SOME NEW REGULARITY CRITERIA FOR THE NAVIER-STOKES EQUATIONS IN TERMS OF ONE DIRECTIONAL DERIVATIVE OF THE VELOCITY FIELD

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Abstract. We establish some regularity criteria for the solutions to the Navier–Stokes equations in the full three-dimensional space in terms of one directional derivative of the velocity field. Revising the method used by Zujin Zhang (2018), we show that a weak solution u is regular on (0, T] provided that $\frac{\partial u}{\partial x_3} \in L^p(0,T;L^q(\mathbb{R}^3))$ with $s = 2$ for $3 \leq q \leq 6$, $\frac{19}{10} \leq s \leq 2$ for $6 \leq q \leq \frac{6}{6s-11}$, and $\frac{11}{6} \leq s \leq \frac{19}{10}$ for $\frac{3}{2}\sqrt{\frac{s^2+2s-3}{(2s-3)^2}} - \frac{3(s-3)}{2(2s-3)} \leq q \leq \frac{6}{6s-11}$ where $s = \frac{2}{p} + \frac{3}{q}$. They improve the known results $\frac{2}{p} + \frac{3}{q} = 2$ for $3 \leq q \leq \frac{19}{6}$ and $\frac{2}{p} + \frac{3}{q} \leq \frac{8}{5} + \frac{9}{11q}$ for $\frac{5}{2} \leq q < \infty$.

1. Introduction and main results

We consider sufficient conditions for the regularity of solutions of the Cauchy problem for the NavierStokes equations in $\mathbb{R}^3$

$$\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0 \qquad \text{in } \mathbb{R}^3 \times (0,\infty) \tag{1.1}$$

$$\nabla \cdot u = 0 \qquad \text{in } \mathbb{R}^3 \times (0,\infty) \tag{1.2}$$

$$u\big|_{t=0} = u_0 \tag{1.3}$$

where $u = (u_1, u_2, u_3) : \mathbb{R}^3 \times (0,T) \to \mathbb{R}^3$ is the velocity field, $p : \mathbb{R}^3 \times (0,T) \to \mathbb{R}$ is a scalar pressure, and u_0 is the initial velocity field. We recall some well-known function spaces, the definitions of weak and strong solutions to (1.1) and introduce some notations before describing the main results. Throughout the paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant C, for $1 \leq q \leq \infty$ we use the well-known Lebesgue spaces $L^q(\mathbb{R}^3)$, with norms $\big\|\cdot\big\|_{L^q(\mathbb{R}^3)} = \|\cdot\|_q$. Further, we use the Bochner spaces $L^s\big(t,T;X)\big)$, $1 \leq s \leq \infty$, where X is a Banach space, if $X = L^q(\mathbb{R}^3)$ then we denote

$$\|\cdot\|_{L^s\left(t,T;L^q(\mathbb{R}^3)\right)} := \left(\int_t^T \|\cdot\|^s_{L^q(\mathbb{R}^3)}\, \mathrm{d}\tau\right)^{1/s} = \big\|\, \|\cdot\|_q \,\big\|_{s;t,T} = \big\|\cdot\big\|_{q,s;t,T}.$$

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We use standard Sobolev space and do not differentiate between scalar and tensor function. We write $u = (u_1, u_2, u_3) \in X$ instead of $u = (u_1, u_2, u_3) \in X^3$. We denote

$$u_h = (u_1, u_2)\,, \nabla_h = (\partial_1, \partial_2)\,, \Delta_h = \partial_1\partial_1 + \partial_2\partial_2.$$

To deal with solenoidal vector fields we introduce the space of divergence-free smooth compactly supported functions $C^\infty_{0,\sigma}(\mathbb{R}^3)) = \{u \in C^\infty_0(\mathbb{R}^3)), \operatorname{div}(u) = 0\}$, and the spaces $L^2_\sigma(\mathbb{R}^3) = \overline{C^\infty_{0,\sigma}(\mathbb{R}^3)}^{\|\cdot\|_2}$. Let $u_0 \in L^2_\sigma(\mathbb{R}^3)$, a weak solution of (1.1)-(1.3) on $[0,T]$ (or $[0,\infty)$ if $T = \infty$) is a function $u : [0,T] \to L^2_\sigma(\mathbb{R}^3)$ in the class $u \in C_w\left([0,T]; L^2_\sigma(\mathbb{R}^3)\right) \cap L^2_{loc}\left(0,T; H^1\left(\mathbb{R}^3\right)\right)$ satisfying

$$(u(t), \varphi(t)) + \int_0^t \left\{-\left(u, \partial_t \varphi\right) + \left(\nabla u, \nabla \varphi\right) + \left(u \cdot \nabla u, \varphi\right)\right\} ds = \left(u_0, \varphi(0)\right)$$

for all $t \in [0,T]$ and all test functions $\varphi \in C^\infty_0\left([0,T) \times \mathbb{R}^3\right)$ with $\nabla \cdot \varphi = 0$. Here $(\cdot,\cdot)$ stands for L^2-inner product, and C_w signifies continuity in the weak topology. For every $u_0 \in L^2\left(\mathbb{R}^3\right)$, there exists a weak solution of (1.1)-(1.3) on $[0,\infty)$ satisfying the following energy inequality (see Leray [5])

$$\|u(\cdot,t)\|^2_{L^2} + 2\nu \int_{t_0}^t \|\nabla u(\cdot,s)\|^2_{L^2} ds \leq \|u_0\|^2_{L^2}\,, \tag{1.4}$$

for almost all $t_0 \in [0,\infty)$ including $t_0 = 0$ and all $t \in [t_0, \infty)$. A Leray-Hopf solution of (1.1)-(1.3) on $[0,T]$ is weak solution on $[0,T]$ satisfying the energy inequality (1.4) for almost all $t_0 \in (0,T)$ and all $t \in [t_0, T]$. A weak solution of the Navier-Stokes equations that belongs to $u \in L^\infty\left([0,T]; H^1\left(\mathbb{R}^3\right)\right) \cap L^2\left(0,T; H^2\left(\mathbb{R}^3\right)\right)$ is called a strong solution. We say that a Leray weak solution is regular if $u \in C^\infty((0,T) \times \mathbb{R}^3)$. It is well-known that if the above Leray weak solution u satisfies

$$u \in L^\infty(0,T; H^1(\mathbb{R}^3)) \cap L^2(0,T; H^2(\mathbb{R}^3)),$$

then such solution is regular.

There are many known sufficient conditions on the velocity which guarantee that a solution is regular for all time. The first such criterion is usually referred to as the Prodi-Serrin condition [10, 17] saying that if

$$u \in L^p(0,T; L^q(\mathbb{R}^3)), \frac{2}{p} + \frac{3}{q} = 1, q \in [3,\infty),$$

then u is regular and is unique in the class of all weak solutions satisfying the energy inequality. Notice that the case $u \in L^\infty(0,T; L^3(\mathbb{R}^3))$ was covered by Escauriaza et al. [3]. A related well-known sufficient condition is due to Beirão da Veiga [1]. Namely, if

$$\nabla u \in L^p(0,T; L^q(\mathbb{R}^3)), \frac{2}{p} + \frac{3}{q} = 2, q \in \Big[\frac{3}{2}, \infty\Big),$$

then u is regular. Note that the case $\nabla u \in L^\infty(0,T; L^{\frac{3}{2}}(\mathbb{R}^3))$ follows immediately from the Sobolev imbedding theorems and [3].

Recently, many authors became interested in the regularity criteria involving only one directional derivative of velocity field, namely $\frac{\partial u}{\partial x_3}$. In [7,9] Penel-Pokorný showed that if

$$\frac{\partial u}{\partial x_3} \in L^p(0,T;L^q(\mathbb{R}^3)), \frac{2}{p}+\frac{3}{q}=\frac{3}{2}, 2 \le q \le \infty$$

then u is regular on (0, T]. Later on, authors proved the regularity criterion in several papers, see [2,4,6,19]. Here, if

$$\frac{\partial u}{\partial x_3} \in L^p(0,T;L^q(\mathbb{R}^3)), \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2} < q \le 3$$

then u is regular on (0, T]. In 2019, Zhang, Yuan, Ganzhou, Zhou and Zhuhai [20] proved two regularity criteria

$$\frac{\partial u}{\partial x_3} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q} \le \frac{8}{5}+\frac{9}{11q}, \quad \frac{5}{2} \le q < \infty$$

or

$$\frac{\partial u}{\partial x_3} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q} \le \frac{14}{11}+\frac{3}{5q}, \quad 4 \le q < \infty.$$

Z. Skalak [12] proved the following regularity criterion

$$\frac{\partial u}{\partial x_3} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q} \le 1+\frac{3}{q}, \ 3 < q \le \frac{10}{3}.$$

Very recently, Skalak [16] showed that if

$$\frac{\partial u}{\partial x_3} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q} = 2, \quad 3 \le q \le \frac{19}{6}$$

then u is regular on (0, T), crossing so for the first time the barrier $q = 3$. In this paper we will focus on the method used in [19] to improve the results from [12,16,20]. The following theorems is the main results of our pager.

Theorem 1.1. *Let $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u = 0, T > 0$. Assume that u is a Leray weak solution of the problem (1.1)-(1.3) with the initial data u_0. Then u is regular on $(0,T)$ if*

$$\frac{\partial u}{\partial x_3} \in L^p(0,T;L^q(\mathbb{R}^3))$$

with

$$\frac{2}{p}+\frac{3}{q}=2, \quad 3 \le q \le 6. \tag{1.5}$$

Theorem 1.2. *Let $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u = 0, T > 0$. Assume that u is a Leray weak solution of the problem (1.1)-(1.3) with the initial data u_0. Then u is regular on $(0,T)$ if*

$$\frac{\partial u}{\partial x_3} \in L^p(0,T;L^q(\mathbb{R}^3))$$

with

$$\frac{19}{10} \le s \le 2, \ 6 \le q \le \frac{6}{6s-11} \tag{1.6}$$

or

$$\frac{11}{6} \leq s \leq \frac{19}{10},\ \frac{3}{2}\sqrt{\frac{s^2+2s-3}{(2s-3)^2}} - \frac{3(s-3)}{2(2s-3)} \leq q \leq \frac{6}{6s-11} \tag{1.7}$$

where

$$s = \frac{2}{p} + \frac{3}{q}.$$

Remark 1.3. The result of Theorem 1.1 improves that of [12, 16]. If $s = \frac{19}{10}$ then the condition (1.6) becomes $6 \leq q \leq 15$ and we have $\frac{19}{10} > \frac{191}{110} \geq \frac{8}{5} + \frac{9}{11q}$, $\frac{19}{10} > \frac{151}{110} \geq \frac{14}{11} + \frac{3}{5q}$; if $s = \frac{11}{6}$ then the condition (1.7) becomes $q > \frac{3\sqrt{145}}{8} + \frac{21}{8} \approx 7.1406$ and we have $\frac{11}{6} > \frac{1}{220}\left(5\sqrt{145}+317\right) \geq \frac{8}{5} + \frac{9}{11q}$, $\frac{11}{6} > \frac{1}{660}\left(11\sqrt{145}+763\right) \geq \frac{14}{11} + \frac{3}{5q}$, therefore the result of Theorem 1.2 improves that of [20].

The following lemmas will be useful in several cases

Lemma 1.4. *(Troisi Inequality). Suppose that $r, p_1, p_2, p_3 \in (1,\infty)$ and*

$$1 + \frac{3}{r} = \sum_{i=1}^{3} \frac{1}{p_i}.$$

Then there exists a constant $C > 0$ such that for every $f \in C_0^\infty(\mathbb{R}^3)$

$$\|f\|_r \leq C \prod_{i=1}^{3} \|\partial_i f\|_{p_i}^{1/3}.$$

Lemma 1.5. *For each $1 \leq s < \infty, 0 < \lambda < \infty$, then exists some constant C such that for each $f \in C_0^\infty(\mathbb{R}^3)$,*

$$\|f\|_{(2\lambda+1)q} \leq C\big\|\partial_i f\big\|_q^{\frac{1}{2\lambda+1}} \big\|\partial_k |f|^\lambda\big\|_2^{\frac{1}{2\lambda+1}} \big\|\partial_j |f|^\lambda\big\|_2^{\frac{1}{2\lambda+1}},$$

where i, j, k is a permutation of 1, 2, 3.

For the proofs of Lemmas 1.4 and 1.5 see [18] and [19], respectively.

Lemma 1.6. (*Hölder's inequality in mixed-norm Lebesgue space).*
Let $1 \leq r, p, q, \bar{r}, \bar{p}, \bar{q} \leq \infty$ and $-\infty \leq t \leq T \leq \infty$ satisfy the relations

$$(\frac{1}{r}, \frac{1}{\bar{r}}) = (\frac{1}{p}, \frac{1}{\bar{p}}) + (\frac{1}{q}, \frac{1}{\bar{q}}).$$

Suppose that $f \in L^{\bar{p}}\big(t,T; L^p(\mathbb{R}^3)\big)$ and $g \in L^{\bar{q}}\big(t,T; L^q(\mathbb{R}^3)\big)$. Then $fg \in L^{\bar{r}}\big(t,T; L^r(\mathbb{R}^3)\big)$ and we have the inequality

$$\big\|fg\big\|_{r,\bar{r};t,T} \leq \big\|f\big\|_{p,\bar{p};t,T}\big\|g\big\|_{q,\bar{q};t,T}.$$

Lemma 1.7. (*Interpolation inequality in mixed-norm Lebesgue space).*
Let $1 \leq r, p, q, \bar{r}, \bar{p}, \bar{q} \leq \infty, -\infty \leq t \leq T \leq \infty$ and $0 \leq \theta \leq 1$ satisfy the relation

$$(\frac{1}{r}, \frac{1}{\bar{r}}) = (1-\theta)(\frac{1}{p}, \frac{1}{\bar{p}}) + \theta(\frac{1}{q}, \frac{1}{\bar{q}}).$$

Suppose that $f \in L^{\bar{p}}\big(t,T;L^p(\mathbb{R}^3)\big) \cap L^{\bar{q}}\big(t,T;L^q(\mathbb{R}^3)\big)$*. Then* $f \in L^{\bar{r}}\big(t,T;L^r(\mathbb{R}^3)\big)$ *and we have the inequality*

$$\|f\|_{r,\bar{r};t,T} \le \|f\|_{p,\bar{p};t,T}^{1-\theta}\|f\|_{q,\bar{q};t,T}^{\theta}.$$

The proofs of Lemmas 1.6 and 1.7 are elementary and may be omitted.

2. The proof of the main result

The proofs of Theorem 1.1 and 1.2 are based on the method used in [19]. We define the quantities $\mathcal{L}$ and $\mathcal{J}$ (see [1,4]) and then prove that $\mathcal{L}$ and $\mathcal{J}$ are uniformly bounded in time. Let $T^* = \sup\{\tau > 0; u \text{ is regular on } (0,\tau)\}$. Since $u_0 \in H^1, u$ is regular on some positive time interval and T^* is either equal to infinity (in which case the proof is finished) or it is a positive number and u is regular on $(0,T^*)$, that is $\nabla u \in L^\infty_{loc}\left([0,T^*)\,;L^2\right)$. It is sufficient to prove that $T^* > T$. We proceed by contradiction and suppose that $T^* \le T$. We take $\epsilon > 0$ sufficiently small (it will be specified later) and fix $t_1 \in (0,T^*)$ such that $\|\partial_3 u\|_{L^p(t_1,T^*;L^q)} < \epsilon$. Taking arbitrarily $t \in (t_1,T^*)$ the proof will be finished if we show that $\|\nabla u(t)\|_2 \le C < \infty$, where C is independent of t. Actually, the standard extension argument then shows that the regularity of u can be extended beyond T^* and it contradicts the definition of T^*. We define

$$\mathcal{J}^2(t) = \max_{\tau\in[t_1,t]}\left(\|\nabla u_h(\tau)\|_2^2 + \|\partial_3 u(\tau)\|_2^2\right) + \int_{t_1}^t \left(\|\Delta u_h(\tau)\|_2^2 + \|\nabla\partial_3 u(\tau)\|_2^2\right)\mathrm{d}\tau$$

$$\mathcal{L}^2(t) = \max_{\tau\in[t_1,t]}\big\|\,|u_3(\tau)|^\lambda\,\big\|_2^2 + \int_{t_1}^t \big\|\nabla(|u_3(\tau)|^\lambda)\big\|_2^2\mathrm{d}\tau, \quad \lambda > \frac{3}{2}.$$

The proof will be finished if we show that $\mathcal{J}(t) + \mathcal{L}(t)$ is bounded on (t_1,T^*).

Proof of Theorem 1.1

We will start with the estimate of $\mathcal{J}$. The NSE may be rewritten as

$$\frac{\partial u_k}{\partial t} - \Delta u_k + \sum_{j=1}^3 u_j\partial_j u_k + \partial_k p = 0 \quad (NSE_k),$$

where $k = 1,2,3$. Now, we proceed as in [6]. For $k = 1,2$, multiplying the equation NSE_k by $-\Delta u_k$ and (1.1) by $\partial_3\partial_3 u$ in $L^2(\mathbb{R}^3)$, respectively, and adding them together we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\nabla u_h\|_2^2 + \|\partial_3 u\|_2^2\Big] + \Big[\|\Delta u_h\|_2^2 + \|\nabla\partial_3 u\|_2^2\Big]\\
&\lesssim \int_{\mathbb{R}^3}|\partial_3 u||\nabla u_h|^2\mathrm{d}x + \int_{\mathbb{R}^3}|u_3||\partial_3 u|(|\Delta u_h| + |\nabla\partial_3 u|)\mathrm{d}x \qquad (2.1)\\
&:= J_1 + J_2,
\end{aligned}$$

where the first inequality was proved in [19], by the Hölder and Gagliardo-Nirenberg inequalities

$$J_1 \le \|\partial_3 u\|_q \big\|\nabla u_h\big\|^2_{\frac{2q}{q-1}} \lesssim \|\partial_3 u\|_q \big\|\nabla u_h\big\|_2^{\frac{2q-3}{q}} \big\|\Delta u_h\big\|_2^{\frac{3}{q}}.$$

Integrating in time and using the Hölder inequality we obtain

$$\int_{t_1}^{t} J_1 \mathrm{d}\tau \le C\big\|\partial_3 u\big\|_{\frac{2q}{2q-3},q;t_1,t}\big\|\nabla u_h\big\|_{\infty,2;t_1,t}^{\frac{2q-3}{q}}\big\|\Delta u_h\big\|_{2,2;t_1,t}^{\frac{3}{q}} \le C\epsilon \mathcal{J}^2(t). \tag{2.2}$$

Using the Hölder inequality to estimate the term J_2

$$\begin{aligned}\int_{t_1}^{t} J_2 \mathrm{d}\tau &= \int_{t_1}^{t}\int_{\mathbb{R}^3} |u_3||\partial_3 u|\big(|\Delta u_h| + |\nabla\partial_3 u|\big)\mathrm{d}x\mathrm{d}\tau \\ &= \Big\||u_3||\partial_3 u|\big(|\Delta u_h| + |\nabla\partial_3 u|\big)\Big\|_{1,1;t_1,t} \\ &\lesssim \|u_3\|_{a,\bar{a};t_1,t}\|\partial_3 u\|_{b,\bar{b};t_1,t}\big\|\Delta u_h, \nabla\partial_3 u\big\|_{2,2;t_1,t},\end{aligned} \tag{2.3}$$

where

$$\big(\frac{1}{a}, \frac{1}{\bar{a}}\big) + \big(\frac{1}{b}, \frac{1}{\bar{\bar{b}}}\big) = \big(\frac{1}{2}, \frac{1}{2}\big). \tag{2.4}$$

Applying the interpolation inequality, Lemma 1.5 and the Hölder inequality in order to obtain

$$\begin{aligned}\|u_3\|_{a,\bar{a};t_1,t} &\le \|u_3\|^{1-v_1}_{2\lambda,\infty;t_1,t}\big\|u_3\big\|^{v_1}_{(2\lambda+1)q,\frac{2(2\lambda+1)q}{4q-3};t_1,t} \\ &\lesssim \|u_3\|^{1-v_1}_{2\lambda,\infty;t_1,t}\Big\|\big\|\nabla_h|u_3|^\lambda\big\|_2^{\frac{2}{2\lambda+1}}\big\|\partial_3 u_3\big\|_q^{\frac{1}{2\lambda+1}}\Big\|^{v_1}_{\frac{2(2\lambda+1)q}{4q-3};t_1,t} \\ &\le \big\||u_3|^\lambda\big\|^{\frac{1-v_1}{\lambda}}_{2,\infty;t_1,t}\big\|\nabla_h|u_3|^\lambda\big\|^{\frac{2v_1}{2\lambda+1}}_{2,2;t_1,t}\big\|\partial_3 u\big\|^{\frac{v_1}{2\lambda+1}}_{q,\frac{2q}{2q-3};t_1,t},\end{aligned}$$

where interpolation inequality with

$$\big(\frac{1}{a}, \frac{1}{\bar{a}}\big) = (1-v_1)\big(\frac{1}{2\lambda}, 0\big) + v_1\big(\frac{1}{(2\lambda+1)q}, \frac{4q-3}{2(2\lambda+1)q}\big), \quad 0 \le v_1 \le 1. \tag{2.5}$$

Using the interpolation inequality, we get

$$\|\partial_3 u\|_{b,\bar{b};t_1,t} \le \big\|\partial_3 u\big\|^{1-v_2}_{q,\frac{2q}{2q-3};t_1,t}\|\partial_3 u\|^{v_2}_{2,\infty;t_1,t}$$

where

$$\big(\frac{1}{b}, \frac{1}{\bar{b}}\big) = (1-v_2)\big(\frac{1}{q}, \frac{2q-3}{q}\big) + v_2\big(\frac{1}{2}, 0\big), \quad 0 \le v_2 \le 1. \tag{2.6}$$

It follows from (2.4), (2.5), and (2.6) that

$$\frac{1-v_1}{2\lambda} + \frac{v_1}{(2\lambda+1)q} + \frac{1-v_2}{q} + \frac{v_2}{2} = \frac{1}{2} \tag{2.7}$$

and

$$\frac{4q-3}{2(2\lambda+1)q}v_1 + \frac{2q-3}{2q}(1-v_2) = \frac{1}{2}. \tag{2.8}$$

From the above estimates we have

$$\begin{aligned}\int_{t_1}^{t} J_2 \mathrm{d}\tau &\lesssim \left\|\partial_3 u\right\|_{q,\frac{2q}{2q-3};t_1,t}^{\frac{v_1}{2\lambda+1}+1-v_2}\left\||u_3|^\lambda\right\|_{2,\infty;t_1,t}^{\frac{1-v_1}{\lambda}}\left\|\nabla|u_3|^\lambda\right\|_{2,2;t_1,t}^{\frac{2v_1}{2\lambda+1}}\left\|\partial_3 u\right\|_{2,\infty;t_1,t}^{v_2}\left\|\Delta u_h,\nabla\partial_3 u\right\|_{2,2;t_1,t}\\ &\lesssim \epsilon^{\frac{v_1}{2\lambda+1}+1-v_2}\mathcal{J}^{1+v_2}(t)\mathcal{L}^{\frac{1-v_1}{\lambda}+\frac{2v_1}{2\lambda+1}}(t).\end{aligned}$$

We deduce that

$$\begin{aligned}\mathcal{J}^2(t) &\leq \|\nabla u_h(t_1)\|_2^2+\|\partial_3 u(t_1)\|_2^2+C\epsilon\mathcal{J}^2(t)\\ &\quad +C\epsilon^{\frac{v_1}{2\lambda+1}+1-v_2}\mathcal{J}^{1+v_2}(t)\mathcal{L}^{\frac{1-v_1}{\lambda}+\frac{2v_1}{2\lambda+1}}(t)\\ &\leq C+C\epsilon\mathcal{J}^2(t)+C\epsilon^{\frac{v_1}{2\lambda+1}+1-v_2}\mathcal{J}^{1+v_2}(t)\mathcal{L}^{\frac{1-v_1}{\lambda}+\frac{2v_1}{2\lambda+1}}(t).\end{aligned} \tag{2.9}$$

We are now prepared to proceed with the estimate of $\mathcal{L}$. proceeding as in [13], multiplying the 3rd equation from (1.1) by $|u_3|^{2\lambda-2}u_3$ and integrating over the whole space we get

$$\frac{d}{dt}\left\||u_3|^\lambda\right\|_2^2+\left(\frac{4\lambda-2}{\lambda}\right)\left\|\nabla|u_3|^\lambda\right\|_2^2=(-2\lambda)\int\partial_3 p\,|u_3|^{2\lambda-2}u_3\equiv L.$$

By the standard procedure one gets

$$-\Delta\partial_3 p=2\sum_{i,j=1}^{3}\partial_i\partial_j\left(u_i\partial_3 u_j\right)=2\sum_{i=1}^{2}\sum_{j=1}^{3}\partial_i\partial_j\left(u_i\partial_3 u_j\right)+2\sum_{j=1}^{3}\partial_3\partial_j\left(u_3\partial_3 u_j\right)$$

and so we have the decomposition of $\partial_3 p$:

$$\partial_3 p=2\sum_{i=1}^{2}\sum_{j=1}^{3}R_iR_j\left(u_i\partial_3 u_j\right)+2\sum_{j=1}^{3}R_3R_j\left(u_3\partial_3 u_j\right)=p_1+p_2$$

where $\mathcal{R}_i=\frac{\partial_i}{\sqrt{-\triangle}}$ is the Riesz transformation, which is bounded from $L^r(\mathbb{R}^3))$ to itself for $1<r<\infty$. We have

$$\int_{t_1}^{t} L\mathrm{d}\tau\lesssim\left\|p_1|u_3|^{2\lambda-2}u_3\right\|_{1,1;t_1,t}+\left\|p_2|u_3|^{2\lambda-2}u_3\right\|_{1,1;t_1,t}:=L_1(t)+L_2(t). \tag{2.10}$$

Using the Hölder inequality we obtain

$$L_1(t)\leq\|u_h\|_{c,\bar{c},t_1,t}\left\|\partial_3 u\right\|_{q,\frac{2q}{2q-3};t_1,t}\left\|u_3\right\|_{d,\bar{d};t_1,t}^{2\lambda-1} \tag{2.11}$$

where

$$\Big(\frac{1}{c},\frac{1}{\bar{c}}\Big)+\Big(\frac{1}{q},\frac{2q-3}{2q}\Big)+(2\lambda-1)\Big(\frac{1}{d},\frac{1}{\bar{d}}\Big)=(1,1). \tag{2.12}$$

Using the interpolation inequality, Lemma 1.4, Sobolev and Hölder inequalities in order to obtain

$$\begin{aligned}\|u_h\|_{c,\bar{c},t_1,t} &\le \big\|u_h\big\|^{1-v_3}_{3q,\frac{6q}{2q-3};t_1,t}\|u_h\|^{v_3}_{6,\infty;t_1,t}\\ &\le \big\|\|\partial_3 u_h\|_q^{\frac{1}{3}}\|\nabla_h u_h\|_2^{\frac{2}{3}}\big\|^{1-v_3}_{\frac{6q}{2q-3};t_1,t}\|\nabla u_h\|^{v_3}_{2,\infty;t_1,t}\\ &\le \big\|\partial_3 u_h\big\|^{\frac{1-v_3}{3}}_{q,\frac{2q}{2q-3};t_1,t}\big\|\nabla_h u_h\big\|^{\frac{2(1-v_3)}{3}}_{2,\infty;t_1,t}\|\nabla u_h\|^{v_3}_{2,\infty;t_1,t}\\ &\le \big\|\partial_3 u\big\|^{\frac{1-v_3}{3}}_{q,\frac{2q}{2q-3};t_1,t}\big\|\nabla u_h\big\|^{\frac{2+v_3}{3}}_{2,\infty;t_1,t},\end{aligned}$$

where interpolation inequality with

$$\left(\frac{1}{c},\frac{1}{\bar{c}}\right) = (1-v_3)\left(\frac{1}{3q},\frac{2q-3}{6q}\right) + v_3\left(\frac{1}{6},0\right), \quad 0 \le v_3 \le 1. \tag{2.13}$$

Applying the interpolation inequality, Lemma 1.5, and the Hölder inequality in order to obtain

$$\begin{aligned}\|u_3\|_{d,\bar{d};t_1,t} &\le \big\|u_3\big\|^{1-v_4}_{2\lambda,\infty;t_1,t}\big\|u_3\big\|^{v_4}_{(2\lambda+1)q,\frac{2(2\lambda+1)q}{4q-3};t_1,t}\\ &\le \big\||u_3|^\lambda\big\|^{\frac{1-v_4}{\lambda}}_{2,\infty;t_1,t}\big\|\|\partial_3 u_3\|_q^{\frac{1}{2\lambda+1}}\|\nabla_h(|u|^\lambda)\|_2^{\frac{2}{2\lambda+1}}\big\|^{v_4}_{\frac{2(2\lambda+1)q}{4q-3};t_1,t}\\ &\le \big\||u_3|^\lambda\big\|^{\frac{1-v_4}{\lambda}}_{2,\infty;t_1,t}\big\|\partial_3 u\big\|^{\frac{v_4}{2\lambda+1}}_{q,\frac{2q}{2q-3};t_1,t}\big\|\nabla_h(|u_3|^\lambda)\big\|^{\frac{2v_4}{2\lambda+1}}_{2,2;t_1,t},\end{aligned}$$

where interpolation inequality with

$$\left(\frac{1}{d},\frac{1}{\bar{d}}\right) = (1-v_4)\left(\frac{1}{2\lambda,0}\right) + v_4\left(\frac{1}{(2\lambda+1)q},\frac{4q-3}{2(2\lambda+1)q}\right), \quad 0 \le v_4 \le 1. \tag{2.14}$$

From the above estimates we have

$$\begin{aligned}L_1(t) &\le C\big\|\partial_3 u\big\|^{1+\frac{1-v_3}{3}+\frac{v_4(2\lambda-1)}{2\lambda+1}}_{q,\frac{2q}{2q-3};t_1,t}\big\|\nabla u_h\big\|^{\frac{2+v_3}{3}}_{2,\infty;t_1,t}\big\||u_3|^\lambda\big\|^{\frac{(1-v_4)(2\lambda-1)}{\lambda}}_{2,\infty;t_1,t}\big\|\nabla(|u_3|^\lambda)\big\|^{\frac{2v_4(2\lambda-1)}{2\lambda+1}}_{2,2;t_1,t}\\ &\le C\epsilon^{\frac{4-v_3}{3}+\frac{v_4(2\lambda-1)}{2\lambda+1}}\mathcal{J}^{\frac{2+v_3}{3}}(t)\mathcal{L}^{\frac{(1-v_4)(2\lambda-1)}{\lambda}+\frac{2v_4(2\lambda-1)}{(2\lambda+1)}}(t).\end{aligned}$$

Applying the Hölder, interpolation, and Sobolev inequalities in order to obtain

$$\begin{aligned}L_2(t) &\le \big\|u_3\big\|_{\frac{2\lambda q}{q-1},\frac{4\lambda q}{3};t_1,t}\big\|\partial_3 u\big\|_{q,\frac{2q}{2q-3};t_1,t}\big\|u_3\big\|^{2\lambda-1}_{\frac{2\lambda q}{q-1},\frac{4\lambda q}{3};t_1,t}\\ &= \big\||u_3|^\lambda\big\|^2_{\frac{2q}{q-1},\frac{4q}{3};t_1,t}\big\|\partial_3 u\big\|_{q,\frac{2q}{2q-3};t_1,t}\\ &\le \big\||u_3|^\lambda\big\|^{\frac{2q-3}{q}}_{2,\infty;t_1,t}\big\||u_3|^\lambda\big\|^{\frac{3}{q}}_{6,2;t_1,t}\big\|\partial_3 u\big\|_{q,\frac{2q}{2q-3};t_1,t}\\ &\lesssim \big\||u_3|^\lambda\big\|^{\frac{2q-3}{q}}_{2,\infty;t_1,t}\big\|\nabla(|u_3|^\lambda)\big\|^{\frac{3}{q}}_{2,2;t_1,t}\big\|\partial_3 u\big\|_{q,\frac{2q}{2q-3};t_1,t}\\ &\le \epsilon\mathcal{L}^2(t).\end{aligned} \tag{2.15}$$

Thus

$$\mathcal{L}^2(t) \le C + L_1(t) + L_2(t) \tag{2.16}$$
$$\le C + C\epsilon^{\frac{4-v_3}{3}+\frac{v_4(2\lambda-1)}{2\lambda+1}} \mathcal{J}^{\frac{2+v_3}{3}}(t)\mathcal{L}^{\frac{(1-v_4)(2\lambda-1)}{\lambda}+\frac{2v_4(2\lambda-1)}{(2\lambda+1)}}(t) + C\epsilon\mathcal{L}^2(t).$$

It follows from (2.12), (2.13), and (2.14) that

$$\frac{1-v_3}{3q} + \frac{v_3}{6} + \Big(\frac{1-v_4}{2\lambda} + \frac{v_4}{q(2\lambda+1)}\Big)(2\lambda-1) + \frac{1}{q} = 1 \tag{2.17}$$

and

$$\frac{2q-3}{2q}\Big(1 + \frac{1-v_3}{3} + \frac{v_4(2\lambda-1)}{2\lambda+1}\Big) + \frac{v_4(2\lambda-1)}{(2\lambda+1)} = 1. \tag{2.18}$$

Now, solve (2.7), (2.8), (2.17), and (2.18) we obtain

$$v_1 = -\frac{(1+2\lambda)(3-2q-2\lambda+q\lambda)}{-3+2q+\lambda}; \qquad v_2 = -\frac{6-5q-3\lambda+2q\lambda}{-3+2q+\lambda}; \tag{2.19}$$

$$v_3 = -\frac{3+10q-4\lambda-4q\lambda}{-3+2q+\lambda}; \qquad v_4 = \frac{(1+2\lambda)(9-6q+2q\lambda)}{3(-3+2q+\lambda)(-1+2\lambda)}. \tag{2.20}$$

Reducing $0 \le v_1 \le 1, 0 \le v_2 < 1, 0 \le v_3 \le 1, 0 \le v_4 \le 1, \lambda > \frac{3}{2}, q \ge 3$, yields

$$3 \le q \le \frac{3}{4}(\sqrt{5}+3), \frac{10q+3}{4q+4} \le \lambda \le \frac{12q}{4q+3}$$

or

$$\frac{3}{4}(\sqrt{5}+3) < q \le 6, \frac{10q+3}{4q+4} \le \lambda \le \frac{2q-3}{q-2}.$$

Hence, the range of q is $[3,6]$. From the above estimates of $\mathcal{J}$ and $\mathcal{L}$ we have

$$\mathcal{J}^2(t) \le C + C\varepsilon\mathcal{J}^2(t) + C\varepsilon^{\frac{v_1}{2\lambda+1}+1-v_2}\mathcal{J}^{j_1}(t)\mathcal{L}^{l_1}(t) \tag{2.21}$$

and

$$\mathcal{L}^2(t) \le C + C\varepsilon^{\frac{4-v_3}{3}+\frac{v_4(2\lambda-1)}{2\lambda+1}}\mathcal{J}^{j_2}(t)\mathcal{L}^{l_2}(t) + C\varepsilon\mathcal{L}^2(t) \tag{2.22}$$

where

$$\begin{aligned} j_1 &= 1 + v_2, \quad l_1 = \tfrac{1-\vartheta_1}{\lambda} + \tfrac{2\vartheta_1}{2\lambda+1} \\ j_2 &= \tfrac{2+\vartheta_3}{3}, \qquad l_2 = \tfrac{(2\lambda-1)(1-\vartheta_4)}{\lambda} + \tfrac{2(2\lambda-1)\vartheta_4}{2\lambda+1}. \end{aligned} \tag{2.23}$$

Since $0 \le v_2 < 1$, it is obvious that $1 \le j_1 < 2$, and we may apply Hölder inequality to (2.21)

$$\mathcal{J}^2(t) \le C + C\varepsilon\mathcal{J}^2(t) + \frac{1}{2}\mathcal{J}^2(t) + C\mathcal{L}^{\frac{2l_1}{2-j_1}}(t)$$
$$\le C + C\varepsilon\mathcal{J}^2(t) + \frac{1}{2}\mathcal{J}^2(t) + C\mathcal{L}^{\frac{2}{2\lambda-3}}(t).$$

Now choose $0 < \varepsilon \ll 1$ sufficiently small such that

$$C\varepsilon \le \frac{1}{4},$$

we have

$$\mathcal{J}(t) \le C + C\mathcal{L}^{\frac{1}{2\lambda-3}}(t). \tag{2.24}$$

Plugging (2.24) into (2.22), and choosing ε such that

$$C\varepsilon^{\frac{4-\vartheta_3}{3}+\frac{\vartheta_4(2\lambda-1)}{2\lambda+1}} \le \frac{1}{4},$$

we obtain

$$\mathcal{L}^2(t) \le C + \frac{1}{4}\mathcal{L}^{\frac{1}{2\lambda-3}j_2+l_2}(t) + \frac{1}{2}\mathcal{L}^2(t) = C + \frac{3}{4}\mathcal{L}^2(t).$$

Thus

$$\mathcal{L}^2(t) \le C. \tag{2.25}$$

It follows from (2.24) and (2.25) that $\|\nabla u_h(t)\|_2$ is uniformly bounded on $t \in [t_1, t)$ as desired. The proof of Theorem 1.1 is completed. $\square$

Proof of Theorem 1.2

We use two different evaluation methods to prove Theorem 1.2.

The first evaluation method

We will start with the estimate of $\mathcal{J}$. To estimate $\mathcal{J}$ we need to evaluate J_1 and J_2 defined by (2.1). From the estimate (2.2) and using Hölder inequality we get

$$\begin{aligned}\int_{t_1}^{t} J_1 \mathrm{d}\tau &\le C\big\|\partial_3 u\big\|_{\frac{2q}{2q-3},q;t_1,t}\|\nabla u_h\|_{\infty,2;t_1,t}^{\frac{2q-3}{q}}\|\Delta u_h\|_{2,2;t_1,t}^{\frac{3}{q}}\\ &\le C(t-t_1)^{\frac{s-2}{2}}\big\|\partial_3 u\big\|_{\frac{2q}{sq-3},q;t_1,t}\|\nabla u_h\|_{\infty,2;t_1,t}^{\frac{2q-3}{q}}\|\Delta u_h\|_{2,2;t_1,t}^{\frac{3}{q}} \le C\epsilon\mathcal{J}^2(t).\end{aligned}$$

We use the estimate (2.3) to evaluate J_2. Applying the interpolation inequality to obtain

$$\begin{aligned}\|u_3\|_{a,\bar{a};t_1,t} &\le \|u_3\|_{2\lambda,\infty;t_1,t}^{1-\alpha_1}\|u_3\|_{2,\infty;t_1,t}^{\alpha_1} \le \big\||u_3|^{\lambda}\big\|_{2,\infty;t_1,t}^{\frac{1-\alpha_1}{\lambda}}\|u_3\|_{2,\infty;t_1,t}^{\alpha_1}\\ &\le \big\||u_3|^{\lambda}\big\|_{2,\infty;t_1,t}^{\frac{1-\alpha_1}{\lambda}}\|u_0\|_2^{\alpha_1} \lesssim \big\||u_3|^{\lambda}\big\|_{2,\infty;t_1,t}^{\frac{1-\alpha_1}{\lambda}},\end{aligned} \tag{2.26}$$

$$\|\partial_3 u\|_{b,\bar{b}} \le \big\|\partial_3 u\big\|_{q,\frac{2q}{sq-3};t_1,t}^{1-\alpha_2}\|\partial_3 u\|_{2,\infty}^{\alpha_2} \tag{2.27}$$

with

$$\Big(\frac{1}{a},\frac{1}{\bar{a}}\Big) = (1-\alpha_1)\Big(\frac{1}{2\lambda},0\Big) + \alpha_1\Big(\frac{1}{2},0\Big), \quad 0 \le \alpha_1 \le 1 \tag{2.28}$$

and

$$\Big(\frac{1}{b},\frac{1}{\bar{b}}\Big) = (1-\alpha_2)\Big(\frac{1}{q},\frac{sq-3}{q}\Big) + \alpha_2\Big(\frac{1}{2},0\Big), \quad 0 \le \alpha_2 \le 1. \tag{2.29}$$

It follows from (2.4), (2.28), and (2.29) that

$$\frac{1-\alpha_1}{2\lambda} + \frac{\alpha_1}{2} + \frac{1-\alpha_2}{q} + \frac{\alpha_2}{2} = \frac{1}{2} \tag{2.30}$$

and

$$(1-\alpha_2)\frac{sq-3}{2q} = \frac{1}{2}. \tag{2.31}$$

Using the estimates (2.3), (2.26), and (2.27) to obtain

$$\int_{t_1}^{t} J_2 \mathrm{d}\tau \leq C \big\| |u_3|^\lambda \big\|_{2,\infty;t_1,t}^{\frac{1-\alpha_1}{\lambda}} \big\| \partial_3 u \big\|_{q,\frac{2q}{sq-3};t_1,t}^{1-\alpha_2} \| \partial_3 u \|_{2,\infty;t_1,t}^{\alpha_2} \| \Delta u_h, \nabla \partial_3 u \|_{2,2;t_1,t}. \tag{2.32}$$

Thus

$$\mathcal{J}^2(t) \leq C + C\epsilon \mathcal{J}^2(t) + C\epsilon^{1-\alpha_2} \mathcal{J}^{1+\alpha_2}(t) \mathcal{L}^{\frac{1-\alpha_1}{\lambda}}(t). \tag{2.33}$$

To estimate $\mathcal{L}$ we need to evaluate L_1 and L_2 defined by (2.10). Using the Hölder inequality we obtain

$$L_1(t) \leq \|u_h\|_{c,\bar{c},t_1,t} \big\| \partial_3 u \big\|_{q,\frac{2q}{sq-3};t_1,t} \|u_3\|_{d,\bar{d};t_1,t}^{2\lambda-1} \tag{2.34}$$

where

$$(\frac{1}{c}, \frac{1}{\bar{c}}) + (\frac{1}{q}, \frac{sq-3}{2q}) + (2\lambda - 1)(\frac{1}{d}, \frac{1}{\bar{d}}) = (1, 1). \tag{2.35}$$

Invoke the Gagliardo-Nirenberg, Hölder inequalities, and Lemma 1.4 to bound as

$$\begin{aligned} \|u_h\|_{c,\bar{c};t_1,t} &\leq \|u_h\|_{3q,\frac{6q}{qs-3};t_1,t}^{1-\alpha_3} \|\Delta u_h\|_{2,2;t_1,t}^{\alpha_3} \\ &\leq \Big\| \|\partial_3 u_h\|_q^{\frac{1}{3}} \|\nabla_h u_h\|_2^{\frac{2}{3}} \Big\|_{\frac{6q}{qs-3};t_1,t}^{1-\alpha_3} \|\Delta u_h\|_{2,2;t_1,t}^{\alpha_3} \qquad (2.36) \\ &\leq \|\partial_3 u\|_{q,\frac{2q}{sq-3};t_1,t}^{\frac{1-\alpha_3}{3}} \|\nabla u_h\|_{2,\infty;t_1,t}^{2\frac{1-\alpha_3}{3}} \|\Delta u_h\|_{2,2;t_1,t}^{\alpha_3} \qquad (2.37) \end{aligned}$$

where interpolation inequality with

$$(\frac{1}{c}, \frac{1}{\bar{c}}) = (1 - \alpha_3)(\frac{1}{3q}, \frac{sq-3}{6q}) + v_3(\frac{1}{2} - \frac{2}{3}, \frac{1}{2}) \tag{2.38}$$

and

$$0 \leq v_3 \leq 1, c > 0.$$

Applying the interpolation inequality to obtain

$$\|u_3\|_{d,\bar{d};t_1,t} \leq \|u_3\|_{2\lambda,\infty;t_1,t}^{(1-\alpha_4)} \|u_3\|_{2,\infty;t_1,t}^{\alpha_4} \leq \big\| |u_3|^\lambda \big\|_{2,\infty;t_1,t}^{\frac{(1-\alpha_4)}{\lambda}} \|u_0\|_2^{\alpha_4} \lesssim \big\| |u_3|^\lambda \big\|_{2,\infty;t_1,t}^{\frac{(1-\alpha_4)}{\lambda}} \tag{2.39}$$

with

$$(\frac{1}{d}, \frac{1}{\bar{d}}) = (1 - v_4)\big(\frac{1}{2\lambda}, 0\big) + v_4\big(\frac{1}{2}, 0\big), \quad 0 \leq v_4 \leq 1. \tag{2.40}$$

Using the estimates (2.34), (2.37), and (2.39) to obtain

$$L_1(t) \leq C \big\| \partial_3 u \big\|_{q,\frac{2q}{sq-3};t_1,t}^{\frac{4-\alpha_3}{3}} \big\| \nabla u_h \big\|_{2,\infty;t_1,t}^{2\frac{1-\alpha_3}{3}} \|\Delta u_h\|_{2,2;t_1,t}^{\alpha_3} \big\| |u_3|^\lambda \big\|_{2,\infty;t_1,t}^{\frac{(1-\alpha_4)(2\lambda-1)}{\lambda}}.$$

From the estimate (2.15), it follows that

$$\begin{aligned} L_2(t) &\lesssim \big\| |u_3|^\lambda \big\|_{2,\infty;t_1,t}^{\frac{2q-3}{q}} \big\| \nabla(|u_3|^\lambda) \big\|_{2,2;t_1,t}^{\frac{3}{q}} \big\| \partial_3 u \big\|_{q,\frac{2q}{2q-3};t_1,t} \\ &\leq \big\| |u_3|^\lambda \big\|_{2,\infty;t_1,t}^{\frac{2q-3}{q}} \big\| \nabla(|u_3|^\lambda) \big\|_{2,2;t_1,t}^{\frac{3}{q}} (t - t_1)^{\frac{s-2}{2}} \big\| \partial_3 u \big\|_{q,\frac{2q}{sq-3};t_1,t} \qquad (2.41) \\ &\lesssim \epsilon \mathcal{L}^2(t). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{L}^2(t) &\le C + L_1(t) + L_2(t) \\ &\le C + C\epsilon^{\frac{4-\alpha_3}{3}} \mathcal{J}^{\frac{2+\alpha_3}{3}}(t) \mathcal{L}^{\frac{(1-\alpha_4)(2\lambda-1)}{\lambda}}(t) + C\epsilon \mathcal{L}^2(t). \end{aligned} \tag{2.42}$$

It follows from (2.35), (2.38), and (2.40) that

$$\frac{sq-3}{2q}(1+\frac{1-\alpha_3}{3}) + \frac{\alpha_3}{2} = 1 \tag{2.43}$$

and

$$\frac{1-\alpha_3}{3q} - \frac{\alpha_3}{6} + (\frac{1-\alpha_4}{2\lambda} + \frac{\alpha_4}{2})(2\lambda - 1) + \frac{1}{q} = 1. \tag{2.44}$$

From (2.30), (2.31), we can solve

$$\alpha_1 = \frac{3 - qs - 2\lambda + q\lambda}{(-3+qs)(-1+\lambda)}, \quad \alpha_2 = \frac{-3-q+qs}{qs-3} \tag{2.45}$$

and from (2.43), (2.44), we have

$$\alpha_3 = \frac{2(-6-3q+2qs)}{-3-3q+qs}, \quad \alpha_4 = \frac{9+9q-3qs+6q\lambda-4qs\lambda}{3(3+3q-qs)(-1+\lambda)(-1+2\lambda)} \tag{2.46}$$

and

$$c = -\frac{3q(qs-3q-3)}{2q^2 s - 3q^2 + 3qs - 9q - 9}. \tag{2.47}$$

From the above estimates of $\mathcal{J}$ and $\mathcal{L}$ we have

$$\mathcal{J}^2(t) \le C + C\varepsilon \mathcal{J}^2(t) + C\varepsilon^{1-\alpha_2} \mathcal{J}^{j_1}(t) \mathcal{L}^{l_1}(t) \tag{2.48}$$

and

$$\mathcal{L}^2(t) \le C + C\varepsilon^{\frac{4-\alpha_3}{3}} \mathcal{J}^{j_2}(t) \mathcal{L}^{l_2}(t) + C\varepsilon \mathcal{L}^2(t) \tag{2.49}$$

where

$$\begin{aligned} j_1 &= 1 + \alpha_2, \quad & l_1 &= \tfrac{1-\alpha_1}{\lambda} \\ j_2 &= \tfrac{2+\alpha_3}{3}, \quad & l_2 &= \tfrac{(1-\alpha_4)(2\lambda-1)}{\lambda}. \end{aligned} \tag{2.50}$$

If $\alpha_2 < 1$ then $1 \le j_1 < 2$, and we may apply Hölder inequality to (2.48)

$$\mathcal{J}^2(t) \le C + C\varepsilon \mathcal{J}^2(t) + \frac{1}{2}\mathcal{J}^2(t) + C\mathcal{L}^{\frac{2l_1}{2-j_1}}(t).$$

Now choose $0 < \varepsilon \ll 1$ sufficiently small such that

$$C\varepsilon \le \frac{1}{4},$$

we have

$$\mathcal{J}(t) \le C + C\mathcal{L}^{\frac{2l_1}{2-j_1}}(t). \tag{2.51}$$

Plugging (2.51) into (2.49), and choosing ε such that

$$C\varepsilon^{\frac{4-\alpha_3}{3}} \le \frac{1}{4},$$

we obtain

$$\mathcal{L}^2(t) \le C + \frac{1}{4}\mathcal{L}^{\frac{2l_1}{2-j_1}j_2+l_2}(t) + \frac{1}{2}\mathcal{L}^2(t) = C + \frac{1}{4}\mathcal{L}^{\gamma}(t) + \frac{1}{2}\mathcal{L}^2(t) \tag{2.52}$$

where

$$\gamma = \frac{l-\alpha_1}{\lambda(1-\alpha_2)}\frac{2+\alpha_3}{3} + \frac{(1-\alpha_4)(2\lambda-1)}{\lambda}. \tag{2.53}$$

Plugging (2.45) and (2.46) into (2.53) to obtain

$$\gamma = \frac{q^2\left(-18\lambda+6s^2+(6\lambda-31)s+45\right)-3q(6\lambda+8s-19)+18}{3(\lambda-1)q(q(s-3)-3)}. \tag{2.54}$$

If $\gamma \le 2$ then from (2.52) we deduce that

$$\mathcal{L}^2(t) \le C. \tag{2.55}$$

It follows from (2.51) and (2.55) that $\|\nabla u_h(t)\|_2$ is uniformly bounded on $t \in [t_1, t)$ as desired. To finish the proof we need to discuss the range of s, q, and λ for which our proof works. Reducing $0 \le \alpha_1 \le 1, 0 \le \alpha_2 < 1, 0 \le \alpha_3 \le 1, 0 \le \alpha_4 \le 1, c > 0$, $\frac{11}{6} < s \le 2$, $q \ge 6, \lambda > \frac{3}{2}, \gamma \le 2$ yields

$$\frac{11}{6} < s \le \frac{19}{10}, \frac{3}{2}\sqrt{\frac{s^2+2s-3}{(2s-3)^2}} - \frac{3(s-3)}{2(2s-3)} < q \le \frac{6}{2s-3}, \frac{qs-3}{q-2} \le \lambda \le \frac{-3qs+9q+9}{4qs-6q}$$

or

$$\frac{19}{10} < s \le 2,\ 6 \le q \le \frac{6}{2s-3},\ \frac{qs-3}{q-2} \le \lambda \le \frac{-3qs+9q+9}{4qs-6q}.$$

Therefore the range of s and q for which our proof works is

$$\frac{11}{6} < s < \frac{19}{10},\ \frac{3}{2}\sqrt{\frac{s^2+2s-3}{(2s-3)^2}} - \frac{3(s-3)}{2(2s-3)} < q < \frac{6}{2s-3} \tag{2.56}$$

or

$$\frac{19}{10} \le s \le 2,\ 6 \le q \le \frac{6}{2s-3}. \tag{2.57}$$

The second evaluation method

We still use (2.48) for the estimate of $\mathcal{J}$, where α_1 and α_2 are given by (2.45) but we use another way for estimate of $\mathcal{L}$. To estimate $\mathcal{L}$ we need to evaluate L_1 and L_2 defined by (2.10). We use the estimate (2.34) to evaluate L_1. Invoke the Gagliardo-Nirenberg, Hölder inequalities, and Lemma 1.4 to bound as

$$\|u_h\|_{c,\bar{c};t_1,t} \le \|u_h\|^{1-\alpha_3}_{3q,\frac{6q}{qs-3};t_1,t}\|\nabla u_h\|^{\alpha_3}_{2,\infty;t_1,t}$$

$$\le \left\|\|\partial_3 u_h\|_q^{\frac{1}{3}}\|\nabla_h u_h\|_2^{\frac{2}{3}}\right\|^{1-\alpha_3}_{\frac{6q}{qs-3};t_1,t}\|\nabla u_h\|^{\alpha_3}_{2,\infty;t_1,t} \tag{2.58}$$

$$\le \|\partial_3 u\|^{\frac{1-\alpha_3}{3}}_{q,\frac{2q}{sq-3};t_1,t}\|\nabla u_h\|^{2\frac{1-\alpha_3}{3}}_{2,\infty;t_1,t}\|\nabla u_h\|^{\alpha_3}_{2,\infty;t_1,t} = \|\partial_3 u\|^{\frac{1-\alpha_3}{3}}_{q,\frac{2q}{sq-3};t_1,t}\|\nabla u_h\|^{\frac{2+\alpha_3}{3}}_{2,\infty;t_1,t} \tag{2.59}$$

where interpolation inequality with

$$\left(\frac{1}{c}, \frac{1}{\bar{c}}\right) = (1-\alpha_3)\left(\frac{1}{3q}, \frac{sq-3}{6q}\right) + v_3\left(\frac{1}{2}-\frac{1}{3}, 0\right), \quad 0 \le v_3 \le 1. \tag{2.60}$$

From the estimates (2.34), (2.39) and (2.59) estimates we have

$$L_1(t) \le C\big\|\partial_3 u\big\|_{q,\frac{2q}{sq-3};t_1,t}^{\frac{4-\alpha_3}{3}}\big\|\nabla u_h\big\|_{2,\infty;t_1,t}^{\frac{2+\alpha_3}{3}}\big\||u_3|^\lambda\big\|_{2,\infty;t_1,t}^{\frac{(1-\alpha_4)(2\lambda-1)}{\lambda}}.$$

Using the above estimate and (2.41)

$$\begin{aligned}\mathcal{L}^2(t) &\le C + L_1(t) + L_2(t)\\ &\le C + C\epsilon^{\frac{4-\alpha_3}{3}}\mathcal{J}^{\frac{2+\alpha_3}{3}}(t)\mathcal{L}^{\frac{(1-\alpha_4)(2\lambda-1)}{\lambda}}(t) + C\epsilon\mathcal{L}^2(t).\end{aligned}\tag{2.61}$$

It follows from (2.35), (2.40), and (2.60) that

$$\frac{1-\alpha_3}{3q} + \alpha_3(\frac{1}{2}-\frac{1}{3}) + (\frac{1-\alpha_4}{2\lambda}+\frac{\alpha_4}{2})(2\lambda-1) + \frac{1}{q} = 1\tag{2.62}$$

and

$$\frac{sq-3}{2q}(1+\frac{1-\alpha_3}{3}) = 1.\tag{2.63}$$

Solving (2.62) and (2.63) to obtain

$$\alpha_3 = \frac{2(-6-3q+2qs)}{-3+qs},\quad \alpha_4 = \frac{-9+3qs+6q\lambda-4qs\lambda}{3(-3+qs)(-1+\lambda)(-1+2\lambda)}.\tag{2.64}$$

Proceeding $\mathcal{J}$ and $\mathcal{L}$ as the first evaluation method, if $\alpha_2 < 1$ and choose $0 < \varepsilon \ll 1$ sufficiently small then

$$\mathcal{L}^2(t) \le C + \frac{1}{4}\mathcal{L}^\gamma(t) + \frac{1}{2}\mathcal{L}^2(t)$$

where γ is given by (2.53), plugging (2.45) and (2.64) into (2.53) to obtain

$$\gamma = \frac{-6\lambda q - 6qs + 17q + 6}{3q - 3\lambda q}.$$

If $\gamma \le 2$ then we deduce that $\mathcal{L}^2(t) \le C$ and $\|\nabla u_h(t)\|_2$ is uniformly bounded on $t \in [t_1, t)$ as desired. Reducing $0 \le \alpha_1 \le 1, 0 \le \alpha_2 < 1, 0 \le \alpha_3 \le 1, 0 \le \alpha_4 \le 1$, $\frac{11}{6} < s \le 2$, $q \ge 6, \lambda > \frac{3}{2}, \gamma \le 2$ yields

$$\frac{11}{6} \le s \le 2, \frac{6}{2s-3} \le q \le \frac{6}{6s-11}, \frac{qs-3}{q-2} \le \lambda \le \frac{3qs-9}{4qs-6q}.$$

Therefore the range of s and q for which our proof works is

$$\frac{11}{6} \le s \le 2, \frac{6}{2s-3} \le q \le \frac{6}{6s-11}.\tag{2.65}$$

We get from (2.56), (2.57), and (2.65) the conditions (1.6) and (1.7) for which u is regular on $(0, T)$. The proof of Theorem 1.2 is completed. □

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References

[1] H. Beirão da Veiga, *A new regularity class for the Navier-Stokes equations in* $\mathbb{R}^n$. Chin. Ann. Math., Ser. B 16 (1995), no. 4, 407-412.

[2] C. Cao, *Sufficient conditions for the regularity to the 3D Navier–Stokes equations*, Discrete Contin. Dyn. Syst. 26 (2010) 1141–1151.

[3] L. Escauriaza, G. A. Serëgin, V. Šverák, $L_{3,\infty}$ *solution of Navier-Stokes equations and backward uniqueness*. Russian Math. Surveys 58 (2003), 211-250.

[4] I. Kucavica, M. Ziane, *Navier-Stokes equations with regularity in one direction*. J. Math. Phys. 48 (2007), no. 6, 10 pp.

[5] J. Leray, *Sur le mouvement dun liquide visqueux emplissant lespace*. Acta Math. 63 (1934) 193-248.

[6] Y. Namlyeyeva, Z. Skalak *The optimal regularity criterion for the Navier-Stokes equations in terms of one directional derivative of the velocity*. ZAMM 100 (2019) 17.

[7] J. Neustupa, A. Novotný, P. Pene, *An interior regularity of a weak solution to the Navier-Stokes equations in dependence on one component of velocity*. Topics in mathematical fluid mechanics, 163-183, Quad. Mat., 10, Dept. Math., Seconda Univ. Napoli, Caserta, 2002.

[8] P. Penel, M. Pokorny, *Some new regularity criteria for the Navier–Stokes equations containing gradient of the velocity*, Appl. Math. 49 (2004) 483–493.

[9] M. Pokorný, *On the result of He concerning the smoothness of solutions to the Navier-Stokes equations*. Electron. J. Differential Equations 2003, no. 11 , 1-8.

[10] J. Serrin, *The initial value problem for the Navier-Stokes equations*. Nonlinear Problem, Proc. Symposium, Madison, Wisconsin, University of Wisconsin Press, Madison, Wisconsin (1963), pp. 69 -98.

[11] Z. Skalak, *On the regularity of the solutions to the Navier-Stokes equations via the gradient of one velocity component*. Nonlinear Anal. 104 (2014), 84 -89.

[12] Z. Skalak, *A note on a regularity criterion for the Navier–Stokes equations*, Ann. Polon. Math. 122 (2019) 193–199.

[13] Z. Skalak, *The end-point regularity criterion for the Navier–Stokes equations in terms of* $\partial_3 u$, Nonlinear Analysis RWA 55(2020), 103120.

[14] Z. Skalak, *A note on the regularity of the solutions to the Navier-Stokes equations via the gradient of one velocity component*. J. Math. Phys. 55 (2014), no. 12, 121506, 6 pp.

[15] Z. Skalak, *A note on a regularity criterion for the Navier–Stokes equations*, Ann. Polon. Math. 122 (2019) 193–199.

[16] Z. Skalak, *An optimal regularity criterion for the Navier-Stokes equations proved by a blow-up argument*. Nonlinear Anal. 58 (2021), 103207

[17] H. Sohr, *The Navier-Stokes Equations*. An elementary function analytic Approach, Birkhäuser Verlag, Basel, Boston, Berlin, (2001).

[18] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotrosi*. Ric. Mat. 18 (1969), 3-24.

[19] Z. Zhang, *An improved regularity criterion for the Navier-Stokes equations in terms of one directional derivative of the velocity field*. Bull. Math. Sci. 8 (2018), no. 1, 33-47.

[20] Z. Zhang, W. Yuan, Ganzhou, Y. Zhou, Zhuhai *Some remarks on the Navier-Stokes with regularity in one direction*. Applications of Mathematics. 64 (2019), no. 3, 301-308.

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