CONNECTIONS ON TRIVIAL VECTOR BUNDLES

INDRANIL BISWAS, PHÙNG HÔ HAI, AND JOÃO PEDRO DOS SANTOS

Abstract. Over a smooth and proper complex scheme, the differential Galois group of an integrable connection may be obtained as the closure of the transcendental monodromy representation. In this paper, we employ a completely algebraic variation of this idea by restricting attention to connections on trivial vector bundles and replacing the fundamental group by a certain Lie algebra constructed from the regular forms. In more detail, we show that the differential Galois group is a certain "closure" of the aforementioned Lie algebra. This is then applied to construct connections on curves with prescribed differential Galois group.

1. Introduction

A fundamental result of modern differential Galois theory affirms that, for a proper ambient variety, the differential Galois group might be obtained as the Zariski closure of the monodromy group. Our objective here is to make a synthesis of results by other mathematicians and use this to throw light on a similar finding in the realm of connections on trivial vector bundles. In this case, the role of the fundamental group is played by a certain Lie algebra (see Definition 2.4) and the role of the Zariski closure by the group-envelope (see Definition 4.8).

Let us be more precise: consider a field K of characteristic zero, a smooth, geometrically connected and proper K-scheme X, and a K-point of $x_0 \in X$. In the special case $K = \mathbb{C}$, it is known, mainly due to GAGA, that the category of integrable connections on X is equivalent to the category of complex representations of the transcendental object $\pi_1(X(\mathbb{C}), x_0)$. In addition, for any such connection $(\mathscr{E}, \nabla)$, the differential Galois group at the point x_0 (Definition 3.2) is the Zariski closure of the image $\mathrm{Im}(M_{\mathscr{E}})$, where $M_{\mathscr{E}} : \pi_1(X(\mathbb{C}), x_0) \longrightarrow \mathbf{GL}(\mathscr{E}|_{x_0})$ is the transcendental monodromy representation.

In this work, we wish to draw attention to the fact that the category of integrable connections $(\mathscr{E}, \nabla)$ on trivial vector bundles (that is, $\mathscr{E} \simeq \mathscr{O}_X^{\oplus r}$) is equivalent, not to a category of representations of a group, but of a Lie algebra $\mathfrak{L}_X$. Then, in the same spirit as the previous paragraph, the differential Galois group of $(\mathscr{E}, \nabla)$ at the point x_0 will be the "closure of the image of $\mathfrak{L}_X$" in $\mathbf{GL}(\mathscr{E}|_{x_0})$ (see Definition 4.8). The advantage here is that, contrary to what happens to the computation of the monodromy representation in case $K = \mathbb{C}$, the image of $\mathfrak{L}_X$ is immediately visible. See Theorem 5.1.

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Once the above results have been put up, we apply our findings to construct connections on curves with prescribed differential Galois groups. For this goal, we make use of the fact that semi-simple Lie algebras can be generated by solely two elements, see Corollary 6.1. Using the push-forward operation on connections, we show that on the projective line minus three points, it is possible to find regular-singular connections having differential Galois groups with arbitrary connected components; see Corollary 6.2. In addition, these connections allow a simple determination. The paper ends, see Section 7, by securing some side results which are unfortunately not recorded in writing in the necessary generality.

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Some notation and conventions. In all that follows, K is a field of characteristic zero. Vector spaces, associative algebras, Lie algebras, Hopf algebras, etc are always to be considered over K.

(1) The category of finite dimensional vector spaces (over K) is denoted by **vect**.
(2) The category of Lie algebras is denoted by **LA**.
(3) All group schemes are to be affine; **GS** is the category of affine group schemes. Given $G \in \mathbf{GS}$, we let $\mathbf{Rep}\, G$ stand for the category of finite dimensional representations of G.
(4) If $\mathfrak{A}$ stands for an associative algebra, we let $\mathfrak{A}$**-mod** be the category of left $\mathfrak{A}$-modules which are of finite dimension over K. The same notation is invoked for Lie algebras.
(5) An *ideal* of an associative algebra is a two-sided ideal. The *tensor algebra* on a vector space V is denoted by $\mathbf{T}(V)$. The *free algebra* on a set $\{s_i\}_{i\in I}$ is denoted by $K\{s_i\}$.
(6) A *curve* C is a one dimensional, integral and smooth K-scheme.
(7) A *vector bundle* is none other than a locally free coherent sheaf of finite rank. A *trivial* vector bundle on X is a direct sum of a finite number of copies of $\mathscr{O}_X$.

2. Construction of a Hopf algebra

Let Φ and Ψ be two finite dimensional vector spaces, and let

$$\beta \ : \ \Phi \otimes \Phi \longrightarrow \Psi$$

be a K-linear arrow with transpose $\beta^* : \Psi^* \longrightarrow \Phi^* \otimes \Phi^*$. Let

$$\mathfrak{I}_\beta \ = \ \text{Ideal in } \mathbf{T}(\Phi^*) \text{ generated by } \operatorname{Im} \beta^*,$$

and define

$$\mathfrak{A}_\beta \ = \ \mathbf{T}(\Phi^*)/\mathfrak{I}_\beta. \tag{1}$$

It is useful at this point to note that $\mathfrak{I}_\beta$ is a homogeneous ideal so that $\mathfrak{A}_\beta$ has a natural grading. In more explicit terms, fix a basis $\{\varphi_i\}_{i=1}^r$ of Φ and a basis $\{\psi_i\}_{i=1}^s$ of Ψ. Write $\{\varphi_i^*\}_{i=1}^r$ and $\{\psi_i^*\}_{i=1}^s$ for the respective dual bases. If

$$\beta(\varphi_k \otimes \varphi_\ell) = \sum_{i=1}^{s} \beta_i^{(k\ell)} \cdot \psi_i,$$

then

$$\beta^*(\psi_i^*) = \sum_{1\le k,\ell\le r} \beta_i^{(k\ell)} \cdot \varphi_k^* \otimes \varphi_\ell^*.$$

Consequently, $\mathfrak{A}_\beta$ in (1) is the quotient of the free algebra $K\{t_1, \ldots, t_r\}$ by the ideal generated by the s elements

$$\sum_{1\le k,\ell\le r} \beta_i^{(k\ell)} t_k t_\ell, \qquad i = 1, \ldots, s.$$

In particular, given $V \in$ **vect** and elements $A_1, \ldots, A_r \in \mathrm{End}(V)$, the association $t_i \longmapsto A_i$ defines a representation of $\mathfrak{A}_\beta$ if and only if

$$\sum_{1\le k,\ell\le r} \beta_i^{(k\ell)} \cdot A_k A_\ell = 0, \qquad \forall\ i = 1, \ldots, s.$$

It is worth pointing out that if β is alternating, then

$$\sum_{1\le k,\ell\le r} \beta_i^{(k\ell)} t_k t_\ell = \sum_{1\le k<\ell\le r} \beta_i^{(k\ell)} [t_k,\, t_\ell]. \tag{2}$$

This reformulation has useful consequences for the structure of $\mathfrak{A}_\beta$.

From now on, β is always assumed to be alternating.

Let $\mathbf{L}(\Phi^*)$ be the free Lie algebra on the vector space Φ^* so that $\mathbf{T}(\Phi^*)$ is the universal enveloping algebra of $\mathbf{L}(\Phi^*)$ [BLie, II.3.1, p. 32, Theorem 1]. Clearly

$$\sum_{1\le k<\ell\le r} \beta_i^{(k\ell)} [t_k,\, t_\ell] \in \mathbf{L}(\Phi^*), \qquad \forall\ i = 1, \ldots, s.$$

Let

$$\mathfrak{K}_\beta = \begin{array}{c}\text{Lie ideal of } \mathbf{L}(\Phi^*) \text{ generated by the } s \\ \text{elements } \{\sum_{1\le k<\ell\le r} \beta_i^{(k\ell)} [t_k,\, t_\ell]\}_{i=1}^s \text{ in (2).}\end{array}$$

Proposition 2.1. *The algebra $\mathfrak{A}_\beta$ in* (1) *is the universal enveloping algebra of $\mathbf{L}(\Phi^*)/\mathfrak{K}_\beta$.*

Proof. The left ideal $\mathbf{T}(\Phi^*) \cdot \mathfrak{K}_\beta$ is in fact a two sided ideal [BLie, I.2.3]; it is easily seen to agree with $\mathfrak{I}_\beta$ in (1). Now we use [BLie, I.2.3, Proposition 3] to complete the proof. □

Definition 2.2. The Lie algebra $\mathbf{L}(\Phi^*)/\mathfrak{K}_\beta$ shall be denoted by $\mathfrak{L}_\beta$.

A simple remark should be recorded here.

Lemma 2.3. *The above Lie algebra $\mathfrak{L}_\beta$ is a quotient of the free Lie algebra $\mathbf{L}(\Phi^*)$. In particular, $\mathfrak{L}_\beta$ is generated by the image of Φ^*.*

Recall that for a Lie algebra L, the universal enveloping algebra $\mathbf{U}L$ has a natural structure of Hopf algebra [Sw69, 3.2.2, p. 58] and hence from Proposition 2.1 it follows that $\mathfrak{A}_\beta$ has the structure of a Hopf algebra. Analogously, $\mathbf{T}(\Phi^*)$ is also a Hopf algebra and the quotient map

$$\mathbf{T}(\Phi^*) \longrightarrow \mathfrak{A}_\beta \tag{3}$$

is an arrow of Hopf algebras.

In what follows, we give the category $\mathfrak{A}_\beta$-**mod** the tensor product explained in [Mo93, 1.8.1, p. 14]. It turns out that the canonical equivalence

$$\mathfrak{L}_\beta\textbf{-mod} \stackrel{\sim}{\longrightarrow} \mathfrak{A}_\beta\textbf{-mod} \tag{4}$$

is actually a tensor equivalence.

The only case in which $\mathfrak{A}_\beta$ will interest us is that of:

Definition 2.4. Let X be a smooth, connected and projective K-scheme. Let

$$\beta \ : \ H^0(X,\, \Omega_X^1) \otimes H^0(X,\, \Omega_X^1) \longrightarrow H^0(X,\, \Omega_X^2)$$

be the wedge product of differential forms. We put

$$\mathfrak{A}_\beta = \mathfrak{A}_X \quad \text{and} \quad \mathfrak{L}_\beta = \mathfrak{L}_X.$$

3. Connections

We shall begin this section by establishing the notation and pointing out structural references. We fix a smooth and connected K-scheme X. Soon, we shall assume X to be projective.

Definition 3.1. We let **MC** be the category of K–linear connections on coherent $\mathscr{O}_X$–modules and **MIC** the full subcategory of **MC** whose objects are integrable connections [Ka70, 1.0]. We let $\mathbf{MC}^{\mathrm{tr}}$ be the full subcategory of **MC** having as objects pairs $(\mathscr{E},\, \nabla)$ in which $\mathscr{E}$ is a trivial vector bundle. The category $\mathbf{MIC}^{\mathrm{tr}}$ is defined analogously: it is the full subcategory of **MIC** having as objects pairs $(\mathscr{E},\, \nabla)$ in which $\mathscr{E}$ is a trivial vector bundle.

A fundamental result of the theory of connections is that for each $(\mathscr{E},\, \nabla)$, the coherent sheaf $\mathscr{E}$ is actually locally free [Ka70, Proposition 8.8] and hence, given $x_0 \in X(K)$, the functor "taking the fibre at x_0" defines a tensor equivalence

$$\bullet\,|_{x_0} \ : \ \mathbf{MIC} \stackrel{\sim}{\longrightarrow} \mathbf{Rep}\,\Pi(X,\, x_0), \tag{5}$$

where $\Pi(X,\, x_0)$ is a group scheme over K; see [DM82, Theorem 2.11]. This group scheme is sometimes called the "differential fundamental group scheme of X at x_0". It is in rare cases that $\Pi(X,\, x_0)$ will be an algebraic group, and hence it is important to turn it into a splice of smaller pieces. This motivates the following definition, which at the end gives a name to the main object of study of the present work.

Definition 3.2 (The differential Galois group)**.** Let $(\mathscr{E},\, \nabla) \in$ **MIC** be given, and let $\rho_{\mathscr{E}} \ : \ \Pi(X,\, x_0) \longrightarrow \mathbf{GL}(\mathscr{E}|_{x_0})$ be the representation associated to $\mathscr{E}$ via the equivalence in (5). The image of $\rho_{\mathscr{E}}$ in $\mathbf{GL}(\mathscr{E}|_{x_0})$ is the differential Galois group of $(\mathscr{E},\,\nabla)$ at the point x_0.

Remark 3.3. For $(\mathscr{E},\,\nabla)\;\in\;\mathbf{MIC}$, the category of representations of the differential Galois group of $(\mathscr{E},\,\nabla)$ at x_0 is naturally a full subcategory of $\mathbf{MIC}$; it is not difficult to see that it is

$$\langle(\mathscr{E},\,\nabla)\rangle_{\otimes}\;=\;\left\{\mathscr{M}'/\mathscr{M}''\in\mathbf{MIC}\;:\;\begin{array}{c}\text{there exist } a_i, b_i\in\mathbb{N}\text{ such that}\\ \mathscr{M}''\subset\mathscr{M}'\subset\bigoplus_i\mathscr{E}^{\otimes a_i}\otimes\check{\mathscr{E}}^{\otimes b_i}\end{array}\right\}.$$

From now on, X is in addition *projective*. Let us be more explicit about objects in $\mathbf{MC}^{\mathrm{tr}}$. Fix $E\in\mathbf{vect}$ and let

$$\begin{aligned}A\;\in\;&\mathrm{Hom}_{K\text{-}\mathbf{alg}}(\mathbf{T}(H^0(X,\,\Omega^1_X)^*),\;\mathrm{End}(E))\\ &=\mathrm{Hom}(H^0(X,\,\Omega^1_X)^*,\;\mathrm{End}(E))\\ &=\mathrm{End}(E)\otimes H^0(X,\,\Omega^1_X).\end{aligned}$$

Hence, A gives rise to a $\mathrm{End}(E)$–valued 1–form on X which, in turn, gives rise to a connection

$$d_A\;:\;\mathscr{O}_X\otimes E\;\longrightarrow\;(\mathscr{O}_X\otimes E)\underset{\mathscr{O}_X}{\otimes}\Omega^1_X \tag{6}$$

on the trivial vector bundle $\mathscr{O}_X\otimes E$. Explicitly, if $\{\theta_i\}_{i=1}^g$ is a basis of $H^0(X,\,\Omega^1_X)$ with dual basis $\{\varphi_i\}_{i=1}^g$ of $H^0(X,\,\Omega^1_X)^*$ and $A_i\;:=\;A(\varphi_i)\;\in\;\mathrm{End}(E)$, we arrive at

$$d_A(1\otimes e)\;=\;\sum_{i=1}^g(1\otimes A_i(e))\otimes\theta_i$$

for all $e\;\in\;E$.

Definition 3.4. The above pair consisting of $(\mathscr{O}_X\otimes E,\,d_A)$ shall be denoted by $\mathscr{V}(E,\,A)$.

Let now $\{\sigma_i\}_{i=1}^h$ be a basis of $H^0(X,\,\Omega^2_X)$ and write

$$\theta_k\wedge\theta_\ell\;=\;\sum_{i=1}^h\beta_i^{(k\ell)}\cdot\sigma_i\,;$$

recall that $\{\theta_i\}_{i=1}^g$ is a basis of $H^0(X,\,\Omega^1_X)$. Since X is proper, Hodge theory tells us that all global 1-forms are closed [Del68, Theorem 5.5] and hence the curvature

$$R_{d_A}\;:\;\mathscr{O}_X\otimes E\;\longrightarrow\;(\mathscr{O}_X\otimes E)\otimes_{\mathscr{O}_X}\Omega^2_X$$

of the connection d_A in (6) satisfies

$$R_{d_A}(1\otimes e)\;=\;\sum_{i=1}^h\sum_{1\le k,\ell\le g}\Big(1\otimes\beta_i^{(k\ell)}A_kA_\ell(e)\Big)\otimes\sigma_i.$$

Hence, $R_{d_A}\;=\;0$ if and only if

$$\sum_{1\le k,\ell\le g}\beta_i^{(k\ell)}A_kA_\ell\;=\;0$$

for each $i \in \{1, \ldots, h\}$. Also, since β in Definition 2.4 is alternating, we conclude that $R_{d_A} = 0$ if and only if for each $i \in \{1, \ldots, h\}$,

$$\sum_{1 \le k, \ell \le g} \beta_i^{(k\ell)} A_k A_\ell = \sum_{1 \le k < \ell \le g} \beta_i^{(k\ell)} [A_k, A_\ell] = 0.$$

These considerations form the main points of the proof of the following result, whose through verification is left to the interested reader. (It is worth recalling that $K = H^0(X, \mathscr{O}_X)$.)

Proposition 3.5. *The functor*

$$\mathscr{V} : \mathbf{T}(H^0(\Omega_X^1)^*)\text{-}\mathbf{mod} \longrightarrow \mathbf{MC}^{\mathrm{tr}}$$

is an equivalence of K-linear categories. Under this equivalence, $\mathscr{V}(E, A)$ lies in $\mathbf{MIC}^{\mathrm{tr}}$ if and only if (E, A) is in fact a representation of $\mathfrak{A}_X$ (see Definition 2.4). □

Let us now discuss tensor products. Given representations

$$A : \mathbf{T}(H^0(\Omega_X^1)^*) \longrightarrow \mathrm{End}(E) \quad \text{and} \quad B : \mathbf{T}(H^0(\Omega_X^1)^*) \longrightarrow \mathrm{End}(F),$$

we obtain a new representation of $A \boxtimes B : \mathbf{T}(H^0(\Omega_X^1)^*) \longrightarrow \mathrm{End}(E \otimes F)$ by putting

$$A \boxtimes B(\varphi) = A(\varphi) \otimes \mathrm{id}_F + \mathrm{id}_E \otimes B(\varphi), \qquad \forall \ \varphi \in H^0(\Omega_X^1)^*.$$

This is of course only the tensor structure on the category $\mathbf{T}(H^0(\Omega_X^1)^*)$-**mod** defined by the Hopf algebra structure of $\mathbf{T}(H^0(\Omega_X^1)^*)$ [Sw69, p. 58]. With this, it is not hard to see that the canonical isomorphism of $\mathscr{O}_X$-modules

$$\mathscr{O}_X \otimes (E \otimes F) \xrightarrow{\sim} (\mathscr{O}_X \otimes E) \otimes_{\mathscr{O}_X} (\mathscr{O}_X \otimes F)$$

is horizontal with respect to the tensor product connection on the right [Ka70, Section 1.1] and the connection $d_{A \boxtimes B}$ on the left (it is the connection induced by the connections d_A and d_B). We then arrive at equivalences of tensor categories:

Theorem 3.6.

(i) The functor

$$\mathscr{V} : \mathbf{T}(H^0(\Omega_X^1)^*)\text{-}\mathbf{mod} \longrightarrow \mathbf{MC}^{\mathrm{tr}}$$

is an equivalence of K-linear tensor categories.

(ii) The restriction

$$\mathscr{V} : \mathfrak{A}_X\text{-}\mathbf{mod} \longrightarrow \mathbf{MIC}^{\mathrm{tr}}$$

is also an equivalence of K-linear tensor categories. In addition, the composition $(\bullet|_{x_0}) \circ \mathscr{V}$ is naturally isomorphic to the forgetful functor, where $\bullet|_{x_0}$ is constructed in (5) (see Definition 2.4 for $\mathfrak{A}_X$).

*(iii) The composition of the equivalence in (4) with $\mathscr{V} : \mathfrak{A}_X$-**mod** $\longrightarrow \mathbf{MIC}^{\mathrm{tr}}$ defines a K-linear tensor equivalence*

$$\mathfrak{L}_X\text{-}\mathbf{mod} \xrightarrow{\sim} \mathbf{MIC}^{\mathrm{tr}}$$

(see Definition 2.4 for $\mathfrak{L}_X$).

□

Making use again of the main theorem of (categorical) Tannakian theory, [DM82, p. 130, Theorem 2.11], we obtain an equivalence of abelian tensor categories:

$$\bullet\,|_{x_0} \;:\; \mathbf{MIC}^{\mathrm{tr}} \xrightarrow{\sim} \mathbf{Rep}\,\Theta(X,\,x_0), \tag{7}$$

where $\Theta(X,\,x_0)$ is a group scheme. In addition, the inclusion map

$$\mathbf{MIC}^{\mathrm{tr}} \longrightarrow \mathbf{MIC}$$

defines a morphism

$$\mathbf{q}_X \;:\; \Pi(X,\,x_0) \longrightarrow \Theta(X,\,x_0),$$

where $\Pi(X,\,x_0)$ and $\Theta(X,\,x_0)$ are constructed in (5) and (7) respectively. Along the line of Proposition 3.1 of [B$^+$21], we have:

Proposition 3.7. *The above morphism $\mathbf{q}_X$ is in fact a quotient morphism.*

Proof. Let $\mathscr{E} \longrightarrow \mathscr{Q}$ be an epimorphism of **MIC** with $\mathscr{E} \in \mathbf{MIC}^{\mathrm{tr}}$; write e for the rank of $\mathscr{E}$ and q for that of $\mathscr{Q}$.

Let G stand for the Grassmann variety of q–dimensional quotients of $K^{\oplus e}$, and let $\mathscr{O}_G^{\oplus e} \longrightarrow \mathscr{U}$ stand for the universal epimorphism [Ni05, 5.1.6]. We then obtain a morphism $f \;:\; X \longrightarrow G$ such that $f^*\mathscr{U} = \mathscr{Q}$. For each proper curve $\gamma \;:\; C \longrightarrow X$, the vector bundle $\gamma^*\mathscr{Q} = (f\circ\gamma)^*\mathscr{U}$ has degree zero [BS06, Remark 3.3]; in particular, $(f\circ\gamma)^*\det\mathscr{U}$ has also degree zero. As $\det\mathscr{U}$ is a very ample invertible sheaf on G, from $\mathrm{degree}((f\circ\gamma)^*\det\mathscr{U}) = \mathrm{degree}((f\circ\gamma)^*\mathscr{U}) = 0$ we conclude that $(f\circ\gamma)^*\det\mathscr{U}$ is trivial, and hence the schematic image of $f\circ\gamma$ is a (closed) point [Liu02, p. 331, Exercise 8.1.7(a)]. Now, Ramanujam's Lemma (see Remark 3.8 below) can be applied to show that any two closed points x_1 and x_2 of X belong to the image of a morphism $\gamma \;:\; C \longrightarrow X$ from a proper curve.

Hence, the schematic image of X under f is a single point. Since $H^0(X,\,\mathscr{O}_X) = K$, this point must be a K-rational point and hence f factors through the structural morphism $X \longrightarrow \operatorname{Spec} K$. Consequently, $f^*\mathscr{U} = \mathscr{Q}$ is a trivial vector bundle. The standard criterion for a morphism of group schemes to be a quotient morphism (see [DM82, p. 139, Proposition 2.21(a)] for the criterion) can be applied to complete the proof. ☐

Remark 3.8 (Ramanujam's Lemma)**.** Let Z be a geometrically integral projective K-scheme and $z_1,\,z_2$ are two closed points on it. We contend that there exists a proper curve C together with a morphism $\gamma \;:\; C \longrightarrow X$ such that z_1 and z_2 belong to the image of γ. The proof is the same as in [Mu70, p. 56], but the Bertini theorem necessary for our purpose comes from [Jou83, Cor. 6.11].

If $\dim Z = 1$, it is sufficient to chose C to be the normalisation of Z. Let $\dim Z := d \geq 2$ and suppose that the result holds for all geometrically integral and projective schemes of dimension strictly smaller than d. We only need now to find a geometrically integral closed subscheme $Y \subset Z$ containing z_1 and z_1 and having dimension strictly smaller than d. Let $\pi \;:\; Z' \longrightarrow Z$ be the blow up of the closed subscheme $\{z_1,\,z_2\}$. Note that Z' is geometrically integral [Liu02, 8.1.12(c) and (d), p. 322]. In addition, the fibres of π above z_1 and z_2 are Cartier subschemes of Z' and hence of dimension at least one [Liu02, 2.5.26, p. 74]. Let $Z' \longrightarrow \mathbb{P}^N$ be a closed immersion, and let $H \subset \mathbb{P}^N$ be a hyperplane of codimension one such that

- $Z' \cap H$ is geometrically integral [Jou83, 6.11 (2)-(3)] of dimension $\dim Z - 1$, and
- $Z' \cap \pi^{-1}(z_i) \neq \emptyset$, loc. cit, (1)-(b).

Then, the schematic image of $\pi : Z' \cap H \longrightarrow Z$ is the scheme Y that we are seeking.

Remark 3.9. In [Liu02, Ch. 8, p. 331, Exercise 1.5], the reader shall find a useful, but slightly weaker version of Ramanujam's Lemma.

Remark 3.10. The idea to consider certain connections as representations of a Lie algebra can be found at least on [Del87, 12.2–5].

4. The Tannakian envelope of a Lie algebra

All that follows in this section is, in essence, due to Hochschild [Ho59]; since he expressed himself without using group schemes and his ideas are spread out in several papers, we shall briefly condense his theory in what follows. The reader should also consult [Na02], where some results reviewed here also appear.

Our objective in this section is to give a construction of the affine envelope of a Lie algebra. One can, of course, employ the categorical Tannakian theory [DM82, p. 130, Theorem 2.11] to the category **$\mathfrak{L}$-mod** to obtain such a construction, but we prefer to draw the reader's attention to something which is less widespread than [DM82] and more concrete.

Let $\mathfrak{L}$ be a Lie algebra with universal enveloping algebra $\mathbf{U}\mathfrak{L}$. Note that $\mathbf{U}\mathfrak{L}$ is not only an algebra, but also a cocommutative Hopf algebra; see [Sw69, p. 58, Section 3.2.2] and [Mo93, p. 72, Example 1.5.4]. Consequently, the Hopf dual $(\mathbf{U}\mathfrak{L})^\circ$ [Sw69, VI] is a commutative Hopf algebra (see [Sw69, Section 6.2, pp. 122-3] or [Mo93, Theorem 9.1.3]). This means that

$$\mathbf{G}(\mathfrak{L}) := \mathrm{Spec}\ (\mathbf{U}\mathfrak{L})^\circ$$

is a group scheme, which we call the *affine envelope of* $\mathfrak{L}$. Let us show that this construction gives a left adjoint to the functor

$$\mathrm{Lie} : \mathbf{GS} \longrightarrow \mathbf{LA}.$$

We start by noting that $\mathbf{G}$ is indeed a functor; given an arrow $\mathfrak{G} \longrightarrow \mathfrak{H}$, the associated arrow $(\mathbf{U}\mathfrak{H})^* \longrightarrow (\mathbf{U}\mathfrak{G})^*$ gives rise to a morphism of coalgebras $(\mathbf{U}\mathfrak{H})^\circ \longrightarrow (\mathbf{U}\mathfrak{G})^\circ$; see [Sw69, p. 114, Remark 1]. The fact that the algebra structures are also preserved is indeed a consequence of the fact that $\mathbf{U}\mathfrak{G} \longrightarrow \mathbf{U}\mathfrak{H}$ is also an arrow of coalgebras.

Let G be a group scheme, and let $\rho : \mathfrak{L} \longrightarrow \mathrm{Lie}\, G$ be a morphism of $\mathbf{LA}$; we write ρ for the arrow induced between universal algebras as well. Interpreting elements in $\mathrm{Lie}\, G$ as elements of $\mathrm{End}_K(\mathscr{O}(G))$, we obtain a morphism of K-algebras $\mathbf{U}(\mathrm{Lie}\, G) \longrightarrow \mathrm{End}_K(\mathscr{O}(G))$. Since $\mathscr{O}(G)$ is a locally finite $\mathrm{Lie}\, G$–module, it is also a locally finite $\mathfrak{L}$–module and a fortiori a locally finite $\mathbf{U}\mathfrak{L}$–module. Let

$$\varphi_\rho : \mathscr{O}(G) \longrightarrow (\mathbf{U}\mathfrak{L})^*$$

be defined by

$$\varphi_\rho(a) \,:\, u \longmapsto \varepsilon(\rho(u)(a)), \tag{8}$$

where

$$\varepsilon \,:\, \mathscr{O}(G) \longrightarrow K \tag{9}$$

is the coidentity and $u \in \mathbf{U}\mathfrak{L}$.

Lemma 4.1.

(1) For each $a \in \mathscr{O}(G)$, the element $\varphi_\rho(a)$ in (8) *lies in the Hopf dual* $(\mathbf{U}\mathfrak{L})^\circ$.
(2) The arrow φ_ρ is a morphism of Hopf algebras.

Proof. (1) Let a belong to the finite dimensional $\mathbf{U}\mathfrak{L}$–submodule V of $\mathscr{O}(G)$. Let $I \subset \mathbf{U}\mathfrak{L}$ be the kernel of the induced arrow of K-algebras $\mathbf{U}\mathfrak{L} \longrightarrow \mathrm{End}(V)$; it follows that $I \subset \mathrm{Ker}\,\varphi_\rho(a)$ and $\varphi_\rho(a) \in (\mathbf{U}\mathfrak{L})^\circ$.

(2) This verification is somewhat lengthy, but straightforward once the right path has been found. We shall only indicate the most important ideas. Let us write φ instead of φ_ρ and consider elements of $\mathbf{U}\mathfrak{L}$ as G-invariant linear operators [Wa79, Section 12.1] on $\mathscr{O}(G)$. In what follows, we shall use freely the symbol Δ to denote comultiplication on different coalgebras.

Compatibility with multiplication. We must show that

$$[\varphi(a) \otimes \varphi(b)]\,(\Delta(u)) \,=\, \varphi(ab)(u) \tag{10}$$

for all a, $b \in \mathscr{O}(G)$ and $u \in \mathbf{U}\mathfrak{L}$. Obviously, formula (10) holds for $u \in K \subset \mathbf{U}\mathfrak{L}$. In case $u \in \mathfrak{L}$, the validity of (10) is an easy consequence of the fact that $u : \mathscr{O}(G) \longrightarrow \mathscr{O}(G)$ is a derivation and $\Delta u = u \otimes 1 + 1 \otimes u$. We then prove that if (10) holds for u then, for any given $\delta \in \mathfrak{L}$, formula (10) holds for $u\delta$. Since $\mathbf{U}\mathfrak{L}$ is generated by $\mathfrak{L}$, we are done.

Compatibility with comultiplication. For $\zeta \in (\mathbf{U}\mathfrak{L})^\circ$, we know that $\Delta_{(\mathbf{U}\mathfrak{L})^\circ}(\zeta)$ is defined by

$$u \otimes v \longmapsto \zeta(uv)$$

for u, $v \in \mathbf{U}\mathfrak{L}$. We need to prove that

$$\varepsilon(uv(a)) \,=\, \varphi \otimes \varphi \circ \Delta a$$

for every triple u, $v \in \mathbf{U}\mathfrak{L}$ and $a \in \mathscr{O}(G)$, where ε is the homomorphism in (9). This follows from the invariance formulas $\Delta u \,=\, (\mathrm{id} \otimes u)\Delta$.

Compatible with unity and co-unity. This is much simpler and we omit its verification.

Compatibility with antipode. A bialgebra map between Hopf algebras is automatically a Hopf algebra map [Sw69, Lemma 4.0.4]. □

Proposition 4.2. *The above construction establishes a bijection*

$$\begin{aligned} \varphi \,:\, \mathrm{Hom}_{\mathbf{LA}}(\mathfrak{L},\ \mathrm{Lie}\,G) \quad &\longrightarrow \quad \mathrm{Hom}_{\mathbf{Hpf}}(\mathscr{O}(G),\ (\mathbf{U}\mathfrak{L})^\circ) \\ &= \mathrm{Hom}_{\mathbf{GS}}(\mathbf{G}(\mathfrak{L}),\ G), \end{aligned}$$

rendering $\mathbf{G} : \mathbf{LA} \longrightarrow \mathbf{GS}$ *a left adjoint to* $\mathrm{Lie} : \mathbf{GS} \longrightarrow \mathbf{LA}$.

Proof. We construct the inverse of φ and leave the reader with all verifications. Let $f : \mathscr{O}(G) \longrightarrow (\mathbf{U}\mathfrak{L})^\circ$ be a morphism of Hopf algebras. Let $x \in \mathfrak{L}$ be given, and define

$$\psi_f(x) : \mathscr{O}(G) \longrightarrow K, \quad a \longmapsto f(a)(x). \tag{11}$$

It is a simple matter to show that $\psi_f(x)$ is an ε-derivation, which is then interpreted as an element of $\operatorname{Lie} G$ in a standard fashion [Wa79, 12.2]. In addition, $\psi_f : \mathfrak{L} \longrightarrow \operatorname{Lie} G$ gives a morphism of Lie algebras (the reader might use the bracket as explained in [Wa79, Section 12.1, p. 93]). Then $f \longmapsto \psi_f$ and $\rho \longmapsto \varphi_\rho$ are mutually inverses; the verification of this fact consists of a chain of simple manipulations and we contend ourselves in giving some elements of the equations to be verified. That $\psi_{\varphi_\rho} = \rho$ is in fact immediate. On the other hand, the verification of

$$\varphi_{\psi_f}(a)(u) = f(a)(u), \qquad \forall\ a \in \mathscr{O}(G), \forall\ \ u \in \mathbf{U}\mathfrak{L}$$

requires the ensuing observations. (We shall employ Sweedler's notation for the Hopf algebra $\mathscr{O}(G)$ [Sw69, Section 1.2, 10ff].)

(1) For $\delta \in \mathfrak{L}$, the derivation $\mathscr{O}(G) \longrightarrow \mathscr{O}(G)$ associated to $\psi_f(\delta)$ is determined by $a \longmapsto \sum_{(a)} a_{(1)} \cdot [f(a_{(2)})(\delta)]$.
(2) The axioms show that $\sum_{(a)} \varepsilon(a_{(1)})a_{(2)} = a$.
(3) Suppose that for $u \in \mathbf{U}\mathfrak{L}$ and $\delta \in \mathbf{U}\mathfrak{L}$ we know that, for all $a \in \mathscr{O}(G)$,

$$\varphi_{\psi_f}(a)(u) = f(a)(u) \quad \text{and} \quad \varphi_{\psi_f}(a)(\delta) = f(a)(\delta).$$

Then $\varphi_{\psi_f}(a)(u\delta) = f(a)(u\delta)$ because of the equations

$$f(a)(xy) = \sum_{(a)} f(a_{(1)})(x) \cdot f(a_{(2)})(y), \qquad \forall\ \ x, y \in \mathbf{U}\mathfrak{L},$$

which is a consequence of the fact that f is a map of Hopf algebras.

This completes the proof. □

In the proof of Proposition 4.2 we defined a bijection

$$\psi : \operatorname{Hom}(\mathbf{G}(\mathfrak{L}), G) \longrightarrow \operatorname{Hom}(\mathfrak{L}, \operatorname{Lie} G)$$

by means of eq. (11). (We are here slightly changing the notation employed previously by using arrows between schemes and not algebras on the domain; this shall cause no confusion.) In case $G = \mathbf{GL}(V)$ and in the light of the identification $\operatorname{Lie} \mathbf{GL}(V) = \mathfrak{gl}(V)$, ψ has a rather useful description. Let $f : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(V)$ be a representation and let $c_f : V \to V \otimes (\mathbf{U}\mathfrak{L})^\circ$ be the associated comodule morphism. It then follows that

$$(\mathrm{id}_V \otimes \text{evaluate at } x) \circ c_f = \psi_f(x). \tag{12}$$

Corollary 4.3. *Let V be a finite dimensional vector space and $f : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(V)$ a representation. Write $\psi_f : \mathfrak{L} \longrightarrow \mathfrak{gl}(V)$ for the morphism of* **LA** *mentioned above. Then, this gives rise to a K-linear equivalence of tensor categories*

$$\mathbf{Rep}\, \mathbf{G}(\mathfrak{L}) \longrightarrow \mathfrak{L}\text{-}\mathbf{mod}.$$

Proof. To define a functor $\mathbf{Rep}\,\mathbf{G}(\mathfrak{L}) \longrightarrow \mathfrak{L}\text{-}\mathbf{mod}$ it is still necessary to define the maps between sets of morphisms.

Let $f : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(V)$ and $g : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(W)$ be representations, and let $T \in \mathrm{Hom}_K(V, W)$. We shall show that $T \in \mathrm{Hom}_{\mathbf{G}(\mathfrak{L})}(V, W)$ if and only if $T \in \mathrm{Hom}_{\mathfrak{L}}(V, W)$.

Consider $\widehat{T} = \begin{pmatrix} I & 0 \\ T & I \end{pmatrix} \in \mathrm{GL}(V \oplus W)$ and denote by

$$C_T : \mathbf{GL}(V \oplus W) \longrightarrow \mathbf{GL}(V \oplus W)$$

the conjugation by $\widehat{T}$. Then T is G–equivariant if and only if $C_T \circ \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$. Similarly let us write $c_T : \mathfrak{gl}(V \oplus W) \longrightarrow \mathfrak{gl}(V \oplus W)$ to denote conjugation by $\widehat{T}$. Then, for given representations $\rho : \mathfrak{L} \longrightarrow \mathfrak{gl}(V)$ and $\sigma : \mathfrak{L} \longrightarrow \mathfrak{gl}(W)$, the arrow T is a morphism of $\mathfrak{L}$-modules if and only if $c_T \begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & \sigma \end{pmatrix}$. Employing equation (12), we verify readily that

$$\begin{array}{ccc} \mathrm{Hom}(\mathbf{G}(\mathfrak{L}), \mathbf{GL}(V \oplus W)) & \xrightarrow{C_T \circ (-)} & \mathrm{Hom}(\mathbf{G}(\mathfrak{L}), \mathbf{GL}(V \oplus W)) \\ \downarrow \psi & & \downarrow \psi \\ \mathrm{Hom}(\mathfrak{L}, \mathfrak{gl}(V \oplus W)) & \xrightarrow[c_T \circ (-)]{} & \mathrm{Hom}(\mathfrak{L}, \mathfrak{gl}(V \oplus W)), \end{array}$$

commutes. We then see that ψ becomes a functor, which is K-linear, exact and fully-faithful.

Let us now deal with the tensor product. Given representations $f : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(V)$ and $g : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(W)$, let us write

$$f \,\square\, g : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(V \otimes W)$$

for the tensor product representation. We then obtain on $V \otimes W$ the a structure of a $\mathfrak{L}$-module via $\psi_{f \,\square\, g}$ and it is to be shown that this is precisely the $\mathfrak{L}$-module structure coming from the tensor product of $\mathfrak{L}$-modules. In other words, we need to show that for any $x \in \mathfrak{L}$, the equation $\psi_f(x) \otimes \mathrm{id}_W + \mathrm{id}_V \otimes \psi_g(x) = \psi_{f \,\square\, g}(x)$ holds. We make use of eq. (12) again. Let $v \in V$ and $w \in W$ be such that $c_f(v) = \sum_i v_i \otimes f_i$ and $c_g(w) = \sum_j w_j \otimes g_j$. Then $c_{f \,\square\, g}(v \otimes w) = \sum_{i,j} v_i \otimes w_j \otimes f_i g_j$ and hence

$$\psi_{f \,\square\, g}(x)(v \otimes w) = \sum_{i,j} v_i \otimes w_j \cdot (f_i(x)\varepsilon(g_j) + \varepsilon(f_i)g_j(x)),$$

where $\varepsilon : (\mathbf{U}\mathfrak{L})^\circ \longrightarrow K$ is the co-unit defined by evaluating at $1 \in \mathbf{U}\mathfrak{L}$, and we have used that $\Delta x = x \otimes 1 + 1 \otimes x$. Now, $\sum_i v_i \varepsilon(f_i) = v$ and $\sum_j w_j \varepsilon(g_j) = w$. Hence, $\psi_{f \,\square\, g}(x)(v \otimes w) = \sum_i f_i(x)v_i \otimes w + \sum_j v \otimes g_j(x)w_j$, as we wanted. □

Corollary 4.4. *Let G be an algebraic group scheme, and let $\mathbf{G}(\mathfrak{L}) \longrightarrow G$ be a quotient morphism. Then G is connected. Said differently, $\mathbf{G}(\mathfrak{L})$ is pro-connected.*

Proof. For the finite etale group scheme $\pi_0(G)$ [Wa79, Section 6.7], the set

$$\mathrm{Hom}_{\mathbf{LA}}(\mathfrak{L}, \mathrm{Lie}\,(\pi_0 G))$$

is a singleton and hence $\mathrm{Hom}_{\mathbf{GS}}(\mathbf{G}(\mathfrak{L}),\ \pi_0(G))$ is a singleton. It then follows that $\pi_0(G)$ is trivial and G is connected [Wa79, Theorem of 6.6]. □

In what follows, we denote by

$$\chi\ :\ \mathfrak{L} \longrightarrow \mathrm{Lie}\,\mathbf{G}(\mathfrak{L}) \tag{13}$$

the morphism of Lie algebras corresponding to the identity of $\mathrm{Hom}_{\mathbf{GS}}(\mathbf{G}(\mathfrak{L}),\ \mathbf{G}(\mathfrak{L}))$ under the bijection in Proposition 4.2. This is, of course, the unit of the adjunction [Mac70, IV.1]. Let us profit to note that, as explained in [Mac70, IV.1, eq. (5)], for each $f \in \mathrm{Hom}_{\mathbf{GS}}(\mathbf{G}(\mathfrak{L}), G)$, the equation

$$\psi_f\ =\ (\mathrm{Lie}\, f) \circ \chi. \tag{14}$$

is valid.

One fundamental property of χ needs to be expressed in terms of "algebraic density" [Ho74, p. 175].

Definition 4.5. Let G be a group scheme. A morphism $\rho : \mathfrak{L} \longrightarrow \mathrm{Lie}\,G$ is *algebraically dense* if the only closed subgroup scheme $H \subset G$ such that $\rho(\mathfrak{L}) \subset \mathrm{Lie}\,H$ is G itself.

Proposition 4.6. *The morphism* $\chi : \mathfrak{L} \longrightarrow \mathrm{Lie}\,\mathbf{G}(\mathfrak{L})$ *in* (13) *is algebraically dense.*

Before proving Proposition 4.6, we shall require:

Lemma 4.7. *Let G be a group scheme. Then there exists a projective system of algebraic group schemes*

$$\{G_i,\, u_{ij}\ :\ G_j \longrightarrow G_i\}$$

where each u_{ij} is faithfully flat and an isomorphism

$$G \simeq \varprojlim_i G_i.$$

In addition, all arrows $\mathrm{Lie}\,u_{ij} : \mathrm{Lie}\,G_j \longrightarrow \mathrm{Lie}\,G_i$ *are surjective.*

Proof. This is a simple exercise once the correct arguments in the literature are brought to light. To find the projective system with the desired properties, we employ [Wa79, Corollary in 3.3, p. 24] and [Wa79, Theorem of 14.1, p. 109]. Then, [Mi17, Corollary 3.25, p. 72] and [Mi17, Proposition 1.63, p. 25] show that the arrows $\mathrm{Lie}\,u_{ij}$ are always surjective. □

Proof of Proposition 4.6. Let $u : H \longrightarrow \mathbf{G}(\mathfrak{L})$ be a closed immersion and let $\rho : \mathfrak{L} \longrightarrow \mathrm{Lie}\,H$ be an arrow of Lie algebras such that

$$(\mathrm{Lie}\,u) \circ \rho\ =\ \chi.$$

Let $f : \mathbf{G}(\mathfrak{L}) \longrightarrow H$ be an arrow from **GS** such that $\rho = \psi_f$. From eq. (14), we have

$$\rho\ =\ (\mathrm{Lie}\,f) \circ \chi.$$

Hence, $\chi = (\mathrm{Lie}(uf)) \circ \chi$, which proves that $u \circ f = \mathrm{id}_{\mathbf{G}(\mathfrak{L})}$ (see eq. (14)). In particular, $\mathrm{Lie}\,u$ is surjective.

Let us now write

$$\mathbf{G}(\mathfrak{L}) = \varprojlim_i G_i$$

as in Lemma 4.7. Define H_i as being the image of H in G_i; a moment's thought shows that

$$H = \varprojlim_i H_i,$$

and that the the transition arrows of the projective system $\{H_i\}$ are also faithfully flat. This being so, the arrows between Lie algebras in the projective system $\{H_i\}$ are all surjective; see [Mi17, Corollary 3.25, p. 72] and [Mi17, Proposition 1.63, p. 25]. Consequently, the obvious arrows $\operatorname{Lie}\mathbf{G}(\mathfrak{L}) \longrightarrow \operatorname{Lie} G_i$ and $\operatorname{Lie} H \longrightarrow \operatorname{Lie} H_i$ are always surjective. Hence, the natural arrows $\operatorname{Lie} H_i \longrightarrow \operatorname{Lie} G_i$ are always surjective.

Using [DG70, Proposition II.6.2.1, p. 259] and the fact that each G_i is connected, we conclude that $H_i = G_i$, and $H = \mathbf{G}(\mathfrak{L})$. This proves Proposition 4.6. □

Let G be an algebraic group scheme with Lie algebra $\mathfrak{g}$. Recall that a *Lie subalgebra of* $\mathfrak{g}$ *is algebraic* if it is the Lie subalgebra of a closed subgroup scheme of G [DG70, Definition II.6.2.4]. As argued in [DG70, II.6.2, p. 262], given an arbitrary Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, there exists a smallest algebraic Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$: it is the (algebraic) *envelope* of $\mathfrak{h}$ inside $\mathfrak{g}$. Allied with [DG70, II.6.2.1a, p. 259], it then follows that there exists a *smallest closed and connected subgroup scheme* of G whose Lie algebra contains $\mathfrak{h}$. This group carries no name in [DG70], so we shall allow ourselves to put forward:

Definition 4.8. Let G be an algebraic group scheme and $\mathfrak{h} \subset \operatorname{Lie} G$ a Lie subalgebra. The *group-envelope* of $\mathfrak{h}$ is the smallest closed subgroup scheme of G whose Lie algebra contains $\mathfrak{h}$. We also define the *group-envelope* of a subspace $V \subset \operatorname{Lie} G$ as being the group-envelope of the Lie algebra generated by V in $\operatorname{Lie} G$.

Theorem 4.9 ([Ho59, Theorem 1, § 3])**.** *Let* $f : \mathbf{G}(\mathfrak{L}) \longrightarrow \mathbf{GL}(E)$ *be the representation associated to the* $\mathfrak{L}$*–module* $\rho : \mathfrak{L} \longrightarrow \mathfrak{gl}(E)$*, that is,* $\psi_f = \rho$*. Then the image* $I = \operatorname{image}(f)$ *of* $\mathbf{G}(\mathfrak{L})$ *in* $\mathbf{GL}(E)$ *is the group-envelope of* $\rho(\mathfrak{L}) \subset \mathfrak{gl}(E)$*.*

Proof. Consider a factorization $\rho : \mathfrak{L} \longrightarrow \operatorname{Lie} H$, where $H \subset \mathbf{GL}(E)$. Because $\rho = (\operatorname{Lie} f) \circ \chi$ (see (14)), it then follows that $\chi(\mathfrak{L}) \subset (\operatorname{Lie} f)^{-1}(\operatorname{Lie} H)$. But $(\operatorname{Lie} f)^{-1}(\operatorname{Lie} H)$ is just $\operatorname{Lie} f^{-1}(H)$. Indeed, in case $\mathbf{G}(\mathfrak{L})$ is an algebraic group, this can be deduced easily from [DG70, p. 259, Proposition II.2.6.1] and an argument using the graph of f, while the general case follows from this one and Lemma 4.7. Hence, $f^{-1}(H) = \mathbf{G}(\mathfrak{L})$ because $\chi : \mathfrak{L} \longrightarrow \operatorname{Lie}\mathbf{G}(\mathfrak{L})$ is algebraically dense. This implies that $I \subset H$. Because $\rho(\mathfrak{L}) = \operatorname{Lie}(f) \circ \chi(\mathfrak{L})$, we deduce that $\operatorname{Lie} I \supset \rho(\mathfrak{L})$, so that I is the group-envelope. □

5. The differential Galois group

In this section, X is assumed to be a proper, connected and smooth K-scheme and x_0 a K-point of X. We recall that $\Theta(X, x_0)$ is the group scheme constructed in eq. (7).

Using the tensor equivalences

$$\mathbf{Rep}(\mathbf{G}(\mathfrak{L}_X)) \xrightarrow{\sim} \mathfrak{L}_X\text{-}\mathbf{mod} \xrightarrow{\sim} \mathfrak{A}_X\text{-}\mathbf{mod} \xrightarrow{\mathscr{V}} \mathbf{MIC}^{\mathrm{tr}} \xrightarrow{\bullet|_{x_0}} \mathbf{Rep}\,\Theta(X, x_0)$$

obtained by eq. (4), Theorem 3.6 and Corollary 4.3, we derive an isomorphism

$$\gamma : \Theta(X,\, x_0) \xrightarrow{\ \sim\ } \mathbf{G}(\mathfrak{L}_X)$$

such that the corresponding functor $\gamma^{\#} : \mathbf{Rep}\,\mathbf{G}(\mathfrak{L}_X) \longrightarrow \Theta(X,\, x_0)$ is naturally isomorphic to the above composition.

Let $(E,\, A) \in \mathfrak{A}_X$-**mod** be given. With an abuse of notation, we shall let A denote the linear arrow $H^0(\Omega^1_X)^* \longrightarrow \mathrm{End}(E)$, the morphism of associative algebras $\mathfrak{A}_X \longrightarrow \mathrm{End}(E)$ or the morphism of Lie algebras $\mathfrak{L}_X \longrightarrow \mathrm{End}(E)$.

Theorem 5.1. *The differential Galois group of $\mathscr{V}(E,\, A) = (\mathscr{O}_X \otimes E,\, d_A)$ is the group-envelope of $A(H^0(X,\, \Omega^1_X)^*)$.*

Said otherwise, given a trivial vector bundle $\mathscr{E}$ or rank r with global basis $\{e_i\}_{i=1}^r$, an integrable connection

$$\nabla : \mathscr{E} \longrightarrow \mathscr{E} \otimes \Omega^1_X$$

and a basis $\{\theta_j\}_{j=1}^g$ of $H^0(X,\, \Omega^1_X)$, define matrices $A_k = (a_{ij}^{(k)})_{1\le i,j\le r} \in \mathrm{M}_r(K)$ by

$$\nabla e_j = \sum_{k=1}^{g}\sum_{i=1}^{r} a_{ij}^{(k)} \cdot e_i \otimes \theta_k.$$

Then, the differential Galois group of $\mathscr{E}$ at the point x_0 is isomorphic to the group-envelope in $\mathbf{GL}_r(K)$ of the Lie algebra generated by $\{A_k\}_{k=1}^g$.

Proof. We note that the Lie subalgebra of $\mathrm{End}(E)$ generated by $A(H^0(\Omega^1_X)^*)$ is the image of $A(\mathfrak{L}_X)$; indeed, as a Lie algebra, $\mathfrak{L}_X$ is generated by $H^0(\Omega^1_X)^*$ (see Lemma 2.3). Now we apply Theorem 4.9 to conclude that the image of $\mathbf{G}(\mathfrak{L}_X)$ in $\mathbf{GL}(E)$ is the group-envelope of the Lie algebra generated by $A(H^0(\Omega^1_X)^*)$. Because of Proposition 3.7, the image of $\Theta(X,\, x_0) \simeq \mathbf{G}(\mathfrak{L}_X)$ is the image of $\Pi(X,\, x_0)$, which is the differential Galois group. □

Remark 5.2. In "birational" differential Galois theory, one can find a result reminiscent of Theorem 5.1; see [PS03, p. 25, Remarks 1.33].

Remark 5.3. Theorem 5.1 is certainly false if X is not proper: take $X = \mathrm{Spec}\, K[x,\, x^{-1}]$ and define $(\mathscr{O}_X e,\, \nabla)$ by $\nabla e = ke \otimes \dfrac{dx}{x}$ for any given $k \in \mathbb{Z}$.

6. Applications

Fixing generators of the Lie algebra of an algebraic subgroup scheme of some general linear group allows us to construct connections with a prescribed differential Galois group.

The following is deduced using Theorem 5.1.

Corollary 6.1. *Let X be a proper curve (smooth and integral, by assumption) over K of genus $g \ge 2$ and carrying a point $x_0 \in X(K)$. Let G be a semi-simple linear algebraic group. Then there exists a trivial vector bundle with a connection having differential Galois group G.*

Proof. We know that $\mathfrak{g} := \mathrm{Lie}(G)$ is semi-simple [Mi17, p. 476]. According to Kuranishi's theorem (see [Ku49, Theorem 1] or [BLie, VIII.2, p. 221, Exercise 8]), there exists a two dimensional vector space $V \subset \mathfrak{g}$ generating $\mathfrak{g}$. Let $G \longrightarrow \mathbf{GL}(E)$ be a closed immersion and regard V as a vector space of matrices in $\mathrm{End}(E)$. We then construct any arrow of vector spaces $A : H^0(X, \Omega^1_X)^* \longrightarrow \mathrm{End}(E)$ such that $\mathrm{Im}(A) = V$. Note that G is the group-envelope of V. □

In what follows, we wish to study connections on

$$P = \mathbf{P}^1_K \quad \text{and} \quad P^* = P \smallsetminus \{0, 1, \infty\}.$$

To carry over this task, we give ourselves a point $p \in P^*(K)$ and put forward the following construction. Let z be the inhomogeneous coordinate on P and, for each integer $n \geq 3$, define X_n as being the smooth, projective and connected curve whose field of rational functions is given by

$$K(z, w), \qquad w^n = [p(p-1)]^{n-1} \cdot z(z-1).$$

Here, of course, we have abused notation and identified p and the value $z(p)$. If

$$f_n : X_n \longrightarrow P$$

is the induced morphism, then its restriction to

$$X_n^* := f_n^{-1}(P^*)$$

is etale and the fibre $f_n^{-1}(p)$ possesses an obvious K-rational point, which shall be denoted by x. In particular, $H^0(\mathscr{O}_{X_n}) = K$ and, according to [Go03, Exercise 3.8],

$$\mathrm{genus}(X_n) = \begin{cases} (n-1)/2, & \text{if } n \text{ is odd,} \\ (n-2)/2, & \text{if } n \text{ is even.} \end{cases}$$

It is not difficult to see that $X_n^* \longrightarrow P^*$ is a principal μ_n-bundle.

Let H be any connected linear algebraic group and define

$$s(H) = \begin{array}{c}\text{minimal number of} \\ \text{generators of } \mathrm{Lie}(H).\end{array}$$

Corollary 6.2. *Let $n \in \mathbb{N}$ be such that* $\mathrm{genus}(X_n) \geq s(H)$. *There exists a logarithmic connection on P* [Ka70, 4.0–4.3], *call it $(\mathscr{M}, \nabla)$, having poles on $\{0, 1, \infty\}$, such that the connected component of the differential Galois group of $\mathscr{M}|_{P^*}$ at the point p is H. In addition, the group of connected components of the differential Galois group is μ_n.*

Proof. Provided that n is such that

$$\mathrm{genus}(X_n) \geq s(H), \tag{$*$}$$

it is possible to find a connection $(\mathscr{N}, \nabla^0)$ of the form $\mathscr{V}(E, A)$ whose differential Galois group at the point x is H. Let $\mathscr{M} = f_*\mathscr{N}$ be endowed with its canonical logarithmic connection, with poles on $\{0, 1, \infty\}$, induced by ∇^0; see Remark 6.3. It is a known fact that the restriction

$$\langle \mathscr{N} \rangle_\otimes \longrightarrow \langle \mathscr{N}|_{X_n^*} \rangle_\otimes$$

is an equivalence of categories; see for example [Kin15, p. 6464, Lemma 2.5], where the proof, written for the case of positive characteristic and $\mathscr{D}$-modules, can be used literally in the present context. (The uninitiated reader might also profit from knowing that the category of $\mathscr{D}$-modules and integrable connections are one and the same [BO78, 2.15] in characteristic zero.) We hence know that the differential Galois group of $\mathscr{N}|_{X_n^*}$ at the point x is also H (see Remark 3.3). Consequently, letting G stand for the differential Galois group of $\mathscr{M}|_{P^*}$, we obtain a closed immersion

$$\iota \,:\, H \longrightarrow G$$

by means of the pull-back functor

$$\langle \mathscr{M}|_{P^*}\rangle_\otimes \longrightarrow \langle \mathscr{N}|_{X_n^*}\rangle_\otimes .$$

From Proposition 7.2, we conclude that $\mathrm{Coker}(\iota) = \mu_n$. □

The following Remark was used in the proof of Corollary 6.2; it should be well-known, but we were unable to find a suitable reference.

Remark 6.3. Let $f : Y \longrightarrow X$ be a surjective morphism between smooth projective curves. We recall some facts from [Ha77, IV.2]. The ramification divisor D^f for f is the divisor of the cokernel $\Omega^1_{Y/X}$ of the natural morphism $f^*\Omega^1_X \longrightarrow \Omega^1_Y$, so

$$\Omega^1_Y = (f^*\Omega^1_X) \otimes \mathscr{O}_Y(D^f). \tag{15}$$

Set $\widetilde{D} = (D^f)_{\mathrm{red}}$ and $D = f(\widetilde{D})_{\mathrm{red}}$. We note that $f^*D \geq D^f$ (in fact, $f^*D - \widetilde{D} \geq D^f$), so using (15) we have an injective homomorphism of coherent sheaves

$$\Omega^1_Y \longrightarrow (f^*\Omega^1_X) \otimes \mathscr{O}_Y(f^*D) = f^*(\Omega^1_X \otimes \mathscr{O}_X(D))\,. \tag{16}$$

Let $\nabla : E \longrightarrow E \otimes \Omega^1_Y$ be a connection on a vector bundle E over Y. Using (16), this gives

$$\nabla : E \longrightarrow E \otimes f^*(\Omega^1_X \otimes \mathscr{O}_X(D))\,.$$

Taking direct image of it and using the projection formula, we get

$$f_*\nabla : f_*E \longrightarrow f_*(E \otimes f^*(\Omega^1_X \otimes \mathscr{O}_X(D))) = (f_*E) \otimes \Omega^1_X \otimes \mathscr{O}_X(D),$$

which is a logarithmic connection on f_*E.

We now want to elaborate on a statement similar to that of Corollary 6.2 in which we can actually assure that the differential Galois group is connected. The price to pay for this seems to be a loss of control on the "constructibility" of our solution and on the types of group attained; it is no longer possible to say exactly which connection produces the desired differential Galois group.

Let us now suppose that H *is reductive*. Letting Z stand for its center and $\mathbf{Out}(H)$ for the sheaf of external automorphisms [SGA3, Exposé XXIV], we *assume* that Z and $\mathbf{Out}(H)$ *are finite*. (Conditions for the validity of this last assumption are spelled out by Corollary 1.6 of [SGA3, XXIV].)

Lemma 6.4. *Let $n \geq 3$ be prime to $\#\mathbf{Out}(H)$ and $\#Z$, and let G be a group scheme over K fitting into a short exact sequence*

$$1 \longrightarrow H \longrightarrow G \longrightarrow \mu_n \longrightarrow 1. \qquad (e)$$

Then $G \simeq H \times \mu_n$ and, in particular, H is a quotient of G.

Proof. By hypothesis, the morphism $\kappa_e : \mu_n \longrightarrow \mathbf{Out}(H)$ associated to (e) (cf. Section 7.1) is trivial. The action of μ_n on Z is also trivial because of this. Let us now adopt the notations of [DG70, III.6] to employ this reference. (We shall mainly require the set $\widetilde{\mathrm{Ex}}^1$, introduced on p. 435 of [DG70].) From [DG70, III.6.4.6, p. 448, Corollary], we learn that $\widetilde{\mathrm{Ex}}^1(\mu_n, Z) = 0$. (In the context of finite groups, this is the "Schur-Zassenhaus" theorem.) Then, Theorem 3.1 of [FLA19] may be applied (see also section 7.1) to conclude that there exists one unique (equivalence class of) extension of μ_n by H whose outer action $\mu_n \longrightarrow \mathbf{Out}(H)$ is trivial. This implies that $G \simeq H \times \mu_n$. □

Corollary 6.5. *Suppose that H is reductive and that both Z and $\mathbf{Out}(H)$ are finite group schemes. Then, there exists a logarithmic connection on P, with poles on $\{0, 1, \infty\}$, whose restriction to P^* has differential Galois group H.* □

Remark 6.6. We find useful to end this section by giving the interested reader some information on the "inverse problem of differential Galois theory", i.e., the problem of describing the linear algebraic groups which are differential Galois groups. A fore-running result appears in [TT79]: each linear algebraic group over $\mathbb{C}$ is a differential Galois group of a connection on *some open* subset of $\mathbf{P}^1$. The proof in op. cit. makes use of the fundamental group, the solution of Hilbert's 21st problem, and a result saying that each linear algebraic group is the closure of a finitely generated subgroup of its points. Another result worth mentioning here is that of Mitschi and Singer in [MS96], where it is proved that, provided K is algebraically closed, any connected linear algebraic group is the differential Galois group of some linear differential system of the form

$$y' = \left(\frac{A_1}{z-\alpha_1} + \cdots + \frac{A_d}{z-\alpha_d} + A_\infty\right) y,$$

where A_j are constant matrices. (Note that the system is regular-singular, except perhaps at ∞.) As this is a theme with many contributions and there is no chance of doing justice to all its developments in this remark, we refer to [PS03, Chapter 11].

7. Appendix

We use this appendix to record some subsidiary results which are unfortunately only explained in insufficient generality in the literature.

7.1. **Extensions of group schemes.** Let H be a group scheme over a field, with center Z and sheaf of automorphisms $\mathbf{Aut}(H)$. Write $c : H \longrightarrow \mathbf{Aut}(H)$ for the conjugation morphism. By definition, $\mathrm{Ker}(c) = Z$, so that H/Z may be seen as a subsheaf of $\mathbf{Aut}(H)$; it is usually called the sheaf of *inner automorphisms of H* and denoted by $\mathbf{Inn}(H)$. The quotient $\mathbf{Aut}(H)/\mathbf{Inn}(H)$ is called *the sheaf of outer automorphisms* of H and is denoted by $\mathbf{Out}(H)$. In summary, we arrive at the tautological short exact sequence

$$1 \longrightarrow H/Z \xrightarrow{\bar{c}} \mathbf{Aut}(H) \longrightarrow \mathbf{Out}(H) \longrightarrow 1. \qquad (\tau)$$

These constructions agree with the ones in [SGA3, XXIV, 1.1].

We now consider an exact sequence of group schemes

$$1 \longrightarrow H \xrightarrow{\iota} E \xrightarrow{p} G \longrightarrow 1. \qquad (e)$$

If $c : E \longrightarrow \mathbf{Aut}(H)$ is the morphism obtained by conjugation, passing to the induced morphism of quotients, we derive a morphism of sheaves of groups

$$\kappa_e : G \longrightarrow \mathbf{Out}(H),$$

called the *canonical outer action*.

Another relevant action coming from (e) is the following. Since Z is characteristic in H, we deduce a morphism of sheaves of groups $r : \mathbf{Aut}(H) \longrightarrow \mathbf{Aut}(Z)$, and $\mathbf{Inn}(Z)$ is certainly in the kernel of r. This allows us to define a morphism of sheaves $\mathbf{Out}(H) \longrightarrow \mathbf{Aut}(Z)$ and consequently an arrow of sheaves of groups

$$\kappa_Z : G \longrightarrow \mathbf{Aut}(Z).$$

In [FLA19], the authors chose to work with a different interpretation of the canonical outer action, which is to be related to the one introduced above in order that we be able to apply the results of op.cit.

Noting that $Z \triangleleft E$, the sequence (e) produces

$$1 \longrightarrow H/Z \xrightarrow{\bar{\iota}} E/Z \xrightarrow{\bar{p}} G \longrightarrow 1. \qquad (\bar{e})$$

The natural morphism or sheaves of groups

$$\bar{c} : E/Z \longrightarrow \mathbf{Aut}(H)$$

gives rise, passing to the associated quotients, to κ_e. (We are using $\bar{c}$ to denote also the arrow $H/Z \longrightarrow \mathbf{Aut}(H)$, but this should cause no confusion.)

Pulling back the sequence (τ) by κ_e, we arrive at an exact sequence of sheaves

$$1 \longrightarrow H/Z \xrightarrow{(\bar{c},e)} \mathbf{Aut}(H) \underset{\mathbf{Out}(H),\kappa_e}{\times} G \xrightarrow{\mathrm{pr}} G \longrightarrow 1. \qquad (\kappa_e^*\tau)$$

Clearly, the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H/Z & \xrightarrow{\bar{\iota}} & E/Z & \xrightarrow{\bar{p}} & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle(\bar{c},\bar{p})} & & \| & & \\ 1 & \longrightarrow & H/Z & \xrightarrow[(\bar{c},e)]{} & \mathbf{Aut}(H) \underset{\mathbf{Out}(H)}{\times} G & \xrightarrow[\mathrm{pr}]{} & G & \longrightarrow & 1 \end{array}$$

commutes, and hence $(\bar{e})$ is equivalent, as an extension, to $(\kappa_e^*\tau)$.

According to [FLA19, Def. 2.4], an extension

$$1 \longrightarrow H \xrightarrow{\iota'} E' \xrightarrow{p'} G \longrightarrow 1, \qquad (e')$$

which induces the extension

$$1 \longrightarrow H/Z \xrightarrow{\bar{\iota}'} E'/Z \xrightarrow{\bar{p}'} G \longrightarrow 1, \qquad (\bar{e}')$$

has "*the same outer action as* (e)" if there exists an isomorphism

$$\varphi \,:\, E/Z \xrightarrow{\sim} E'/Z$$

such that $(\bar{e})$ and $(\bar{e}')$ are isomorphic via φ *and*

$$\begin{array}{ccc} E/Z & \xrightarrow{\bar{c}} & \mathbf{Aut}(H) \\ {\scriptstyle\varphi}\downarrow & \nearrow_{\bar{c}} & \\ E'/Z & & \end{array}$$

commutes.

Lemma 7.1. *The extension* (e') *has the same outer action as* (e)*, in the terminology of* [FLA19]*, if and only if* $\kappa_e = \kappa_{e'}$*.*

Proof. $(\Rightarrow)$ This is a simple consequence of the commutative diagram

$$\begin{array}{ccccc} H/Z & \xrightarrow{\bar{\iota}} & E/Z & \xrightarrow{\bar{c}} & \mathbf{Aut}(H) \\ \| & & {\scriptstyle\varphi}\downarrow & \nearrow_{\bar{c}} & \\ H/Z & \xrightarrow[\bar{\iota}']{} & E'/Z. & & \end{array}$$

$(\Leftarrow)$ This is quite simple given the isomorphisms of extensions $(\bar{e}) \simeq (\kappa_e^*\tau)$ and $(\bar{e}') \simeq (\kappa_{e'}^*\tau)$ explained above. $\square$

7.2. **Generalising a result of Katz.** In [Ka87, Proposition 1.3.2], Katz shows how to compare the differential Galois group of a connection and of its push-forward along a Galois etale covering. In this section, we extend this result to the case of a torsor under an *etale group scheme*.

Let X be a smooth and connected K-scheme and x_0 a K-point of X. Given a finite group scheme G, a G-torsor

$$f \,:\, Y \longrightarrow X$$

and a K-point y_0 of Y above x_0, we can say that:

Proposition 7.2. *Let* $\mathscr{N}$ *be a vector bundle on* Y *with an integrable connection, and let* $\mathscr{M}$ *be its push-forward connection on* X*. Then the connection* $f^*(\mathscr{M})$ *is isomorphic to a direct sum of the connection* $\mathscr{N}$*. In addition, if* $G_{\mathscr{M}}$ *(respectively,* $G_{\mathscr{N}}$*) is the differential Galois of* $\mathscr{M}$ *(respectively,* $\mathscr{N}$*) at the point* x_0 *(respectively,* y_0*), then the natural pull-back functor*

$$f^* \,:\, \langle \mathscr{M} \rangle_{\otimes} \longrightarrow \langle \mathscr{N} \rangle_{\otimes}$$

induces a closed immersion $\iota \,:\, G_{\mathscr{M}} \longrightarrow G_{\mathscr{N}}$ *with cokernel* G*.*

Proof. The first claim is very simple and we offer no proof. We thus concentrate on the relation between the differential Galois groups. The essence of the argument we offer is explained in [BHdS18, Proposition 2.2], but we reproduce the details for the convenience of the reader. It is clear that ι is a closed immersion [DM82, p. 139, Proposition 2.21(a)]. Let $\mathfrak{T}$ stand for the full subcategory of $\langle \mathscr{M} \rangle_\otimes$ having for objects those $\mathscr{E} \in \langle \mathscr{M} \rangle_\otimes$ such that $f^*\mathscr{E}$ is a *trivial connection*. The main theorem of Tannakian duality [DM82, p. 130, Theorem 2.11] assures the existence of a group scheme Q such that the functor x_0^* gives us an equivalence

$$x_0^* : \mathfrak{T} \longrightarrow \mathbf{Rep}(Q).$$

In addition, the inclusion $\mathfrak{T} \subset \langle \mathscr{M} \rangle_\otimes$ produces a quotient morphism $G_{\mathscr{M}} \longrightarrow Q$.

We set out to prove that (a), (b) and (c) of [EHS08, Theorem A.1(iii)], henceforth called simply conditions (a), (b) and (c), are satisfied for the diagram $G_{\mathscr{N}} \longrightarrow G_{\mathscr{M}} \longrightarrow Q$. Condition (a) is assured by the construction of Q from its category of representations. Condition (c) is guaranteed by the fact that the counit $f^*f_*\mathscr{F} \longrightarrow \mathscr{F}$ is an epimorphism for each $\mathscr{F} \in \langle \mathscr{N} \rangle_\otimes$. We only need to show that (b) holds. The verification employs the following.

Let $n = \#G$. Since $\mathscr{M}^{\oplus n} \simeq f_*(\mathscr{N}^{\oplus n})$ and $\mathscr{N}^{\oplus n} \simeq f^*\mathscr{M}$, the projection formula gives

$$\begin{aligned} \mathscr{M}^{\oplus n} &\simeq f_*f^*(\mathscr{M}) \\ &\simeq \mathscr{M} \otimes f_*(\mathscr{O}_Y). \end{aligned}$$

As $\mathscr{O}_X$ is a quotient of $\mathscr{M}^\vee \otimes \mathscr{M}$, it then follows that the connection $f_*(\mathscr{O}_Y)$ belongs to $\langle \mathscr{M} \rangle_\otimes$ and hence to $\mathfrak{T}$, since $f^*f_*(\mathscr{O}_Y)$ is certainly trivial. In addition, it is not difficult to see that $\mathfrak{T} \subset \langle f_*(\mathscr{O}_Y) \rangle_\otimes$ which assures the equality

$$\mathfrak{T} = \langle f_*(\mathscr{O}_Y) \rangle_\otimes.$$

Let now $\mathscr{E} \in \langle \mathscr{M} \rangle_\otimes$ be given. Denote by $\mathscr{P}$ the connection $(f_*\mathscr{O}_Y)^\vee \in \mathfrak{T}$ and, for each horizontal arrow $\varphi : \mathscr{P} \longrightarrow \mathscr{E}$, let $\mathscr{P}_\varphi$ stand for its image. Taking the sum over all such φ, we obtain a subobject $\mathscr{E}_0$ of $\mathscr{E}$; needless to say, $\mathscr{E}_0$ is actually a quotient of a finite number of copies of $\mathscr{P}$. Since $f^*\mathscr{P}$ is a trivial connection, we see that $f^*\mathscr{E}_0$ is also trivial. Let $s \in \Gamma(Y, f^*\mathscr{E})$ be horizontal and consider the evaluation map $s^\vee : f^*(\mathscr{E}^\vee) \longrightarrow \mathscr{O}_Y$. This gives rise to $t : \mathscr{E}^\vee \longrightarrow f_*(\mathscr{O}_Y)$ and hence to $t^\vee : \mathscr{P} \longrightarrow \mathscr{E}$. Now, it is not difficult to see that, for each open affine subset U where $\mathscr{P}|_U$ and $\mathscr{E}|_U$ are free $\mathscr{O}_U$-modules, the restriction $s|_U$ belongs to the image of $f^*(t^\vee) : f^*\mathscr{P} \longrightarrow f^*\mathscr{E}$. Hence $s \in \Gamma(Y, f^*\mathscr{E}_0)$ and (b) is assured.

To end, it must be observed that $Q \simeq G$, and this follows from the fact that the associated bundle construction $Y \times^G (-)$ gives us an equivalence between $\mathbf{Rep}(G)$ and $\langle f_*(\mathscr{O}_Y) \rangle_\otimes$. □

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: indranil@math.tifr.res.in

Institute of Mathematics, Vietnam Academy of Science and Technology, Hanoi, Vietnam

E-mail address: phung@math.ac.vn

Institut de Mathématiques de Jussieu – Paris Rive Gauche, 4 place Jussieu, Case 247, 75252 Paris Cedex 5, France

E-mail address: joao_pedro.dos_santos@yahoo.com