A COMPARISON PRINCIPLE FOR PARABOLIC COMPLEX MONGE-AMPÈRE EQUATIONS

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Dedicated to Professor Ahmed Zeriahi on the occasion of his retirement.

Abstract. In this paper, we study the Cauchy-Dirichlet problem for Parabolic complex Monge-Ampère equations on strongly pseudoconvex domains using the viscosity method. We prove a comparison principle for Parabolic complex Monge-Ampère equations and use it to study the existence and uniqueness of viscosity solution in certain cases where the sets $\{z \in \Omega : f(t, z) = 0\}$ may be pairwise disjoint.

Contents

1. Introduction

In Algebraic Geometry, the Minimal Model Program is known as a process of simplifying algebraic varieties through algebraic surgeries in birational geometry. In [11, 10], Song and Tian gave a conjectural picture to approach the Minimal Model Program via the Kähler-Ricci flow. This approach requires a theory of weak solutions for certain degenerate parabolic complex Monge-Ampère equations.

A viscosity approach for parabolic Monge-Ampère (PMA) equations has been developed by Eyssidieux-Guedj-Zeriahi both on domains [4] and on compact Kähler manifolds [5, 6] (see also [2] and [12] for some generalizations). In another direction, a theory of pluripotential solutions for PMA equations has been developed in [7, 8]. Under suitable conditions, the notions of these weak solutions are equivalent [9]. Besides having applications for the Minimal Model Program, the theories of weak solutions for PMA equations are interesting topics in themself. The aim of this paper is to study the theory of viscosity solutions for PMA equations in domains of $\mathbb{C}^n$.

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Let $\Omega \subset \mathbb{C}^n$ be a strongly pseudoconvex domain and let $T \in (0, \infty)$. We consider the following Cauchy-Dirichlet problem:

$$\begin{cases} (dd^c u)^n = e^{\partial_t u + F(t,z,u)} \mu(t,z) & \text{in} \quad \Omega_T, \\ u = \varphi \quad \text{in} \quad [0,T) \times \partial\Omega, \\ u(0,z) = u_0(z) \quad \text{in} \quad \bar{\Omega}, \end{cases} \tag{1}$$

where

- $\Omega_T = (0,T) \times \Omega$.
- $F(t,z,r)$ is continuous in $[0,T] \times \bar{\Omega} \times \mathbb{R}$ and non-decreasing in r.
- $\mu(t,z) = f(t,z)dV$, where dV is the standard volume form in $\mathbb{C}^n$ and $f \geq 0$ is a bounded continuous function in $[0,T] \times \Omega$.
- $\varphi(t,z)$ is a continuous function in $[0,T] \times \partial\Omega$.
- $u_0(z)$ is continuous in $\bar{\Omega}$ and plurisubharmonic in Ω such that $u_0(z) = \varphi(0,z)$ in $\partial\Omega$.

In [4], Eyssidieux-Guedj-Zeriahi proved that if $(u_0, \mu(0,z))$ is admissible (see Definition 2.10) and F, f, φ do not depend on t then (1) has a unique viscosity solution. In [2], this result has been extended to the case where F, f, φ depend on t and f satisfies some additional conditions under which $\{z \in \Omega : f(t,z) = 0\} \subset \{z \in \Omega : f(s,z) = 0\}$ for $0 < s < t < T$ (see [2, Theorem 4.13]). In the general case, with f is merely a non-negative, bounded, continuous function, the question about the existence and uniqueness of viscosity solution to (1) is still open.

In this paper, we prove a comparison principle for (1) and use it to study the existence and uniqueness of viscosity solution to (1) in certain cases where the sets $\{z \in \Omega : f(t,z) = 0\}$ may be pairwise disjoint. Specifically, we assume that $\Phi : (-1,1) \times \Omega \to \mathbb{C}^n$ is a continuous mapping satisfying the following conditions:

- the mapping $z \mapsto \Phi(s,z)$ is holomorphic in Ω for every $s \in (-1,1)$;
- for each $U \Subset \Omega$, there exists $C_U > 0$ such that
$$|\Phi(s,z) - z| \leq C_U |s|, \tag{2}$$
for every $(s,z) \in (-1,1) \times U$. In particular, $\Phi(0,z) = z$ and for every $U \Subset \Omega$, there exists δ_U such that $\Phi(s,z) \in \Omega$ for all $(s,z) \in (-\delta_U, \delta_U) \times U$.

Our main result is as follows:

Theorem 1.1. *Suppose that for every $0 < R < S < T$, $K \Subset \Omega$ and $\epsilon > 0$, there exists $0 < \delta < \delta_K$ such that*
$$(1+\epsilon) f(t,z) \geq f(t+s, \Phi(s,z)), \tag{3}$$
for every $z \in K$, $R < t < S$ and $|s| < \delta$. Assume that u and v, respectively, is a bounded viscosity subsolution and a bounded viscosity supersolution to (1). Then, for every $0 < R < S < T$, $K \Subset \Omega$ and $\epsilon > 0$, there exists $0 < \delta < \delta_K$ such that
$$u(t+s_1, \Phi(s_1,z)) < v(t+s_2, \Phi(s_2,z)) + \epsilon,$$
for all $z \in K$, $R < t < S$ and $\max\{|s_1|, |s_2|\} < \delta$.

It is easy to see that if f does not depend on t then f satisfies (3) with $\Phi(s,z) = z$. Some other simple examples are $(f, \Phi) = (g(tz_0 + z), -sz_0 + z)$ and $(f, \Phi) = (g(e^{it}z), e^{-is}z)$, where g is a non-negative continuous function in $\mathbb{C}^n$ and $z_0 \in \mathbb{C}^n$. If f_1, f_2 satisfy (3) for the same Φ then $tf_1 + (T-t)f_2$ satisfies (3).

By using Theorem 1.1, we obtain the following result:

Corollary 1.2. *Assume that* $(u_0, \mu(0,z))$ *is admissible (see Definition 2.10). Suppose that* Φ *and* f *satisfy the conditions in Theorem 1.1. Then* (1) *has a unique viscosity solution.*

2. Preliminaries

In this section, we recall some basic concepts and well-known results about viscosity sub/super-solutions. The reader can find more details in [1], [3] and [2].

Definition 2.1. *(Test functions) Let* $w : \Omega_T \longrightarrow \mathbb{R}$ *be any function defined in* Ω_T *and* $(t_0, z_0) \in \Omega_T$ *a given point. An upper test function (resp. a lower test function) for* w *at the point* (t_0, z_0) *is a* $C^{(1,2)}$*-smooth function* q *(i.e.,* q *is* C^1 *in* t *and* C^2 *in* z*) in a neighbourhood of the point* (t_0, z_0) *such that* $w(t_0, z_0) = q(t_0, z_0)$ *and* $w \le q$ *(resp.* $w \ge q$*) in a neighbourhood of* (t_0, z_0)*.*

Definition 2.2. *1. A function* $u \in USC(\Omega_T)$ *is said to be a (viscosity) subsolution to the parabolic complex Monge-Ampère equation*

$$(dd^c u)^n = e^{\partial_t u + F(t,z,u)} \mu(t,z), \tag{4}$$

in Ω_T *if for any point* $(t_0, z_0) \in \Omega_T$ *and any upper test function* q *for* u *at* (t_0, z_0)*, we have*

$$(dd^c q_{t_0}(z_0))^n \ge e^{\partial_t q(t_0,z_0) + F(t_0,z_0,q(t_0,z_0))} \mu(t_0, z_0).$$

In this case, we also say that u *satisfies the differential inequality*

$$(dd^c u)^n \ge e^{\partial_t u(t,z) + F(t,z,u(t,z))} \mu(t,z),$$

in the viscosity sense in Ω_T*.*

A function $u \in USC([0,T) \times \overline{\Omega})$ *is called a subsolution to the Cauchy-Dirichlet problem* (1) *if* u *is a subsolution to* (4) *satisfying* $u \le \varphi$ *in* $[0,T) \times \partial\Omega$ *and* $u(0,z) \le u_0(z)$ *for all* $z \in \Omega$*.*

2. A function $v \in \Omega_T$ *is said to be a (viscosity) supersolution to the parabolic complex Monge-Ampère equation* (4) *in* Ω_T *if for any point* $(t_0, z_0) \in \Omega_T$ *and any lower test function* q *for* v *at* (t_0, z_0) *such that* $dd^c q_{t_0}(z_0) \ge 0$*, we have*

$$(dd^c q_{t_0})^n(z_0) \le e^{\partial_t q(t_0,z_0) + F(t_0,z_0,q(t_0,z_0))} \mu(t_0, z_0).$$

In this case we also say that v *satisfies the differential inequality*

$$(dd^c v)^n \le e^{\partial_t v(t,z) + F(t,z,v(t,z))} \mu(t,z),$$

in the viscosity sense in Ω_T*.*

A function $v \in LSC([0,T) \times \overline{\Omega})$ *is called a supersolution to* (1) *if* v *is a supersolution to* (4) *satisfying* $v \ge \varphi$ *in* $[0,T) \times \partial\Omega$ *and* $v(0,z) \ge u_0(z)$ *for all* $z \in \Omega$*.*

3. A function u *is said to be a (viscosity) solution to* (4) *(respectively,* (1)*) if it is a subsolution and a supersolution to* (4) *(respectively,* (1)*).*

Remark 2.3. *a) By the same argument as in the proof of* [3, Proposition 1.3]*, if* u *is a subsolution to* (4) *and* q *is an upper test function for* u *at* $(t_0, z_0) \in \Omega_T$ *then* $dd^c q_{t_0}(z_0) \ge 0$*;*
b) If u *is a subsolution to* (4) *then* $u(t,z)$ *is plurisubharmonic in* z *for every* $t \in (0,T)$ *(see* [4, Corollary 3.7]*).*

For each function $u : \Omega_T \longrightarrow \mathbb{R}$ and for every $(t_0, z_0) \in \Omega_T$, we define by $\mathcal{P}^{2,+}u(t_0, z_0)$ the set of $(\tau, p, Q) \in \mathbb{R} \times \mathbb{R}^{2n} \times \mathcal{S}_{2n}$ satisfying

$$(5)\quad u(t, z) \leq u(t_0, z_0) + \tau(t - t_0) + o(|t - t_0|) + \langle p, z - z_0\rangle + \frac{1}{2}\langle Q(z - z_0), z - z_0\rangle + o(|z - z_0|^2),$$

and denote by $\bar{\mathcal{P}}^{2,+}u(t_0, z_0)$ the set of $(\tau, p, Q) \in \mathbb{R} \times \mathbb{R}^{2n} \times \mathcal{S}_{2n}$ satisfying: $\exists (t_m, z_m) \to (t_0, z_0)$ and $(\tau_m, p_m, Q_m) \in \mathcal{P}^{2,+}u(t_0, z_0)$ such that $(\tau_m, p_m, Q_m) \to (\tau, p, Q)$ and $u(t_m, z_m) \to u(t_0, z_0)\}$.

We define in the same way the sets $\mathcal{P}^{2,-}u(t_0, z_0)$ and $\bar{\mathcal{P}}^{2,-}u(t_0, z_0)$ by

$$\mathcal{P}^{2,-}u(t_0, z_0) = -\mathcal{P}^{2,+}(-u)(t_0, z_0),$$

and

$$\bar{\mathcal{P}}^{2,-}u(t_0, z_0) = -\bar{\mathcal{P}}^{2,+}(-u)(t_0, z_0).$$

Since F and f are continuous, by [3, Proposition 2.6], we have:

Proposition 2.4.
1. An upper semi-continuous function $u : \Omega_T \longrightarrow \mathbb{R}$ is a subsolution to the parabolic equation

$$(6)\qquad (dd^c u)^n = e^{\partial_t u + F(t,z,u)}\mu(t, z),$$

if and only if for all $(t_0, z_0) \in \Omega_T$ and $(\tau, p, Q) \in \bar{\mathcal{P}}^{2,+}u(t_0, z_0)$, we have $dd^c Q \geq 0$ and

$$(7)\qquad (dd^c Q)^n \geq e^{\tau + F(t_0, z_0, u(t_0, z_0))}\mu(t_0, z_0).$$

Here $dd^c Q := (dd^c\langle Qz, z\rangle)$, $z \in \mathbb{C}^n = \mathbb{R}^{2n}$.
2. A lower semi-continuous function $v : \Omega_T \longrightarrow \mathbb{R}$ is a supersolution to the parabolic equation (6) if and only if for all $(t_0, z_0) \in \Omega_T$ and $(\tau, p, Q) \in \bar{\mathcal{P}}^{2,-}u(t_0, z_0)$ such that $dd^c Q \geq 0$, we have

$$(8)\qquad (dd^c Q)^n \leq e^{\tau + F(t_0, z_0, v(t_0, z_0))}\mu(t_0, z_0).$$

The following theorem is the parabolic Jensen-Ishiis maximum principle which plays an important role in the theory of viscosity solution:

Theorem 2.5. [1, Theorem 8.3] *Let $u \in USC(\Omega_T)$ and $v \in LSC(\Omega_T)$. Let ϕ be a function defined in $(0, T) \times \Omega^2$ such that $(t, \xi, \eta) \longmapsto \phi(t, \xi, \eta)$ is continuously differentiable in t and twice continuously differentiable in (ξ, η).*

Assume that the function $(t, \xi, \eta) \longmapsto u(t, \xi) - v(t, \eta) - \phi(t, \xi, \eta)$ has a local maximum at some point $(\hat{t}, \hat{\xi}, \hat{\eta}) \in (0, T) \times \Omega^2$.

Assume furthermore that both $w = u$ and $w = -v$ satisfy:

$$(2.5)\quad \begin{cases} \forall (s, z) \in \Omega \quad \exists r > 0 \text{ such that } \forall M > 0\ \exists C \text{ satisfying} \\ \left.\begin{array}{l} |(t, \xi) - (s, z)| \leq r, \\ (\tau, p, Q) \in \mathcal{P}^{2,+}w(t, \xi) \\ |w(t, \xi)| + |p| + |Q| \leq M \end{array}\right\} \Longrightarrow \tau \leq C. \end{cases}$$

Then for any $\kappa > 0$, there exists $(\tau_1, p_1, Q^+) \in \bar{\mathcal{P}}^{2,+}u(\hat{t}, \hat{\xi})$, $(\tau_2, p_2, Q^-) \in \bar{\mathcal{P}}^{2,-}v(\hat{t}, \hat{\eta})$ such that

$$\tau_1 = \tau_2 + D_t\phi(\hat{t}, \hat{\xi}, \hat{\eta}),\ \ p_1 = D_\xi\phi(\hat{t}, \hat{\xi}, \hat{\eta}),\ \ p_2 = -D_\eta\phi(\hat{t}, \hat{\xi}, \hat{\eta})$$

and

$$-\left(\frac{1}{\kappa} + \|A\|\right) I \leq \begin{pmatrix} Q^+ & 0 \\ 0 & -Q^- \end{pmatrix} \leq A + \kappa A^2,$$

where $A := D^2_{\xi,\eta}\phi(\hat{t},\hat{\xi},\hat{\eta}) \in \mathcal{S}_{4n}$.

The following lemma is deduced by combining Proposition 2.4 and Theorem 2.5:

Lemma 2.6. *Let* (u_τ) *be a locally uniformly bounded family of real valued functions defined in* Ω_T.

1. Assume that for every τ, u_τ *is a viscosity subsolution to the equation*

$$(dd^c w)^n = e^{\partial_t w + F(t,z,w)}\mu(t,z), \tag{9}$$

in Ω_T. *Then* $\overline{u} = (\sup_\tau u_\tau)^*$ *is a subsolution to* (9) *in* Ω_T.

2. Assume that for every τ, u_τ *is a viscosity supersolution to* (9). *Then* $\underline{u} = (\inf_\tau u_\tau)_*$ *is a supersolution to* (9) *in* Ω_T.

3. If $\tau \in \mathbb{N}$ *then 1. and 2. hold for* $\overline{u} = (\limsup_{\tau\to\infty} u_\tau)^*$ *and* $\underline{u} = (\liminf_{\tau\to\infty} u_\tau)_*$.

In the theory of viscosity solution, the comparison principle and Perron method are two key tools for studying the existence and uniqueness of solution. The following comparison principle has been established in [4]:

Theorem 2.7. [4, pages 949-953] *Let* u *(resp.* v*) be a bounded subsolution (resp. supersolution) to the parabolic complex MongeAmpère equation* (4) *in* Ω_T. *Assume that one of the following conditions is satisfied*

a) $\mu(t,z) > 0$ *for every* $(t,z) \in (0,T)\times\Omega$.
b) μ *is independent of* t.
c) *Either* u *or* v *is locally Lipschitz in* t *uniformly in* z.

Then

$$\sup_{\Omega_T}(u-v) \le \sup_{\partial_P(\Omega_T)} (u-v)_+,$$

where u *(resp.* v*) has been extended as an upper (resp. a lower) semicontinuous function to* $\overline{\Omega_T}$.

In order to recall the Perron method, we need the concepts of ϵ-sub/super-barrier.

Definition 2.8. *a) A function* $u \in USC([0,T)\times\bar{\Omega})$ *is called* ϵ*-subbarrier for* (1) *if* u *is subsolution to* (4) *in the viscosity sense such that* $u_0 - \epsilon \le u_* \le u \le u_0$ *in* $\{0\}\times\bar{\Omega}$ *and* $\varphi - \epsilon \le u_* \le u \le \varphi$ *in* $[0,T)\times\partial\Omega$.
b) A function $u \in LSC([0,T)\times\bar{\Omega})$ *is called* ϵ*-superbarrier for* (1) *if* u *is supersolution to* (4) *in the viscosity sense such that* $u_0+\epsilon \ge u^* \ge u \ge u_0$ *in* $\{0\}\times\bar{\Omega}$ *and* $\varphi+\epsilon \ge u^* \ge u \ge \varphi$ *in* $[0,T)\times\partial\Omega$.

Proposition 2.9. [2, Proposition 4.1] *For all* $\epsilon > 0$, *there exists a continuous* ϵ*-subbarrier for* (1) *which is Lipschitz in* t.

Definition 2.10. *We say that* $(u_0, \mu(0,.))$ *is* admissible *if for all* $\epsilon > 0$, *there exist* $u_\epsilon \in C(\bar{\Omega})$ *and* $C_\epsilon > 0$ *such that* $u_0 \le u_\epsilon \le u_0 + \epsilon$ *and* $(dd^c u_\epsilon)^n \le e^{C_\epsilon}\mu(0,z)$ *in the viscosity sense.*

Proposition 2.11. [2, Theorem 1.3] *If* $(u_0, \mu(0,.))$ *is admissible then the function* u_ϵ *in the definition 2.10 can be taken to be psh in* Ω.

Proposition 2.12. [2, Proposition 4.3] *If* $(u_0(z), \mu(0,z))$ *is admissible then for all* $\epsilon > 0$, *there exists a continuous* ϵ*-superbarrier for* (1) *which is Lipschitz in* t.

Lemma 2.13. *(Perron method)*[2, Lemma 2.12] *Assume that for every $\epsilon > 0$, the problem (1) admits a continuous ϵ-superbarrier which is Lipschitz in t and a continuous ϵ-subbarrier. Denote by S the family of all continuous subsolutions to (1). Then $\phi_S = \sup\{v : v \in S\}$ is a discontinuous viscosity solution to (1), i.e., $(\phi_S)^*$ is a subsolution and $(\phi_S)_*$ is a supersolution.*

3. Some useful lemmas

Throughout this section, we always suppose that Φ and f satisfy the conditions as in Theorem 1.1. Given a bounded function $u : \Omega_T \to \mathbb{R}$ and a constant $A > 2osc_{\Omega_T}(u)$. For every relatively compact open subet U of Ω and for each constant $0 < \delta \ll 1$ satisfying $\Phi([-\delta, \delta] \times U) \subset \Omega)$, we define

$$u^k(t, z) = \sup\{u(t + s, \Phi(s, z)) - k|s| : |s| \leq \frac{A}{k}\},$$

and

$$u_k(t, z) = \inf\{u(t + s, \Phi(s, z)) - k|s| : |s| \leq \frac{A}{k}\},$$

for every $k > \max\{\frac{A}{\delta}, \frac{2A}{T}\}$ and $(t, z) \in (A/k, T - A/k) \times U$.

We have the following modified version of [4, Lemma 3.5]:

Lemma 3.1. *Assume that u is a bounded upper semicontinuous function in Ω_T. Then*

(i) *u^k is upper semicontinuous in $(A/k, T - A/k) \times U$;*

(ii) *for all $(t, z) \in (A/k, T - A/k) \times U$,*
$$u(t, z) \leq u^k(t, z) \leq \sup_{|s|\leq A/k} u(t + s, \Phi(s, z));$$

(iii) *if (t, z) and $(t + s, \Phi(s, z))$ belong in $(A/k, T - A/k) \times U$ then*
$$|u^k(t, z) - u^k(t + s, \Phi(s, z))| \leq k|s|;$$

(iv) *if $(dd^c u)^n \geq e^{\partial_t u + F(t,z,u)}\mu(t, z)$ in the viscosity sense in Ω_T then, for every $0 < \epsilon < 1$, there exists $k_\epsilon > 0$ such that, for every $k > k_\epsilon$,*

$$(dd^c u^k)^n \geq (1 - \epsilon)e^{\partial_t u^k + F_k(t,z,u^k)} f_k(t, z)dV, \tag{10}$$

in the viscosity sense in $(\delta, T-\delta)\times U$, where $F_k(t, z, r) = \inf_{|s|\leq A/k} F(t+s, \Phi(s, z), r)$ and $f_k(t, z) = \inf_{|s|\leq A/k} f(t + s, \Phi(s, z))$.

Proof. (i) Let $(t_0, z_0) \in (A/k, T - A/k) \times U$. We will show that

$$u^k(t_0, z_0) \geq \limsup_{(t,z)\to(t_0,z_0)} u^k(t, z).$$

Assume that $(t_m, z_m) \in (A/k, T - A/k) \times U$ satisfies $(t_m, z_m) \to (t_0, z_0)$ as $m \to \infty$ and

$$\limsup_{(t,z)\to(t_0,z_0)} u^k(t, z) = \lim_{m\to\infty} u^k(t_m, z_m).$$

Since u is usc and Φ is continuous, by the definition of u^k, we have

$$u^k(t_m, z_m) = u(s_m + t_m, \Phi(s_m, z_m)) - k|s_m|,$$

for some $|s_m| \leq A/k$. Let $\{s_{m_l}\}$ be a subsequence of $\{s_m\}$ which converges to a point $s_0 \in [-A/k, A/k]$. Then

$$\begin{aligned}
\limsup_{(t,z)\to(t_0,z_0)} u^k(t,z) &= \lim_{m_l\to\infty} u^k(t_{m_l}, z_{m_l}) \\
&= \lim_{m_l\to\infty} (u(s_{m_l}+t_{m_l}, \Phi(s_{m_l}, z_{m_l})) - k|s_{m_l}|) \\
&= \lim_{m_l\to\infty} u(s_{m_l}+t_{m_l}, \Phi(s_{m_l}, z_{m_l})) - k|s_0| \\
&\le u(s_0+t_0, \Phi(s_0,z_0)) - k|s_0| \\
&\le u^k(t_0,z_0).
\end{aligned}$$

Hence, u^k is usc in $(A/k, T-A/k)\times U$.

(ii) Obvious.

(iii) Let $|s_0|\le A/k$ such that $u^k(t,z) = u(t+s_0, \Phi(s_0,z)) - k|s_0|$. If $|s-s_0|>A/k$ then

$$\begin{aligned}
u^k(t,z) = u(t+s_0,\Phi(s_0,z)) - k|s_0| &\le u(t+s,\Phi(s,z)) + 2osc_{\Omega_T}u - k|s_0| \\
&\le u^k(t+s,\Phi(s,z)) + A - k|s_0| \\
&\le u^k(t+s,\Phi(s,z)) + k|s-s_0| - k|s_0| \\
&\le u^k(t+s,\Phi(s,z)) + k|s|.
\end{aligned}$$

If $|s-s_0|\le A/k$ then

$$\begin{aligned}
u^k(t,z) = u(t+s_0,\Phi(s_0,z)) - k|s_0| &\le u^k(t+s,\Phi(s,z)) + k|s-s_0| - k|s_0| \\
&\le u^k(t+s,\Phi(s,z)) + k|s|.
\end{aligned}$$

Hence

$$u^k(t,z) - u^k(t+s,\Phi(s,z)) \le k|s|.$$

By the same argument, we also have

$$u^k(t+s,\Phi(s,z)) - u^k(t,z) \le k|s|.$$

Therefore

$$|u^k(t,z) - u^k(t+s,\Phi(s,z))| \le k|s|.$$

(iv) Let $r_0>0$ such that $V := U + r_0\mathbb{B}^{2n} \Subset \Omega$. Since $\Phi(t,z)$ is holomorphic in z and converges locally uniformly to Id as $t\to 0$, we have $\frac{\partial\Phi_\alpha}{\partial z_\beta}(t,z)$ converges uniformly in V to $\delta_{\alpha\beta}$ as $t\to 0$ for every $1\le\alpha,\beta\le n$. Hence, for every $0<\epsilon<1$, there exists $0<r_1<\delta$ such that $\Phi([-r_1,r_1]\times V)\Subset\Omega$ and

$$\det\left(\frac{\partial\Phi_j}{\partial z_k}(t,z)\right) > 1-\epsilon, \tag{11}$$

for every $(t,z)\in[-r_1,r_1]\times V$. Denote $k_\epsilon = \max\{A/r_1, 2A/T\}$. We will show that (10) holds in the viscosity sense in $(\delta, T-\delta)\times U$ for every $k>k_\epsilon$.

Let $(t_0,z_0)\in(\delta,T-\delta)\times U$, $s_0\in(-A/k,A/k)$ and let q be an upper test function of $u_{s_0}(t,z) := u(t+s_0,\Phi(s_0,z)) - k|s_0|$ at (t_0,z_0). Then $\hat{q}(t,z) := q(t-s_0,\Phi^{-1}(s_0,z)) + k|s_0|$ is an upper test function of u at $(\hat{t},\hat{z}) = (t_0+s_0,\Phi(s_0,z_0))$. Since $(dd^cu)^n \ge e^{\partial_t u + F(t,z,u)}\mu(t,z)$ in the viscosity sense, we have

$$(dd^c\hat{q}(\hat{t},\xi))^n|_{\xi=\hat{z}} \ge e^{\partial_t\hat{q}(\hat{t},\hat{z}) + F(\hat{t},\hat{z},\hat{q}(\hat{t},\hat{z}))}\mu(\hat{t},\hat{z}). \tag{12}$$

Note that $\partial_t\hat{q}(\hat{t},\hat{z}) = \partial_t q(t_0,z_0)$ and

$$(dd^cq(t_0,\xi))^n|_{\xi=z_0} = \left|\det\left(\frac{\partial\Phi_j}{\partial z_k}(t,z)\right)\right|^2 (dd^c\hat{q}(\hat{t},\xi))^n|_{\xi=\hat{z}}.$$

Therefore, by (11) and (12), we have

$$\begin{aligned}(dd^c q(t_0,\xi))^n|_{\xi=z_0} &\geq (1-\epsilon)e^{\partial_t q(t_0,z_0)+F(\hat{t},\hat{z},\hat{q}(\hat{t},\hat{z}))}\mu(\hat{t},\hat{z})\\ &\geq (1-\epsilon)e^{\partial_t q(t_0,z_0)+F(\hat{t},\hat{z},q(t_0,z_0))}\mu(\hat{t},\hat{z})\\ &\geq (1-\epsilon)e^{\partial_t q(t_0,z_0)+F_k(t_0,z_0,q(t_0,z_0))}f_k(t_0,z_0)dV.\end{aligned}$$

Since (t_0, z_0) and q are arbitrary, we get u_{s_0} is a subsolution to the equation

$$(dd^c w)^n = (1-\epsilon)e^{\partial_t w+F_k(t,z,w)}f_k(t,z)dV, \tag{13}$$

in $(\delta, T-\delta)\times U$. Then, it follows from Lemma 2.6 that the function

$$u^k = \sup_{|s_0|\leq A/k} u_{s_0} = (\sup_{|s_0|\leq A/k} u_{s_0})^*$$

is a subsolution to (13) in $(\delta, T-\delta)\times U$.

The proof is completed. □

By the same argument, we have

Lemma 3.2. *Assume that u is a bounded lower semicontinuous function in Ω_T. Then*

(i) *u_k is lower semicontinuous in $(A/k, T-A/k)\times U$;*

(ii) *for all $(t,z)\in(A/k, T-A/k)\times U$,*

$$u(t,z)\geq u_k(t,z)\geq \inf_{|s|\leq A/k} u(t+s,\Phi(s,z));$$

(iii) *if (t,z) and $(t+s,\Phi(s,z))$ belong in $(A/k, T-A/k)\times U$ then*

$$|u_k(t,z)-u_k(t+s,\Phi(s,z))|\leq k|s|;$$

(iv) *if $(dd^c u)^n\leq e^{\partial_t u+F(t,z,u)}\mu(t,z)$ in the viscosity sense in Ω_T then, for every $0<\epsilon<1$, there exists $k_\epsilon>0$ such that, for every $k>k_\epsilon$,*

$$(dd^c u_k)^n\leq (1+\epsilon)e^{\partial_t u_k+F^k(t,z,u_k)}f^k(t,z)dV,$$

in the viscosity sense in $(\delta, T-\delta)\times U$, where $F^k(t,z,r)=\sup_{|s|\leq A/k}F(t+s,\Phi(s,z),r)$ and $f^k(t,z)=\sup_{|s|\leq A/k}f(t+s,\Phi(s,z))$.

In [4], by applying the maximal principle, Eyssidieux-Guedj-Zeriahi has proved the comparison principle for the case where either the given subsolution or the given supersolution is Lipschitz in t. Using the same method as in [4], we obtain the following lemma:

Lemma 3.3. *Let $u\in USC\cap L^\infty([0,T)\times\overline{\Omega})$ and $v\in LSC\cap L^\infty([0,T)\times\overline{\Omega})$ be, respectively, a subsolution and a supersolution to the equation*

$$(dd^c w)^n = e^{\partial_t w+F(t,z,w)}\mu(t,z), \tag{14}$$

in Ω_T. Assume that, for $w=u,v$, the following condition holds: for every $U\Subset\Omega$ and $0<\delta<T/2$, there exists $k(U,\delta)>0$ such that if $(t,z)\in(\delta,T-\delta)\times U$ then

$$|w(t,z)-w(t+s,\Phi(s,z))|\leq k(U,\delta)|s|, \tag{15}$$

for $0<|s|\ll 1$. Then

$$\sup_{\Omega_T}(u-v)\leq \sup_{\partial_P\Omega_T}(u-v)_+.$$

Proof. Let $\delta>0$ be an arbitrary positive constant and denote

$$h(t,z)=u(t,z)-v(t,z)-\frac{\delta}{T-t}+\delta(|z|^2-C),$$

where $C = \sup_{z\in\Omega} |z|^2$. We will show that

$$\max_{\Omega_T} h \le \max_{\partial_P \Omega_T} h_+. \tag{16}$$

Assume that (16) is false. Then, there exists $(t_0, z_0) \in \Omega_T$ such that

$$M := h(t_0, z_0) = \max_{[0,T)\times\overline{\Omega}} h > \max_{\partial_P\Omega_T} h_+.$$

For every $N > 0$, we denote

$$h_N(t, \xi, \eta) = u(t, \xi) + \delta(|z|^2 - C) - v(t, \eta) - \frac{\delta}{T - t} - \frac{N|\xi - \eta|^2}{2},$$

and let $(t_N, \xi_N, \eta_N) \in [0, T) \times \overline{\Omega}^2$ such that

$$h_N(t_N, \xi_N, \eta_N) = \max_{[0,T)\times\overline{\Omega}^2} h_N =: M_N.$$

By [1, Proposition 3.7], we have $\lim_{N\to\infty} N|\xi_N - \eta_N|^2 = 0$ and we can assume that $\xi_N, \eta_N \to z_0$, $t_N \to t_0$ as $N \to \infty$. In particular, there exists $N_0 > 0$ such that $(t_N, \xi_N, \eta_N) \in (0, T) \times \Omega^2$ for all $N \ge N_0$.

By Lemma 3.4, the functions $w_1 = u + \delta(|z|^2 - C)$ and $w_2 = -v$ satisfy the condition (2.5) in Theorem 2.5. Then, it follows from Theorem 2.5 that, for every $N > N_0$, there exist $(\tau_{N1}, p_{N1}, Q^+_N) \in \overline{\mathcal{P}}^{2,+} w_1(t_N, \xi_N)$ and $(\tau_{N2}, p_{N2}, Q^-_N) \in \overline{\mathcal{P}}^{2,-} v(t_N, \eta_N)$ such that

$$\tau_{N1} = \tau_{N2} + \frac{\delta}{(T - t_N)^2}, p_{N1} = p_{N2} = 0, \tag{17}$$

and $Q^-_N \ge Q^+_N$ (i.e., $\langle Q^+_N\zeta, \zeta\rangle \ge \langle Q^-_N\zeta, \zeta\rangle$ for every $\zeta \in \mathbb{R}^{2n}$). In particular, we have

$$dd^c Q^-_N \ge dd^c Q^+_N \ge \delta\omega > 0, \tag{18}$$

where $\omega = dd^c|z|^2$. The second inequality holds due to Proposition 2.4. Moreover, it follows from Proposition 2.4 that

$$e^{\tau_{N1}+F(t_N,\xi_N,u(t_N,\xi_N))}\mu(t_N, \xi_N) \le (dd^c Q^+_N - \delta\omega)^n < (dd^c Q^+_N)^n. \tag{19}$$

and

$$e^{\tau_{N2}+F(t_N,\eta_N,u(t_N,\eta_N))}\mu(t_N, \eta_N) \ge (dd^c Q^-_N)^n. \tag{20}$$

Combining (18), (19) and (20), we get

$$e^{\tau_{N2}+F(t_N,\eta_N,u(t_N,\eta_N))}\mu(t_N, \eta_N) > e^{\tau_{N1}+F(t_N,\xi_N,u(t_N,\xi_N))}\mu(t_N, \xi_N).$$

Therefore, by (17), we have

$$e^{F(t_N,\eta_N,u(t_N,\eta_N))}\mu(t_N, \eta_N) > e^{\frac{\delta}{(T-t_N)^2}+F(t_N,\xi_N,u(t_N,\xi_N))}\mu(t_N, \xi_N).$$

Letting $N \to \infty$, we get

$$e^{F(t_0,z_0,u(t_0,z_0))}\mu(t_0, z_0) > e^{\frac{\delta}{(T-t_N)^2}+F(t_0,z_0,u(t_0,z_0))}\mu(t_0, z_0).$$

This is a contradiction. Then (16) holds. Letting $\delta \searrow 0$, we obtain

$$\sup_{\Omega_T}(u - v) \le \sup_{\partial_P\Omega_T}(u - v)_+.$$

The proof is completed. $\square$

Lemma 3.4. *Let w be a bounded usc function in Ω_T satisfying the following condition: for every $U \Subset \Omega$ and $0 < \delta < T/2$, there exists $k(U, \delta) > 0$ such that if $(t, z) \in (\delta, T - \delta) \times U$ then*

$$|w(t, z) - w(t + s, \Phi(s, z))| \leq k(U, \delta)|s|, \tag{21}$$

for $0 < |s| \ll 1$. Then w satisfies the condition (2.5) in Theorem 2.5.

Proof. Assume that q is an upper test function for w at $(t_0, z_0) \in (\delta, T - \delta) \times U$. By (21), for $0 < |s| \ll 1$, we have

$$q(s + t_0, \Phi(s, z_0)) - q(t_0, z_0) \geq -k(U, \delta)|s|.$$

Then, for $0 < |s| \ll 1$,

$$\begin{aligned}\frac{q(s + t_0, \Phi(s, z_0)) - q(t_0, \Phi(s, z_0))}{|s|} &\geq -k(U, \delta) - \frac{|q(t_0, \Phi(s, z_0)) - q(t_0, z_0)|}{|s|} \\ &\geq -k(U, \delta) - \sup_{|\xi| \leq C_U |s|} \|Dq(t_0, \Phi(\xi, z_0))\| \frac{|\Phi(s, z_0) - z_0|}{|s|} \\ &\geq -k(U, \delta) - C_U. \sup_{|\xi| \leq C_U |s|} \|Dq(t_0, \Phi(\xi, z_0))\|,\end{aligned}$$

where $C_U > 0$ is a constant satisfying (2). Letting $s \to 0^+$ and $s \to 0^-$, we get

$$|\partial_t q(t_0, z_0)| \leq k(U, \delta) + C_U.\|Dq(t_0, \Phi(t_0, z_0))\|. \tag{22}$$

Note that $(\tau, p, Q) \in \mathcal{P}^{2,+} w(t_0, z_0)$ iff there exists an upper test function q for w at (t_0, z_0) such that $(\tau, p, Q) = (\partial_t q(t_0, z_0), Dq(t_0, z_0), D^2 q(t_0, z_0))$. Then, by (22), it is easy to see that w satisfies the condition (2.5) in Theorem 2.5. □

4. Proof of Theorem 1.1 and Corollary 1.2

For the reader's convenience, we recall the Theorem 1.1:

Theorem 4.1. *Let $\Phi : (-1, 1) \times \Omega \to \mathbb{C}^n$ be a continuous mapping satisfying the following conditions:*

- *the mapping $z \mapsto \Phi(s, z)$ is holomorphic in Ω for every $s \in (-1, 1)$;*
- *$\Phi(0, z) = z$ for every $z \in \Omega$;*
- *for each $U \Subset \Omega$, there exists $C_U > 0$ such that*

$$|\Phi(s, z) - z| \leq C_U |s|, \tag{23}$$

for every $(s, z) \in (-1, 1) \times U$.

Suppose that for every $0 < R < S < T$, $K \Subset \Omega$ and $\epsilon > 0$, there exists $0 < \delta \ll 1$ such that

$$(1 + \epsilon) f(t, z) \geq f(t + s, \Phi(s, z)), \tag{24}$$

for every $z \in K$, $R < t < S$ and $|s| < \delta$. Assume that u and v, respectively, is a bounded viscosity subsolution and a bounded viscosity supersolution to the Cauchy-Dirichlet problem (1). Then, for every $0 < R < S < T$, $K \Subset \Omega$ and $\epsilon > 0$, there exists $0 < \delta \ll 1$ such that

$$u(t + s_1, \Phi(s_1, z)) < v(t + s_2, \Phi(s_2, z)) + \epsilon, \tag{25}$$

for all $z \in K$, $R < t < S$ and $\max\{|s_1|, |s_2|\} < \delta$.

Proof. Let $0 < R < S < T$ and denote $S_1 = (S+T)/2$. First, we show that for every $\epsilon > 0$ there exists $\delta_1 = \delta_1(\epsilon) \in (0, \min\{1, R\})$ such that $K \Subset \Omega_{\delta_1}$ and

$$u(t,\eta) < v(s,\xi) + \epsilon, \tag{26}$$

for every $(t,\eta), (s,\xi) \in \Omega_{S_1} \setminus ([\delta_1, S_1] \times \Omega_{\delta_1})$ with $|t-s| + |\eta - \xi| < \delta_1$. Here $\Omega_{\delta_1} = \{z \in \Omega : d(z, \partial\Omega) > \delta_1\}$.

We consider the mapping

$$\begin{aligned} G : \overline{\Omega_{S_1}} \times \overline{\Omega_{S_1}} &\longrightarrow \mathbb{R} \\ (t,\eta,s,\xi) &\mapsto u(t,\eta) - v(s,\xi). \end{aligned}$$

Since G is upper semicontinuous, the set

$$U = \{(t,\eta,s,\xi) \in \overline{\Omega_{S_1}}^2 : G(t,\eta,s,\xi) < \epsilon\},$$

is relatively open in $\overline{\Omega_{S_1}}^2$. Denote

$$A = \{(t,\eta,t,\eta) : (t,\eta) \in ([0,S_1] \times \partial\Omega) \cup (\{0\} \times \overline{\Omega})\}.$$

We have A is compact and $G \le 0$ on A. Hence there exists $0 < \delta_1 < \min\{1, R, dist(K, \partial\Omega)\}$ such that

$$\overline{\Omega_{S_1}}^2 \cap (3\delta_1 \mathbb{B}_{4n+2} + A) \subset U, \tag{27}$$

where $\mathbb{B}_{4n+2}$ is the unit ball in the Euclidean space $\mathbb{R}^{4n+2}$. Then, we have

$$\{(t,\eta,t,\eta) : (t,\eta) \in \Omega_{S_1} \setminus ([\delta_1, S_1] \times \Omega_{\delta_1})\} \subset 2\delta_1 \mathbb{B}_{4n+2} + A. \tag{28}$$

Note that if $|t-s| + |\eta-\xi| < \delta_1$ then $(t,\eta,s,\xi) - (t,\eta,t,\eta) \in \delta_1 \mathbb{B}_{4n+2}$. Therefore, it follows from (28) that

$$\{(t,\eta,s,\xi) \in (\Omega_{S_1} \setminus ([\delta_1, S_1] \times \Omega_{\delta_1}))^2 : |t-s| + |\eta-\xi| < \delta_1\} \subset 3\delta_1 \mathbb{B}_{4n+2} + A. \tag{29}$$

Combining (27) and (29), we obtain (26). By the condition $0 < \delta_1 < \min\{1, R, dist(K, \partial\Omega)\}$, we also have $K \Subset \Omega_{\delta_1}$.

By the assumption (24), there exists $\delta_2 = \delta_2(\epsilon) \in (0, \frac{\min\{\delta_1, T-S\}}{4})$ such that

$$8(C_{\Omega_{\delta_1/2}} + 1)\delta_2 < \delta_1, \tag{30}$$

and

$$(1+\epsilon) f(t,z) \ge f(t+s, \Phi(s,z)), \tag{31}$$

for every $z \in \Omega_{\delta_1/2}$, $\delta_1/2 < t < S_1$ and $|s| < 2\delta_2$. Here $C_{\Omega_{\delta_1/2}} > 0$ is defined by (23). Since F is continuous, we can choose δ_2 small enough such that

$$|F(t,z,r) - F(t+s, \Phi(s,z), r)| < \epsilon, \tag{32}$$

for every $z \in \Omega_{\delta_1/2}$, $\delta_1/2 < t < S_1$, $|s| < 2\delta_2$ and $|r| < M$, where $M = \max\{\sup_{\Omega_T} |u|, \sup_{\Omega_T} |v|\}$.

For every $k > \dfrac{2M}{\delta_2}$ and $(t,z) \in [\delta_1/2, S_1] \times \Omega_{\delta_1/2}$, we consider

$$u^k(t,z) = \sup\{u(t+s, \Phi(s,z)) - k|s| : |s| \le \frac{2M}{k}\},$$

and

$$v_k(t,z) = \inf\{v(t+s, \Phi(s,z)) - k|s| : |s| \le \frac{2M}{k}\}.$$

It follows from (26) and (30) that

$$u^k(t+\tau_1,\Phi(\tau_1,z))\leq v_k(t+\tau_2,\Phi(\tau_2,z))+\epsilon, \tag{33}$$

for every $(t,z)\in\partial_P((3\delta_1/4,S)\times\Omega_{3\delta_1/4})$ and $\max\{|\tau_1|,|\tau_2|\}<\delta_2$. Denote $u^k_{\tau_1}(t,z)=u^k(t+\tau_1,\Phi(\tau_1,z))$ and $v_{k,\tau_2}(t,z)=v_k(t+\tau_2,\Phi(\tau_2,z))$. By Lemma 3.1 and Lemma 3.2, there exists $k_\epsilon>\dfrac{2M}{\delta_2}$ such that

$$(dd^cu^k_{\tau_1})^n\geq(1-\epsilon)e^{\partial_tu^k_{\tau_1}+F_k(t+\tau_1,\Phi(\tau_1,z),u^k_{\tau_1})}f_k(t+\tau_1,\Phi(\tau_1,z))dV, \tag{34}$$

and

$$(dd^cv_{k,\tau_2})^n\leq(1+\epsilon)e^{\partial_tv_{k,\tau_2}+F^k(t+\tau_2,\Phi(\tau_2,z),v_{k,\tau_2})}f^k(t+\tau_2,\Phi(\tau_2,z))dV, \tag{35}$$

in the viscosity sense in $(3\delta_1/4,S)\times\Omega_{3\delta_1/4}$ for every $k>k_\epsilon$ and $\max\{|\tau_1|,|\tau_2|\}<\delta_2$. Here, $F_k(t,z,r)=\inf_{|s|\leq 2M/k}F(t+s,\Phi(s,z),r)$, $f_k(t,z)=\inf_{|s|\leq 2M/k}f(t+s,\Phi(s,z))$, $F^k(t,z,r)=\sup_{|s|\leq 2M/k}F(t+s,\Phi(s,z),r)$ and $f^k(t,z)=\sup_{|s|\leq 2M/k}f(t+s,\Phi(s,z))$.

Moreover, it follows from (31) and (32) that

$$(1+\epsilon)f_k(t+\tau_1,\Phi(\tau_1,z))\geq f(t,z)\geq\frac{f^k(t+\tau_2,\Phi(\tau_2,z))}{1+\epsilon}, \tag{36}$$

and

$$F_k(t+\tau_1,\Phi(\tau_1,z),r)+\epsilon\geq F(t,z,r)\geq F^k(t+\tau_2,\Phi(\tau_2,z),r)-\epsilon, \tag{37}$$

for every $(t,z)\in(3\delta_1/4,S)\times\Omega_{3\delta_1/4}$, $\max\{|\tau_1|,|\tau_2|\}<\delta_2$, $|r|\leq M$ and $k>k_\epsilon$.

Combining (34), (35), (36) and (37), we get

$$(dd^cu^k_{\tau_1})^n\geq(1-\epsilon)^2e^{\partial_tu^k_{\tau_1}+F(t,z,u^k_{\tau_1})-\epsilon}f(t,z)dV, \tag{38}$$

and

$$(dd^cv_{k,\tau_2})^n\leq(1+\epsilon)^2e^{\partial_tv_{k,\tau_2}+F(t,z,v_{k,\tau_2})+\epsilon}f(t,z)dV, \tag{39}$$

in the viscosity sense in $(3\delta_1/4,S)\times\Omega_{3\delta_1/4}$ for every $k>k_\epsilon$ and $\max\{|\tau_1|,|\tau_2|\}<\delta_2$. Therefore, $w_1:=u^k_{\tau_1}+3\epsilon t$ and $w_2:=v_{k,\tau_2}-3\epsilon t$ is respectively a subsolution and a supersolution to the equation

$$e^{\partial_tw+F(t,z,w)}\mu(t,z)=(dd^cw)^n,$$

in $(3\delta_1/4,S)\times\Omega_{3\delta_1/4}$. Note that, by Lemma 3.1 and Lemma 3.2, the functions w_1 and w_2 satisfy the condition (15) in Lemma 3.3. Then, by using Lemma 3.3, we have

$$\sup_{(3\delta_1/4,S)\times\Omega_{3\delta_1/4}}(w_1-w_2)\leq\sup_{\partial_P((3\delta_1/4,S)\times\Omega_{3\delta_1/4})}(w_1-w_2)_+\leq\epsilon+6\epsilon S,$$

where the last inequality holds due to (33). Then

$$u^k(t+\tau_1,\Phi(\tau_1,z))-v_k(t+\tau_2,\Phi(\tau_2,z))\leq-6\epsilon t+\epsilon+6\epsilon S\leq(6S+1)\epsilon,$$

in $(3\delta_1/4,S)\times\Omega_{3\delta_1/4}$ for every $k>k_\epsilon$ and $|\tau|<\delta_2$. Letting $k\to\infty$, we get

$$u(t+\tau_1,\Phi(\tau_1,z))-v(t+\tau_2,\Phi(\tau_2,z))\leq(6S+1)\epsilon$$

in $(R,S)\times K\subset(3\delta_1/4,S)\times\Omega_{3\delta_1/4}$ for every $\max\{|\tau_1|,|\tau_2|\}<\delta_2$. Choosing $\delta=\delta(\epsilon)=\delta_2(\epsilon/(6S+1))$, we obtain (25).

The proof is completed. □

Corollary 4.2. *Assume that* Ω *is a smooth strictly pseudoconvex domain and* $(u_0, \mu(0,z))$ *is admissible. Suppose that* Φ *and* f *satisfy the conditions in Theorem 1.1. Then the Cauchy-Dirichlet problem* (1) *has a unique viscosity solution.*

Proof. By Propositions 2.9 and 2.12, for every $\epsilon > 0$, the problem (1) admits a continuous ϵ-superbarrier which is Lipschitz in t and a continuous ϵ-subbarrier. Then, it follows from Lemma 2.13 that

$$u := \sup\{v : v \text{ is a continuous subsolution to } (1)\},$$

is a discontinuous solution to (1), i.e., u^* is a subsolution and $u = u_*$ is a supersolution.

Let $(t,z) \in \Omega_T$ be an arbitrary point. By Theorem 1.1, for every $\epsilon > 0$, there exists $0 < \delta \ll 1$ such that $(t+s, \Phi(s,z)) \in \Omega_T$ and

$$u(t,z) + \epsilon > u^*(t+s, \Phi(s,z)) \geq u(t+s, \Phi(s,z)),$$

for every $|s| < \delta$. Then $\limsup_{s\to 0} u(t+s, \Phi(s,z) \leq u(t,z) + \epsilon$. Letting $\epsilon \searrow 0$, we get $\limsup_{s\to 0} u(t+s, \Phi(s,z) \leq u(t,z)$. Therefore, since u is lower semicontinuous, we have

$$\lim_{s\to 0} u(t+s, \Phi(s,z) = u(t,z). \tag{40}$$

Moreover, it follows from Theorem 1.1 that for every $\epsilon > 0$ there exists $0 < \delta \ll 1$ such that $(t+s, \Phi(s,z)) \in \Omega_T$ and

$$u(t+s, \Phi(s,z)) + \epsilon > u^*(t,z),$$

for every $|s| < \delta$. Then

$$\liminf_{s\to 0} u(t+s, \Phi(s,z) + \epsilon \geq u^*(t,z). \tag{41}$$

Combining (40) and (41), we have

$$u(t,z) + \epsilon \geq u^*(t,z).$$

Letting $\epsilon \searrow 0$, we obtain $u(t,z) \geq u^*(t,z)$, and then $u = u_* = u^*$. Hence, u is a viscosity solution to (1).

Now, we assume that u_1 and u_2 are two viscosity solutions to (1). By Theorem 1.1, for every $(t,z) \in \Omega_T$ and $\epsilon > 0$, there exists $0 < \delta \ll 1$ such that $(t+s, \Phi(s,z)) \in \Omega_T$ and

$$u_1(t,z) + \epsilon > u_2(t+s, \Phi(s,z)),$$

for every $|s| < \delta$. Since u_2 is continuous, it implies that

$$u_1(t,z) + \epsilon \geq \lim_{s\to 0} u_2(t+s, \Phi(s,z)) = u_2(t,z).$$

Letting $\epsilon \searrow 0$, we get $u_1(t,z) \geq u_2(t,z)$. By the same argument, we also have $u_1(t,z) \leq u_2(t,z)$. Then $u_1 \equiv u_2$.

Thus (1) admits a unique viscosity solution. □

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