

Semigroup property of fractional differential operators and its applications

Nguyen Dinh Cong*

28 July 2021

Abstract

We establish partial semigroup property of families of Riemann-Liouville and Caputo fractional differential operators. Using this result we prove theorems on reduction of multi-term fractional differential systems to single-term and multi-order systems. As an application we obtain existence and uniqueness of solution to multi-term Caputo fractional differential systems.

1 Introduction

The theory of fractional differential equations has been developed fast in the last few decades due to the contributions of scientists from various fields of research (see [4, 7, 10] and the references therein). With its long memory and nonlocal feature the fractional differential equations serve better for modeling real world processes in many applications from physics, engineering to biology and finance. It is of great importance, both from the theoretical and practical point of view, to understand the fundamental properties of fractional differential operators, especially those similar to the well investigated properties of the classical integer order differential operators.

In this paper we deal with one of the fundamental properties of the fractional differential operators, namely we study the semigroup property of the families of Riemann-Liouville and Caputo fractional differential operators. It is known that one of the important property of the classical differentiation is its semigroup property: if a function $f \in C^m[a, b]$, $[a, b] \subset \mathbb{R}$, $m \in \mathbb{N}$, i.e. f is m times continuously differentiable on $[a, b]$, then for all $n, l \in \mathbb{N}$, $n + l \leq m$ we have $f \in C^n[a, b]$, $D^n f \in C^l[a, b]$ and $D^l(D^n f) = D^{n+l} f$;

*ndcong@math.ac.vn, Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, 10307 Ha Noi, Viet Nam.

and vice versa. This semigroup property allows us to reduce a high order ordinary differential equation to a high-dimensional system of ordinary differential equations of order one, for which a well established theory is already developed. As for the fractional differential operators it is well known that the semigroup property does, in general, not hold (see [4, 10]). Diethelm [4, Lemma 3.13] has proved a kind of weak semigroup property of the Caputo fractional differential operators and used it to derive a theorem on reduction of a multi-term Caputo fractional differential equation to a single-term Caputo fractional differential equation. This result is a useful tool for studying multi-term fractional differential equations, since it allows us to reduce a multi-term system to a single-term system for which many tools of investigation are available. There are a number of works dealing with this problem, especially the reduction to single-term fractional differential equations is used to study the asymptotic behavior of solutions to multi-order fractional differential systems [4, 5, 1, 3, 8, 11]. However, for a proof of this weak semigroup property of Caputo fractional differential operators, Diethelm [4] requires a strong assumption of smoothness of the functions under differentiation. This high smoothness assumption seems unnatural, and can hardly be fulfilled for a general Caputo fractional differential equation. For a simple example let us consider an initial value problem with Caputo fractional differential equation

$${}^C D_0^\alpha x(t) - \lambda x(t) = 0, \quad t \in [0, 1], 0 < \alpha < 1, \lambda \in \mathbb{R}, x(0) = x_0 \neq 0.$$

The solution to this problem is the Mittag-Leffler function $x_0 E_\alpha(\lambda t^\alpha)$, which is continuous on $[0, 1]$ but its derivative, $\frac{d}{dt} x_0 E_\alpha(\lambda t^\alpha) = x_0 \lambda t^{\alpha-1} E_{\alpha, \alpha}(\lambda t^\alpha)$, has a singularity at 0, hence the solution $x_0 E_\alpha(\lambda t^\alpha) \notin C^1[0, 1]$, thus does not have the degree of smoothness required by Diethelm [4] for this Caputo fractional differential equation. The problem is more visible if we replace the constant λ in the above equation by a piecewise constant or piecewise continuous function $\lambda(t)$, what can easily lead to non-differentiability of the solution in points in the interior of the interval $[0, 1]$. This shows that the strong assumption of smoothness is hardly satisfied, so that the results of Diethelm [4, Lemma 3.13, Theorems 8.1, 8.2, 8.9] have limited applicability to the Caputo fractional differential equations. This example also shows that a natural requirement for the solutions of initial value problems with Caputo fractional differential equations is continuous Caputo fractional differentiability of order α —the order of the fractional differential equation, and that is what we do in this paper.

In this paper we will prove a (partial) semigroup property of the families of Riemann-Liouville and Caputo fractional differential operators under

natural assumptions of fractional differentiability of the function under the operators.

The paper is organized as follow. In Section 2 we present the definitions of Riemann-Liouville and Caputo fractional differential operators which will be used in this paper. We present two approaches to the definition of the operators: the integrable and the continuous setting. In Section 3 we present the main results of the paper: theorems on (partial) semigroup property of the families of Riemann-Liouville and Caputo fractional differential operators. In Section 4 we present two important applications of the results on semigroup property derived in Section 3, namely the reduction of multi-term Caputo fractional differential equations to single-term and multi-order Caputo fractional differential systems, and a proof of the existence and uniqueness of solution to multi-term Caputo fractional differential equation. Section 5 presents two illustrative examples for the results of the preceding sections.

2 Preliminaries

Let $[a, b] \subset \mathbb{R}$, $-\infty < a < b < \infty$, be a closed finite interval of the real line. Denote by $C[a, b]$ the space of continuous functions, by $C^m[a, m]$, $1 \leq m \in \mathbb{N}$, the space of m times continuously differentiable functions, and by $L_1[a, b]$ the space of integrable functions on $[a, b]$. Recall the notion of Riemann-Liouville fractional integral operator (see e.g. [4]). For $\alpha > 0$, the operator J_a^α defined on $L_1[a, b]$ by

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt,$$

for $a \leq x \leq b$ is called *Riemann-Liouville fractional integral operator of order α* . For $\alpha = 0$, we set $J_a^0 = I$, the identity operator.

The integral operator J_a^α has semigroup property with respect to its parameter $\alpha \geq 0$,

$$J_a^\alpha J_a^\beta = J_a^{\alpha+\beta} \quad \text{for all } \alpha > 0, \beta > 0.$$

Furthermore, clearly $J_a^\alpha f(x) \in C[a, b]$, hence the range of the operator J_a^α , $\alpha > 0$, is in $C[a, b]$. Clearly, $C[a, b] \subset L_1[a, b]$ hence J_a^α can also be considered as an operator from $C[a, b]$ to $C[a, b]$ for all $\alpha \geq 0$.

In this paper we deal with two families of well-known fractional differential operators, Riemann-Liouville and Caputo fractional differential operators, which are defined based on the Riemann-Liouville integral operator.

Like the integral operator we can consider the differential operators, which are inverses of the integral operator, in two settings: integrable and continuous, what we do below (cf. [4, 12]).

Denote by $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$ the sets of real numbers, rational numbers, integers and natural numbers, respectively. For a real number $\alpha \in \mathbb{R}$ we denote by $\lceil \alpha \rceil$ the ceiling of α , i.e. smallest integer no smaller α , by $\lfloor \alpha \rfloor$ the floor of α , i.e. the greatest integer no greater than α .

Definition 1 (Riemann-Liouville fractional derivative). Let $0 < \alpha \in \mathbb{R}$ and $m := \lceil \alpha \rceil$. The operator ${}^{RL}D_a^\alpha$ defined on $L_1[a, b]$ by

$${}^{RL}D_a^\alpha f := D^m J_a^{m-\alpha} f,$$

where D^m denotes the ordinary differential operator of integer order m , is called the *Riemann-Liouville fractional operator of order α* . For $\alpha = 0$, we set ${}^{RL}D_a^0 = I$, the identity operator.

If the function $D^m J_a^{m-\alpha} f$ exists and is in the class $L_1[a, b]$ then we say that the function f is *Riemann-Liouville α -differentiable*.

If $f \in C[a, b]$ and $D^m J_a^{m-\alpha} f$ exists and is in the class $C[a, b]$ then we say that f is *continuously Riemann-Liouville α -differentiable*.

So, in the integrable setting we consider ${}^{RL}D_a^\alpha : L_1[a, b] \rightarrow L_1[a, b]$, whereas in the continuous setting ${}^{RL}D_a^\alpha : C[a, b] \rightarrow C[a, b]$. Clearly, for each fixed $\alpha > 0$, there are functions $f \in L_1[a, b]$ which are not Riemann-Liouville α -differentiable, and there are functions $f \in C[a, b]$ which are neither continuously Riemann-Liouville α -differentiable nor Riemann-Liouville α -differentiable.

Now we recall a definition of Caputo fractional differential differentiation in both the integrable and continuous settings (cf. [4, 12]).

Definition 2 (Caputo fractional derivative). Let $0 < \alpha \in \mathbb{R}$ and $m := \lceil \alpha \rceil$. Moreover, assume that $f \in C^{m-1}[a, b]$ is such that ${}^{RL}D_a^\alpha f$ exists and belongs to $L_1[a, b]$. The operator ${}^CD_a^\alpha$ defined on $C^{m-1}[a, b]$ by

$${}^CD_a^\alpha f := {}^{RL}D_a^\alpha (f - T_{m-1}[f; a]),$$

where $T_{m-1}[f; a] := \sum_{k=0}^{m-1} \frac{(D^k f)(a)}{k!} (t - a)^k$ denotes the Taylor polynomial of order $m-$ of f centered at a , is called the *Caputo fractional operator of order α* . For $\alpha = 0$, we set ${}^CD_a^0 = I$, the identity operator.

If the function ${}^CD_a^\alpha f$ exists and is in the class $L_1[a, b]$ then we say that the function f is *Caputo α -differentiable*.

If ${}^CD_a^\alpha f$ exists and is in the class $C[a, b]$ then we say that f is *continuously Caputo α -differentiable*.

In parallel with the Riemann-Liouville fractional differentiation we see that in the integrable setting ${}^C D_a^\alpha : C^{[\alpha]-1}[a, b] \rightarrow L_1[a, b]$, whereas in the continuous setting ${}^C D_a^\alpha : C^{[\alpha]-1}[a, b] \rightarrow C[a, b]$.

Notice that in Definition 2 we require the function $f \in C[a, b]$ to be of class $C^{[\alpha]-1}[a, b]$, whereas some other authors (including Caputo in his original work [2]) require the function f to be of class $C^{[\alpha]}[a, b]$ to define its Caputo α -derivative by a formula other than that of Definition 2, hence in their approach ${}^C D_a^\alpha : C^{[\alpha]}[a, b] \rightarrow C[a, b]$. The first approach is a generalization of the second one, and in case $f \in C^{[\alpha]}[a, b]$ the two definitions coincide.

Unlike the fractional integral operators J_a^α , $0 \leq \alpha \in \mathbb{R}$, and differential operators of integer orders D^m , $m \in \mathbb{N}$, the fractional differential operators, both Riemann-Liouville ${}^{RL} D_a^\alpha f$ and Caputo ${}^C D_a^\alpha f$, $0 \leq \alpha \in \mathbb{R}$, do not have semigroup property with respect to the parameter α of differentiation. They may possess limited semigroup property under some assumptions (see Diethelm [4] and Section 3 below).

3 Semigroup property of Riemann-Liouville and Caputo fractional differential operators

In this section we investigate in details semigroup property of the families of Riemann-Liouville and Caputo fractional differential operators. The semigroup property derived is partial because it holds only for a particular subset of the set of orders of differentiation $0 \leq \alpha \in \mathbb{R}$. However, the theorems we get are true for all the functions (under differentiation) of suitable α -differentiability, similar to the case of differentiation of integer order.

3.1 Semigroup property of the family of Riemann-Liouville fractional differential operators

Theorem 3 (Partial semigroup property of Riemann-Liouville fractional differential operators). I. Let $0 \leq \beta < \alpha < 1$ be arbitrary, $f \in C[a, b]$ be continuously Riemann-Liouville α -differentiable. Then

(i) f is continuously Riemann-Liouville β -differentiable. Moreover, we have $({}^{RL} D_a^\beta f)(a) = 0$.

(ii) The derivative ${}^{RL} D_a^\beta f$ is continuously Riemann-Liouville $(\alpha - \beta)$ -differentiable. Moreover, ${}^{RL} D_a^{\alpha-\beta} ({}^{RL} D_a^\beta f) = {}^{RL} D_a^\alpha f$.

(iii) If $f \in C^1[a, b]$ and $f(a) = 0$ then for any $0 \leq \beta \leq 1$ the function f is continuously Riemann-Liouville β -differentiable. Moreover, the

derivative ${}^{RL}D_a^\beta f$ is continuously Riemann-Liouville $(1 - \beta)$ -differentiable and ${}^{RL}D_a^{1-\beta}({}^{RL}D_a^\beta f) = Df$.

II. Conversely, let $0 < \beta < 1$, $\gamma > 0$ be arbitrary. Assume that a function $h \in C[a, b]$ is continuously Riemann-Liouville β -differentiable and the derivative ${}^{RL}D_a^\beta h$ is continuously Riemann-Liouville γ -differentiable, then h is continuously Riemann-Liouville $(\beta + \gamma)$ -differentiable and ${}^{RL}D_a^\gamma({}^{RL}D_a^\beta h) = {}^{RL}D_a^{\beta + \gamma} h$.

Proof. I. (i) Since f is continuously Riemann-Liouville α -differentiable we have $J_a^{1-\alpha} f \in C^1[a, b]$. Put $g := J_a^{1-\alpha} f$. Then $g \in C^1[a, b]$, and $g(a) = 0$ because $1 - \alpha > 0$. Since $\alpha - \beta > 0$, by Miller and Ross [9, Theorem 2(b)] we have

$$D(J_a^{\alpha-\beta} g) = J_a^{\alpha-\beta}(Dg) + \frac{(t-a)^{\alpha-\beta-1}g(a)}{\Gamma(\alpha-\beta)} = J_a^{\alpha-\beta}(Dg).$$

Consequently, $J_a^{\alpha-\beta} g \in C^1[a, b]$. Due to the group property of the family of integral operators J_a^r , $0 \leq r \in \mathbb{R}$, we have

$$J_a^{\alpha-\beta} g = J_a^{\alpha-\beta} J_a^{1-\alpha} f = J_a^{1-\beta} f,$$

Hence $J_a^{1-\beta} f$ is of $C^1[a, b]$, thus f is Riemann-Liouville β -differentiable.

Now, since

$${}^{RL}D_a^\beta f = D(J_a^{1-\beta} f) = D(J_a^{\alpha-\beta} g) = J_a^{\alpha-\beta}(Dg)$$

we have $({}^{RL}D_a^\beta f)(a) = J_a^{\alpha-\beta}(Dg)(a) = 0$ because $Dg \in C[a, b]$ and $\alpha - \beta > 0$.

(ii) Put $f_1 := {}^{RL}D_a^\beta f = J_a^{\alpha-\beta}(Dg)$. Since $g \in C^1[a, b]$ and $g(a) = 0$, due to the group property of the fractional Riemann-Liouville integral, we have

$$J_a^{1-(\alpha-\beta)} f_1 = J_a^{1-(\alpha-\beta)} J_a^{\alpha-\beta}(Dg) = J_a^1(Dg) = g \in C^1[a, b].$$

Consequently, $J_a^{1-(\alpha-\beta)} f_1 \in C^1[a, b]$, hence f_1 is continuously Riemann-Liouville $(\alpha - \beta)$ -differentiable. Now, we have

$${}^{RL}D_a^{(\alpha-\beta)} f_1 = D(J_a^{1-(\alpha-\beta)} f_1) = Dg = D(J_a^{1-\alpha} f) = {}^{RL}D_a^\alpha f.$$

(iii) If $\beta = 1$ or $\beta = 0$ then (iii) is obviously true. Suppose that $0 < \beta < 1$. From the assumption $f \in C^1[a, b]$ and $f(a) = 0$, due to Diethelm [4, Lemma 2.12], it implies that for any $0 < \beta < 1$ the function f is continuously Riemann-Liouville β -differentiable. Using the arguments of (i) and (ii) above one can easily get the assertions of (iii).

II. By assumption $J_a^{1-\beta}h \in C^1[a, b]$. Put $h_1 := {}^{RL}D_a^\beta h = D(J_a^{1-\beta}h)$. Obviously, $(J_a^{1-\beta}h)(a) = 0$. Put $k := \lceil \gamma \rceil \geq 1$ then $k \leq \lceil \beta + \gamma \rceil \leq k+1$. From the assumption that h_1 is continuously Riemann-Liouville γ -differentiable it follows that

$$J_a^{k-\gamma}h_1 \in C^k[a, b] \quad \text{and} \quad {}^{RL}D_a^\gamma h_1 = D^k J_a^{k-\gamma}h_1. \quad (1)$$

Taking into account the fact that $J_a^{1-\beta}h \in C^1[a, b]$ and $(J_a^{1-\beta}h)(a) = 0$, by Miller and Ross [9, Theorem 2(b)] and the semigroup property of the operators J_a^r , $0 \leq r \in \mathbb{R}$, we have

$$J_a^{k-\gamma}h_1 = J_a^{k-\gamma}D(J_a^{1-\beta}h) = D J_a^{k-\gamma}(J_a^{1-\beta}h) = D(J_a^{k+1-(\beta+\gamma)}h). \quad (2)$$

Now, if $\lceil \beta + \gamma \rceil = k + 1$ then from (1)–(2) it follows that

$$D J_a^{\lceil \beta + \gamma \rceil - (\beta + \gamma)}h = D J_a^{k+1-(\beta+\gamma)}h = J_a^{k-\gamma}h_1 \in C^k[a, b],$$

implying $D^{k+1} J_a^{\lceil \beta + \gamma \rceil - (\beta + \gamma)}h = D^k J_a^{k-\gamma}h_1$, hence ${}^{RL}D_a^{\beta+\gamma}h = {}^{RL}D_a^\gamma h_1$. If $\lceil \beta + \gamma \rceil = k$ then from (1)–(2) it follows that

$$J_a^{\lceil \beta + \gamma \rceil - (\beta + \gamma)}h = D(J_a^{k+1-(\beta+\gamma)}h) = J_a^{k-\gamma}h_1 \in C^k[a, b],$$

implying $D^k J_a^{\lceil \beta + \gamma \rceil - (\beta + \gamma)}h = D^k J_a^{k-\gamma}h_1$, hence ${}^{RL}D_a^{\beta+\gamma}h = {}^{RL}D_a^\gamma h_1$.

The theorem is proved. \square

Theorem 3 provides us with partial semigroup property of the family of Riemann-Liouville fractional differential operators, similar to what of the family of differential operators of integer orders, but limited to the range of the orders of fractional differentiation $0 \leq \alpha \leq 1$.

If the order of differentiation is an integer then the Riemann-Liouville differential operator coincides with the classical differential operator and possesses some distinctive feature of the later. Next theorem takes into account this circumstance.

Theorem 4. Let $\alpha > 1$ be arbitrary, and $f \in C[a, b]$ be continuously Riemann-Liouville α -differentiable. Then

(i) For those $0 \leq \beta < \alpha$ such that $\alpha - \beta \in \mathbb{N}$, the function f is continuously Riemann-Liouville β -differentiable, the derivative ${}^{RL}D_a^\beta f$ is continuously Riemann-Liouville $(\alpha - \beta)$ -differentiable and $D^{\alpha-\beta} {}^{RL}D_a^\beta f = {}^{RL}D_a^\alpha f$.

(ii) For any $0 \leq \beta < \alpha - \lfloor \alpha \rfloor$, the function f is continuously Riemann-Liouville β -differentiable and $({}^{RL}D_a^\beta f)(a) = 0$. Moreover, for any $0 < \gamma \leq \alpha - \lfloor \alpha \rfloor - \beta$ the derivative ${}^{RL}D_a^\beta f \in C[a, b]$ is continuously Riemann-Liouville γ -differentiable and ${}^{RL}D_a^\gamma {}^{RL}D_a^\beta f = {}^{RL}D_a^{\beta+\gamma} f$.

Proof. (i) If $\alpha \in \mathbb{N}$ then $\beta \in \mathbb{N}$, hence (i) is clearly true due to the properties of the classical differentiation. Suppose that $\alpha \notin \mathbb{N}$. Put $2 \leq m := \lceil \alpha \rceil \in \mathbb{N}$, $n := \lceil \beta \rceil \in \mathbb{N}$. By assumption, $m > n$ and $m - \alpha = n - \beta$. Since f is continuously Riemann-Liouville α -differentiable we have $J_a^{m-\alpha} f \in C^m[a, b]$, hence $J_a^{n-\beta} f \in C^n[a, b]$. Consequently, f is continuously Riemann-Liouville β -differentiable. Using $J_a^{m-\alpha} f \in C^m[a, b]$ and $\alpha - \beta = m - n$ we get

$${}^{RL}D_a^\alpha f = D^m J_a^{m-\alpha} f = D^{m-n} D^n J_a^{m-\alpha} f = D^{m-n} D^n J_a^{n-\beta} f = D^{\alpha-\beta} {}^{RL}D_a^\beta f.$$

(ii) If $\alpha \in \mathbb{N}$ then $\alpha - \lfloor \alpha \rfloor = 0$ and (ii) is obviously true. Suppose that $\alpha \notin \mathbb{N}$. Put $\alpha_1 := \alpha - \lfloor \alpha \rfloor$, $\alpha_2 := \beta + \gamma$. Then $0 < \alpha_2 < \alpha_1 < 1$. By (i) above f is continuously Riemann-Liouville α_1 -differentiable. By Theorem 3(i) the function f is continuously Riemann-Liouville α_2 -differentiable. Applying Theorem 3 to the function f , which is continuously Riemann-Liouville α_2 -differentiable, we complete the proof of part (ii).

The theorem is proved. \square

Proposition 5. (i) Let $\alpha \geq 1$, $0 < \beta < \alpha$ be such that $\alpha - \beta \notin \mathbb{N}$ and $\beta > \alpha - \lfloor \alpha \rfloor$. Then there exists a function $f \in C[a, b]$ such that f is continuously Riemann-Liouville α -differentiable, but f is not continuously Riemann-Liouville β -differentiable. Moreover, if $\beta > (\alpha - \lfloor \alpha \rfloor) + 1$ then f is not Riemann-Liouville β -differentiable.

(ii) Let $\alpha > 1$, $0 \leq \beta < \alpha - \lfloor \alpha \rfloor < 1$ be arbitrary. Then for any $\gamma > (\alpha - \lfloor \alpha \rfloor) - \beta$, $(\beta + \gamma) - \alpha + \lfloor \alpha \rfloor \notin \mathbb{N}$, there exists a continuous function $f \in C[a, b]$ such that f is continuously Riemann-Liouville α -differentiable, hence continuously Riemann-Liouville β -differentiable, but the derivative ${}^{RL}D_a^\beta f$ is not continuously Riemann-Liouville γ -differentiable.

Proof. (i) Put $c := \alpha - \lfloor \alpha \rfloor$. Then $0 \leq c < 1$. We choose $f := (t - a)^c \in C[a, b]$. By Diethelm [4, Example 2.1], in case $c > 0$ we have

$$J_a^{\lfloor \alpha \rfloor - \alpha} f = J_a^{\lfloor \alpha \rfloor - \alpha} (t - a)^c = (t - a) \Gamma(c + 1) \in C^\infty[a, b];$$

and in case $c = 0$ we have $J_a^{\lfloor \alpha \rfloor - \alpha} f \equiv 1 \in C^\infty[a, b]$. Hence f is continuously Riemann-Liouville α -differentiable with $({}^{RL}D_a^\alpha f) \in C[a, b]$ for any $\alpha > 0$.

From assumption we have $\beta - c \notin \mathbb{N}$, hence by Diethelm [4, Example 2.4] we have

$${}^{RL}D_a^\beta f = \frac{\Gamma(c + 1)}{\Gamma(c + 1 - \beta)} (t - a)^{c - \beta}.$$

Furthermore, we have $\beta > c$, hence ${}^{RL}D_a^\beta f \notin C[a, b]$. Consequently, f is not continuously Riemann-Liouville β -differentiable.

Now, suppose that $\beta > (\alpha - \lfloor \alpha \rfloor) + 1$, then $\beta - c > 1$. In this case we have ${}^{RL}D_a^\beta f \notin L_1[a, b]$. Consequently, f is not Riemann-Liouville β -differentiable.

(ii) If $\alpha \in \mathbb{N}$ then there is no β and (ii) is true.

Suppose that $\alpha \notin \mathbb{N}$. We use the same function f as (i) above. Put $f_1 := {}^{RL}D_a^\beta f$. By Diethelm [4, Example 2.4], since $(\beta + \gamma) - c \notin \mathbb{N}$ we have

$${}^{RL}D_a^\gamma f_1 = {}^{RL}D_a^\gamma ({}^{RL}D_a^\beta f) = \frac{\Gamma(c+1)}{\Gamma(c+1 - (\beta + \gamma))} (t-a)^{c-(\beta+\gamma)}.$$

Since $c - (\beta + \gamma) < 0$ we have ${}^{RL}D_a^\gamma f_1 \notin C[a, b]$ and (ii) is proved. \square

Remark 6. (i) In the proof of Theorem 3 we need to use essentially the assumption that $f \in C[a, b]$ is continuously Riemann-Liouville α -differentiable. Note that this assumption implies that $f(a) = 0$ (use $\beta = 0$ in part (i) of the theorem).

(ii) Part I of Theorem 3 is similar to Theorem 2.13 in Diethelm [4], which is formulated for integrable setting.

(iii) Theorems 3–4 and Proposition 5 provide us with a complete description of partial semigroup property of the family of Riemann-Liouville fractional differential operators.

3.2 Semigroup property of the family of Caputo fractional differential operators

Since Caputo differentiation is defined via Riemann-Liouville differentiation it is not difficult to adapt the result of Subsection 3.1 above to the case of the Caputo fractional differentiation. However, the Caputo differentiation has its distinctive feature what allows us to prove partial semigroup property of the family of Caputo fractional differential operators ${}^C D_a^\alpha f$, $0 \leq \alpha \in \mathbb{R}$, for a bigger set of the parameter α than the Riemann-Liouville case.

First we use Theorem 3 to prove a similar result for Caputo fractional differentiation.

Proposition 7. Let $0 < \beta < \alpha < 1$ be arbitrary, $f \in C[a, b]$ be continuously Caputo α -differentiable. Then

(i) f is continuously β -differentiable. Moreover, $({}^C D_a^\beta f)(a) = 0$.

(ii) The derivative ${}^C D_a^\beta f$ is continuously Caputo $(\alpha - \beta)$ -differentiable.

Moreover, ${}^C D_a^{\alpha-\beta} {}^C D_a^\beta f = {}^C D_a^\alpha f$.

Proof. (i) Put $g := f - f(a)$. Since $0 < \beta < \alpha < 1$, by definition ${}^C D_a^\alpha f = {}^{RL}D_a^\alpha g$. Therefore g is continuously Riemann-Liouville α -differentiable. By

Theorem 3, g is continuously Riemann-Liouville β -differentiable, and $({}^{RL}D_a^\beta g)(a) = 0$. Consequently, f is continuously Caputo β -differentiable, ${}^CD_a^\beta f = {}^{RL}D_a^\beta g$ and $({}^CD_a^\beta f)(a) = ({}^{RL}D_a^\beta g)(a) = 0$.

(ii) For $f_1 := {}^CD_a^\beta f \in C[a, b]$, by (i) above we have $f_1(a) = ({}^CD_a^\beta f)(a) = 0$, and also $f_1 = {}^{RL}D_a^\beta g$. Since g is continuously Riemann-Liouville α -differentiable, by Theorem 3, f_1 is continuously Riemann-Liouville $(\alpha - \beta)$ -differentiable and ${}^{RL}D_a^{\alpha-\beta} f_1 = {}^{RL}D_a^\alpha g$. Consequently, since ${}^{RL}D_a^{\alpha-\beta} f_1 = {}^CD_a^{\alpha-\beta} f_1$ because $0 < \alpha - \beta < 1$ and $f_1(a) = 0$, taking account the equality ${}^{RL}D_a^\alpha g = {}^CD_a^\alpha f$ we get ${}^CD_a^{\alpha-\beta} f_1 = {}^CD_a^\alpha f$. \square

Unlike the Riemann-Liouville case where the semigroup property is lost as soon as $\alpha > 1$ as shown in Proposition 5, in the Caputo case we still have some limited semigroup property of the operator of Caputo continuous fractional differentiation for $\alpha > 1$ as asserted in the theorems below. To derive the theorem we will use the results by Vainikko [12].

Recall that for a smooth function $f \in C^m[a, b]$, $m \in \mathbb{N}$, we denote by $T_m[f; a] := \sum_{k=0}^m \frac{(D^k f)(a)}{k!} (t-a)^k$ its Taylor polynomial of order m centered at a . For $m \in \mathbb{N}$ we put

$$C_0^m[a, b] := \{v \in C^m[a, b] : v^k(0) = 0, k = 0, 1, \dots, m-1\}.$$

It is easily seen that for $m \in \mathbb{N}$ the range of the Riemann-Liouville fractional integral operator J_a^m , in the continuous setting, is

$$J_a^m C[a, b] = \{v \in C^m[a, b] : v^k(0) = 0, k = 0, 1, \dots, m-1\} = C_0^m[a, b].$$

We denote by D_0^α , $0 \leq \alpha \in \mathbb{R}$, the inverse of the Riemann-Liouville fractional integral operator J_a^α , i.e.

$$D_0^\alpha v = (J_a^\alpha)^{-1} v, \quad \text{for } v \in J_a^\alpha C[a, b];$$

that means D_0^α is defined on $J_a^\alpha C[a, b]$, and for $v \in J_a^\alpha C[a, b]$, hence exists $u \in C[a, b]$ such that $J_a^\alpha u = v$, we set $D_0^\alpha v = u$. Clearly the range of D_0^α is a subset of $C[a, b]$. Using the semigroup property of the family of integral operators J_a^α , $0 \leq \alpha \in \mathbb{R}$, one can prove that the domain of D_0^α is a subset of $C_0^{[\alpha]}[a, b] \subset C[a, b]$; and the family of differential operators D_0^α , $0 \leq \alpha \in \mathbb{R}$, has semigroup property with respect to the parameter α , like the family of classical integer order differential operators D^n , $n \in \mathbb{N}$, does. For more details on the operator D_0^α and its properties as well as its relation to Caputo and Riemann-Liouville derivatives we refer the reader to Vainikko [12].

Theorem 8. Let $\alpha > 0$, $[\alpha] - 1 \leq \beta < \alpha$ be arbitrary, $f \in C^{[\alpha]-1}[a, b]$ be continuously Caputo α -differentiable. Then

(i) f is continuously β -differentiable. Moreover, if $\beta \notin \mathbb{N}$ then $({}^C D_a^\beta f)(a) = 0$.

(ii) The derivative ${}^C D_a^\beta f$ is continuously Caputo $(\alpha - \beta)$ -differentiable. Moreover, ${}^C D_a^{\alpha-\beta} {}^C D_a^\beta f = {}^C D_a^\alpha f$.

Proof. (i) Put $m := [\alpha]$. Then $m \geq 1$, $f \in C^{m-1}[a, b]$. Put $g := T_{m-1}[f; a] = \sum_{k=0}^{m-1} \frac{(D^k f)(a)}{k!} (t-a)^k$. We consider two possible cases.

1. First we consider the case $\alpha \notin \mathbb{N}$. By Vainikko [12, Proposition 5.1], f is continuously Caputo α -differentiable if and only if $f - g$ is D_0^α -differentiable. This implies that $f - g$ is D_0^β -differentiable because $0 < \beta < \alpha$ and the family of operators D_0^r , $r > 0$, has semigroup property with respect to the parameter r . Then, again by virtue of Vainikko [12, Proposition 5.1] we get that f is continuously Caputo β -differentiable. By Vainikko [12, Theorem 5.2], since $f \in C^{m-1}[a, b]$ is continuously Caputo α -differentiable, there exists a finite limit

$$\lim_{t \rightarrow a} (t-a)^{m-1-\alpha} ((D^{m-1} f)(t) - (D^{m-1} f)(a)) =: \gamma_m, \quad (3)$$

and $({}^C D_a^\alpha f)(a) = \Gamma(\alpha + 2 - m) \gamma_m$. Similarly, if $\beta \notin \mathbb{N}$ there exists a finite limit

$$\lim_{t \rightarrow a} (t-a)^{m-1-\beta} ((D^{m-1} f)(t) - (D^{m-1} f)(a)) =: \gamma'_m, \quad (4)$$

and $({}^C D_a^\beta f)(a) = \Gamma(\beta + 2 - m) \gamma'_m$. Since $\alpha - \beta > 0$, from (3)-(4) we get

$$\begin{aligned} \gamma'_m &= \lim_{t \rightarrow a} (t-a)^{m-1-\beta} ((D^{m-1} f)(t) - (D^{m-1} f)(a)) \\ &= \lim_{t \rightarrow a} (t-a)^{\alpha-\beta} (t-a)^{m-1-\alpha} ((D^{m-1} f)(t) - (D^{m-1} f)(a)) \\ &= 0 \times \gamma_m = 0. \end{aligned}$$

Thus $({}^C D_a^\beta f)(a) = 0$ if $\beta \notin \mathbb{N}$.

If $\beta \in \mathbb{N}$ then $\beta = m - 1$ and clearly $f \in C^\beta[a, b]$, hence is continuously β -differentiable.

2. Now we consider the case $\alpha \in \mathbb{N}$. We have $\alpha = [\alpha] = m$ and $m - 1 \leq \beta < m$. Since f is continuously Caputo α -differentiable, $f \in C^m[a, b]$. Therefore, by Diethelm [4, Lemma 3.11], f is continuously Caputo β -differentiable with continuous derivative $({}^C D_a^\beta f) \in C[a, b]$ and, moreover, $({}^C D_a^\beta f)(a) = 0$ in case $\beta \notin \mathbb{N}$. If $\beta \in \mathbb{N}$ then $\beta = m - 1$ and clearly $f \in C^\beta[a, b]$, hence is continuously β -differentiable. Part (i) of the theorem is proved.

(ii) We prove this part in considering all four possible cases.

1. First we consider the case $\alpha \notin \mathbb{N}$ and $\beta \notin \mathbb{N}$, thus $[\alpha] = m$ and $\beta > m - 1$. Put $\hat{f} := f - g$, where $g = T_{m-1}[f; a] = \sum_{k=0}^{m-1} \frac{(D^k f)(a)}{k!} (t - a)^k$. Since f is continuously Caputo α -differentiable, by Vainikko [12, Proposition 5.1] the function $\hat{f} = f - g$ is D_0^α -differentiable and $D_0^\alpha \hat{f} = {}^C D_a^\alpha f$. Using semigroup property of the family of operators D_0^r , $r > 0$, with respect to the parameter r , since $0 < \beta < \alpha$ the function \hat{f} is D_0^β -differentiable and $D_0^\beta \hat{f}$ is $D_0^{\alpha-\beta}$ -differentiable, furthermore

$$D_0^{\alpha-\beta}(D_0^\beta \hat{f}) = D_0^\alpha \hat{f} = {}^C D_a^\alpha f. \quad (5)$$

Due to part (i) above f is continuously Caputo β -differentiable, hence the function $\hat{f} = f - g$ is D_0^β -differentiable with derivative $D_0^\beta(f - g) \in C[a, b]$ and ${}^C D_a^\beta f = D_0^\beta(f - g)$, that is

$${}^C D_a^\beta f = D_0^\beta \hat{f}, \quad (6)$$

and by (5) the derivative $f_1 := {}^C D_a^\beta f$ is continuously $D_0^{\alpha-\beta}$ -differentiable. Since $0 < \alpha - \beta < 1$, by Vainikko [12, Proposition 5.1] we have

$$D_0^{\alpha-\beta}(D_0^\beta \hat{f}) = {}^C D_a^{\alpha-\beta}((D_0^\beta \hat{f}) - (D_0^\beta \hat{f})(a)). \quad (7)$$

Since f is continuously Caputo β -differentiable the function $\hat{f} = f - g$ is continuously Caputo β -differentiable, because g is a polynomial of degree not greater than $[\beta] - 1$ hence ${}^C D_a^\beta g = 0$. Therefore, by Vainikko [12, Proposition 5.1], the function $\hat{f} - T_{m-1}[\hat{f}; a]$ has continuous derivative $D_0^\beta(\hat{f} - T_{m-1}[\hat{f}; a]) \in C[a, b]$ and ${}^C D_a^\beta \hat{f} = D_0^\beta(\hat{f} - T_{m-1}[\hat{f}; a])$. Moreover, since $\beta \notin \mathbb{N}$, ${}^C D_a^\beta \hat{f}(a) = 0$. Since $g = \sum_{k=0}^{m-1} \frac{(D^k f)(a)}{k!} (t - a)^k$ we have $T_{m-1}[\hat{f}; a] = 0$, hence

$${}^C D_a^\beta \hat{f} = D_0^\beta \hat{f} \quad \text{and} \quad D_0^\beta \hat{f}(a) = {}^C D_a^\beta \hat{f}(a) = 0. \quad (8)$$

From (6)–(8) it follows that the function f_1 is continuously Caputo $(\alpha - \beta)$ -differentiable; and moreover, by virtue of (5) we have

$${}^C D_a^\alpha f = D_0^{\alpha-\beta}(D_0^\beta \hat{f}) = {}^C D_a^{\alpha-\beta} f_1.$$

2. The case $\alpha \notin \mathbb{N}$ and $\beta \in \mathbb{N}$, thus $\beta = m - 1 < \alpha < m$. Put $f_2 := D^{m-1} f$, then we have $f_2 \in C[a, b]$ and $f_2 = {}^C D_a^\beta f$. By Vainikko [12, Theorem 5.2] the continuous Caputo derivative of order α of f is given by

$$\begin{aligned} {}^C D_a^\alpha f(t) &= \frac{1}{\Gamma(m - \alpha)} \left((t - a)^{m-1-\alpha} (f_2(t) - f_2(a)) \right. \\ &\quad \left. + (\alpha - m + 1) \int_a^t (t - s)^{m-\alpha-2} (f_2(t) - f_2(s)) ds \right), \quad a \leq t \leq b. \end{aligned}$$

We have $0 < \alpha - \beta = \alpha - m + 1 < 1$ and $\lceil \alpha - \beta \rceil = 1$. Applying Vainikko [12, Theorem 5.2] to the function f_2 we get that f_2 is continuously Caputo $(\alpha - \beta)$ -differentiable and the Caputo derivative of order $(\alpha - \beta)$ of f_2 is given by

$$\begin{aligned} {}^C D_a^{\alpha-\beta} f_2(t) &= \frac{1}{\Gamma(m-\alpha)} \left((t-a)^{m-1-\alpha} (f_2(t) - f_2(a)) \right. \\ &\quad \left. + (\alpha - m + 1) \int_a^t (t-s)^{m-\alpha-2} (f_2(t) - f_2(s)) ds \right), \quad a \leq t \leq b. \end{aligned}$$

Consequently, ${}^C D_a^{\alpha-\beta} {}^C D_a^\beta f = {}^C D_a^{\alpha-\beta} f_2 = {}^C D_a^\alpha f$.

3. The case $\alpha \in \mathbb{N}$ and $\beta \notin \mathbb{N}$, thus $\alpha = m$ and $\lceil \beta \rceil = m$. In this case $f \in C^m[a, b]$, and by Diethelm [4, Lemma 3.13] we have ${}^C D_a^{\alpha-\beta} {}^C D_a^\beta f = {}^C D_a^\alpha f$.

4. The case $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$, thus $\alpha = m$ and $\beta = m - 1$. In this case the Caputo differentiation coincide with the classical differentiation. Since f is continuously Caputo α -differentiable we have $f \in C^m[a, b]$ and clearly ${}^C D_a^{\alpha-\beta} {}^C D_a^\beta f = {}^C D_a^\alpha f$.

Part (ii) is proved. The theorem is proved completely. \square

The following theorem asserts the partial semigroup property of the family of Caputo fractional differential operators. This theorem includes Theorem 8 as a particular case, whereas Theorem 8 is a generalization of Proposition 7.

Theorem 9 (Partial semigroup property of Caputo fractional differential operators). I. Let $\alpha > 0$ be arbitrary and $f \in C^{\lceil \alpha \rceil - 1}[a, b]$ be continuously Caputo α -differentiable. Then

(i) For any $0 < \beta < \alpha$, f is continuously Caputo β -differentiable. Furthermore, if $\beta \notin \mathbb{N}$ then $({}^C D_a^\beta f)(a) = 0$.

(ii) For any $0 < \gamma \leq 1$ such that $\beta + \gamma \leq \alpha$ and $\lceil \beta + \gamma \rceil - \lfloor \beta \rfloor = 1$, the derivative ${}^C D_a^\beta f$ is continuously Caputo γ -differentiable, and ${}^C D_a^\gamma {}^C D_a^\beta f = {}^C D_a^{\beta+\gamma} f$.

II. Conversely, let $\beta > 0$, $0 < \gamma \leq 1$ be such that $\lceil \beta + \gamma \rceil - \lfloor \beta \rfloor = 1$. Assume that a continuous function $h \in C^{\lceil \beta \rceil - 1}[a, b]$ is continuously Caputo β -differentiable such that ${}^C D_a^\beta h$ is continuously Caputo γ -differentiable, and assume additionally that ${}^C D_a^\beta h(a) = 0$ if $\beta \notin \mathbb{N}$. Then h is continuously Caputo $(\beta + \gamma)$ -differentiable, and ${}^C D_a^\gamma ({}^C D_a^\beta h) = {}^C D_a^{\beta+\gamma} h$.

Proof. I. (i) If $\lceil \alpha \rceil - 1 \leq \beta < \alpha$ then we are done by using Theorem 8(i). If $\beta < \lceil \alpha \rceil - 1$, then $\lfloor \beta \rfloor + 1 \leq \lceil \alpha \rceil - 1$, hence $f \in C^{\lfloor \beta \rfloor + 1}[a, b]$. Consequently f is continuously Caputo $(\lfloor \beta \rfloor + 1)$ -differentiable. Therefore, by Theorem 8, f is

continuously Caputo β -differentiable. Moreover, if $\beta \notin \mathbb{N}$ then $({}^C D_a^\beta f)(a) = 0$.

(ii) By (i) above, since $\beta + \gamma \leq \alpha$ the function f is continuously Caputo $(\beta + \gamma)$ -differentiable.

Now, since $\gamma > 0$ and $[\beta + \gamma] - [\beta] = 1$ we have $[\beta + \gamma] - 1 \leq \beta < \beta + \gamma$. Therefore, by Theorem 8, the derivative ${}^C D_a^\beta f$ is continuously Caputo γ -differentiable, and ${}^C D_a^\gamma {}^C D_a^\beta f = {}^C D_a^{\beta + \gamma} f$.

II. We prove this part in considering all possible three cases.

Case 1: $\beta \notin \mathbb{N}$. We put $\hat{h} := h - T_{[\beta]-1}[h; a]$. Since h is continuously Caputo β -differentiable, by Vainikko [12, Proposition 5.1] the function \hat{h} is D_0^β -differentiable and $D_0^\beta \hat{h} = {}^C D_a^\beta h =: h_1$. Note that, in this case $0 < \gamma < 1$. Therefore, since by assumption h_1 is continuously Caputo γ -differentiable and $h_1(a) = 0$, the function h_1 is D_0^γ -differentiable and ${}^C D_a^\gamma h_1 = D_0^\gamma h_1$. Consequently, by the semigroup property of the operators D_0^r , $r > 0$, with respect to the parameter r , the function \hat{h} is $D_0^{\beta + \gamma}$ -differentiable and we have

$$D_0^{\beta + \gamma} \hat{h} = D_0^\gamma D_0^\beta \hat{h} = D_0^\gamma h_1 = {}^C D_a^\gamma h_1.$$

Since $[\beta] = [\beta + \gamma]$ and \hat{h} is $D_0^{\beta + \gamma}$ -differentiable, in case $\beta + \gamma \notin \mathbb{N}$ by Vainikko [12, Proposition 5.1] the function h is continuously Caputo $(\beta + \gamma)$ -differentiable, and ${}^C D_a^{\beta + \gamma} h = D_0^{\beta + \gamma} \hat{h}$, hence ${}^C D_a^{\beta + \gamma} h = {}^C D_a^\gamma h_1$. Otherwise, if $\beta + \gamma \in \mathbb{N}$ then, since $T_{[\beta]-1}[h; a]$ is a polynomial of degree not greater than $\beta + \gamma - 1$, we have

$$D_0^{\beta + \gamma} \hat{h} = D^{\beta + \gamma} \hat{h} = D^{\beta + \gamma} (h - T_{[\beta]-1}[h; a]) = D^{\beta + \gamma} h = {}^C D_a^{\beta + \gamma} h,$$

hence ${}^C D_a^{\beta + \gamma} h = {}^C D_a^\gamma h_1$.

Case 2: $\beta \in \mathbb{N}$, $\beta + \gamma \notin \mathbb{N}$. In this case we have $[\beta + \gamma] = \beta + 1$, $\gamma < 1$. The function $h \in C^\beta[a, b]$ and $h_1 = {}^C D_a^\beta h = D^\beta h \in C[a, b]$ because h is continuously Caputo β -differentiable. By Vainikko [12, Theorem 5.2], since h_1 is continuously Caputo γ -differentiable, $0 < \gamma < 1$, we have the finite limit $\lim_{t \rightarrow a} (t - a)^{-\gamma} (h_1(t) - h_1(a)) =: c_1$ exists and

$$\sup_{a < t \leq b} \left| \int_{\theta t}^t (t - s)^{-\gamma - 1} (h_1(t) - h_1(s)) ds \right| \rightarrow 0 \quad \text{as } \theta \uparrow 1.$$

Now apply Vainikko [12, Theorem 5.2] to the function $h \in C^\beta[a, b]$ with $D^\beta h = h_1$ we get that the function h is continuously Caputo $(\beta + \gamma)$ -differentiable and ${}^C D_a^{\beta + \gamma} h(a) = \Gamma(\gamma + 1)c_1 = {}^C D_a^\gamma h_1(a)$. Moreover, for

$a < t \leq b$ we have

$$\begin{aligned} {}^C D_a^{\beta+\gamma} h(t) &= \frac{1}{\Gamma(1-\gamma)} \left((t-a)^{-\gamma} (h_1(t) - h_1(a)) \right. \\ &\quad \left. + \gamma \int_a^t (t-s)^{-\gamma-1} (h_1(t) - h_1(s)) ds \right) \\ &= {}^C D_a^\gamma h_1(t). \end{aligned}$$

Thus, ${}^C D_a^{\beta+\gamma} h = {}^C D_a^\gamma h_1$.

Case 3: $\beta \in \mathbb{N}$, $\beta + \gamma \in \mathbb{N}$. Our assertion is true and this case due to the rule of classical integer order differentiation. \square

Next theorem concerns the distinctive case of one order of Caputo differentiation being an integer.

Theorem 10. Let $\alpha > 0$ be arbitrary and $f \in C^{[\alpha]-1}[a, b]$ be continuously Caputo α -differentiable. Then for any $l \in \mathbb{N}$, $0 \leq l \leq [\alpha] - 1$ the function f is continuously Caputo l -differentiable. The derivative $D^l f$ is continuously Caputo $(\alpha - l)$ -differentiable. Moreover, ${}^C D_a^{\alpha-l} D^l f = {}^C D_a^\alpha f$.

Proof. In case $\alpha \in \mathbb{N}$ we are back to the classical integer order differentiation and the theorem trivially holds. Assume that $\alpha \notin \mathbb{N}$.

Put $m := [\alpha]$, $g := T_{m-1}[f; a]$, $f_1 := D^l f$, $f_2 := D^{m-1} f$. Since f is continuously Caputo α -differentiable, by Vainikko [12, Theorem 5.2] the continuous Caputo derivative of order α of f is given by

$$\begin{aligned} {}^C D_a^\alpha f(t) &= \frac{1}{\Gamma(m-\alpha)} \left((t-a)^{m-1-\alpha} (f_2(t) - f_2(a)) \right. \\ &\quad \left. + (\alpha - m + 1) \int_a^t (t-s)^{m-\alpha-2} (f_2(t) - f_2(s)) ds \right), \quad a \leq t \leq b. \end{aligned}$$

Clearly, $f_2 = D^{m-1-l} f_1$, hence by Vainikko [12, Theorem 5.2] the function f_1 is continuously Caputo $(\alpha - l)$ -differentiable, and its continuous Caputo derivative of order $(\alpha - l)$ is given by

$$\begin{aligned} {}^C D_a^{\alpha-\beta} f_1(t) &= \frac{1}{\Gamma(m-\alpha)} \left((t-a)^{m-1-\alpha} (f_2(t) - f_2(a)) \right. \\ &\quad \left. + (\alpha - m + 1) \int_a^t (t-s)^{m-\alpha-2} (f_2(t) - f_2(s)) ds \right), \quad a \leq t \leq b. \end{aligned}$$

Consequently, ${}^C D_a^{\alpha-l} f_1 = {}^C D_a^\alpha f$, thus ${}^C D_a^{\alpha-l} D^l f = {}^C D_a^\alpha f$. \square

Proposition 11. (i) For any $0 < \alpha, \beta \in \mathbb{R}$ such that $\beta \notin \mathbb{N}$, $[\beta] < \alpha$, there exists a continuous function $f \in C[a, b]$ such that f is continuously Caputo α -differentiable hence continuously Caputo β -differentiable, but the derivative ${}^C D_a^\beta f$ is not continuously Caputo $(\alpha - \beta)$ -differentiable.

(ii) For any $0 < \alpha, \beta \in \mathbb{R}$ such that $\beta \notin \mathbb{N}$, $[\beta] + 1 < \alpha$, there exists a continuous function $f \in C[a, b]$ such that f is continuously Caputo α -differentiable hence continuously Caputo β -differentiable, but the derivative ${}^C D_a^\beta f$ is not Caputo $(\alpha - \beta)$ -differentiable.

(iii) For any $\beta, \gamma > 0$, $\beta \notin \mathbb{N}$, and $\gamma \notin \mathbb{N}$, there exists a continuous function $h \in C^{[\beta]-1}[a, b]$ which is continuously Caputo β -differentiable such that ${}^C D_a^\beta h$ is continuously Caputo γ -differentiable, but ${}^C D_a^\gamma ({}^C D_a^\beta h) \neq {}^C D_a^{\beta+\gamma} h$.

Proof. Put $m := [\alpha]$, $n := [\beta]$. Choose $f := (t - a)^n$ for both parts (i)-(ii) of the proposition. By assumption, $m > n$ and $n - 1 < \beta < n$. Since $f \in C^\infty[a, b]$ the function f is continuously Caputo β -differentiable as well as continuously Caputo α -differentiable. A simple calculation gives

$$\begin{aligned} {}^C D_a^\alpha f &= J_a^{m-\alpha} D^m f = 0, \\ f_1 := {}^C D_a^\beta f &= J_a^{n-\beta} D^n f = J_a^{n-\beta} n! = \frac{\Gamma(n+1)}{\Gamma(n-\beta+1)} (t-a)^{n-\beta}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} {}^C D_a^{\alpha-\beta} f_1 &= {}^C D_a^{\alpha-\beta} \frac{\Gamma(n+1)}{\Gamma(n-\beta+1)} (t-a)^{n-\beta} \\ &= \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} (t-a)^{n-\alpha}. \end{aligned}$$

(i) Since by assumption $\alpha > [\beta] = n$ we have $n - \alpha < 0$. Therefore $(t - a)^{n-\alpha} \notin C[a, b]$, hence $f_1 = {}^C D_a^\beta f$ is not continuously Caputo $(\alpha - \beta)$ -differentiable. Furthermore, in case $-1 < n - \alpha < 0$ we have $(t - a)^{n-\alpha} \in L_1[a, b]$ hence f_1 is Caputo $(\alpha - \beta)$ -differentiable, but ${}^C D_a^\alpha f \neq {}^C D_a^{\alpha-\beta} f_1$.

(ii) Since by assumption $\alpha > [\beta] + 1$ we have $n - \alpha < -1$. Therefore $(t - a)^{n-\alpha} \notin L_1[a, b]$, hence ${}^C D_a^\beta f$ is not Caputo $(\alpha - \beta)$ -differentiable.

(iii) Take $h := (t - a)^\beta$ then h is continuously Caputo β -differentiable with derivative $h_1 = {}^C D_a^\beta h = \Gamma(\beta + 1) \neq 0$. The function h_1 is continuously Caputo γ -differentiable, and ${}^C D_a^\gamma h_1 = 0$. On the other hand,

$${}^C D_a^{\beta+\gamma} h = {}^C D_a^{\beta+\gamma} (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(1-\gamma)} (t-a)^{-\gamma}.$$

Therefore, ${}^C D_a^\gamma ({}^C D_a^\beta h) = {}^C D_a^\gamma h_1 \neq {}^C D_a^{\beta+\gamma} h$. \square

Remark 12. (i) The assumption that f is continuously Caputo α -differentiable in Proposition 7 does not imply that $f(a) = 0$ as in the Riemann-Liouville case (cf. Remark 6(i)).

(ii) Lemma 3.13 in Diethelm [4] is similar to Theorem 10, but it requires a stronger smoothness condition on f .

(iii) Theorem 9–10 and Proposition 11 provide us with a complete description of a (partial) semigroup property of the family of Caputo fractional differential operators.

4 Applications

In this section we present some applications of the semigroup property of the family of Caputo fractional differential operators derived in the preceding section. Namely we will present our results on reduction of a multi-term Caputo fractional differential equation to a single-term or to a multi-order Caputo fractional differential system, and a theorem on existence and uniqueness of solution to a general multi-term Caputo fractional differential system.

In this section, speaking of a solution of Caputo fractional differential equation we mean a function which is continuously Caputo differentiable of the desired order and satisfies the equation in consideration; and for the vector case we do the differentiation and computation component-wise. Note that this is a natural definition of a solution of a differential equation, and it allows us to show that the Caputo fractional differential equation with continuous coefficients is equivalent to the corresponding Volterra integral equation (see Diethelm, Siengmund and Tuan [5, Definition 2.1], Diethelm [4, Chapter 6])

4.1 Reduction of a multi-term system to a single-term or multi-order system of Caputo fractional differential equations

In this subsection we deal with the problem of reduction of multi-term Caputo fractional differential equation to a system of single-term or multi-order Caputo fractional differential equations.

In certain cases we have to deal with Caputo fractional differential equations of order higher than 1 or equation of many differential term of different orders, for example in the problem of fractional relaxation and oscillation (see Gorenflo et al [6, Section 8.1.2]), or the Bagley-Torvik equation (see Diethelm [4, Chapter 8]). The results in this paper provide us with a tool

to deal with such kind of equations by reducing a multi-term equation to a single or multi-order fractional differential system. This is achieved by the method of change of variable. This approach is simple and similar to the one in the classical theory of ordinary differential equations, and is treated in the book by Diethelm [4, Chapter 8]. The main key here is the connection of the Caputo fractional derivatives of different orders—the semigroup property—provided by Theorem 9.

Since the semigroup property derived in Theorem 9 is partial, i.e. true for only limited set of parameter of differentiation, the reduction we are able to do is not simple as its counterpart in the classical theory of ordinary differential equations.

First, we present a reduction theorem in the case of rational orders of Caputo differentiation. Let us emphasize that while our Theorem 13 below looks like Theorem 8.1 of Diethelm [4] the important difference is the fact that using our result on semigroup property in Section 3 we are able to drop the assumption on high smoothness of the solution, thus our theorem provides a natural equivalence of the systems under consideration.

Theorem 13 (cf. Theorem 8.1 of Diethelm [4]). Consider the equation

$${}^C D_a^{\alpha_k} x(t) = f(t, x(t), {}^C D_a^{\alpha_1} x(t), {}^C D_a^{\alpha_2} x(t), \dots, {}^C D_a^{\alpha_{k-1}} x(t)), \quad (9)$$

$t \in [a, b]$, subject to the initial conditions

$$(D^j x)(a) = x_a^{(j)}, \quad j = 0, 1, \dots, \lceil \alpha_k \rceil - 1, \quad (10)$$

where $\alpha_k > \alpha_{k-1} > \dots > 0$, $\alpha_j - \alpha_{j-1} \leq 1$ for all $j = 2, 3, \dots, k$ and $0 < \alpha_1 \leq 1$, $f : [a, b] \times \mathbb{R}^k \rightarrow \mathbb{R}$ is a continuous function. Assume that $\alpha_j \in \mathbb{Q}$ for all $j = 1, 2, \dots, k$, define M to be the least common multiple of the denominators of $\alpha_1, \alpha_2, \dots, \alpha_k$ and set $\gamma = 1/M$ and $N = M\alpha_k$. Then this initial value problem is equivalent to the system of equations

$$\begin{aligned} {}^C D_a^\gamma x_0(t) &= x_1(t), \\ {}^C D_a^\gamma x_1(t) &= x_2(t), \\ &\vdots \\ {}^C D_a^\gamma x_{N-2}(t) &= x_{N-1}(t), \\ {}^C D_a^\gamma x_{N-1}(t) &= f(t, x_0(t), {}^C D_a^{\alpha_1} x_{\alpha_1/\gamma}(t), \dots, {}^C D_a^{\alpha_{k-1}} x_{\alpha_{k-1}/\gamma}(t)), \end{aligned} \quad (11)$$

together with the initial conditions

$$x_j(a) = \begin{cases} x_a^{(j/M)}, & \text{if } j/M \in \mathbb{N}, \\ 0, & \text{if } j/M \notin \mathbb{N}, \end{cases} \quad (12)$$

in the following sense.

1. Whenever the vector function $X := (x_0, x_1, \dots, x_{N-1})^T$ is the solution of the system (11)-(12), the function $x := x_0$ solves the initial value problem (9)-(10) on $[a, b]$.

2. Whenever a function $x \in C^{[\alpha_k]-1}$ is a solution the initial value problem (9)-(10), the vector function

$$X := (x_0, x_1, \dots, x_{N-1})^T := (x, {}^C D_a^\gamma x, {}^C D_a^{2\gamma} x, \dots, {}^C D_a^{(N-1)\gamma} x)^T$$

solves the multidimensional initial value problem (11)-(12).

Proof. Note that the only difference between our theorem and Theorem 8.1 of Diethelm [4] is the assumption of smoothness of solution: we require the solution of the problem (9)-(10) to be continuously Caputo α_k -differentiable whereas in Theorem 8.1 of [4] one needs a stronger assumption of $C^{[\alpha_k]}[a, b]$.

It is easily seen that same arguments as in the proof of Theorem 8.1 of [4] with the change of the usage of Lemma 3.13 there to the usage of our Theorem 9 furnish the proof of the theorem. \square

Corollary 14. In the situation of Theorem 13, assume additionally that the function f is uniformly Lipschitz in the x variable, i.e. there exists $L > 0$ such that

$$|f(t, y) - f(t, z)| \leq L \|y - z\|, \quad \text{for all } t \in [a, b], y, z \in \mathbb{R}^k.$$

Then both the initial value problems (9)-(10) and (11)-(12) have unique solutions and the solutions agree in the sense of Theorem 13.

Proof. By Theorem 2.3 of Diethelm, Siengmund and Tuan [5] for any initial value (12) the equation (11) has unique solution. Therefore, by Theorem 13 for any initial value (10) the equation (9) has unique solution and that solution agrees with the solution of (11)-(12). \square

As noted by Diethelm [4], Theorem 15 has limitation in applications: it is applicable only to the case of rational indices of differentiation or commensurate indices if all the indices are not greater than 1. Furthermore, the reduced single-term system may be of very high dimension because the parameter γ there can be very small. A way to tackle these drawbacks would be the use of a multi-order Caputo fractional system instead of single-term Caputo fractional system what we present below.

Let us consider Caputo fractional differential equation (9) on $[a, b]$ subject to the initial conditions (10). Put $\beta_1 := \alpha_1$, $\beta_j := \alpha_j - \alpha_{j-1}$ for

$j = 2, \dots, k$, $x_1 := x$, $x_j := {}^C D_a^{\alpha_j - 1} x$ for $j = 2, \dots, k$. Without loss of generality we assume further that there is no integer lying between two consecutive numbers α_j , $j = 1, \dots, k$, because we may add an integer derivative of x to the set of variables of f without changing the function f in (9). Note that by the assumption on α_j , $j = 1, \dots, k$, we have $0 < \beta_j \leq 1$ for all j . We have the following equivalence result.

Theorem 15 (cf. Theorem 8.9 of Diethelm [4]). Subject to the above conditions, the multi-term equation (9) with the initial conditions (10) is equivalent to the system

$$\begin{aligned} {}^C D_a^{\beta_1} x_1(t) &= x_2(t), \\ {}^C D_a^{\beta_2} x_2(t) &= x_3(t), \\ &\vdots \\ {}^C D_a^{\beta_{k-1}} x_{k-1}(t) &= x_k(t), \\ {}^C D_a^{\beta_k} x_k(t) &= f(t, x_1(t), x_2(t), \dots, x_k(t)) \end{aligned} \tag{13}$$

with the initial conditions

$$x_j(a) = \begin{cases} x_a^{(0)}, & \text{if } j = 1, \\ x_a^{(l)}, & \text{if } j = l \in \mathbb{N}, \\ 0, & \text{else,} \end{cases} \tag{14}$$

in the following sense.

1. Whenever the function $x \in C^{\lceil \alpha_k \rceil - 1}$ is a solution of the multi-term Caputo fractional differential equation (9) with the initial conditions (10), the vector function $X := (x_1, x_2, \dots, x_k)^T$ with

$$x_j(t) := \begin{cases} x(t), & \text{if } j = 1, \\ {}^C D_a^{\alpha_j - 1} x(t), & \text{if } j \geq 2, \end{cases} \tag{15}$$

is a solution of the multi-order Caputo fractional differential equation (13) with initial conditions (14).

2. Whenever the vector function $X := (x_1, x_2, \dots, x_k)^T$ is a solution of the multi-order Caputo fractional differential system (13) with initial conditions (14), the function $x := x_1$ solves the multi-term Caputo fractional differential equation (9) with the initial conditions (10) on $[a, b]$.

Proof. Similar to the proof of Theorem 13 above. \square

Remark 16. (i) Theorems 13 and 15 are still valid if we considers the equations on the infinite time interval $[a, \infty)$.

(ii) In the situation of Theorem 13, if $\alpha_k \leq 1$ then one needs not assume the indices α_j , $j = 1, \dots, k$ to be rational numbers but only the assumption that they are commensurate, i.e. $\alpha_i/\alpha_j \in \mathbb{Q}$ for all j, j , suffices.

(iii) In Theorems 13 and 15 we do not assume neither existence nor the uniqueness of solutions of the systems in the considerations.

(iv) The smoothness conditions on solutions of the systems considered in Theorems 13 and 15 are natural enough such that it is automatically provided by all the theorems on the existence and uniqueness of the solutions of those systems under a natural requirement of Lipschitz continuity of the coefficients of the systems.

4.2 Existence and uniqueness of solution to multi-term Caputo fractional differential systems

Based on our results above and the available partial results in the literature we are able to give a concise proof of a theorem on existence and uniqueness of solutions to a general multi-term and general multi-order Caputo fractional differential system. Notice that a theorem on existence and uniqueness of solution to multi-term is proved by Diethelm [4, Chapter 8] using a fairly complicated method of approximation.

Theorem 17. Consider the equation

$${}^C D_a^{\alpha_k} x(t) = f(t, x(t), {}^C D_a^{\alpha_1} x(t), {}^C D_a^{\alpha_2} x(t), \dots, {}^C D_a^{\alpha_{k-1}} x(t)), \quad (16)$$

$t \in [a, b]$, subject to the initial conditions

$$(D^j x)(a) = x_a^{(j)}, \quad j = 0, 1, \dots, \lceil \alpha_k \rceil - 1, \quad (17)$$

where $\alpha_k > \alpha_{k-1} > \dots > 0$, $\alpha_j - \alpha_{j-1} \leq 1$ for all $j = 2, 3, \dots, k$ and $0 < \alpha_1 \leq 1$, $f : [a, b] \times \mathbb{R}^k \rightarrow \mathbb{R}$ is a continuous function. Assume that f satisfies an uniform Lipschitz condition, i.e. there exists a positive constant $L > 0$ such that

$$|f(t, y) - f(t, z)| \leq L \|y - z\|, \quad \text{for all } t \in [a, b], y, z \in \mathbb{R}^k.$$

Then for any $x_0 \in \mathbb{R}^k$ there exists unique continuous solution $x : [a, b] \rightarrow \mathbb{R}^k$ to the equation (16) that satisfies the initial conditions (17).

Proof. Notice that the system (16)-(17) is exactly the system (9)-(10) treated in Theorem 15. Use Theorem 15 to reduce the multi-term Caputo fractional differential system (16)-(17) to a multi-order system (13)-(14). By Theorem 15 all the orders of Caputo differentiation in the derived multi-order

Caputo fractional differential system (13)-(14) are in $(0, 1]$. Therefore, due to the assumption of Lipschitz continuity of f , Theorem 2.3 of Diethelm, Siengmund and Tuan [5] is applicable to the multi-order system (13)-(14), implying that for any initial value (14) the multi-order system (13) has unique solution. By Theorem 15, this implies that for all initial value (17) the original multi-order system (16) has unique solution. \square

5 Illustrative examples

In this section we give two examples. The first is an example of a diagonal system to show non-applicability of Theorem 8.1 of Diethelm [4] due to the strong smoothness requirements, and in the meantime Theorem 13 is applicable to that system. The second example is application of the reduction technique to investigate Lyapunov stability of a linear multi-order Caputo fractional system; this example is taken from [5].

Example 18 (α -differentiability but no $C^{[\alpha]}$ -smoothness). Consider the multi-term Caputo FDE,

$${}^C D_0^\alpha x - \lambda {}^C D_0^\beta x = 0, \quad (18)$$

$t \in [0, 2]$, $0 < \beta < \alpha$. Choose $\lambda = 1$, $\beta = 1$, $\alpha = 1.5$ so $\alpha - \beta = 0.5 =: \gamma$, $\beta = 2\gamma$, $\alpha = 3\gamma$. According to Kilbas et al. [7, Theorem 5.13] this equation has solution

$$\begin{aligned} x_0(t) &= E_{\alpha-\beta}(\lambda t^{\alpha-\beta}) - \lambda t^{\alpha-\beta} E_{\alpha-\beta, \alpha-\beta+1}(\lambda t^{\alpha-\beta}) = 1, \\ x_1(t) &= t E_{\alpha-\beta, 2}(\lambda t^{\alpha-\beta}) = t E_{\gamma, 2}(t^\gamma), \end{aligned}$$

where $E_{\alpha, \beta}(\cdot)$ denotes the two parameter Mittag-Leffler function, and $E_\alpha(\cdot) := E_{\alpha, 1}(\cdot)$ denotes the one parameter Mittag-Leffler function. We have $x_0(0) = 1$, $Dx_0(0) = 0$ and $x_1(0) = 0$, $Dx_1(0) = 1$. These two solutions form a fundamental system of solutions to the linear equation (18).

A simple computation gives $Dx_1 = E_\gamma(t^\gamma) + E_{\gamma, \gamma}(t^\gamma)t^\gamma$ on $[0, 2]$, and for $0 < t \leq 2$

$$D^2 x_1 = E_{\gamma, \gamma}(t^\gamma)t^\gamma + (DE_{\gamma, \gamma}(t^\gamma))t^\gamma + E_{\gamma, \gamma}(t^\gamma)\gamma t^{\gamma-1},$$

hence $x_1 \in C^1[0, 2] \cap C^2(0, 2]$ but $x_1 \notin C^2[0, 2]$.

Now we make a change of variables to apply Theorem 13 to reduce the Caputo fractional equation (18) to the case of linear system of single-term Caputo fractional differential equations. Put

$$y_1(\cdot) := x(\cdot), \quad y_2(\cdot) := {}^C D_0^\gamma y_1, \quad y_3(\cdot) := {}^C D_0^\gamma y_2(\cdot).$$

Then the FDE (18) becomes

$$\begin{cases} {}^C D_0^\gamma y_1(t) &= y_2(t), \\ {}^C D_0^\gamma y_2(t) &= y_3(t), \\ {}^C D_0^\gamma y_3(t) &= y_3(t), \end{cases} \quad (19)$$

or, in the vector form,

$${}^C D_0^\gamma y = Ay; \quad y := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3}.$$

The characteristic polynomial of the matrix A of the linear time independent single-term Caputo fractional differential equation (19) is

$$\Delta_A(s) = \det(\text{diag}(\lambda I - A)) = \lambda^3 - \lambda^2. \quad (20)$$

The matrix A has eigenvalue 0 with multiplicity 2 and 1 with multiplicity 1. Hence, by Diethelm [4, Theorem 7.14] the three-dimensional linear time independent Caputo fractional differential equation (19) has general solution with the first coordinate given by

$$y_1(t) = a_1 + a_2 t^\gamma + a_3 E_\gamma(t^\gamma), \quad a_1, a_2, a_3 \in \mathbb{R}.$$

Since $0 < \gamma < 1$ we must have ${}^C D_0^\gamma y_1(0) = 0$ hence $a_3 = -a_2 \Gamma(1 + \gamma)$. Therefore the general solution of (19) is

$$\begin{cases} y_1(t) &= a_1 + a_2 t^\gamma - a_2 \Gamma(1 + \gamma) E_\gamma(t^\gamma), \\ y_2(t) &= a_2 \Gamma(1 + \gamma) - a_2 \Gamma(1 + \gamma) E_\gamma(t^\gamma), \\ y_3(t) &= -a_2 \Gamma(1 + \gamma) E_\gamma(t^\gamma), \end{cases}$$

for arbitrary $a_1, a_2 \in \mathbb{R}$. By choosing $(a_1 = 1, a_2 = 0)$ and $(a_1 = 0, a_2 = -1/\Gamma(1 + \gamma))$ we get two vectors

$$\hat{y}_1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{y}_2(t) = \begin{pmatrix} -1 - t^\gamma + \Gamma(1 + \gamma) E_\gamma(t^\gamma) \\ -\Gamma(1 + \gamma) + \Gamma(1 + \gamma) E_\gamma(t^\gamma) \\ \Gamma(1 + \gamma) E_\gamma(t^\gamma) \end{pmatrix}$$

forming a fundamental system of solution of (19). Notice that all the solution of (19) are continuously Caputo fractional α -differentiable, but the first coordinate of the solution $\hat{y}_2(t)$ is does not belong to $C^2[0, 2]$, hence Theorem

8.1 of Diethelm [4] is not applicable to the solution \hat{y}_2 of (19), whereas Theorem 13 is applicable to the solution \hat{y}_2 of (19) allowing us to conclude that the first coordinate of the vector \hat{y}_2 is a solution of the equation (18).

By the way one may verify that the solutions given by the application of Theorem 13 and the ones given by the direct computation above agree, i.e.

$$-1 - t^\gamma + \Gamma(1 + \gamma)E_\gamma(t^\gamma) = tE_{\gamma,2}(t^\gamma).$$

This equality can be verified by using series presentation of $E_\gamma(t^\gamma)$ and $E_{\gamma,2}(t^\gamma)$.

Example 19 (Stability study by reduction to a single-term system). Recall Example 3.6 of Diethelm, Siegmund and Tuan [5]. Consider the system

$$y := \begin{pmatrix} {}^C D_0^{1/2} x_1(t) \\ {}^C D_0^{1/4} x_2(t) \end{pmatrix} = Ax(t), \quad \text{where } A = \begin{pmatrix} 0.00001 & 1 \\ -0.0022 & 0.1 \end{pmatrix}. \quad (21)$$

This two-dimensional system can be written as a three-dimensional system of order 1/4 in the form

$${}^C D_0^{1/4} x^*(t) = A^* x^*(t), \quad \text{with } A^* = \begin{pmatrix} 0 & 1 & 0 \\ 0.00001 & 0 & 1 \\ -0.0022 & 0 & 0.1 \end{pmatrix}. \quad (22)$$

By Theorem 15 system (21) is equivalent to the system (22). Since A^* has eigenvalues $-0.1.03917$ and $0.101958 \pm 0.10385i$, the system (22) is stable we conclude that the system (21) is stable (see [5]). Notice that the stability of (21) is not easily deducted without its reduction to the single-term system (22). Furthermore we also note that the solution of (22) is not of the class C^1 , hence Theorem 8.1 of Diethelm [4] is not application to this situation.

Acknowledgement

The author thanks Prof. D. T. Son and Dr. H. T. Tuan for helpful discussion. This research is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED).

References

- [1] P. Badri and M. Sojoodi. Stability and Stabilization of Fractional-Order Systems with Different Derivative Orders. *Asian Journal of Control*, **21**(2019), 1–10.

- [2] M. Caputo. Linear models of dissipation whose Q is almost frequency independent – II. *Geophys. J. Roy. Astron. Soc.*, Vol. 13 (1967), 529–539; reprinted in *Fract. Calc. Appl. Anal.* 11, 4–14 (2008).
- [3] W. Deng, C. Li and Q. Guo. Analysis of fractional differential equations with multi-orders. *Fractals*, **15**(2007), 173–182.
- [4] K. Diethelm. *The Analysis of Fractional Differential Equations. An Application-oriented Exposition Using Differential Operators of Caputo Type. Lecture Notes in Mathematics* **2004**. Springer-Verlag, Berlin, 2010.
- [5] K. Diethelm, S. Siegmund and H. T. Tuan. Asymptotic behavior of solutions of linear multi-order fractional differential systems. *Fractional calculus and applied analysis*, **20**(2017), 1165–1195.
- [6] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin. *Mittag-Leffler Functions, Related Topics and Applications*. Springer-Verlag, Berlin, 2014.
- [7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations. North-Holland mathematics studies, Vol. 204*. Elsevier, Amsterdam, 2006.
- [8] C. Li, F. Zhang, J. Kurths and F. Zeng. Equivalent system for a multiple-rational-order fractional differential system. *Phil. Trans. R. Soc. A* 371:20120156, (2013), <http://dx.doi.org/10.1098/rsta.2012.0156>.
- [9] K. S. Miller, B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York, 1993.
- [10] I. Podlubny. *Fractional Differential equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of Their Applications*. Mathematics in Science and Engineering, **198**. Academic Press, Inc., CA, 1999.
- [11] J. Rebenda. Application of Differential Transform to Multi-Term Fractional Differential Equations with Non-Commensurate Orders. *Symmetry*, **11**(2019), 1390.
- [12] G. Vainikko. Which functions are fractionally differentiable? *Zeitschrift für Analysis und ihre Anwendungen*. Volume 35 (2016), 465–487.