R/LR$ with $k[x_0, x_1]$. Denote by $\overline{()}$ the residue class modulo LR. Then $\overline{I}^{\text{sat}}$ is a principal ideal of $\overline{R}$. Moreover, any homogeneous generator of $\overline{I}^{\text{sat}}$ can be written as

$$\overline{h} = x_1^e(\alpha x_0^{i+1} + g) + LR,$$

where $\alpha \in k \setminus 0, g \in k[x_0, x_1], d_0(g) \le i$, and $e \ge 1$.

Before proving this result, we show how to deduce Lemma 2.3 from it. Below, if M is a finitely generated (graded) module over a noetherian local ring (a standard graded k-algebra), $e(M)$ denotes its Hilbert-Samuel multiplicity (Hilbert multiplicity, respectively). An immediate corollary of Proposition A.1 is

Corollary A.2. *With the hypotheses of Proposition A.1, there is an equality*

$$e\left(\frac{R}{I+LR}\right) - e\left(\left(\frac{R}{I+LR}\right)_{\mathfrak{q}}\right) = i+1.$$

Proof. Take any generator $\overline{h}$ of $\overline{I}^{\text{sat}}$. By Proposition A.1, we can write

$$\overline{h} = x_1^e(\alpha x_0^{i+1} + g) + LR,$$

where $\alpha \in k \setminus 0, g \in k[x_0, x_1], d_0(g) \le i$ and $e \ge 1$. It remains to observe that

$$e(\overline{R}/\overline{I}) = e(\overline{R}/\overline{I}^{\text{sat}}) = e(\overline{R}/\overline{h}) = e + i + 1,$$

and $(R/(I+LR))_{\mathfrak{q}} \cong (k[x_0, x_1]/(x_1^e))_{(x_1)}$ has multiplicity e. □

Corollary A.3. *Let $X \subseteq \mathbb{P}^N$ be a closed subscheme with saturated defining ideal I, such that $q = (1 : 0 : \cdots : 0) \in X$. Keep using Notation 2.1. Let $z \in \mathbb{P}^{N-1}$ be a closed point such that $z \notin V(K_\infty(I))$. Let $L \subseteq S$ be the defining ideal of z, and $\mathfrak{q} = (x_1, \dots, x_N)$. Then the following statements are equivalent:*

(i) $z \in V(K_i(I))$,

(ii) $e\left(\frac{R}{I+LR}\right) - e\left(\left(\frac{R}{I+LR}\right)_{\mathfrak{q}}\right) \ge i+1$,

(iii) $\operatorname{length}((X \setminus q) \cap \langle q, z \rangle) \ge i+1$.

Proof. By change of coordinates, we may assume that $z = (1 : 0 : \cdots : 0) \in \mathbb{P}^{N-1}$, i.e. $L = (x_2, \dots, x_N)$. Note that $X \cap \langle q, z\rangle = X \cap L$ is defined by $(I + LR)^{\text{sat}}$. The desired assertions are immediate from Corollary A.2 and the fact that length$((X \setminus q) \cap \langle q, z\rangle) = e\,(R/(I + LR)) - e\left((R/(I + LR))_{\mathfrak{q}}\right)$. □

Proof of Lemma 2.3. Directly follows from (i) $\Longleftrightarrow$ (iii) in Corollary A.3. □

It remains to prove Proposition A.1. A key step in the proof is a result motivated by the theory of Sylvester matrices and resultants. Let S be a commutative ring with unit, and $R = S[x]$ a polynomial extension.

Notation A.4 (Killing the leading term for two polynomials)**.** Consider elements $u = \sum_{i=0}^{d} x^{d-i}u_i, z = \sum_{j=0}^{e} x^{e-j}z_j$ in R, where $u_i, z_j \in S$. We define the element $R(u, z)$ as follows. If $d \le e$, we let $R(u, z) = x^{e-d}z_0u - u_0z$, otherwise let $R(u, z) = z_0u - x^{d-e}u_0z$.

Hence $R(u, z)$ belongs to (u, z) and introducing $R(u, z)$ "kills the leading term" of the element with higher x-degree among u and z.

The notion of partial elimination ideals in Notation 2.1 can be extended easily to more general base rings.

Notation A.5. Let I be an ideal of $R = S[x]$. For each $i \ge 0$, let

$$K_i(I) = \{f \in S : \text{ for some } g \in S, \deg_x g \le i - 1, \text{ we have } x^i f + g \in I\}.$$

Clearly $K_i(I)$ is an ideal of S and $K_0(I) = I \cap S \subseteq K_1(I) \subseteq K_2(I) \subseteq \cdots \subseteq S$. By convention, set $K_j(I) = (0)$ for $j \le -1$.

A crucial step in the proof of Proposition A.1 is accomplished by the following elementary result.

Lemma A.6 (Subresultant Lemma)**.** *Let S be a commutative ring with unit, $R = S[x]$ a polynomial extension. Let c, d, e be integers such that $c \ge 0, \min\{d, e\} \ge c+1$. Let $u = \sum_{i=0}^{d} x^{d-i}u_i, z = \sum_{j=0}^{e} x^{e-j}z_j$ be elements in R, where $u_i, z_j \in S$. Assume that $I \subseteq R, L \subseteq S$ are ideals containing (u, z) and $(u_0, \dots, u_{d-c-1})$, respectively.*

Then there exist $w \in K_c(I)$ and integers $\alpha, \beta \ge 0$ such that $w - z_0^{\alpha}u_{d-c}^{\beta} \in L$.

Sketch of proof. The idea is to induct on $d + e$, and each time replace either u or z by $R(u, z)$ to lower the x-degree.

First consider the case $d = c + 1$. We prove the stronger statement that there exists $w \in K_c(I)$ such that $w - z_0u_1^{e-c} \in L$. Note $d = c + 1 \le e$, so

$$\begin{aligned} R(u, z) &= z_0x^{e-d}u - u_0z \\ &= (z_0u_1 - z_1u_0)x^{e-1} + (z_0u_2 - z_2u_0)x^{e-2} + \cdots + (z_0u_d - z_du_0)x^{e-d} - \\ &\quad - z_{d+1}u_0x^{e-d-1} - \cdots - z_eu_0. \end{aligned}$$

If $e = c + 1$, then the above expression shows that $w = z_0u_1 - z_1u_0 \in K_c(I)$. By the hypothesis, $u_0 \in L$ so $w - z_0u_1 \in L$.

If $e \ge c + 2$, then $e - 1 \ge c + 1$. The induction hypothesis for u and $z' = R(u, z)$ yields $w \in K_c(I)$ such that $w - z_0'u_1^{e-1-c} = w - (z_0u_1 - z_1u_0)u_1^{e-1-c} \in L$. As $u_0 \in L$, the last inclusion yields $w - z_0u_1^{e-c} \in L$.

Now assume that $d \ge c + 2$. Consider two cases.

Case 1: If $d \geq e$, let $u' = R(u, z)$. The induction hypothesis for u' and z gives $w \in K_c(I)$ and $\alpha, \beta \geq 0$ such that $w - z_0^{\alpha} {u'}_{d-1-c}^{\beta} \in L$. Set $z_i = 0$ if $i > e$. Then $u'_{d-1-c} = z_0 u_{d-c} - u_0 z_{d-c} \equiv z_0 u_{d-c}$ modulo L, consequently $w - z_0^{\alpha+\beta} u_{d-c}^{\beta} \in L$.

Case 2: $d \leq e-1$. This is the most subtle case. Define inductively the elements $u^{(i)}$ as follows: $u^{(0)} = u$, and $u^{(i+1)} = R(u^{(i)}, z)$ for each $i \geq 0$. Then $u^{(i)}$ starts with a monomial of form $u_0^{(i)} x^{e-1}$ for each $i \geq 1$, and more importantly, for $i = d-c$, one can show that $u_0^{(d-c)} \equiv z_0^{d-c} u_{d-c}$ modulo L. The last follows from the hypothesis $(u_0, \dots, u_{d-1-c}) \subseteq L$.

Now consider u and $\widetilde{z} = u^{(d-c)}$. The induction hypothesis applies to give $w \in K_c(I)$, $\alpha, \beta \geq 0$ such that $w - \widetilde{z}_0^{\alpha} u_{d-c}^{\beta} \in L$. By the above observation on $u_0^{(d-c)}$, working modulo L, $w \equiv \widetilde{z}_0^{\alpha} u_{d-c}^{\beta} \equiv z_0^{\alpha(d-c)} u_{d-c}^{\alpha+\beta}$. This finishes the induction, and the proof of the lemma. □

We are ready to prove the main technical result of this appendix.

Proof of Proposition A.1. Since $K_\infty(I) \not\subseteq L$, I is not contained in LR, hence $\overline{I} \neq (0)$. Moreover as $I \subseteq \mathfrak{q}$, it implies that $\overline{I} \subseteq (x_1)$. Hence $\operatorname{ht} \overline{I} = 1 = \operatorname{ht} \overline{I}^{\text{sat}}$. In particular, $\overline{R}/\overline{I}^{\text{sat}}$ is Cohen-Macaulay of dimension 1. Using for example the Auslander-Buchsbaum formula, we conclude that $\overline{I}^{\text{sat}}$ is a principal ideal of $k[x_0, x_1]$.

Any generator $\overline{h}$ of $\overline{I}^{\text{sat}}$ has the form $\overline{h} = x_1^e(\alpha x_0^c + g) + LR$, where $\alpha \in k \setminus 0, g \in k[x_0, x_1]$ homogeneous of degree $c \geq 0$ such that $d_0(g) \leq c-1$, and $e \geq 1$. We have to show that $c = i+1$.

Step 1: We claim that $c \leq i+1$.

Take $j \leq c-1$, $f \in K_j(I)_\ell$, it suffices to show that $f \in L$.

By definition, there exists $f' \in (\mathfrak{q}^{\ell+1})_{\ell+j}$ with $d_0(f') \leq j-1$ such that $x_0^j f + f' \in I_{\ell+j} \cap (\mathfrak{q}^\ell)_{\ell+j}$. Working modulo LR,

$$\overline{x_0^j f + f'} \in \overline{I}^{\text{sat}}_{\ell+j} \cap (\overline{\mathfrak{q}}^\ell)_{\ell+j} = (x_1^{\max\{\ell,e\}}(\alpha x_0^c + g))_{\ell+j}.$$

Since $\max\{\ell, e\} + c \geq \ell + c \geq \ell + j + 1$, we have $\overline{x_0^j f + f'} = 0$, namely $x_0^j f + f' \in LR$. If $f \notin L$ then $j = d_0(x_0^j f + LR) = d_0(f' + LR) \leq d_0(f') \leq j-1$, a contradiction. Hence $f \in L$, as desired.

Step 2: We show that $c \geq i+1$. Assume the contrary, that $c \leq i$. Since $\overline{h} \in \overline{I}^{\text{sat}}$, for $d \gg 0$, $x_1^d \overline{h} \in \overline{I}$. In particular, there exists $v \in LR$ such that $u = x_1^{d+e}(\alpha x_0^c + g) - v \in I$.

Case 2a: $d_0(v) \leq c$. In this case $v = x_0^c v' + v''$, where $v' \in S, v'' \in R, d_0(v'') \leq c-1$. Clearly $d_0(u) \leq c$. From $v \in LR$, we deduce $v' \in L, v'' \in LR$. Now

$$u = x_1^{d+e}(\alpha x_0^c + g) - v = x_0^c(\alpha x_1^{d+e} - v') + x_1^{d+e} g - v''.$$

As $\alpha \neq 0$, $\alpha x_1^{d+e} \notin L$, so $\alpha x_1^{d+e} - v' \neq 0$ and it does not belong to L. In particular, $K_c(I) \not\subseteq L$. This contradicts the fact that $K_i(I) \subseteq L$, and the assumption $c \leq i$.

Case 2b: $d_0(v) \geq c+1$. In this case $d_0(u) = d_0(v) = d_1 \geq c+1$. Moreover, if we write $u = x_0^{d_1} u_0 + x_0^{d_1-1} u_1 + \cdots + x_0 u_{d_1-1} + u_{d_1}$, where $u_j \in S$, then $u_0, \dots, u_{d_1-c-1}$ all belong to L, while $u_{d_1-c} \equiv \alpha x_1^{d+e}$ (modulo L) does not.

As $K_\infty(I) \not\subseteq L$, there exists a $z \in I$ such that $d_0(z) = e_1$, and in the expression $z = x_0^{e_1} z_0 + x_0^{e_1-1} z_1 + \cdots + x_0 z_{e_1-1} + z_{e_1}$ (where $z_j \in S$), the relation $z_0 \notin L$ holds. The definition of i yields $e_1 \geq i+1 \geq c+1$.

Thus $(u, v) \subseteq I$, $(u_0, \ldots, u_{d_1-c-1}) \subseteq L$, and $\min\{d_1, e_1\} \geq c+1$. The Subresultant Lemma A.6 yields $w \in K_c(I)$ and $\alpha_1, \beta_1 \geq 0$ such that $w - z_0^{\alpha_1} u_{d_1-c}^{\beta_1} \in L$. Since L is a prime ideal, z_0, u_{d_1-c} are not in L, neither is w. This again leads to the contradiction $K_c(I) \not\subseteq L$. Therefore $c \geq i+1$, and the proof is completed. □

Acknowledgments

Parts of this work were done during visits of H.D. Nguyen to Korea Advanced Institute of Science and Technology (KAIST), and to Vietnam Institute for Advanced Study in Mathematics (VIASM). He is grateful to the hospitality and generous support of colleagues and staff members at KAIST and VIASM. The authors are grateful to the anonymous referee for many useful suggestions on the manuscript.

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