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Optimal Economic Growth Models with Nonlinear Utility Functions*

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Abstract We study a class of finite horizon optimal economic growth problems with nonlinear utility functions and linear production functions. By using a maximum principle in the optimal control theory and employing the special structure of the problems, we are able to explicitly describe the unique solution via input parameters. Economic interpretations of the obtained results and an open problem about the case where the total factor productivity falls into a bounded open interval defined by the growth rate of labor force, the real interest rate, and the exponent of the utility function are also expressed.

Keywords Optimal economic growth · optimal control · maximum principle

Mathematics Subject Classification (2000) 91B62 · 49J15 · 37N40 · 46N10 · 91B55

1 Introduction

Models of economic growth allow ones to analyze, plan, and predict relations among global factors, which include capital, labor force, production technology, and national product, of a particular economy in a given planning time interval. The interval can be finite or infinite as well. One major issue regarding a growth of an economy is the *optimal economic growth problem*, which asks to define the amount of saving at each time moment *to maximize a certain target of consumption satisfaction* while fulfilling the given economic relations. Classical concepts and results on economic growth can be found in the works

* Dedicated to Professor Pham Huu Sach on the occasion of his 80th birthday.

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of Ramsey [1], Harrod [2], Domar [3], Solow [4], Swan [5], Cass [6], and Koopmans [7]. For subsequent studies on economic growth models, the interested reader is referred to [8–14] and the references therein.

Following Takayama [8, Chapter 5], Huong [15] has considered the problem of *optimal growth of an aggregative economy* and proved several theorems on the solution existence of finite horizon optimal economic growth problems. The solution existence is established in [15] not only for general problems but also for typical ones with the production function and the utility function being either the AK function or the Cobb–Douglas one. Interestingly, focusing on the linearity/nonlinearity of the functions involved, Huong [15, Section 4] has shown that the typical optimal economic growth problems can be classified into four types. This classification is very useful, because it puts the optimal control problems into different groups depending on the levels of difficulty. Since all the problems have global solutions, it is natural to try to find the solutions, i.e., to get a synthesis of the optimal processes based on the given data set. Results in this direction would greatly enrich our knowledge on economic growth models and certainly have profound economic meanings. It turns out that the desired syntheses have no analogues in the literature and they are not easy to be obtained.

Assuming that both the production function and the utility function are linear, Huong [16] and Huong *et al.* [17] have solved the optimal economic growth problem with six parameters formulated in [15] by using a maximum principle for optimal control problems. The next natural question arises: *Are there some analogues of the results of* [16, 17] *for optimal economic growth problems with nonlinear utility functions and linear production functions, or not?* The present paper aims at solving this question. In other words, we are interested in studying a class of finite horizon optimal economic growth problems with nonlinear utility functions and linear production functions. Each problem from the class depends on seven parameters.

Thanks to the maximum principle in [18, Theorem 6.2.1] and a series of seven technical lemmas employing the special structure of the problems, we are able to explicitly describe the unique solution of the problem, provided that some additional conditions on the input parameters are fulfilled. Like the preceding syntheses of optimal solutions given in [16, 17], our results have very clear economic interpretations.

There are certain syntheses of optimal solutions for optimal economic growth problems in both abstract and specified contexts; see, e.g., [13, 14, 10, 11]. But, our synthesis of optimal solutions are presented by explicit formulas, while those in the just-mentioned papers are in a feedback form via the solution of the Hamilton-Jacobi-Bellman equation associated with the given problem.

It it worthy to stress that the economic meanings of *all* the concepts, variables, functions, parameters, and relations concerning the finite horizon optimal economic growth problems and the transformation of the latter to constrained optimal control problems were explained thoroughly in Section 2 of [15]. Hence, in this paper, only brief explanations of the economic meanings of the concepts in question will be given.

The paper is organized as follows. Section 2 describes the optimal economic growth models of interest. Section 3 recalls a maximum principle from [18], which is the main tool in our study. The main results are obtained in Section 4, while the relevant economic interpretations and an open problem are given in Section 5. The last section is devoted to several concluding remarks.

2 Optimal Economic Growth Models

By $\mathbb{R}$ (resp., $\mathbb{R}_+$) we denote the set of real numbers (resp., the set of nonnegative real numbers). The Euclidean norm in the n-dimensional space $\mathbb{R}^n$ is denoted by $\|.\|$. By $\mathbb{R}^n_+$ we denote the nonnegative orthant in $\mathbb{R}^n$. The open ball (resp., the closed ball) centered at $a \in \mathbb{R}^n$ with radius $\mu > 0$ is denoted by $B(a,\mu)$ (resp., $\bar{B}(a,\mu)$). The *Sobolev space* $W^{1,1}([t_0,T],\mathbb{R}^n)$ (see, e.g., [19, p. 21]) is the linear space of the *absolutely continuous functions* $x : [t_0,T] \to \mathbb{R}^n$ equipped with the norm $\|x\|_{W^{1,1}} = \|x(t_0)\| + \int_{t_0}^T \|\dot{x}(t)\|dt$.

The interested reader is referred to the books [8, Chapter 5], [20, Chapters 5, 7, 10, and 11], [21, Chapter 20], and [12, Chapters 7 and 8] for systematical expositions of optimal economic growth models. Note that the problem of *optimal growth of an aggregative economy* with all the related economic concepts is presented in detail in [15, Subsection 2.1] and recalled in Huong, Yao and Yen [17, Section 2]. In the sequel, $k(t)$ and $s(t)$ respectively are the *capital-to-labor ratio* and the *propensity to save* at a time moment t in the *planning interval* $[t_0,T]$. The values $\phi(k)$, $k \geq 0$, of the *per capita production function* $\phi(\cdot)$ express the *outputs per capita*. The *utility function* $\omega(\cdot)$ depends on the variable c, which is the *per capita consumption*. The problem is as follows:

$$\text{Maximize } \ I(k,s) := \int_{t_0}^{T} \omega[(1-s(t))\phi(k(t))]e^{-\lambda t}dt \tag{1}$$

over $k \in W^{1,1}([t_0,T],\mathbb{R})$ and measurable functions $s : [t_0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{k}(t) = s(t)\phi(k(t)) - \sigma k(t), & \text{a.e. } t \in [t_0,T] \\ k(t_0) = k_0 \\ s(t) \in [0,1], & \text{a.e. } t \in [t_0,T] \\ k(t) \geq 0, & \forall t \in [t_0,T], \end{cases} \tag{2}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\omega : \mathbb{R}_+ \to \mathbb{R}$ are given functions, $0 \leq t_0 < T < +\infty$, $\lambda \geq 0$, $\sigma > 0$, and $k_0 > 0$ are given as parameters. The problem (1)–(2) is denoted by (GP).

The integral $\displaystyle\int_{t_0}^{T} \omega(c(t))e^{-\lambda t}dt$, where $c(t)$ is the *per capita consumption* of the economy at time t and $\lambda \geq 0$ is the *real interest rate*, represents the total amount of the utility gained by the society on the time period $[t_0,T]$. Since $c(t) = (1-s(t))\phi(k(t))$ (see [15, equation (9)]) for all $t \geq 0$, the just-mentioned integral equals to the value $I(k,s)$ defined in (1). The parameters σ and k_0 in (2) respectively stand for the *growth rate of labor force* and the *initial capital-to-labor ratio*.

According to [22], (GP) is a *finite horizon optimal control problem of the Lagrange type*, where s and k play the roles of *control variable* and *state variable*, respectively. Besides, due to the appearance of the *state constraint* $k(t) \geq 0$ for $t \in [t_0,T]$, (GP) belongs to the class of *optimal control problems with state constraints*.

Several results on the existence of global solutions $(\bar{k},\bar{s})$ for (GP) were obtained in [15] under some mild conditions on $\phi(\cdot)$ and $\omega(\cdot)$. In addition, the solution existence for the problem (GP) with $\phi(k) = Ak^\alpha$ and $\omega(c) = c^\beta$, where $A > 0$ and $\alpha, \beta \in]0,1]$ are parameters, was also considered in that paper. Economically, the choosing of α and β either in $]0,1[$ or equal to 1 expresses typical problems with the production function and the utility function being either the *Cobb–Douglas function* or the *AK function* (see, e.g., [9] and [8]). Meanwhile, as in [15], we can mathematically classify these typical problems into four types depending on the displacement of α and β on $]0,1]$:

(GP_1) $\phi(k) = Ak$ and $\omega(c) = c$ (the "linear-linear" problem: both per capita production function and utility functions are linear);

(GP_2) $\phi(k) = Ak$ and $\omega(c) = c^\beta$ with $\beta \in]0,1[$ (the "linear-nonlinear" problem: the per capita production function is linear, but the utility function is nonlinear);

(GP_3) $\phi(k) = Ak^\alpha$ and $\omega(c) = c$ with $\alpha \in]0,1[$ (the "nonlinear-linear" problem: the per capita production function is nonlinear, but the utility function is linear);

(GP_4) $\phi(k) = Ak^\alpha$ and $\omega(c) = c^\beta$ with $\alpha \in]0,1[$ and $\beta \in]0,1[$ (the "nonlinear-nonlinear" problem: both the per capita production function and the utility function are nonlinear).

By [15, Theorem 4.1], we know that any problem belonging to one of these classes has a global solution. The above classification arranges the difficulties of solving the four types of problems into a reasonable scale. Clearly, the first problem is the easiest one, while the fourth one is the most difficult.

If the data triple A, σ, λ satisfy certain strict linear inequalities, then explicit formulas for the unique global solution of the problem (GP_1) can be given. For more details, the reader is referred to [16] and [17].

We will focus on solving (GP_2), which is considered as a parametric problem depending on seven parameters: $A > 0$, $\beta \in]0,1[$, $\sigma > 0$, $\lambda \geq 0$, $k_0 > 0$, and t_0, T with $0 \leq t_0 < T$. So, we will have deal with the problem

$$\text{Maximize} \quad \int_{t_0}^{T} [1 - s(t)]^\beta k^\beta(t) e^{-\lambda t} dt \tag{3}$$

over $k \in W^{1,1}([t_0,T], \mathrm{I\!R})$ and measurable functions $s : [t_0,T] \to \mathrm{I\!R}$ satisfying

$$\begin{cases} \dot{k}(t) = (As(t) - \sigma)k(t), & \text{a.e. } t \in [t_0,T] \\ k(t_0) = k_0 \\ s(t) \in [0,1], & \text{a.e. } t \in [t_0,T] \\ k(t) \geq 0, & \forall\, t \in [t_0,T], \end{cases} \tag{4}$$

which has been denoted by (GP_2).

The following theorem is a special case of [15, Theorem 4.1] and [16, Theorem A.3].

Theorem 2.1 *For any $A > 0$, $\beta \in]0,1[$, $\sigma > 0$, $\lambda \geq 0$, $k_0 > 0$, and and t_0, T with $0 \leq t_0 < T$, (GP_2) has a global solution.*

3 A Maximum Principle

Now we present a simplified version of a maximum principle given in the book of Vinter [18]. Consider the following *finite horizon optimal control problem of the Mayer type*, denoted by $\mathcal{M}$,

$$\text{Minimize} \quad g(x(t_0), x(T)) \tag{5}$$

over $x \in W^{1,1}([t_0,T], \mathrm{I\!R}^n)$ and measurable functions $u : [t_0,T] \to \mathrm{I\!R}^m$ satisfying

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & \text{a.e. } t \in [t_0,T] \\ (x(t_0), x(T)) \in C \\ u(t) \in U(t), & \text{a.e. } t \in [t_0,T] \\ h(t, x(t)) \leq 0, & \forall t \in [t_0,T], \end{cases} \tag{6}$$

where $[t_0, T]$ is a given interval, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $h : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}$ are given functions, $C \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed set, and $U : [t_0, T] \rightrightarrows \mathbb{R}^m$ is a set-valued map.

A measurable function $u : [t_0, T] \to \mathbb{R}^m$ satisfying $u(t) \in U(t)$ for almost every $t \in [t_0, T]$ is called a *control function.* A *process* (x, u) consists of a control function u and an arc $x \in W^{1,1}([t_0, T]; \mathbb{R}^n)$ that is a solution to the differential equation in (6). A *state trajectory* x is the first component of some process (x, u). A process (x, u) is called *feasible* if the state trajectory satisfies the *endpoint constraint* $(x(t_0), x(T)) \in C$ and the *state constraint* $h(t, x(t)) \leq 0$, $t \in [t_0, T]$.

Due to the appearance of the *state constraint* $h(t, x(t)) \leq 0, t \in [t_0, T]$, the problem $\mathcal{M}$ in (5)–(6) is said to be an *optimal control problem with state constraints.* But, if the state constraint is fulfilled for any state trajectory x with $(x(t_0), x(T)) \in C$, i.e., the state constraint can be removed from (6), then one says that $\mathcal{M}$ an *optimal control problem without state constraints.*

Definition 3.1 A feasible process $(\bar{x}, \bar{u})$ is called a $W^{1,1}$ *local minimizer* for $\mathcal{M}$ if there exists $\delta > 0$ such that $g(\bar{x}(t_0), \bar{x}(T)) \leq g(x(t_0), x(T))$ for any feasible process (x, u) satisfying $\|\bar{x} - x\|_{W^{1,1}} \leq \delta$.

Definition 3.2 The *Hamiltonian* $\mathcal{H} : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ of (6) is defined by

$$\mathcal{H}(t, x, p, u) := \langle p, f(t, x, u) \rangle = \sum_{i=1}^{n} p_i f_i(t, x, u). \tag{7}$$

Theorem 6.2.1 from [18] is a maximum principle for nonsmooth problems, where the limiting normal cone and the limiting subdifferential are used. It is well known that for continuously differentiable functions, the limiting subdifferential is a singleton consisting of the gradient of the function at the point in question. Also, for convex sets, the limiting normal cone coincides with the normal cone of the set in question. We refer to [23, 24] for comprehensive treatments of the limiting normal cone, the limiting subdifferential, and the related calculus rules. To analyze the problem (GP_2), we need the next smooth version of [18, Theorem 6.2.1].

Theorem 3.1 *Suppose that $\mathcal{M}$ is an optimal control problem without state constraints, C is a closed convex set, and $U(t) = U$ for all $t \in [t_0, T]$ with U being a compact subset of $\mathbb{R}^m$. Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$ local minimizer for $\mathcal{M}$. Assume that*

(A1) *The function $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and, for some $\delta > 0$, $f(t, \cdot, u)$ is continuously differentiable on the ball $B(\bar{x}(t), \delta)$ for all $(t, u) \in [t_0, T] \times U$, and the derivative $f_x(t, x, u)$ of $f(t, \cdot, u)$ at x is continuous on the set of vectors (t, x, u) satisfying $(t, u) \in [t_0, T] \times U$ and $x \in B(\bar{x}(t), \delta)$;*
(A2) *g is continuously differentiable.*

Then there exist $p \in W^{1,1}([t_0, T]; \mathbb{R}^n)$ and $\gamma \geq 0$ such that $(p, \gamma) \neq (0, 0)$ and the following holds true:

(i) $-\dot{p}(t) = \mathcal{H}_x(t, \bar{x}(t), p(t), \bar{u}(t))$ *a.e., where $\mathcal{H}_x(t, \bar{x}(t), p(t), \bar{u}(t))$ means the derivative of the function $\mathcal{H}(t, \cdot, p(t), \bar{u}(t))$ at $\bar{x}(t)$;*
(ii) $(p(t_0), -p(T)) \in \gamma \nabla g(\bar{x}(t_0), \bar{x}(T)) + N_C(\bar{x}(t_0), \bar{x}(T))$, *where*

$$N_C(\bar{z}) := \left\{ z^* \in \mathbb{R}^{2n} \ : \ \langle z^*, z - \bar{z} \rangle \ \forall z \in C \right\}$$

is the normal cone to C at a point $\bar{z} \in C$;

(iii) $\mathcal{H}(t, \bar{x}(t), p(t), \bar{u}(t)) = \max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), p(t), u)$ *a.e.*.

Let us show that the assumptions made in Theorem 3.1 guarantee that the hypothesis (H1) in [18, Theorem 6.2.1] is fulfilled. To do so, take any $\delta' \in]0, \delta[$ and note that the set of all (t, x, u) satisfying $(t, u) \in [t_0, T] \times U$ and $x \in \bar{B}(\bar{x}(t), \delta')$, denoted by $\mathcal{A}$, is compact. Hence, assumption (A1) in Theorem 3.1 implies that the number $\gamma = \max\{\|f_x(t, x, u)\| \, : \, (t, x, u) \in \mathcal{A}\}$ is well defined. By the mean value theorem for vector-valued functions (see, e.g., [19, p. 27]) we have

$$\|f(t, x, u) - f(t, x', u)\| \leq \gamma \|x - x'\|, \ \ \forall t \in [t_0, T], \, x, x' \in \bar{B}(\bar{x}(t), \delta'), \, u \in U.$$

Thus, condition (H1) in [18, Theorem 6.2.1] is satisfied.

4 Partial Synthesis of the Optimal Processes

In order to apply Theorem 3.1 for finding optimal processes for (GP_2), we have to interpret the Lagrange problem (GP_2) in the form of the Mayer problem $\mathcal{M}$. To do so, for $t \in [0, T]$, we set $x(t) = (x_1(t), x_2(t))$, where $x_1(t)$ plays the role of $k(t)$ in (3)–(4) and

$$x_2(t) := -\int_{t_0}^{t} [1 - s(\tau)]^{\beta} x_1^{\beta}(\tau) e^{-\lambda \tau} d\tau \tag{8}$$

Thus, (GP_2) is equivalent to the following problem, denoted by (GP_{2a}):

$$\text{Minimize } \ x_2(T)$$

over $x = (x_1, x_2) \in W^{1,1}([t_0, T], \mathbb{R}^2)$ and measurable functions $s : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}_1(t) = (As(t) - \sigma) x_1(t), & \text{a.e. } t \in [t_0, T] \\ \dot{x}_2(t) = -[1 - s(t)]^{\beta} x_1^{\beta}(t) e^{-\lambda t}, & \text{a.e. } t \in [t_0, T] \\ (x(t_0), x(T)) \in \{(k_0, 0)\} \times \mathbb{R}^2 & \\ s(t) \in [0, 1], & \text{a.e. } t \in [t_0, T] \\ x_1(t) \geq 0, & \forall t \in [t_0, T]. \end{cases} \tag{9}$$

To see (GP_{2a}) in the form of the Mayer problem $\mathcal{M}$, we choose $n = m = 1$, $C = \{(k_0, 0)\} \times \mathbb{R}^2$, $U(t) = [0, 1]$ for $t \in [t_0, T]$, $g(x, y) = y_2$ for $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, $h(t, x) = -x_1$ for $(t, x) \in [t_0, T] \times \mathbb{R}^2$. Besides, we let $f(t, x, s) := ((Ax_1 s - \sigma) x_1, -|1 - s|^{\beta} |x_1|^{\beta} e^{-\lambda t})$ for $(t, x, s) \in [t_0, T] \times \mathbb{R}^2 \times \mathbb{R}$. Then, $f(.)$ is well-defined on $[t_0, T] \times \mathbb{R}^2 \times \mathbb{R}$ and

$$f(t, x, s) = ((Ax_1 s - \sigma) x_1, -(1 - s)^{\beta} x_1^{\beta} e^{-\lambda t}) \tag{10}$$

for every $(t, x, s) \in [t_0, T] \times (\mathbb{R}_+ \times \mathbb{R}) \times [0, 1]$. Moreover, by (7), the Hamiltonian of (GP_{2a}) is given by $\mathcal{H}(t, x, p, s) = (As - \sigma) x_1 p_1 - |1 - s|^{\beta} |x_1|^{\beta} p_2 e^{-\lambda t}$ for all $(t, x, p, s) \in [t_0, T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$. In particular,

$$\mathcal{H}(t, x, p, s) = (As - \sigma) x_1 p_1 - (1 - s)^{\beta} x_1^{\beta} p_2 e^{-\lambda t} \tag{11}$$

for each $(t, x, p, s) \in [t_0, T] \times (\mathbb{R}_+ \times \mathbb{R}) \times \mathbb{R}^2 \times [0, 1]$. Thus, at such a point (t, x, p, s) with $x_1 > 0$, $\mathcal{H}(\cdot)$ is continuously differentiable in x and, one has

$$\mathcal{H}_x(t, x, p, u) = \big\{ ((As - \sigma) p_1 - \beta (1 - s)^{\beta} x_1^{\beta - 1} p_2 e^{-\lambda t}, 0) \big\}. \tag{12}$$

The relationships between a control function $s(\cdot)$ and the corresponding trajectory $x(\cdot)$ of (9) can be described as follows.

Lemma 4.1 *For each measurable function $s(\cdot) : [t_0, T] \to \mathbb{R}$ with $s(t) \in [0, 1]$, there exists a unique function $x = (x_1, x_2) \in W^{1,1}([t_0, T], \mathbb{R}^2)$ such that (x, s) is a feasible process of (GP_{2a}). Moreover, for every $\tau \in [t_0, T]$, one has*

$$x_1(t) = x_1(\tau)e^{\int_\tau^t (As(z)-\sigma)dz}, \quad \forall t \in [t_0, T]. \tag{13}$$

In particular, $x_1(t) > 0$ for all $t \in [t_0, T]$.

Proof Given a function $s(\cdot)$ satisfying the assumptions of the lemma, consider the Cauchy problem of finding absolutely functions $x_1(\cdot) : [t_0, T] \to \mathbb{R}$ such that

$$\begin{cases} \dot{x}_1(t) = [As(t) - \sigma]x_1(t), & \text{a.e. } t \in [t_0, T] \\ x_1(t_0) = k_0. \end{cases} \tag{14}$$

Since $s(\cdot)$ is measurable and bounded on $[t_0, T]$, so is $t \mapsto As(t) - \sigma$. In particular, the latter is Lebesgue integrable on $[t_0, T]$. Hence, by the lemma in [25, pp. 121–122] on the solution existence and uniqueness of the Cauchy problem for linear differential equations, one knows that (14) has a unique solution. Thus, $x_1(\cdot)$ is defined uniquely via $s(\cdot)$. This and (8) imply the uniqueness of the absolutely continuous function $x_2(\cdot) : [t_0, T] \to \mathbb{R}$ such that (x, s) is a feasible process of (GP_{2a}).

To prove the second assertion, put

$$\Omega(t, \tau) = e^{\int_\tau^t (As(z)-\sigma)dz}, \quad \forall t, \tau \in [t_0, T]. \tag{15}$$

By the Lebesgue integrability of the function $t \mapsto As(t) - \sigma$ on $[t_0, T]$, $\Omega(t, \tau)$ is well defined on $[t_0, T] \times [t_0, T]$, and by [29, Theorem 8, p. 324] one has

$$\frac{d}{dt}\left(\int_\tau^t (As(z) - \sigma)dz\right) = As(t) - \sigma, \quad \text{a.e. } t \in [t_0, T]. \tag{16}$$

Therefore, from (15) and (16) it follows that $\Omega(\cdot, \tau)$ is the solution of the Cauchy problem

$$\begin{cases} \frac{d}{dt}\Omega(t, \tau) = (As(t) - \sigma)\Omega(t, \tau), & \text{a.e. } t \in [t_0, T] \\ \Omega(\tau, \tau) = 1. \end{cases}$$

In other words, the real-valued function $\Omega(t, \tau)$ of the variables t and τ is the *principal matrix solution* (see [25, p. 123]) specialized to the homogeneous differential equation in (14). Hence, by the theorem in [25, p. 123] on the solution of linear differential equations, we obtain (13). Since $x_1(t_0) = k_0 > 0$, applying (13) for $\tau = t_0$ implies that $x_1(t) > 0$ for all $t \in [t_0, T]$. □

By Lemma 4.1, any process satisfying the first four conditions in (9) automatically satisfies the state constraint $x_1(t) \geq 0$, $t \in [t_0, T]$. Thus, the latter can be omitted in the problem formulation. This means that now we can apply the maximum principle for optimal control problems without state constraints in Theorem 3.1 to derive necessary optimality conditions for (GP_{2a}).

From now on, let $(\bar{x}, \bar{s})$ be a fixed $W^{1,1}$ local minimizer for (GP_{2a}).

By Lemma 4.1, one can find a constant $\bar{\varepsilon} > 0$ such that $\bar{x}_1(t) \geq \bar{\varepsilon}$ for all $t \in [t_0, T]$. Since (10) is valid for any $(t, x, s) \in [t_0, T] \times (\mathbb{R}_+ \times \mathbb{R}) \times [0, 1]$, we can find some $\delta > 0$ such that the assumption (A1) of Theorem 3.1 is satisfied. The fulfillment of (A2) is obvious. So, by Theorem 3.1 we can find $p \in W^{1,1}([t_0, T]; \mathbb{R}^2)$ and $\gamma \geq 0$ such that $(p, \gamma) \neq (0, 0)$ and the conditions (i)–(iii) hold true.

For (GP_{2a}), the conditions (i)–(iii) in Theorem 3.1 imply the following.

Condition (i): By (12), (i) implies that

$$-\dot{p}(t) = ((A\bar{s}(t)-\sigma)p_1(t) - \beta(1-\bar{s}(t))^{\beta}\bar{x}_1^{\beta-1}(t)p_2(t)e^{-\lambda t}, 0), \quad \text{a.e. } t \in [t_0, T].$$

Hence, $p_2(t)$ is a constant for all $t \in [t_0, T]$ and

$$\dot{p}_1(t) = -(A\bar{s}(t)-\sigma)p_1(t) + \beta(1-\bar{s}(t))^{\beta}\bar{x}_1^{\beta-1}(t)p_2(t)e^{-\lambda t}, \text{ a.e. } t \in [t_0, T]. \quad (17)$$

Condition (ii): By the formulas for g and C, $\partial g(\bar{x}(t_0), \bar{x}(T)) = \{(0,0,0,1)\}$ and $N_C(\bar{x}(t_0), \bar{x}(T)) = \mathbb{R}^2 \times \{(0,0)\}$. Thus, (iii) yields

$$(p(t_0), -p(T)) \in \{(0,0,0,\gamma)\} + \mathbb{R}^2 \times \{(0,0)\},$$

which means that $p_1(T) = 0$ and $p_2(T) = -\gamma$.

Condition (iii): By (11), from (iii) one gets

$$\begin{aligned}(A\bar{s}(t)-\sigma)\bar{x}_1(t)p_1(t) &- (1-\bar{s}(t))^{\beta}\bar{x}_1^{\beta}(t)p_2(t)e^{-\lambda t} \\ &= \max_{s\in[0,1]}\left[(As-\sigma)\bar{x}_1(t)p_1(t) - (1-s)^{\beta}\bar{x}_1^{\beta}(t)p_2(t)e^{-\lambda t}\right]\end{aligned}$$

for almost every $t \in [t_0, T]$. Equivalently, we have

$$\begin{aligned}A\bar{x}_1(t)p_1(t)\bar{s}(t) &- (1-\bar{s}(t))^{\beta}\bar{x}_1^{\beta}(t)p_2(t)e^{-\lambda t} \\ &= \max_{s\in[0,1]}\left[A\bar{x}_1(t)p_1(t)s - (1-s)^{\beta}\bar{x}_1^{\beta}(t)p_2(t)e^{-\lambda t}\right] \quad (18)\end{aligned}$$

for almost every $t \in [t_0, T]$.

We are going to analyze furthermore the above Conditions (i)–(iii). As $p_2(t)$ is a constant for all $t \in [t_0, T]$ and $p_2(T) = -\gamma$, one must have $p_2(t) = -\gamma$ for all $t \in [t_0, T]$. Substituting $p_2(t) = -\gamma$ into (17) and (18) yields

$$\dot{p}_1(t) = -(A\bar{s}(t)-\sigma)p_1(t) - \gamma\beta(1-\bar{s}(t))^{\beta}\bar{x}_1^{\beta-1}(t)e^{-\lambda t}, \quad \text{a.e. } t \in [t_0, T] \quad (19)$$

and

$$\begin{aligned}A\bar{x}_1(t)p_1(t)\bar{s}(t) &+ \gamma(1-\bar{s}(t))^{\beta}\bar{x}_1^{\beta}(t)e^{-\lambda t} \\ &= \max_{s\in[0,1]}\left[A\bar{x}_1(t)p_1(t)s + \gamma(1-s)^{\beta}\bar{x}_1^{\beta}(t)e^{-\lambda t}\right] \quad (20)\end{aligned}$$

for almost every $t \in [t_0, T]$. Denote by Σ the set of $t \in [t_0, T]$ such that the equality (20) holds. Then, the Lebesgue measure of Σ is $T - t_0$.

The next lemma describes the function $p_1(\cdot)$ corresponding to the control function $\bar{s}(\cdot)$. It is an analogue of Lemma 4.1, which describes the function $x_1(\cdot)$ w.r.t. a control function $s(\cdot)$.

Lemma 4.2 *The Cauchy problem defined by the differential equation* (19) *and the condition* $p_1(T) = 0$ *possesses a unique solution* $p_1(\cdot) : [t_0, T] \to \mathbb{R}$,

$$p_1(t) = \int_t^T c(z)\bar{\Omega}(z,t)dz, \quad \forall t \in [t_0, T], \quad (21)$$

where $\bar{\Omega}(t,\tau)$ *is defined by* (15) *for* $s(t) = \bar{s}(t)$, *i.e.*,

$$\bar{\Omega}(t,\tau) := e^{\int_\tau^t (A\bar{s}(z)-\sigma)dz}, \quad \forall t, \tau \in [t_0, T], \quad (22)$$

and

$$c(t) := \gamma\beta(1-\bar{s}(t))^{\beta}\bar{x}_1^{\beta-1}(t)e^{-\lambda t}, \quad \forall t \in [t_0, T]. \quad (23)$$

In addition, for any fixed value $\tau \in [t_0, T]$, *one has*

$$p_1(t) = p_1(\tau)\bar{\Omega}(\tau,t) + \int_t^\tau c(z)\bar{\Omega}(z,t)dz, \quad \forall t \in [t_0, T]. \quad (24)$$

Proof Since $\bar{s}(\cdot)$ is measurable, bounded, and $\bar{x}_1(\cdot)$ is absolutely continuous, the function $t \mapsto c(t)$ defined by (23) is measurable and bounded on $[t_0, T]$. Moreover, the function $t \mapsto A\bar{s}(t) - \sigma$ is also measurable and bounded on $[t_0, T]$. In particular, both functions $c(\cdot)$ and $A\bar{s}(\cdot) - \sigma$ are Lebesgue integrable on $[t_0, T]$. Hence, by the lemma in [25, pp. 121–122] we can assert that, for any $\tau \in [t_0, T]$ and $\eta \in \mathbb{R}$, the Cauchy problem defined by the linear differential equation (19) and the initial condition $p_1(\tau) = \eta$ has a unique solution $p_1(\cdot)$. As shown in the proof of Lemma 4.1, $\bar{\Omega}(t, \tau)$ given in (22) is the principal solution of the homogeneous equation

$$\dot{\bar{x}}_1(t) = (A\bar{s}(t) - \sigma)\bar{x}_1(t), \quad \text{a.e. } t \in [t_0, T]. \tag{25}$$

Besides, by the form of (19) and by the theorem in [25, p. 123], the solution of (19) is given by (24). Especially, applying this formula for the case $\tau = T$ and note that $p_1(T) = 0$, we obtain (21). □

Looking back to the maximum principle in Theorem 3.1, we see that the objective function g is taken into a full account in condition (iii) only if $\gamma > 0$. In such a situation, the maximum principle is said to be *normal*. The reader is referred to [26–28] for investigations on the normality of maximum principles for optimal control problems. For (GP_{2a}), we can exclude the situation $\gamma = 0$.

Lemma 4.3 *One must have* $\gamma > 0$.

Proof If $\gamma = 0$, then (23) implies that $c(t) \equiv 0$. Hence, from (21) it follows that $p_1(t) \equiv 0$. Combining this with the facts that $p_2(t) = -\gamma = 0$ for all $t \in [t_0, T]$, we get $(p, \gamma) = (0, 0)$. This contradicts the property $(p, \gamma) \neq (0, 0)$ assured by Theorem 3.1. □

Since $\gamma > 0$, the conditions (i)–(iii) in Theorem 3.1 are satisfied with the pair of multipliers (p, γ) being replaced by $(\tilde{p}, \tilde{\gamma}) := \gamma^{-1}(p, \gamma) = (\gamma^{-1}p, 1)$. So, in what follows we will admit that $\gamma = 1$.

Lemma 4.4 *One has* $p_1(t) \geq 0$ *for all* $t \in [t_0, T]$.

Proof Thanks to Lemmas 4.1 and 4.3, we have $\bar{x}_1(t) > 0$ for all $t \in [t_0, T]$ and, as aforesaid, $\gamma = 1$. Thus, (23) yields $c(t) \geq 0$ for all $t \in [t_0, T]$. From this, (21), and (22), we get the desired assertion. □

To investigate the condition (20) in more details, it is convenient for us to consider the functions

$$\psi_1(t) := A\bar{x}_1(t)p_1(t), \quad \psi_2(t) := \bar{x}_1^{\beta}(t)e^{-\lambda t}, \quad \forall t \in [t_0, T], \tag{26}$$

$$\varphi(t, s) := \psi_1(t)s + \psi_2(t)(1 - s)^{\beta}, \quad \forall (t, s) \in [t_0, T] \times] - \infty, 1]. \tag{27}$$

Since $\bar{x}_1(t) > 0$ and $p_1(t) \geq 0$ for $t \in [t_0, T]$, we have $\psi_1(t) \geq 0$ and $\psi_2(t) > 0$ for $t \in [t_0, T]$. In addition, as the functions $\bar{x}_1(\cdot)$ and $p_1(\cdot)$ are absolutely continuous on $[t_0, T]$, so are $\psi_1(\cdot)$ and $\psi_2(\cdot)$.

Condition (20) requires that for almost every $t \in [t_0, T]$, the function $\varphi(t, \cdot)$ attains a maximum on $[0, 1]$ and $\bar{s}(t)$ is one of the maximizers. Fix an arbitrary value $t \in [t_0, T]$. As $\varphi(t, \cdot)$ is continuous on $] - \infty, 1]$ and differentiable on $] - \infty, 1[$, any maximizer of $\varphi(t, \cdot)$ on $[0, 1]$ must be one of the endpoints $0, 1$ of the segment, or one solution of the equation

$$\frac{\partial \varphi(t, s)}{\partial s} = 0, \tag{28}$$

where $s \in]0,1[$. The equation (28) means that

$$(1-s)^{\beta-1} = \frac{\psi_1(t)}{\beta\psi_2(t)}. \tag{29}$$

If $\psi_1(t) = 0$, then (29) has no solution in $]0,1[$. If $\psi_1(t) > 0$, then (29) is equivalent to

$$s = 1 - e^{\rho(t)} \tag{30}$$

with

$$\rho(t) := \frac{1}{\beta-1} \ln \frac{\psi_1(t)}{\beta\psi_2(t)}. \tag{31}$$

From (31) and the condition $\beta \in]0,1[$, it follows that the value s in (30) belongs to $]0,1[$ if and only if $\psi_1(t) > \beta\psi_2(t)$. In that case, we denote this value s, which is the unique solution of (28) in $]0,1[$, by $s^*(t)$.

Consequently, if $\psi_1(t) \leq \beta\psi_2(t)$, then by comparing the values $\varphi(t,0)$, and $\varphi(t,1)$ one finds a maximizer of $\varphi(t,\cdot)$ on $[0,1]$. If $\psi_1(t) > \beta\psi_2(t)$, then one has to compare the values $\varphi(t,s^*(t))$, $\varphi(t,0)$, and $\varphi(t,1)$ to find a maximizer .

At any $t \in [t_0,T]$ with $\psi_1(t) \leq \beta\psi_2(t)$, by the properties $\psi_2(t) > 0$ and $\beta \in]0,1[$, one has $\varphi(t,0) = \psi_2(t) > \beta\psi_2(t) \geq \psi_1(t) = \varphi(t,1)$. Thus, the unique maximizer of $\varphi(t,\cdot)$ on $[0,1]$ is $s = 0$.

At any $t \in [t_0,T]$ with $\psi_1(t) > \beta\psi_2(t)$, the inequality $\psi_2(t) > 0$ yields $\psi_1(t) > 0$. Besides, it follows from (29) that $(1-s^*(t))^\beta = \dfrac{\psi_1(t)}{\beta\psi_2(t)}(1-s^*(t))$. So, substituting this into (27), we have $\varphi(t,s^*(t)) = \psi_1(t)s^*(t) + \dfrac{1}{\beta}\psi_1(t)(1-s^*(t))$. Thus, we get $\varphi(t,s^*(t)) - \varphi(t,1) = \psi_1(t)(1-s^*(t))(\dfrac{1}{\beta}-1)$. Since $\psi_1(t) > 0$, $s^*(t) \in]0,1[$, and $\beta \in]0,1[$, we have $\varphi(t,s^*(t)) > \varphi(t,1)$. So, $s = 1$ cannot be a maximizer of $\varphi(t,\cdot)$ on $[0,1]$. As $\dfrac{\partial^2\varphi(t,s)}{\partial s^2} = \beta(\beta-1)\psi_2(t)(1-s)^{\beta-2} < 0$, for all $s \in]-\infty,1[$ the function $\varphi(t,\cdot)$ is strictly concave on $[0,1)$. Therefore, the equality $\dfrac{\partial\varphi(t,s^*(t))}{\partial s} = 0$ and the inclusion $s^*(t) \in]0,1[$ show that $s = s^*(t)$ is the unique maximizer $\varphi(t,\cdot)$ on $[0,1]$.

In summary, we have proved that for each $t \in [t_0,T]$, the function $\varphi(t,\cdot)$ has a unique maximizer $\hat{s}(t)$ on $[0,1]$, which greatly depends on the sign of the value

$$\psi(t) := \psi_1(t) - \beta\psi_2(t). \tag{32}$$

Namely, one has $\hat{s}(t) = 0$ if $\psi(t) \leq 0$ and $\hat{s}(t) = s^*(t)$ if $\psi(t) > 0$. Thus, by (20) and the construction of the set Σ, for any $t \in \Sigma$, next alternatives are valid:

(a1) *If* $\psi(t) \leq 0$, *then* $\bar{s}(t) = 0$;

(a2) *If* $\psi(t) > 0$, *then* $\bar{s}(t) = s^*(t)$.

This crucial property of the optimal control function $\bar{s}(\cdot)$ motivates our further investigation on the sign of $\psi(t)$ for $t \in [t_0,T]$.

By (26), $\psi(T) = A\bar{x}_1(T)p_1(T) - \beta\bar{x}_1^\beta(T)e^{-\lambda T}$. Since $\beta > 0$, $\bar{x}_1(T) > 0$, and $p_1(T) = 0$ by Condition (ii), this equality implies that $\psi(T) < 0$. Besides, as $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are absolutely continuous on $[t_0,T]$, from (32) it follows that $\psi(\cdot)$ is also absolutely continuous on $[t_0,T]$.

For any subinterval of $[t_0,T]$ where $\psi(\cdot)$ has a fixed sign, formulas for $\bar{s}(\cdot)$, $\bar{x}_1(\cdot)$, and $p_1(\cdot)$ can be given. Namely, the following statement is valid.

Lemma 4.5 *Let* $[t_1,t_2] \subset [t_0,T]$, $t_1 < t_2$, *and* $\tau \in [t_1,t_2]$ *be given arbitrarily.*

(a) *If* $\psi(t) \le 0$ *for every* $t \in [t_1, t_2]$, *then*

$$\bar{s}(t) = 0, \quad a.e.\ t \in [t_1, t_2], \tag{33}$$

$$\bar{x}_1(t) = \bar{x}_1(\tau) e^{-\sigma(t-\tau)}, \quad \forall t \in [t_1, t_2] \tag{34}$$

and

$$p_1(t) = p_1(\tau) e^{\sigma(t-\tau)} + \frac{\beta}{\beta\sigma + \lambda} \bar{x}_1^{\beta-1}(\tau) e^{\sigma(\beta-1)\tau} e^{\sigma t} \big[e^{-(\beta\sigma+\lambda)t} - e^{-(\beta\sigma+\lambda)\tau} \big] \tag{35}$$

for all $t \in [t_1, t_2]$.

(b) *If* $\psi(t) > 0$ *for every* $t \in]t_1, t_2[$, *then*

$$\bar{s}(t) = s^*(t), \quad a.e.\ t \in [t_1, t_2], \tag{36}$$

$$p_1(t) = p_1(\tau) e^{(\sigma - A)(t-\tau)}, \quad \forall t \in [t_1, t_2], \tag{37}$$

and $\bar{x}_1(\cdot)$ *is described as follows:*

(b$_1$) *If* $A\beta = \beta\sigma + \lambda$, *then*

$$\bar{x}_1(t) = \bar{x}_1(\tau) e^{(A-\sigma)(t-\tau)} - A \Big[\frac{A}{\beta} p_1(\tau) e^{(A-\sigma)\tau} \Big]^{\frac{1}{\beta-1}} e^{(A-\sigma)t} (t - \tau) \tag{38}$$

for all $t \in [t_1, t_2]$.

(b$_2$) *If* $A\beta \neq \beta\sigma + \lambda$, *then*

$$\bar{x}_1(t) = \bar{x}_1(\tau) e^{(A-\sigma)(t-\tau)} - \frac{A(\beta-1)}{\beta\sigma + \lambda - A\beta} \Big[\frac{A}{\beta} p_1(\tau) e^{(A-\sigma)\tau} \Big]^{\frac{1}{\beta-1}} \times e^{(A-\sigma)t} \big[e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1} t} - e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1} \tau} \big] \tag{39}$$

for all $t \in [t_1, t_2]$.

Proof To prove assertion (a), suppose that $\psi(t) \le 0$ for every $t \in [t_1, t_2]$. Then, by the alternative **(a1)** we have $\bar{s}(t) = 0$ for every $t \in [t_1, t_2] \cap \Sigma$. Since the set $\Sigma \subset [t_0, T]$ is of full Lebesgue measure, this shows that (33) is fulfilled. To obtain (34), it suffices to substitute $x_1(\cdot) = \bar{x}_1(\cdot)$ and $s(\cdot) = \bar{s}(\cdot)$ into (13). To prove (35), we can use (24) and the obtained formulas of $\bar{s}(\cdot)$ and $\bar{x}_1(\cdot)$ on $[t_1, t_2]$. Namely, by (22) we have $\bar{\Omega}(\tau, t) = e^{\sigma(t-\tau)}$, $\bar{\Omega}(z, t) = e^{\sigma(t-z)}$ for all $t, z \in [t_1, t_2]$. In addition, from (23) and (34) it follows that

$$c(z) = \beta \bar{x}_1^{\beta-1}(z) e^{-\lambda z} = \beta \bar{x}_1^{\beta-1}(\tau) e^{\sigma(\beta-1)\tau} e^{-[\sigma(\beta-1)+\lambda]z}$$

for all $t, z \in [t_1, t_2]$. Substituting these formulas to (24) yields (35).

To prove assertion (b), suppose that $\psi(t) > 0$ for all $t \in [t_1, t_2]$. Then, by the alternative **(a2)** we have $\bar{s}(t) = s^*(t)$ for every $t \in [t_1, t_2] \cap \Sigma$. Hence (36) is valid. To obtain (37), observe from (36), the fact that (29) is fulfilled with $s = s^*(t)$, and (26) that $(1 - \bar{s}(t))^{\beta-1} = \dfrac{A\bar{x}_1(t)p_1(t)}{\beta \bar{x}_1^{\beta}(t) e^{-\lambda t}}$ for a.e. $t \in [t_1, t_2]$. Therefore, by (19), we have

$$\dot{p}_1(t) = -(A\bar{s}(t) - \sigma)p_1(t) - \beta \frac{A\bar{x}_1(t)p_1(t)}{\beta \bar{x}_1^{\beta}(t) e^{-\lambda t}} (1 - \bar{s}(t)) \bar{x}_1^{\beta-1}(t) e^{-\lambda t} = (\sigma - A)p_1(t)$$

for almost every $t \in [t_1, t_2]$. So, (37) is valid. It remains to prove (38) and (39).

As $(\bar{x}, \bar{s})$ is a $W^{1,1}$ local minimizer of (GP_{2a}), it satisfies the constraint system (9). In particular, we have

$$\dot{\bar{x}}_1(t) = (A\bar{s}(t) - \sigma)\bar{x}_1(t), \quad \text{a.e. } t \in [t_1, t_2]. \tag{40}$$

For almost every $t \in [t_1, t_2]$, it follows from (36), (30), (31), and (26) that

$$\bar{s}(t) = 1 - \Big[\frac{A\bar{x}_1(t)p_1(t)}{\beta\bar{x}_1^\beta(t)e^{-\lambda t}}\Big]^{\frac{1}{\beta-1}} = 1 - \frac{1}{\bar{x}_1(t)}\Big[\frac{A}{\beta}p_1(t)e^{\lambda t}\Big]^{\frac{1}{\beta-1}}.$$

Thus, by (40) we have

$$\dot{\bar{x}}_1(t) = (A-\sigma)\bar{x}_1(t) - A\Big[\frac{A}{\beta}p_1(t)e^{\lambda t}\Big]^{\frac{1}{\beta-1}}, \quad \text{a.e. } t \in [t_1, t_2].$$

So, $\bar{x}_1(\cdot)$ is a solution of the linear differential equation

$$\dot{\bar{x}}_1(t) = (A-\sigma)\bar{x}_1(t) + a(t), \quad \text{a.e. } t \in [t_1, t_2], \tag{41}$$

where

$$a(t) := -A\Big[\frac{A}{\beta}p_1(t)e^{\lambda t}\Big]^{\frac{1}{\beta-1}}, \quad t \in [t_1, t_2]. \tag{42}$$

The principal matrix solution $\Omega_1(t,s)$ of the homogeneous differential equation corresponding to (41) is given by $\Omega_1(t,s) = e^{(A-\sigma)(t-s)}$, for all $t, s \in [t_1, t_2]$. Hence, by the theorem from [25, p. 123], (42), and (37), we obtain

$$\begin{aligned}\bar{x}_1(t) &= \bar{x}_1(\tau)\Omega_1(t,\tau) + \int_\tau^t \Omega_1(t,s)a(s)ds \\ &= \bar{x}_1(\tau)e^{(A-\sigma)(t-\tau)} - A\Big[\frac{A}{\beta}p_1(\tau)e^{(A-\sigma)\tau}\Big]^{\frac{1}{\beta-1}} e^{(A-\sigma)t}\int_\tau^t e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1}s}ds\end{aligned}$$

for all $t \in [t_1, t_2]$. If $A\beta = \beta\sigma + \lambda$, then this implies (38). If $A\beta \neq \beta\sigma + \lambda$, then one gets (39) from the just obtained integral expression for $\bar{x}_1(\cdot)$.

The proof is complete. □

Lemma 4.6 *The following assertions hold.*

(a) *If $A \le \beta\sigma + \lambda$, then $\psi(\cdot)$ is nondecreasing on $[t_0, T]$.*

(b) *If $A \ge \sigma + \dfrac{\lambda}{\beta}$, then $\psi(\cdot)$ is nonincreasing on $[t_0, T]$.*

Proof Since $\psi(\cdot)$ is absolutely continuous on $[t_0, T]$, it is Fréchet differentiable almost everywhere on $[t_0, T]$. Moreover, according to [29, Theorem 6, p. 340], the Fréchet derivative $\dot{\psi}(\cdot)$ of $\psi(\cdot)$ is Lebesgue integrable on $[t_0, T]$ and it holds that

$$\psi(t) = \psi(t_0) + \int_{t_0}^t \dot{\psi}(s)ds, \quad \forall t \in [t_0, T]. \tag{43}$$

From formulas (32) and (26), it follows that

$$\dot{\psi}(t) = A\big[\dot{\bar{x}}_1(t)p_1(t) + \bar{x}_1(t)\dot{p}_1(t)\big] - \beta\big[\beta\bar{x}_1^{\beta-1}(t)\dot{\bar{x}}_1(t)e^{-\lambda t} - \lambda\bar{x}_1^\beta(t)e^{-\lambda t}\big]$$

for almost every $t \in [t_0, T]$. Thus, by invoking (19) and (25), we have

$$\dot{\psi}(t) = -\beta\bar{x}_1^\beta(t)e^{-\lambda t}\big[A(1-\bar{s}(t))^\beta + A\beta\bar{s}(t) - (\beta\sigma+\lambda)\big], \quad \text{a.e.}\, t \in [t_0, T].$$

So, by letting

$$z(s) := A(1-s)^\beta + A\beta s - (\beta\sigma + \lambda), \quad s \in]-\infty, 1], \tag{44}$$

we have

$$\dot{\psi}(t) = -\beta\bar{x}_1^\beta(t)e^{-\lambda t}z(\bar{s}(t)), \quad \text{a.e.}\, t \in [t_0, T]. \tag{45}$$

We see that $z(\cdot) :]-\infty, 1] \to \mathbb{R}$ is twice differentiable on $]-\infty, 1[$ and

$$\dot{z}(s) = -A\beta(1-s)^{\beta-1} + A\beta, \quad \forall s \in]-\infty, 1[,$$

$$\ddot{z}(s) = A\beta(\beta-1)(1-s)^{\beta-2}, \quad \forall s \in]-\infty, 1[. \tag{46}$$

As $\beta \in]0,1[$, it follows from (46) that $\ddot{z}(s) < 0$ for all $s \in]-\infty, 1[$. This means that $\dot{z}(\cdot)$ is strictly decreasing on $]-\infty, 1[$. In particular, $\dot{z}(s) < \dot{z}(0) = 0$ for all $z \in]0,1[$, which implies that $z(\cdot)$ is decreasing on $[0, 1)$.

We are now in a position to prove the two assertions of the lemma.

If $A \leq \beta\sigma + \lambda$, then from (44) one has $z(0) = A - (\beta\sigma + \lambda) \leq 0$. Combining this with the facts that $z(\cdot)$ is decreasing on $[0, 1[$ and $\bar{s}(t) \in [0, 1[$ for almost every $t \in [t_0, T]$ yields $z(\bar{s}(t)) \leq z(0) \leq 0$ for almost every $t \in [t_0, T]$. Thus, by (45), one has $\dot{\psi}(t) \geq 0$ for almost every $t \in [t_0, T]$. Therefore, from formula (43) it follows that $\psi(\cdot)$ is nondecreasing on $[t_0, T]$. So, assertion (a) is proved.

Consider the situation where $A \geq \sigma + \dfrac{\lambda}{\beta}$ and let Σ' be the set of $t \in [t_0, T]$ such that (45) holds. Then, $\Sigma' \subset [t_0, T]$ is of full Lebesgue measure, and the subset $\Sigma \cap \Sigma'$ of $[t_0, T]$ also has full Lebesgue measure. For each $t \in \Sigma \cap \Sigma'$, either $\bar{s}(t) = 0$ or $\bar{s}(t) = s^*(t)$ (see alternatives **(a1)** and **(a2)**). Since $A \geq \sigma + \dfrac{\lambda}{\beta}$ and $\beta \in]0,1[$, one has $A > A\beta \geq \beta\sigma + \lambda$. So, if $\bar{s}(t) = 0$, then from (44), one has $z(\bar{s}(t)) = z(0) = A - (\beta\sigma + \lambda) > 0$. If $\bar{s}(t) = s^*(t)$, then **(a2)** occurs; so one must have $\psi(t) > 0$. By (32), the latter means that $\psi_1(t) > \beta\psi_2(t)$. As $\beta \in]0,1[$, A and $\psi_2(t)$ are positive, the last inequality yields

$$A\frac{\psi_1(t)}{\beta\psi_2(t)} > A\beta. \tag{47}$$

Since (29) is fulfilled with $s = s^*(t)$ and $\bar{s}(t) = s^*(t)$, $(1 - \bar{s}(t))^{\beta-1} = \dfrac{\psi_1(t)}{\beta\psi_2(t)}$. As $\bar{s}(t) \in]0,1[$, the last equality is equivalent to $(1-\bar{s}(t))^{\beta} = \dfrac{\psi_1(t)}{\beta\psi_2(t)}(1-\bar{s}(t))$. Thus, by (44), one has $z(\bar{s}(t)) = \Big[A\dfrac{\psi_1(t)}{\beta\psi_2(t)} - A\beta\Big](1-\bar{s}(t)) + A\beta - (\beta\sigma + \lambda)$. So, by the strict inequality in (47) and the facts that $\bar{s}(t) \in]0,1[$, $A\beta \geq \beta\sigma + \lambda$, we can deduce that $z(\bar{s}(t)) > 0$. We have thus shown that $z(\bar{s}(t)) > 0$ for any $t \in \Sigma \cap \Sigma'$. Thus, having in mind that the set $\Sigma \cap \Sigma' \subset [t_0, T]$ is of full Lebesgue measure, we obtain by formula (45), which is valid for all $t \in \Sigma'$, that $\dot{\psi}(t) < 0$ for almost every $t \in [t_0, T]$.

The proof is complete. □

Lemma 4.7 *If the condition*

$$A \neq \beta\sigma + \lambda \tag{48}$$

is fulfilled, then the situation $\psi(t) = 0$ *for all* t *from an open subinterval* $]t_1, t_2[$ *of* $[t_0, T]$ *with* $t_1 < t_2$ *cannot occur.*

Proof To prove by contradiction, suppose that (48) is valid, but there exist $t_1, t_2 \in [t_0, T]$ with $t_1 < t_2$ such that $\psi(t) = 0$ for $t \in]t_1, t_2[$. Then, formula (32) yields $\psi_1(t) = \beta\psi_2(t)$ for $t \in]t_1, t_2[$. Thus, by (26) we have

$$p_1(t) = \frac{\beta}{A}\bar{x}_1^{\beta-1}(t)e^{-\lambda t}, \quad \forall t \in]t_1, t_2[. \tag{49}$$

Combining (49) with (25) gives $\dot{p}_1(t) = \big[(\beta-1)(A\bar{s}(t)-\sigma)-\lambda\big]p_1(t)$, for almost every $t \in]t_1, t_2[$. Thus, by (19) we get

$$-(A\bar{s}(t)-\sigma)p_1(t) - \beta(1-\bar{s}(t))^{\beta}\bar{x}_1^{\beta-1}(t)e^{-\lambda t} = \big[(\beta-1)(A\bar{s}(t)-\sigma)-\lambda\big]p_1(t)$$

for almost every $t \in]t_1, t_2[$. According to (49), this means that

$$-(A\bar{s}(t) - \sigma)p_1(t) - A(1 - \bar{s}(t))^\beta p_1(t) = \big[(\beta - 1)(A\bar{s}(t) - \sigma) - \lambda\big]p_1(t) \quad (50)$$

for almost every $t \in]t_1, t_2[$. Since $\psi(t) = 0$ for all $t \in]t_1, t_2[$, the continuity of $\psi(\cdot)$ yields $\psi(t) = 0$ for all $t \in [t_1, t_2]$. Therefore, by assertion (a) of Lemma 4.5, we have $\bar{s}(t) = 0$ for almost every $t \in [t_1, t_2]$. Thus, (50) implies that

$$(\sigma - A)p_1(t) = (\sigma - \beta\sigma - \lambda)p_1(t), \quad \text{a.e. } t \in [t_1, t_2]. \quad (51)$$

Since $\bar{x}_1(t) > 0$ for every $t \in [t_0, T]$, by (49) we have $p_1(t) > 0$ for all $t \in]t_1, t_2[$. So, from (51) it follows that $A = \beta\sigma + \lambda$. This contradicts (48).

The proof is complete. □

Theorem 4.1 *Consider the problem (GP_2). If the parameters $A, \beta, \lambda, \sigma$ are such that*

$$A \le \beta\sigma + \lambda, \quad (52)$$

then (GP_2) has a unique global solution $(\bar{k}, \bar{s})$, which is given by

$$\bar{s}(t) = 0, \quad a.e.\ t \in [t_0, T], \quad and \quad \bar{k}(t) = k_0 e^{-\sigma(t-t_0)}, \quad \forall t \in [t_0, T]. \quad (53)$$

If $A, \beta, \lambda, \sigma$ satisfy the condition

$$A \ge \sigma + \frac{\lambda}{\beta}, \quad (54)$$

then (GP_2) has a unique global solution $(\bar{k}, \bar{s})$. Moreover, for

$$\rho := \frac{1}{\beta\sigma + \lambda} \ln \frac{A}{A - (\beta\sigma + \lambda)} \quad and \quad \bar{t} := T - \rho, \quad (55)$$

the optimal process $(\bar{k}, \bar{s})$ can be described as follows:

(a) [Short planning period] *If $T - t_0 \le \rho$, then $(\bar{k}, \bar{s})$ is given by* (53).

(b) [Long planning period] *If $T - t_0 > \rho$, then $\bar{s}(\cdot)$ is given by*

$$\bar{s}(t) = \begin{cases} 1 - e^{\frac{\lambda + \sigma - A}{\beta - 1}(t - \bar{t})}, & a.e.\ t \in [t_0, \bar{t}[\\ 0, & a.e.\ t \in [\bar{t}, T] \end{cases}$$

and $\bar{k}(\cdot)$ is defined as follows:

(b$_1$) *If $A = \sigma + \dfrac{\lambda}{\beta}$, then*

$$\bar{k}(t) = \begin{cases} \bar{k}(\bar{t}) e^{(A-\sigma)(t-\bar{t})} \Big[1 + A(\bar{t} - t) e^{\frac{A\beta - \beta\sigma - \lambda}{\beta - 1}\bar{t}}\Big], & t \in [t_0, \bar{t}] \\ \bar{k}(\bar{t}) e^{-\sigma(t - \bar{t})}, & t \in]\bar{t}, T] \end{cases}$$

with

$$\bar{k}(\bar{t}) := \frac{k_0 e^{(A-\sigma)(\bar{t} - t_0)}}{1 + A(\bar{t} - t_0) e^{\frac{A\beta - \beta\sigma - \lambda}{\beta - 1}\bar{t}}}.$$

(b$_2$) *If $A > \sigma + \dfrac{\lambda}{\beta}$, then*

$$\bar{k}(t) = \begin{cases} \bar{k}(\bar{t}) e^{(A-\sigma)(t-\bar{t})} \Big[1 + \frac{A(\beta-1)}{\beta\sigma + \lambda - A\beta}\big(1 - e^{\frac{\beta\sigma + \lambda - A\beta}{\beta - 1}(t - \bar{t})}\big)\Big], & t \in [t_0, \bar{t}] \\ \bar{k}(\bar{t}) e^{-\sigma(t - \bar{t})}, & t \in]\bar{t}, T] \end{cases}$$

with

$$\bar{k}(\bar{t}) = \frac{k_0 e^{(A-\sigma)(\bar{t} - t_0)}}{1 + \frac{A(\beta-1)}{\beta\sigma + \lambda - A\beta}\big(1 - e^{\frac{\beta\sigma + \lambda - A\beta}{\beta - 1}(t_0 - \bar{t})}\big)}.$$

Proof According to Theorem 2.1, (GP_2) has a global solution. Hence (GP_{2a}) also has a global solution. Let $(\bar{x}, \bar{s})$ be a $W^{1,1}$ local minimizer of (GP_{2a}). By Theorem 3.1, there exist $p \in W^{1,1}([t_0, T]; \mathbb{R}^2)$ and $\gamma \geq 0$ with $(p, \gamma) \neq (0, 0)$ such that conditions (i)–(iii) in Theorem 3.1 hold true. In what follows, we will employ the results of the analysis of those conditions, which was given before the formulation of our theorem. All the notations used in the that analysis will be kept. Thanks to Lemma 4.3, we have admitted that $\gamma = 1$. Recall also that $\psi(t)$ is given by (32) and $s = s^*(t)$ is the unique solution in $]0, 1[$ of (28), which is equivalent to (30), when $\psi(t) > 0$. In addition, $\psi(\cdot)$ is absolutely continuous on $[t_0, T]$ with $\psi(T) < 0$.

First, let $A, \beta, \lambda, \sigma$ be such that (52) holds. Then, Lemma 4.6 shows that $\psi(\cdot)$ is nondecreasing on $[t_0, T]$. So, one has $\psi(t) \leq \psi(T) < 0$ for all $t \in [t_0, T]$. Therefore, by choosing $t_1 = t_0$, $t_2 = T$, $\tau = t_0$, and applying assertion (a) of Lemma 4.5, we have $\bar{s}(t) = 0$ for a.e. $t \in [t_0, T]$ and $\bar{x}_1(t) = k_0 e^{-\sigma(t - t_0)}$ for all $t \in [t_0, T]$. It follows that (GP_2) has a unique global solution $(\bar{k}, \bar{s})$, which is given by (53).

Now, suppose that the constants $A, \beta, \lambda, \sigma$ satisfy the condition (54). By Lemma 4.6 we know that $\psi(\cdot)$ is nonincreasing on $[t_0, T]$. We claim that *there is at most one point* $t \in [t_0, T]$ *such that* $\psi(t) = 0$. To prove this, suppose for a while that there are some points t_1 and t_2 from $[t_0, T]$ with $t_1 < t_2$ and $\psi(t_1) = \psi(t_2) = 0$. Since $\psi(\cdot)$ is nonincreasing on $[t_1, t_2]$, we must have $\psi(t) = 0$ for all $t \in]t_1, t_2[$. As $\beta \in]0, 1[$, from (54) one deduces that $A > A\beta \geq \beta\sigma + \lambda$. Hence, by Lemma 4.7 we get a contradiction. Our claim is justified.

There are two possibilities: (P1) $\psi(t_0) \leq 0$; (P2) $\psi(t_0) > 0$. We will consider these situations separately.

If (P1) occurs, then the fact that $\psi(\cdot)$ is nonincreasing on $[t_0, T]$ gives $\psi(t) \leq 0$ for all $t \in [t_0, T]$. Consequently, setting $t_1 = t_0$, $t_2 = T$, $\tau = t_0$ and applying assertion (a) of Lemma 4.5, we get $\bar{s}(t) = 0$ for almost every $t \in [t_0, T]$, $\bar{x}_1(t) = k_0 e^{-\sigma(t - t_0)}$ for all $t \in [t_0, T]$, and

$$p_1(t) = p_1(t_0) e^{\sigma(t - t_0)} + \frac{\beta}{\beta\sigma + \lambda} \bar{x}_1^{\beta - 1}(t_0) e^{\sigma(\beta - 1)t_0} e^{\sigma t} \left[e^{-(\beta\sigma + \lambda)t} - e^{-(\beta\sigma + \lambda)t_0} \right]$$

for all $t \in [t_0, T]$. As $p_1(T) = 0$, it follows that

$$p_1(t_0) e^{\sigma(T - t_0)} + \frac{\beta}{\beta\sigma + \lambda} \bar{x}_1^{\beta - 1}(t_0) e^{\sigma(\beta - 1)t_0} e^{\sigma T} \left[e^{-(\beta\sigma + \lambda)T} - e^{-(\beta\sigma + \lambda)t_0} \right] = 0.$$

Equivalently, we have $p_1(t_0) = \frac{\beta}{\beta\sigma + \lambda} \bar{x}_1^{\beta - 1}(t_0) e^{\sigma\beta t_0} \left[e^{-(\beta\sigma + \lambda)t_0} - e^{-(\beta\sigma + \lambda)T} \right]$. Since $\psi(t_0) \leq 0$, by (32) we get $\psi_1(t_0) \leq \beta\psi_2(t_0)$. Combining this with (26) gives $p_1(t_0) \leq \dfrac{\beta}{A} \bar{x}_1^{\beta - 1}(t_0) e^{-\lambda t_0}$. Substituting the above formula for $p_1(t_0)$ into the last inequality yields

$$\frac{\beta}{\beta\sigma + \lambda} \bar{x}_1^{\beta - 1}(t_0) e^{\sigma\beta t_0} \left[e^{-(\beta\sigma + \lambda)t_0} - e^{-(\beta\sigma + \lambda)T} \right] \leq \frac{\beta}{A} \bar{x}_1^{\beta - 1}(t_0) e^{-\lambda t_0}.$$

As $\bar{x}_1(t_0) = k_0 > 0$, this is equivalent to $e^{(\beta\sigma + \lambda)(t_0 - T)} \geq \dfrac{A - (\beta\sigma + \lambda)}{A}$. Since $A > \beta\sigma + \lambda$, the latter means that $T - t_0 \leq \rho$ with $\rho > 0$ being defined in (55).

If (P2) occurs then, by the inequality $\psi(T) < 0$ and the continuity of $\psi(\cdot)$ on $[t_0, T]$, one can find some $\bar{t} \in]t_0, T[$ such that $\psi(\bar{t}) = 0$. Since there is at most one point $t \in [t_0, T]$ such that $\psi(t) = 0$, this point $\bar{t}$ is uniquely defined. Moreover, one has $\psi(t) > 0$ for all $t \in [t_0, \bar{t}[$ and $\psi(t) < 0$ for all $t \in]\bar{t}, T]$. We will compute $\bar{t}$ and, on that basis, find formulas for the functions $\bar{s}(\cdot)$ and $\bar{x}_1(\cdot)$ on $[t_0, T]$.

As $\psi(\bar t) = 0$, it follows from (26) and (32) that

$$p_1(\bar t) = \frac{\beta}{A}\bar x_1^{\beta-1}(\bar t)e^{-\lambda\bar t}. \tag{56}$$

Choosing $t_1 = \bar t$, $t_2 = T$, and $\tau = \bar t$, by assertion (a) of Lemma 4.5 one gets $\bar s(t) = 0$ for almost every $t \in [\bar t, T]$, $\bar x_1(t) = \bar x_1(\bar t)e^{-\sigma(t-\bar t)}$ for all $t \in [\bar t, T]$, and

$$p_1(t) = p_1(\bar t)e^{\sigma(t-\bar t)} + \frac{\beta}{\beta\sigma+\lambda}\bar x_1^{\beta-1}(\bar t)e^{\sigma(\beta-1)\bar t}e^{\sigma t}\big[e^{-(\beta\sigma+\lambda)t} - e^{-(\beta\sigma+\lambda)\bar t}\big]$$

for all $t \in [\bar t, T]$. Substituting (56) into the last formula yields

$$p_1(t) = \frac{\beta}{A}\bar x_1^{\beta-1}(\bar t)e^{-\lambda\bar t}e^{\sigma(t-\bar t)} + \frac{\beta}{\beta\sigma+\lambda}\bar x_1^{\beta-1}(\bar t)e^{\sigma(\beta-1)\bar t}e^{\sigma t}\big[e^{-(\beta\sigma+\lambda)t} - e^{-(\beta\sigma+\lambda)\bar t}\big]$$

for all $t \in [\bar t, T]$. Equivalently, one has

$$p_1(t) = \frac{\beta}{A}e^{-\lambda\bar t}\bar x_1^{\beta-1}(\bar t)e^{\sigma(t-\bar t)}\Big\{1 + \frac{A}{\beta\sigma+\lambda}\big[e^{(\beta\sigma+\lambda)(\bar t-t)} - 1\big]\Big\} \tag{57}$$

for all $t \in [\bar t, T]$. By (57) and the condition $p_1(T) = 0$, we have

$$\frac{\beta}{A}e^{-\lambda\bar t}\bar x_1^{\beta-1}(\bar t)e^{\sigma(T-\bar t)}\Big\{1 + \frac{A}{\beta\sigma+\lambda}\big[e^{(\beta\sigma+\lambda)(\bar t-T)} - 1\big]\Big\} = 0.$$

As $\bar x_1(\bar t) > 0$, this equality is equivalent to $1 + \frac{A}{\beta\sigma+\lambda}\big[e^{(\beta\sigma+\lambda)(\bar t-T)} - 1\big] = 0$. Since $A > \beta\sigma + \lambda$, the latter means that $\bar t = T - \rho$ with $\rho > 0$ being given in (55). From this formula it is clear that the condition $\bar t \in]t_0, T[$ is satisfied if and only if $T - t_0 > \rho$.

For $t_1 = t_0$, $t_2 = \bar t$, and $\tau = \bar t$, since $\psi(t) > 0$ for all $t \in]t_0, \bar t[$, applying assertion (b) of Lemma 4.5 we get $\bar s(t) = s^*(t)$, for almost every $t \in [t_0, \bar t]$ and

$$p_1(t) = p_1(\bar t)e^{(\sigma-A)(t-\bar t)}, \quad \forall t \in [t_0, \bar t]. \tag{58}$$

The formula for $\bar x_1(\cdot)$ depends on the relationships between A and the constants σ, λ, β. Namely, the following statements are valid.

– If $A = \sigma + \frac{\lambda}{\beta}$, then

$$\bar x_1(t) = \bar x_1(\bar t)e^{(A-\sigma)(t-\bar t)} - A\Big[\frac{A}{\beta}p_1(\bar t)e^{(A-\sigma)\bar t}\Big]^{\frac{1}{\beta-1}}e^{(A-\sigma)t}(t-\bar t), \quad \forall t \in [t_0, \bar t]. \tag{59}$$

– If $A > \sigma + \frac{\lambda}{\beta}$, then

$$\bar x_1(t) = \bar x_1(\bar t)e^{(A-\sigma)(t-\bar t)} - \frac{A(\beta-1)}{\beta\sigma+\lambda-A\beta}\Big[\frac{A}{\beta}p_1(\bar t)e^{(A-\sigma)\bar t}\Big]^{\frac{1}{\beta-1}} \times e^{(A-\sigma)t}\big(e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1}t} - e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1}\bar t}\big) \tag{60}$$

for all $t \in [t_0, \bar t]$.

As $\psi(t) > 0$ for all $t \in [t_0, \bar t[$, (30) is satisfied with $s = s^*(t)$ for all $t \in [t_0, \bar t[$. Thus, from (31) and (26) it follows that

$$s^*(t) = 1 - \frac{1}{\bar x_1(t)}\Big[\frac{A}{\beta}p_1(t)e^{\lambda t}\Big]^{\frac{1}{\beta-1}}, \quad \forall t \in [t_0, \bar t[. \tag{61}$$

Substituting (56) into (58), we get

$$p_1(t) = \frac{\beta}{A}\bar{x}_1^{\beta-1}(\bar{t})e^{-\lambda\bar{t}}e^{(\sigma-A)(t-\bar{t})}, \quad \forall t \in [t_0, \bar{t}]. \tag{62}$$

Combining (62) with (61) yields $s^*(t) = 1 - e^{\frac{\lambda+\sigma-A}{\beta-1}(t-\bar{t})}$ for all $t \in [t_0, \bar{t}[$.

By (56), the next transformations are valid for any $t \in [t_0, \bar{t}]$:

$$\begin{aligned}\left[\frac{A}{\beta}p_1(\bar{t})e^{(A-\sigma)\bar{t}}\right]^{\frac{1}{\beta-1}} e^{(A-\sigma)t} &= \left[\frac{A}{\beta}\frac{\beta}{A}\bar{x}_1^{\beta-1}(\bar{t})e^{-\lambda\bar{t}}e^{(A-\sigma)\bar{t}}\right]^{\frac{1}{\beta-1}} e^{(A-\sigma)\bar{t}}e^{(A-\sigma)(t-\bar{t})}\\ &= \bar{x}_1(\bar{t})e^{\frac{A-\sigma-\lambda}{\beta-1}\bar{t}}e^{(A-\sigma)\bar{t}}e^{(A-\sigma)(t-\bar{t})}.\end{aligned}$$

So, for any $t \in [t_0, \bar{t}]$, one has

$$\left[\frac{A}{\beta}p_1(\bar{t})e^{(A-\sigma)\bar{t}}\right]^{\frac{1}{\beta-1}} e^{(A-\sigma)t} = \bar{x}_1(\bar{t})e^{\frac{A\beta-\beta\sigma-\lambda}{\beta-1}\bar{t}}e^{(A-\sigma)(t-\bar{t})}. \tag{63}$$

If $A = \sigma + \dfrac{\lambda}{\beta}$, then inserting (63) to (59) yields

$$\bar{x}_1(t) = \bar{x}_1(\bar{t})e^{(A-\sigma)(t-\bar{t})} - A\bar{x}_1(\bar{t})e^{\frac{A\beta-\beta\sigma-\lambda}{\beta-1}\bar{t}}e^{(A-\sigma)(t-\bar{t})}(t-\bar{t}), \quad \forall t \in [t_0, \bar{t}].$$

Equivalently, we have

$$\bar{x}_1(t) = \bar{x}_1(\bar{t})e^{(A-\sigma)(t-\bar{t})}\left[1 + A(\bar{t}-t)e^{\frac{A\beta-\beta\sigma-\lambda}{\beta-1}\bar{t}}\right], \quad \forall t \in [t_0, \bar{t}].$$

As $\bar{x}_1(t_0) = k_0$, it follows that $k_0 = \bar{x}_1(\bar{t})e^{(A-\sigma)(t_0-\bar{t})}\left[1 + A(\bar{t}-t_0)e^{\frac{A\beta-\beta\sigma-\lambda}{\beta-1}\bar{t}}\right]$. Since $1 + A(\bar{t}-t_0)e^{\frac{A\beta-\beta\sigma-\lambda}{\beta-1}\bar{t}} > 0$, the last equality implies that

$$\bar{x}_1(\bar{t}) = \frac{k_0 e^{(A-\sigma)(\bar{t}-t_0)}}{1 + A(\bar{t}-t_0)e^{\frac{A\beta-\beta\sigma-\lambda}{\beta-1}\bar{t}}}. \tag{64}$$

If $A > \sigma + \dfrac{\lambda}{\beta}$, then substituting (63) into (60) yields

$$\begin{aligned}\bar{x}_1(t) &= \bar{x}_1(\bar{t})e^{(A-\sigma)(t-\bar{t})}\\ &\quad - \frac{A(\beta-1)}{\beta\sigma+\lambda-A\beta}\bar{x}_1(\bar{t})e^{\frac{A\beta-\beta\sigma-\lambda}{\beta-1}\bar{t}}e^{(A-\sigma)(t-\bar{t})}\left(e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1}t} - e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1}\bar{t}}\right)\end{aligned}$$

for all $t \in [t_0, \bar{t}]$. Equivalently,

$$\bar{x}_1(t) = \bar{x}_1(\bar{t})e^{(A-\sigma)(t-\bar{t})}\left[1 + \frac{A(\beta-1)}{\beta\sigma+\lambda-A\beta}\left(1 - e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1}(t-\bar{t})}\right)\right], \quad \forall t \in [t_0, \bar{t}].$$

Since $\bar{x}_1(t_0) = k_0$, the latter implies that

$$k_0 = \bar{x}_1(\bar{t})e^{(A-\sigma)(t_0-\bar{t})}\left[1 + \frac{A(\beta-1)}{\beta\sigma+\lambda-A\beta}\left(1 - e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1}(t_0-\bar{t})}\right)\right].$$

Since $\frac{A(\beta-1)}{\beta\sigma+\lambda-A\beta}\left(1 - e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1}(t_0-\bar{t})}\right) > 0$, the last equality yields

$$\bar{x}_1(\bar{t}) = \frac{k_0 e^{(A-\sigma)(\bar{t}-t_0)}}{1 + \frac{A(\beta-1)}{\beta\sigma+\lambda-A\beta}\left(1 - e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1}(t_0-\bar{t})}\right)}. \tag{65}$$

Summing up, we have seen that the situation (P1) occurs if and only if $T-t_0 \le \rho$, while the situation (P2) occurs if and only if $T-t_0 > \rho$. Therefore, if

$T - t_0 \le \rho$, then $\bar{s}(t) = 0$ for almost everywhere $t \in [t_0, T]$, $\bar{x}_1(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$. In addition, if $T - t_0 > \rho$, then $\bar{s}(\cdot)$ is given by

$$\bar{s}(t) = \begin{cases} s^*(t), & \text{a.e. } t \in [t_0, \bar{t}] \\ 0, & \text{a.e. } t \in]\bar{t}, T] \end{cases} \quad \text{with} \quad s^*(t) = 1 - e^{\frac{\lambda+\sigma-A}{\beta-1}(t-\bar{t})}, \quad \forall t \in [t_0, \bar{t}[,$$

and $\bar{x}_1(\cdot)$ is described as follows:

- If $A = \sigma + \dfrac{\lambda}{\beta}$, then

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(\bar{t}) e^{(A-\sigma)(t-\bar{t})} \left[1 + A(\bar{t} - t) e^{\frac{A\beta-\beta\sigma-\lambda}{\beta-1}\bar{t}}\right], & t \in [t_0, \bar{t}] \\ \bar{x}_1(\bar{t}) e^{-\sigma(t-\bar{t})}, & t \in]\bar{t}, T] \end{cases}$$

with $\bar{x}_1(\bar{t})$ being given by (64).

- If $A > \sigma + \dfrac{\lambda}{\beta}$, then

$$\bar{x}_1(t) = \begin{cases} \bar{x}_1(\bar{t}) e^{(A-\sigma)(t-\bar{t})} \left[1 + \frac{A(\beta-1)}{\beta\sigma+\lambda-A\beta}\big(1 - e^{\frac{\beta\sigma+\lambda-A\beta}{\beta-1}(t-\bar{t})}\big)\right], & t \in [t_0, \bar{t}] \\ \bar{x}_1(\bar{t}) e^{-\sigma(t-\bar{t})}, & t \in]\bar{t}, T] \end{cases}$$

with $\bar{x}_1(\bar{t})$ being given by (65).

Recalling that the variable $\bar{x}_1$ in (GP_{2a}) plays the role of the variable $\bar{k}$ in (GP_2), from the just obtained results we get all the desired assertions. □

5 Economic Interpretations of Theorem 4.1

To give some clear economic interpretations of Theorem 4.1, we now recall the roles of the parameters $A, \beta, \lambda, \sigma$ in the optimal economic growth problems under consideration. The two constants λ and σ appearing in the general problem (GP) are the *real interest rate* and the *growth rate of labor force*, respectively. The remaining parameters A and β are the characteristic index of the typical problem (GP_2). Namely, A in the per capita function expresses the *total factor productivity*[1] (TFP) and β is the exponent of the Cobb–Douglas utility function. Note that TFP is a measure of economic efficiency that represents the increase in total production which is in excess of the increase that results from increase in inputs and depends on some intangible factors such as technological change, education, research and development, etc. Using σ, λ, and β, we can define two constants $\chi_1 = \beta\sigma + \lambda$ and $\chi_2 = \sigma + \dfrac{\lambda}{\beta}$, which play important roles in the analysis of problem (GP_2). Since σ, λ are positive, $\beta \in]0, 1[$, and $\chi_2 = \dfrac{\chi_1}{\beta}$, one has $0 < \chi_1 < \chi_2$. So, the constants χ_1, χ_2 divide the positive half-line $]0, +\infty[$ into three domains $]0, \chi_1]$, $]\chi_1, \chi_2[$, and $[\chi_2, +\infty[$. Interpreting the half-line $]0, +\infty[$ as the parameter space for the total factor productivity $A > 0$, one may say that "*the total factor productivity is relatively small*" (resp., "*the total factor productivity is enough high*") when $A \in]0, \chi_1]$ (resp., $A \in [\chi_2, +\infty[$).

Adopting the above terminologies, we can read (52) as the condition saying that the total factor productivity is relatively small. Similarly, condition (54) is read as the total factor productivity is enough high. In the case where (54) occurs, we define two more constants by the formulas in (55): $\rho > 0$ is to decide whether the *planning period is short* $(T - t_0 \le \rho)$, or it is *long* $(T - t_0 > \rho)$;

[1] See, e.g., https://en.wikipedia.org/wiki/Total_factor_productivity.

$\bar{t} \in]t_0, T[$ is the *special time* of ceasing the expansion of the production facility when the control process is optimal. So, Theorem 4.1 postulates the following on the model (GP_2):

(a) If the total factor productivity is relatively small, then the expansion of the production facility does not lead to a higher total consumption satisfaction of the society;

(b) If the total factor productivity is enough high and the planning period is short, then expanding the production facility also does not lead to a higher total consumption satisfaction of the society;

(c) If the total factor productivity is enough high and the planning period is relatively long, then the highest total consumption satisfaction of the society is attained only if the expansion of the production facility is ceased after a special time.

The following three figures depict the synthesis of the optimal trajectory $\bar{k}(t)$ and the corresponding optimal control $\bar{s}(t)$, $t \in [t_0, T]$, which is given by Theorem 4.1. Note that the explicit formulas of $\bar{k}$ and $\bar{s}$ can be found in the theorem.

For the optimal economic growth problems with $A \in]0, \chi_1[$, one has $\bar{s}(t) = 0$ for all $t \in [t_0, T]$ and the optimal capital-to-labor ratio function $\bar{k}(t)$ monotonically decreases on the whole planning interval $[t_0, T]$ (see Figure 1).

For the problems with $A = \chi_2$ (see Figure 2), the optimal propensity to save function $\bar{s}(t)$ may be either identically zero on $[t_0, T]$ or monotonically decreasing until the moment $\bar{t}$ and identically zero on $[\bar{t}, T]$. The function $\bar{k}(t)$ monotonically decreases on the whole planning interval $[t_0, T]$, but its shape is different from the one of $\bar{k}(t)$ in Figure 1.

For the problems with $A > \chi_2$ (see Figure 3), the behavior of the optimal capital-to-labor ratio function is rather complicated and interesting. Namely, the function $\bar{s}(t)$ may be either identically zero on $[t_0, T]$ or monotonically decreasing until the moment $\bar{t}$ and identically zero on $[\bar{t}, T]$. However, the function $\bar{k}(t)$ monotonically decreases on the whole planning interval $[t_0, T]$ only if the planning period $T - t_0$ is relatively short. If the period is relatively long, then the function $\bar{k}(t)$ first monotonically increases and monotonically decreases afterwards.

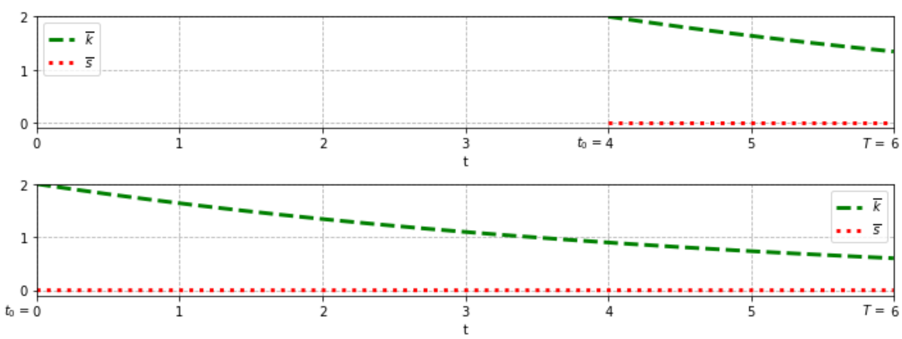

Fig. 1 The optimal processes $(\bar{k}, \bar{s})$ of (GP_2) w.r.t. parameters $A = 0.15$, $\beta = 0.5$, $\lambda = 0.1$, $\sigma = 0.2$, $k_0 = 2$, $T = 6$, and either $t_0 = 4$ or $t_0 = 0$

The obtained results lead us to the following open problem.

Problem 1: *Is it possible to perform a synthesis of the optimal processes of the economic growth problem (GP_2) in the case where the total factor productivity A falls into the bounded open interval $]\chi_1, \chi_2[$ defined by the growth rate of*

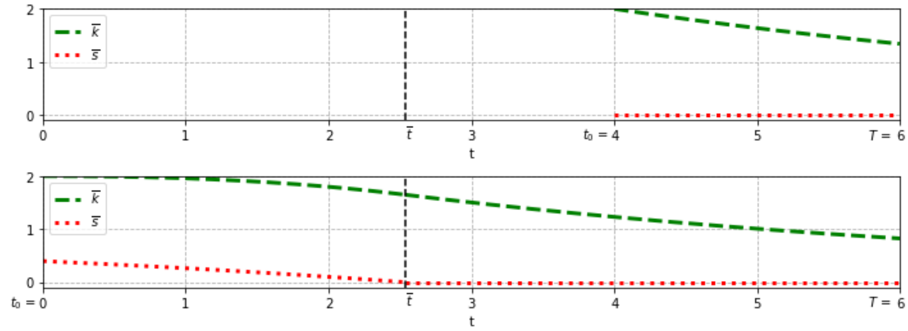

Fig. 2 The optimal processes $(\bar{k}, \bar{s})$ of (GP_2) w.r.t. parameters $A = 0.4$, $\beta = 0.5$, $\lambda = 0.1$, $\sigma = 0.2$, $k_0 = 2$, $T = 6$, and either $t_0 = 4$ $(T - t_0 < \rho)$ or $t_0 = 0$ $(T - t_0 > \rho)$

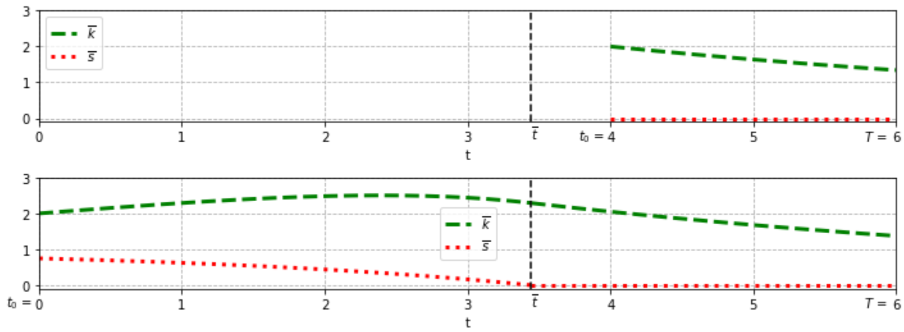

Fig. 3 The optimal processes $(\bar{k}, \bar{s})$ of (GP_2) w.r.t. parameters $A = 0.5$, $\beta = 0.5$, $\lambda = 0.1$, $\sigma = 0.2$, $k_0 = 2$, $T = 6$, and either $t_0 = 4$ $(T - t_0 < \rho)$ or $t_0 = 0$ $(T - t_0 > \rho)$

labor force, the real interest rate, and the exponent of the Cobb–Douglas utility function, or not?

6 Conclusions

We have proved that a parametric optimal economic growth problem with nonlinear utility function and linear production function, which satisfies certain conditions on the input parameters, has a unique global solution. Moreover, we have provided an explicit description of the solution.

Since we still have not been able to deal with the case where the total factor productivity falls into a bounded open interval defined by the growth rate of labor force, the real interest rate, and the exponent of the utility function, further research is needed to give a complete picture about the optimal solutions of a finite horizon optimal economic growth problem with a nonlinear utility function and linear production function. We think that this interesting open question can hardly be solved without invoking new tools and applying different ideas.

A solution of Problem 1 would complement the synthesis of the optimal processes given in Theorem 4.1 and yield a comprehensive analysis of (GP_2).

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