Necessary and sufficient conditions for assignability of dichotomy spectra of continuous time-varying linear systems

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This paper is dedicated to Professor Nguyen Dinh Cong on the occasion of his sixtieth birthday

Abstract

We consider a version of the pole placement problem for continuous time-varying linear systems. Our purpose is to prove that uniform complete controllability is equivalent to possibility of arbitrary assignment of the dichotomy spectrum. The main ingredients of the proof are the reduction of system to upper triangular form and the use of the concept of uniform complete stabilization. To illustrate the theoretical result, we consider scalar continuous time-varying control systems. For these systems we provide a simple necessary and sufficient condition for uniform complete controllability and if this condition holds, then we construct an explicit control to assign the dichotomy spectrum.

1 Introduction

To describe the properties of dynamical systems, a number of numerical characteristics or spectra such as Lyapunov, Bohl, Perron or Sacker-Sell spectra (see [8], [16] or [23] for a survey) have been introduced.

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The most known of them - the Lyapunov spectrum is closely related to exponential stability. The negativity of the largest exponent guarantees exponential stability. The disadvantage of the Lyapunov spectrum is its discontinuity as a function of system parameters and one of the implications of this property is the fact that in case of negative Lyapunov exponents, the stability of nonlinearly perturbed systems is not guaranteed without an additional regularity condition.

In [26] authors developed the Sacker–Sell spectrum theory, which is now also called dichotomy spectrum for non-autonomous differential equations. Now dichotomy spectrum is an important tool in qualitative theory of non-autonomous dynamical systems. This is due the following reasons. The dichotomy spectrum together with the spectral manifolds completely describe the dynamical skeleton of a linear system. This spectrum describes uniform exponential stability as follows: if the dichotomy spectrum lies left of zero, then the uniform exponential stability of nonlinearly perturbed systems is guaranteed [4]. More generally this concept may be used to discuss the existence and the smoothness of invariant manifolds for non-autonomous differential equations, to obtain a version of the Grobman–Hartman theorem for non-autonomous systems (in this context the hyperbolicity is formulated as zero does not belong to the dichotomy spectrum) [5], to characterize the existence of center manifolds [27] and in the theory of Lyapunov regularity [6]. Using the resonance of the dichotomy spectrum to study the normal forms of non-autonomous system, in [29] a finite order normal form were obtained, and in [31] analytic normal forms of a class of analytic non-autonomous differential systems were presented. Finally, information on the fine structure of the dichotomy spectrum allows to classify various types of non-autonomous bifurcations [22].

If we consider systems with control, the goals set for the control system can be achieved by designing control in such a way that the spectrum of the closed system has a predetermined position. This approach is well known in the theory of time-invariant linear systems and is called the pole-placement method. For continuous time-varying systems and the Lyapunov spectrum, a broad discussion of this method is presented in the monograph [19], and for discrete systems and the Lyapunov spectrum such an approach is described in [3]. Assignability of dichotomy spectrum for discrete time-varying systems has been discussed in [11], where it was shown that uniform complete controllability is a sufficient condition for placing the dichotomy spectrum arbitrarily.

The main objective of this paper is to investigate the dichotomy spectrum assignability problem for continuous time-varying system. In fact, we show that uniformly complete controllability is equivalent to assignability of dichotomy spectrum for both one-sided and two-sided continuous time-varying linear control systems. The paper is organized as follows: In Section 2, we introduce the notion of dichotomy spectrum, uniformly complete controllability of linear continuous time-varying control systems and state the main result of this paper. Section 3 is devoted to some preparatory results including the equivalence between uniform complete controllability and uniform complete stabilizability, transforming uniform complete conrollability systems to upper triangular linear systems by suitable linear state feedbacks and dichotomy spectra of a specific class of upper triangular linear systems. We give a proof of the main result in Section 4. Several examples are presented in Section 5 to illustrate the theoretical results of the paper.

Notations: Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_-$ denote the set of real numbers, non-negative real numbers, non-positive real numbers, respectively. Let $\mathcal{KC}_{n,m}(\mathbb{J})$, where $\mathbb{J}$ is either $\mathbb{R}_+$, $\mathbb{R}_-$ or $\mathbb{R}$, denote the set of bounded and piecewise continuous matrix-valued functions $M : \mathbb{J} \to \mathbb{R}^{n\times m}$. We say that a piecewise continuous matrix function $B : \mathbb{J} \to \mathbb{R}^{n\times m}$ is piecewise uniformly continuous on $\mathbb{J}$ if the following conditions are satisfied: there exists a $\Delta_0 > 0$ such that the length of each continuity interval I_j ($j \in J \subset \mathbb{N}$) of the function B satisfies the inequality $|I_j| \geq \Delta_0$, and for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|B(t) - B(s)\| \leq \varepsilon$ for each $j \in J$ and for all $t, s \in I_j$ satisfying the inequality $|t - s| \leq \delta$.

2 Preliminaries and the statement of the main results

2.1 Dichotomy spectra of linear time-varying systems

Consider a linear time-varying system

$$\dot{x} = M(t)x, \tag{1}$$

where $M \in \mathcal{KC}_{n,n}(\mathbb{J})$. Denote by $X_M(\cdot,\cdot) : \mathbb{J}\times\mathbb{J}$ the *transition matrix* of (1), i.e. $X_M(\cdot, s)\xi$ solves (1) with the initial value condition $x(s) = \xi$. Now, we recall the notion of exponential dichotomy which is also known as uniform hyperbolicity for time-varying systems, see also [9], [10], [20].

Definition 1 (Exponential dichotomy) *System* (1) *is said to admit an* exponential dichotomy *(ED) on* $\mathbb{J}$ *if there exist* $K, \varepsilon > 0$ *and an invariant* *family of projections* $P : \mathbb{J} \to \mathbb{R}^{n\times n}$*, i.e.* $P(t)X_M(t,s) = X_M(t,s)P(s)$ *if* $s,t \in \mathbb{J}$*, satisfying the following inequalities*

$$\|X_M(t,s)P(s)\| \le Ke^{-\varepsilon(t-s)} \;\; if \;\; s \le t, \; s,t \in \mathbb{J}, \tag{2}$$

and

$$\|X_M(t,s)\,(I - P(s))\| \le Ke^{\varepsilon(t-s)} \;\; if \;\; t \le s, \;\; s,t \in \mathbb{J}. \tag{3}$$

Based on the notion of ED, we introduce the notion of dichotomy spectrum, see also [26].

Definition 2 (Dichotomy spectrum) *The dichotomy spectrum of* (1) *is defined by*

$$\Sigma_{\mathrm{ED}}^{\mathbb{J}}(M) := \{\gamma \in \mathbb{R} : \; \dot{x} = (M(t) - \gamma I)\, x \\ \text{has no ED on } \mathbb{J}\}\,.$$

The structure of the dichotomy spectrum is described by the following theorem. We refer the readers to [28, 25] for a proof of this theorem.

Theorem 1 (Spectral theory of time-varying linear systems) *The dichotomy spectrum* $\Sigma_{\mathrm{ED}}^{\mathbb{J}}(M)$ *of* (1) *consists of at most* n *disjoint closed intervals i.e.* $\Sigma_{\mathrm{ED}}(M) = [\alpha_1, \beta_1], \dots, [\alpha_\ell, \beta_\ell]$*, where* $\alpha_1 \le \beta_1 < \alpha_2 \le \beta_2 < \cdots < \alpha_\ell \le \beta_\ell$ *and* $\ell \le n$*. Moreover, when* $\mathbb{J} = \mathbb{R}$ *there exists a time-dependent linear decomposition*

$$\mathbb{R}^n = \mathcal{W}_1(s) \oplus \cdots \oplus \mathcal{W}_\ell(s)$$

such that for any $\varepsilon > 0$ *there exists* $K > 1$ *such that for all* $\xi \in \mathcal{W}_i(s)$ *and* $t \ge s$*,* t*,* $s \in \mathbb{J}$ *we have*

$$\frac{1}{K} e^{(t-s)(\alpha_i - \varepsilon)} \|\xi\| \le \|X_M(t,s)\xi\| \le Ke^{(t-s)(\beta_i+\varepsilon)} \|\xi\|\,. \tag{4}$$

2.2 Linear time-varying control systems

Consider a linear time-varying control system described by the following equation

$$\dot{x} = A(t)x + B(t)u, \qquad t \in \mathbb{J}, \tag{5}$$

where $A \in \mathcal{KC}_{n,n}(\mathbb{J})$, $B \in \mathcal{KC}_{n,m}(\mathbb{J})$ and $u \in \mathcal{KC}_{m,1}(\mathbb{J})$ is the control. For $(t_0, x_0) \in \mathbb{J} \times \mathbb{R}^n$ the solution of system (5) satisfying $x(t_0) = x_0$, will be denoted by $x(\cdot, t_0, x_0, u)$. Now we will introduce the definition of uniform complete controllability. This concept was for the first time formulated by Kalman in [17] in a different way but according to Theorem 4 these two formulations are equivalent.

Definition 3 (Uniform complete controllability) *System (5) is called uniformly completely controllable on* $\mathbb{J}$*, where* $\mathbb{J}$ *is* $\mathbb{R}_+$ *or* $\mathbb{R}$*, if there exist* $\alpha, K > 0$ *such that for all* $(t_0, \xi) \in \mathbb{J} \times \mathbb{R}^n$ *there exists a control* $u \in \mathcal{KC}_{m,1}(\mathbb{J})$ *such that* $x(t_0 + K, t_0, 0, u) = \xi$ *and*

$$\|u(t)\| \leq \alpha \|\xi\|, \qquad t \in [t_0, t_0 + K] .$$

If in system (5) we apply a control of the form

$$u(t) = F(t)x(t),$$

where the feedback $F \in \mathcal{KC}_{m,n}(\mathbb{J})$, we obtain a so called closed loop system

$$\dot{x} = (A(t) + B(t)F(t))\, x. \tag{6}$$

Our interest in this paper is to know the possibility of assigning $\Sigma_{\mathrm{ED}}^{\mathbb{J}}(A + BF)$. We have the following definition of assignability of dichotomy spectrum which is a continuous time counterpart of the definition from [11].

Definition 4 (Assignability of dichotomy spectrum) *The dichotomy spectrum* $\Sigma_{\mathrm{ED}}^{\mathbb{J}}(A + BF)$ *of (6) is called* assignable *on* $\mathbb{J}$ *if for arbitrary* $1 \leq \ell \leq n$ *and arbitrary disjoint closed intervals* $[\alpha_1, \beta_1], \ldots, [\alpha_\ell, \beta_\ell]$*, there exists a feedback* $F \in \mathcal{KC}_{m,n}(\mathbb{J})$ *such that* $\Sigma_{\mathrm{ED}}^{\mathbb{J}}(A + BF) = \cup_{i=1}^{\ell} [\alpha_i, \beta_i]$.

Remark 1 *(i) Recall that for system (1) the* Lyapunov exponent *of a nontrivial solution* $X_M(\cdot, 0)\xi$ *of (1) is given by*

$$\chi(\xi) := \limsup_{t\to\infty} \frac{1}{t} \log \|X_M(t, 0)\xi\|.$$

The Lyapunov spectrum of (1) is defined as

$$\Sigma_{\mathrm{Lya}}(M) := \bigcup_{0 \neq \xi \in \mathbb{R}^n} \chi(\xi).$$

It is known that $\Sigma_{\mathrm{Lya}}(M)$ *consists of at most* n *elements (cf. [1, Chapter II]). Furthermore, suppose that* $\Sigma_{\mathrm{ED}}^{\mathbb{R}_+}(M)$ *is represented as a disjoint union of* ℓ *intervals* $\bigcup_{i=1}^{\ell} [a_i, b_i]$*. Then,*

$$\Sigma_{\mathrm{Lya}}(M) \subset \Sigma_{\mathrm{ED}}^{\mathbb{R}_+}(M), \qquad \Sigma_{\mathrm{Lya}}(M) \cap [\alpha_i, \beta_i] \neq \emptyset, \tag{7}$$

for all $i = 1, ..., \ell$*, see, e.g. [24, p. 106] (cf. [16]).*

(ii) Suppose that the dichotomy spectrum of system (5) *is assignable. Now, let* $\{\lambda_1, \ldots, \lambda_\ell\}$ *be an arbitrary set of* ℓ *real numbers, where* $1 \le \ell \le n$. *Let* $\alpha_i = \beta_i = \lambda_i$ *for* $1 \le i \le \ell$. *There exists a feedback* $F \in \mathcal{KC}_{m,n}$ *such that* $\Sigma_{\text{ED}}^{\mathbb{R}_+}(A + BF) = \bigcup_{i=1}^{\ell}\{\lambda_i\}$. *This together with* (7) *implies that*

$$\Sigma_{\text{ED}}^{\mathbb{R}_+}(A + BF) = \Sigma_{\text{Lya}}(A + BF) = \bigcup_{i=1}^{\ell}\{\lambda_i\}.$$

Consequently, for continuous time-varying linear control systems assignability of dichotomy spectrum implies assignability of Lyapunov spectrum. However, the converse statement is, in general, not true, see Example 2 at the end of the paper.

We now state the main result of this paper.

Theorem 2 (Characterization of assignability of dichotomy spectrum) *Consider system (5) on* $\mathbb{J}$, *where* $\mathbb{J}$ *is* $\mathbb{R}_+$ *or* $\mathbb{R}$. *Assume that* $B : \mathbb{J} \to \mathbb{R}^{n\times m}$ *is piecewise uniformly continuous and bounded. Then the dichotomy spectrum of (6) on* $\mathbb{J}$ *is assignable if and only if system (5) is uniformly completely controllable on* $\mathbb{J}$.

3 Preparatory results

3.1 Uniform complete stablization

The concept of uniform complete stabilizability was used for the first time in the paper of Ikeda et al. [15] with $\mathbb{J} = \mathbb{R}_+$ (see also the discussion in Remark 3.10 in [2]).

Definition 5 (Uniform complete stabilizability) *System (5) (*$\mathbb{J}$ *is* $\mathbb{R}_+$ *or* $\mathbb{R}$*) is called uniformly completely stabilizable if for any* $\alpha \in \mathbb{R}_+$, *there exist a feedback* $F \in \mathcal{KC}_{m,n}(\mathbb{J})$ *and* $C \in \mathbb{R}_+$ *such that*

$$\|X_{A+BF}(t_2, t_1)\| \le Ce^{-\alpha(t_2 - t_1)} \tag{8}$$

for all t_2, $t_1 \in \mathbb{J}$, $t_2 \ge t_1$.

The proof of the next theorem for $\mathbb{J} = \mathbb{R}_+$ can be found in [15] and for $\mathbb{J} = \mathbb{R}$ in [30].

Theorem 3 *System (5) is uniformly completely controllable on* $\mathbb{J}$ *(*$\mathbb{J}$ *is* $\mathbb{R}_+$ *or* $\mathbb{R}$ *) if and only if it is uniformly completely stabilizable on* $\mathbb{J}$.

3.2 Conjugating uniform complete controllability systems to upper triangular systems by linear state feedback

We start this subsection by recalling the notion of asymptotical equivalence for continuous time-varying linear systems (this concept is also known in the literature as kinematic similarity, Lyapunov similarity or simply equivalence).

Definition 6 (Asymptotical Equivalence) *Suppose that $T : \mathbb{J} \to \mathbb{R}^{n\times n}$ is a family of invertible matrices such that T is continuously differentiable and T, T^{-1}, $\dot{T}$ are bounded, then T is called a Lyapunov matrix. The linear transformation $y = T(t)x$ is called then the Lyapunov transformation. Two linear continuous time-varying systems*

$$\dot{x} = M(t)x, \qquad \dot{y} = N(t)y,$$

where $M, N \in \mathcal{KC}_{n,n}(\mathbb{J})$, are said to be asymptotically equivalent if there exists a Lyapunov matrix $T : \mathbb{J} \to \mathbb{R}^{n\times n}$ such that

$$\dot{T}(t) = N(t)T(t) - T(t)M(t) \qquad \text{for } t \in \mathbb{J}.$$

Observe that the set of all Lyapunov transformations of $\mathbb{R}^n$ form a group with composition of maps.

Before stating and proving the main result of this section on conjugating uniform complete controllability systems to upper triangular systems by linear state feedback, we recall the following well known characterization of uniform complete controllability (see [17] for $\mathbb{J} = \mathbb{R}_+$ and [19] for $\mathbb{J} = \mathbb{R}$).

Theorem 4 (Kalman's characterization of uniform complete controllability) *System (5) is uniformly completely controllable on $\mathbb{J}$ if and only if there exist positive numbers ρ and ϑ such that the controllability matrix*

$$W(t_0, t_0 + \vartheta) = \int_{t_0}^{t_0+\vartheta} X_A(t_0, s)B(s)B^{\mathrm{T}}(s)X_A^{\mathrm{T}}(t_0, s)ds$$

of system (5) on the interval $[t_0, t_0 + \vartheta]$ satisfies the inequality

$$\xi^{\mathrm{T}} W(t_0, t_0 + \vartheta)\, \xi \geq \rho \, \|\xi\|^2 \tag{9}$$

for any $t_0 \in \mathbb{J}$ and $\xi \in \mathbb{R}^n$.

Theorem 5 *If system (5) is uniformly completely controllable on $\mathbb{J}$ ($\mathbb{J}$ is $\mathbb{R}_+$ or $\mathbb{R}$) and B is piecewise uniformly continuous, then for arbitrary piecewise continuous bounded functions $p_i : \mathbb{J} \to \mathbb{R}$, $i = 1, ..., n$, there exists a feedback $F \in \mathcal{KC}_{m,n}(\mathbb{J})$ such that the closed loop system (6) is asymptotically equivalent to a system with an upper triangular piecewise continuous bounded matrix function whose diagonal coincides with $(p_1, ..., p_n)$.*

The main ingredient of the proof is from paper [21] in which a proof for two-sided time system has been done. It seems that the statement of the proof for two-sided time system can be carried over to the case $\mathbb{J} = \mathbb{R}_+$ by a slight modification of notations of the proof from [21]. However, to make the paper self-contained, we establish below a direct way to use the result in [21] (two-sided time system) to one-sided time system. This way was done by a suitable extension of a one-sided time system to two-sided time systems.

Proof of Theorem 5. We only consider the case $\mathbb{J} = \mathbb{R}_+$. Let us extend the original system $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, $B : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ to a system $\overline{A} : \mathbb{R} \to \mathbb{R}^{n\times n}$, $\overline{B} : \mathbb{R} \to \mathbb{R}^{n\times m}$ as follows

$$\overline{A}(t) = A(-t) \text{ and } \overline{B}(t) = B(-t)$$

for $t \in \mathbb{R}_-$ ($\overline{A}(t) = A(t)$ and $\overline{B}(t) = B(t)$ for $t \in \mathbb{R}_+$). From Theorem 4 it follows that uniform complete controllability of the pair (A, B) implies the uniform complete controllability of $\left(\overline{A}, \overline{B}\right)$. Indeed, suppose that ϑ and ρ are the constants from (9) for the pair (A, B). We will show that

$$\xi^{\mathrm{T}}\overline{W}\left(t_0, t_0 + 2\vartheta\right)\xi \geq \overline{\rho}\,\|\xi\|^2 \tag{10}$$

for any $t_0 \in \mathbb{J}$ and $\xi \in \mathbb{R}^n$, where

$$\overline{W}\left(t_0, t_0 + 2\vartheta\right) = \int_{t_0}^{t_0+2\vartheta} X_{\overline{A}}(t_0, s)\overline{B}(s)\overline{B}^{\mathrm{T}}(s)X_{\overline{A}}^{\mathrm{T}}(t_0, s)ds$$

and

$$\overline{\rho} = \min\left\{\rho, \frac{\rho}{\sup\limits_{s\in[-\vartheta,0]} \|X_A\left(0, s\right)\|^2}\right\}.$$

For $t_0 \geq 0$ and for $t_0 \leq -\,\vartheta$ the inequality (10) follows from the fact that

$\overline{W}(t_0, t_0 + 2\vartheta) \geq W(t_0, t_0 + \vartheta)$. Suppose now that $t_0 \in (-\vartheta, 0)$, then

$$\begin{aligned} \overline{W}(t_0, t_0 + 2\vartheta) &= \int_{t_0}^{t_0+2\vartheta} X_{\overline{A}}(t_0, s)\overline{B}(s)\overline{B}^{\mathrm{T}}(s)X_{\overline{A}}^{\mathrm{T}}(t_0, s)ds \\ &\geq \int_0^{\vartheta} X_{\overline{A}}(t_0, s)\overline{B}(s)\overline{B}^{\mathrm{T}}(s)X_{\overline{A}}^{\mathrm{T}}(t_0, s)ds \\ &= X_{\overline{A}}(t_0, 0)W(0, \vartheta)\, X_{\overline{A}}^{\mathrm{T}}(t_0, 0). \end{aligned}$$

Therefore

$$\begin{aligned} \xi^{\mathrm{T}}\overline{W}(t_0, t_0 + 2\vartheta)\,\xi &\geq \xi^{\mathrm{T}} X_{\overline{A}}(t_0, 0)W(0, \vartheta)\, X_{\overline{A}}^{\mathrm{T}}(t_0, 0)\xi \\ &\geq \rho \left\| X_{\overline{A}}^{\mathrm{T}}(t_0, 0)\xi \right\|^2 \\ &\geq \frac{\rho}{\sup\limits_{s\in[-\vartheta,0]} \left\| X_A(0, s) \right\|^2} \left\| \xi \right\|^2. \end{aligned}$$

This completes the proof of (10). Using the result in [21] to the pair $(\overline{A}, \overline{B})$ and functions $\overline{p}_i : \mathbb{R} \to \mathbb{R}^{n\times m}$, $\overline{p}(t) = 0$ for $t \in \mathbb{R}_-$ and $\overline{p}(t) = p(t)$ for $t \in \mathbb{R}_+$, we obtain a feedback $\overline{F} : \mathbb{R} \to \mathbb{R}^{m\times n}$ whose restriction $\overline{F}\big|_{\mathbb{R}_+}$ is the desired feedback for the pair (A, B). ■

3.3 Dichotomy spectrum of upper-triangular time-varying linear systems

Our aim in this subsection is to develop some results in computing the dichotomy spectrum of upper triangular time-varying linear systems. To do this, we start with the simplest form of linear time-varying system

$$\dot{x} = m(t)x, \tag{11}$$

where $m \in \mathcal{KC}_{1,1}(\mathbb{J})$. A necessary and sufficient condition for possessing an ED of (11) on $\mathbb{J}$ and an explicit form of the dichotomy spectrum of (11) are given by the following Lemma. It was mentioned in the literature (see [7] and Proposition 5 in [14]) but without proof.

Lemma 1 *System* (11) *has an ED if and only if*

$$\alpha := \liminf_{t-s\to\infty} \frac{1}{t-s} \int_s^t m(\tau)\, d\tau > 0 \tag{12}$$

or

$$\beta := \limsup_{t-s\to\infty} \frac{1}{t-s} \int_s^t m(\tau)\, d\tau < 0 \tag{13}$$

Moreover, $\Sigma_{\mathrm{ED}}^{\mathbb{J}}(m) = [\alpha, \beta]$.

Proof. For the transition operator of system (11) we have

$$X_m(t,s) = \exp\left(\int_s^t m(\tau)\, d\tau\right) \qquad \text{for } s,t \in \mathbb{J}$$

Suppose that system (11) has an ED, then $P(s) \equiv 1$ or $P(s) \equiv 0$. In the first case we get from (2)

$$X_m(t,s) = \exp\left(\int_s^t m(\tau)\, d\tau\right) \leq K e^{-\varepsilon(t-s)}$$

and therefore

$$\frac{1}{t-s} \int_s^t m(\tau)\, d\tau \leq \frac{\log K}{t-s} - \varepsilon$$

for all $s \leq t$, $s,t \in \mathbb{J}$ and certain $K > 1$, $\varepsilon > 0$. Taking the upper limit when $t - s \to \infty$ we obtain (13). Similarly, in case of $P(s) \equiv 0$, from (3) we get (12).

Suppose now that (12) holds. Let us fix $\delta > 0$ such that $\alpha - \delta > 0$, then by the definition of lower limit there exists $\Delta > 0$ such that

$$\frac{1}{t-s} \int_s^t m(\tau)\, d\tau \geq \alpha - \delta$$

and therefore

$$\exp\left(\int_s^t m(\tau)\, d\tau\right) \geq e^{(t-s)(\alpha-\delta)}$$

for all $t - s > \Delta$, $s,t \in \mathbb{J}$. Since m is bounded, there exists $K > 0$ such that

$$\exp\left(\int_s^t m(\tau)\, d\tau\right) \geq K e^{(t-s)(\alpha-\delta)}$$

for all $s,\ t \in \mathbb{J}$. The last inequality implies that system (11) has an ED with $P(s) \equiv 0$. Similarly we may show that (13) implies an ED of (11) with $P(s) \equiv 1$. Finally, the equality $\Sigma_{\mathrm{ED}}^{\mathbb{J}}(m) = [\alpha, \beta]$ follows from the first part of the lemma and the definition of dichotomy spectrum. ■

Next, we study upper triangular time-varying systems of the form

$$\dot{x}(t) = U(t)x(t), \tag{14}$$

where $U \in \mathcal{KC}_{n,n}(\mathbb{J})$ and $U(t) = (u_{ij}(t))_{1 \le i,j \le n}$ is an upper triangular matrix for any $t \in \mathbb{J}$. The following theorem gives some relations between the dichotomy spectra of $\Sigma_{\mathrm{ED}}^{\mathbb{J}}(U)$ and the dichotomy spectra of the diagonal of U. A proof of the first part is in [13] Theorem 5.5, see also Proposition 5 in [14]. A proof of the second part can be seen in [7], Section 4.

Theorem 6 *Consider system* (14) *on* $\mathbb{J}$ *with upper triangular matrix* U. *Then the following statements hold:*

(i) If $\mathbb{J} = \mathbb{R}_+$ or $\mathbb{J} = \mathbb{R}_-$ then

$$\Sigma_{\mathrm{ED}}^{\mathbb{J}}(U) = \bigcup_{i=1}^{n} \Sigma_{\mathrm{ED}}^{\mathbb{J}}(u_{ii}),$$

(ii) If $\mathbb{J} = \mathbb{R}$ then

$$\Sigma_{\mathrm{ED}}^{\mathbb{R}}(U) \subset \bigcup_{i=1}^{n} \Sigma_{\mathrm{ED}}^{\mathbb{R}}(u_{ii}). \tag{15}$$

It is worth to notice that the inclusion (15) may be strict as demonstrated by the following example inspired by Example 5.10 in [23].

Example 1 *Consider a two dimensional system* (14) *with*

$$U(t) = \begin{bmatrix} u_{11}(t) & u_{12}(t) \\ 0 & u_{22}(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} & \textit{for } t \ge 0, \\[2ex] \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} & \textit{for } t < 0. \end{cases}$$

A direct calculation shows that $\Sigma_{\mathrm{ED}}^{\mathbb{R}}(U) = \{-1, 1\}$, *whereas* $\Sigma_{\mathrm{ED}}^{\mathbb{R}}(u_{11}) = \Sigma_{\mathrm{ED}}^{\mathbb{R}}(u_{22}) = [-1, 1]$.

Finally, we will show that if the diagonal elements have a special form, then the dichotomy spectrum on $\mathbb{R}$ of an upper triangular system is the union of the dichotomy spectra of the subsystems corresponding to the diagonal entries.

Theorem 7 *Consider system* (14) *on* $\mathbb{R}$ *with upper triangular matrix* U *and suppose that*

$$u_{ii}(t) = u_{ii}(-t) \tag{16}$$

for all $i = 1, ..., n$ *and* $t \in \mathbb{R}$. *Then*

$$\Sigma_{\mathrm{ED}}^{\mathbb{R}}(U) = \bigcup_{i=1}^{n} \Sigma_{\mathrm{ED}}^{\mathbb{R}}(u_{ii}).$$

Proof. According to Theorem 6(ii) to complete the proof it is sufficient to show that

$$\Sigma_{\mathrm{ED}}^{\mathbb{R}}(u_{ii}) \subset \Sigma_{\mathrm{ED}}^{\mathbb{R}_+}(u_{ii}) \cup \Sigma_{\mathrm{ED}}^{\mathbb{R}_-}(u_{ii}) \tag{17}$$

for all $i = 1, ..., n$. For this purpose let us fix $i = 1, ..., n$ and $\gamma \notin \Sigma_{\mathrm{ED}}^{\mathbb{R}_+}(u_{ii})$. Then, by definition of dichotomy spectrum one of the following alternatives holds:

(A1) There exist $K \geq 1$ and $\alpha > 0$ such that

$$\exp\left(\int_s^t u_{ii}(\tau)d\tau\right) \leq$$
$$Ke^{(\gamma-\alpha)(t-s)}$$
$$\text{if } s \leq t,\ s,t \in \mathbb{R}_+.$$

Thus, by (16) we also have

$$\begin{aligned}
\exp\left(\int_s^t u_{ii}(\tau)d\tau\right) &= \exp\left(\int_s^0 u_{ii}(\tau)d\tau + \int_0^t u_{ii}(\tau)d\tau\right) \\
&= \exp\left(\int_0^{-s} u_{ii}(\tau)d\tau + \int_0^t u_{ii}(\tau)d\tau\right) \\
&\leq K^2 e^{(\gamma-\alpha)(t-s)}
\end{aligned}$$

if $s \le 0 \le t,\ s,t \in \mathbb{R}$ and

$$\exp\left(\int_s^t u_{ii}(\tau)d\tau\right) = \exp\left(\int_{-t}^{-s} u_{ii}(\tau)d\tau\right) \le Ke^{(\gamma-\alpha)(t-s)}$$

if $s \le t \le 0,\ s,t \in \mathbb{R}$. It means that $\gamma \notin \Sigma_{\mathrm{ED}}^{\mathbb{R}}(u_{ii})$.

(A2) There exist $K \ge 1$ and $\beta > 0$ such that

$$\exp\left(\int_s^t u_{ii}(\tau)d\tau\right) \le Ke^{(\gamma+\beta)(t-s)} \text{ if } t \le s,\ \ s,t \in \mathbb{R}_+.$$

Thus, by (16) we also have

$$\begin{aligned}\exp\left(\int_s^t u_{ii}(\tau)d\tau\right) &= \exp\left(\int_s^0 u_{ii}(\tau)d\tau + \int_0^t u_{ii}(\tau)d\tau\right)\\ &= \exp\left(\int_0^{-s} u_{ii}(\tau)d\tau + \int_0^t u_{ii}(\tau)d\tau\right)\\ &\le K^2e^{(\gamma+\beta)(t-s)}\end{aligned}$$

if $s \le 0 \le t,\ s,t \in \mathbb{R}$ and

$$\exp\left(\int_s^t u_{ii}(\tau)d\tau\right) = \exp\left(\int_{-t}^{-s} u_{ii}(\tau)d\tau\right) \le Ke^{(\gamma+\beta)(t-s)}$$

if $s \le t \le 0,\ s,t \in \mathbb{R}$. It means that $\gamma \notin \Sigma_{\mathrm{ED}}^{\mathbb{R}}(u_{ii})$.

Since $\gamma \notin \Sigma_{\mathrm{ED}}^{\mathbb{R}_+}(u_{ii})$ is arbitrary it follows that $\Sigma_{\mathrm{ED}}^{\mathbb{R}}(u_{ii}) \subset \Sigma_{\mathrm{ED}}^{\mathbb{R}_+}(u_{ii})$. This shows (17) and the proof is complete. ■

4 Proof of the Main result

The proof uses the obvious fact that the ED and therefore $\Sigma_{\mathrm{ED}}^{\mathbb{J}}$ of system (5) is preserved under Lyapunov transformations (see also Theorem 3.1 in Chapter IV in [12]).

Proof of Theorem 2. We separate the proof into two cases: $\mathbb{J} = \mathbb{R}_+$ or $\mathbb{J} = \mathbb{R}$:

Case 1: $\mathbb{J} = \mathbb{R}_+$. Suppose that dichotomy spectrum of (5) is assignable on $\mathbb{R}_+$. Let us fix $\alpha \in \mathbb{R}_+$. Then there exists a bounded feedback $F \in \mathcal{K}_{n,m}(\mathbb{J})$ such that

$$\Sigma_{\text{ED}}^{\mathbb{R}_+}(A + BF) = \{-\alpha'\}, \qquad \text{where } \alpha' > \alpha.$$

Thus, $(-\alpha', \infty) \in \rho_{\text{ED}}^{\mathbb{R}_+}(A+BF)$, where $\rho_{\text{ED}}^{\mathbb{R}_+}(A+BF) := \mathbb{R} \setminus \Sigma_{\text{ED}}^{\mathbb{R}_+}(A+BF)$. Then for each $\gamma \in (-\alpha', \infty)$ the system

$$\dot{x}(t) = (A(t) + B(t)F(t) - \gamma I_n)x(t)$$

exhibits an exponential dichotomy with an invariant family of projections $P_\gamma : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$. Thanks to [25, Lemma 6.5], we have

$$\text{im}P_{\gamma_1}(t) = \text{im}P_{\gamma_2}(t) \qquad \text{for } \gamma_1, \gamma_2 \in (-\alpha', \infty). \tag{18}$$

Since $A(t) + B(t)F(t)$ is bounded on $\mathbb{R}_+$, there exist $K, \beta > 0$ such that

$$\|X_{A+BF}(t, s)\| \le Ke^{\beta(t-s)} \quad \text{ for } t \ge s \ge 0.$$

In other words, $P_\beta(t) = I_n$ for all $t \in \mathbb{R}_+$. This together with (18) implies that $P_{-\alpha'}(t) = I_n$ for all $t \in \mathbb{R}_+$. Then, there exist $C, \varepsilon > 0$ such that

$$\|X_{A+BF}(t, s)\| \le Ce^{(-\alpha'-\varepsilon)(t-s)} \quad \text{ for } t \ge s \ge 0.$$

Since $\alpha \in \mathbb{R}_+$ is arbitrary, then (5) is uniformly completely stabilizable and therefore by Theorem 3 it is uniformly completely controllable.

Conversely, suppose now that (5) is uniformly completely controllable. Let us fix ℓ, $1 \le \ell \le n$, and arbitrary disjoint closed intervals $[\alpha_1, \beta_1], \ldots, [\alpha_\ell, \beta_\ell]$. Consider any piecewise continuous bounded functions $u_{ii} : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, ..., l$ such that

$$\alpha_i = \liminf_{t-s\to\infty} \frac{1}{t-s} \int_\tau^t u_{ii}(\tau)\, d\tau$$

and

$$\beta_i = \limsup_{t-s\to\infty} \frac{1}{t-s} \int_s^t u_{ii}(\tau)\, d\tau.$$

Moreover for $\ell+1 \le i \le n$ we define $u_{ii} = u_{11}$. According to Theorem 5, there exists a feedback $F \in \mathcal{KC}_{m,n}(\mathbb{J})$ such that the closed system (6) is asymptotically equivalent to a system with an upper triangular piecewise continuous bounded matrix function whose diagonal coincides with $(u_{11}, ..., u_{nn})$. The assignability of the dichotomy spectrum on $\mathbb{R}_+$ now follows from Theorem 6.

Case 2: $\mathbb{J} = \mathbb{R}$. The fact that assignability of the dichotomy spectrum on $\mathbb{R}$ implies uniformly completely controllable may be proved exactly in the same as in the case of $\mathbb{J} = \mathbb{R}_+$. Proving the opposite implication is enough to extend the functions $u_{ii} : \mathbb{R}_+ \to \mathbb{R}$ to functions $u_{ii} : \mathbb{R} \to \mathbb{R}$ by setting $u_{ii}(-t) = u_{ii}(t)$ for $t \in \mathbb{R}_+$ and then we may use Theorem 7. ■

Let us identify each system (1) with the function $M \in \mathcal{KC}_{n,n}(\mathbb{J})$. A. M. Lyapunov in his famous paper [18] considered quantities or properties of systems that are preserved under the action of the group of Lyapunov transformations. These quantities or properties are nowadays called Lyapunov invariants. Examples of Lyapunov invariants serve Lyapunov exponents, dichotomy spectrum, asymtotic stability. Consider some set of Lyapunov invariants $\iota_\alpha, \alpha \in A$. A possible value of this set is defined as a set $\{\widehat{\iota}_\alpha\}_{\alpha \in A}$ for which there exists a system $\widehat{M} \in \mathcal{KC}_{n,n}(\mathbb{J})$ such that $\iota_\alpha\left(\widehat{M}\right) = \widehat{\iota}_\alpha$ for all $\alpha \in A$; here $\iota_\alpha\left(\widehat{M}\right)$ is the value of the invariant ι_α for the system $\widehat{M}$. We say ([21]) that system (6) has the property of simultaneous global assignability of the set of invariants $\{\iota_\alpha\}_{\alpha \in A}$ if, for any possible value $\{\widehat{\iota}_\alpha\}_{\alpha \in A}$ of this set, there exists a feedback $F \in \mathcal{KC}_{m,n}(\mathbb{J})$ such that $\iota_\alpha(A + BF) = \widehat{\iota}_\alpha$ for each $\alpha \in A$. Let $\mathcal{I}$ be the set of all Lyapunov invariants of systems in $\mathcal{KC}_{n,n}(\mathbb{J})$ determined by the diagonal approximation systems for systems with triangular matrices; i.e., $\iota \in \mathcal{I}$ if and only if $\iota(M) = \iota(diag(m_{11}, ..., m_{nn}))$ for arbitrary system with upper triangular matrix $M(t) = (m_{ij}(t))_{1 \le i,\, j \le n}$.The set $\mathcal{I}$ is nonempty and contains, for example, the dichotomy spectrum (see Theorem 6) in case of $\mathbb{J} = \mathbb{R}_+$ and $\mathbb{J} = \mathbb{R}_-$.

Remark 2 *The proof of the fact that uniform complete controllability on $\mathbb{R}_+$ implies the assignability of the dichotomy spectrum on $\mathbb{R}_+$ may be also concluded from Corollary 2 in [21], where it has been shown that system (6) has the property of simultaneous global assignable of any set of Lyapunov invariants in $\mathcal{I}$. But this line of reasoning is not longer true when we consider $\mathbb{J} = \mathbb{R}$ since then the dichotomy spectrum of an upper triangular matrix-valued linear system may be a proper subset of the union of the dichotomy spectra of its diagonal system as it was demonstrated in Example 1.*

5 Examples

This section is devoted to study scalar continuous time-varying control system

$$\dot{x}(t) = a(t)x + b(t)u(t), \tag{19}$$

where $a, b \in \mathcal{KC}_{1,1}(\mathbb{R}_+)$. The following result consists of two parts. In the first part, a simple necessary and sufficient condition for which (19) is uniformly completely controllable is provided. Next, in the second part we construct an explicit control to assign a given dichotomy spectral interval.

Proposition 1 *(i) System* (19) *is uniformly completely controllable iff there exist* $\widehat{\rho} > 0$ *and* $\vartheta > 0$ *such that*

$$\int_t^{t+\vartheta} b(s)^2 \, ds \geq \widehat{\rho} \tag{20}$$

for all $t > 0$.

(ii) Suppose that (20) *holds. Let* $[\alpha, \beta]$ *be an arbitrary given spectral interval. Define* $\mathcal{I} := \cup_{n \in \mathbb{Z}_{\geq 0}} \{2^{2n}, 2^{2n}+1, \ldots, 2^{2n+1}-1\}$ *and*

$$f(t) := \begin{cases} b(t) \frac{\alpha\vartheta - \int_{k\vartheta}^{(k+1)\vartheta} a(s)\, ds}{\int_{k\vartheta}^{(k+1)\vartheta} b(s)^2 \, ds} & \text{for} \quad \begin{array}{c} t \in [k\vartheta, (k+1)\vartheta), \\ k \in \mathcal{I}, \end{array} \\ b(t) \frac{\beta\vartheta - \int_{k\vartheta}^{(k+1)\vartheta} a(s)\, ds}{\int_{k\vartheta}^{(k+1)\vartheta} b(s)^2 \, ds} & \text{for} \quad \begin{array}{c} t \in [k\vartheta, (k+1)\vartheta) \\ k \in \mathbb{Z}_{\geq 0} \setminus \mathcal{I}. \end{array} \end{cases}$$

Then, $f \in \mathcal{KC}_{1,1}(\mathbb{R}_+)$ *and the dichotomy spectrum* $\Sigma_{\mathrm{ED}}^{\mathbb{R}_+}(a + bf)$ *of the closed loop system*

$$\dot{x} = (a(t) + b(t)f(t))x(t) \tag{21}$$

satisfies $\Sigma_{\mathrm{ED}}^{\mathbb{R}_+}(a + bf) = [\alpha, \beta]$.

Proof. (i) A direct computation yields that the controllability matrix of (19) is given by

$$W(t_1, t_2) = \int_{t_1}^{t_2} e^{2\int_{t_1}^{s} a(\tau)\, d\tau} b(s)^2 \, ds. \tag{22}$$

Suppose that (19) is uniformly completely controllable. Then, by virtue of Theorem 4 there exist positive numbers ρ and ϑ such that $W(t_0, t_0 + \vartheta) \geq \rho$ for all $t_0 \in \mathbb{R}_+$. Let $\kappa := \sup_{t \in \mathbb{R}_+} |a(t)|$. Thus, by (22), we arrive at

$$e^{2\kappa\vartheta} \int_t^{t+\vartheta} b(s)^2 \, ds \geq W(t, t+\vartheta) \geq \rho,$$

which proves (20) with $\widehat{\rho} := \frac{\rho}{e^{2\kappa\vartheta}}$. Conversely, suppose that there exist $\widehat{\rho} > 0$ and $\vartheta > 0$ such that (20) holds. Then, for $\kappa := \sup_{t\in\mathbb{R}_+} |a(t)|$ we have for all $t \in \mathbb{R}_+$

$$\begin{aligned} W(t, t+\vartheta) &= \int_t^{t+\vartheta} e^{2\int_t^s a(\tau)\, d\tau} b(s)^2\, ds \\ &\geq e^{-2\kappa\vartheta} \int_t^{t+\vartheta} b(s)^2\, ds \geq e^{-2\kappa\vartheta}\rho. \end{aligned}$$

Hence, in light of Theorem 4, system (19) is uniformly completely controllable. The proof of this part is complete.
(ii) By (i), it is obvious that $f \in \mathcal{KC}_{1,1}(\mathbb{R}_+)$. Now, it remains to compute $\Sigma_{\mathrm{ED}}^{\mathbb{R}_+}(a+bf)$. For this purpose, let $X_{a+bf}(\cdot,\cdot)$ denote the transition matrix associated with (21). Then, for $s, t \in \mathbb{R}_+$

$$X_{a+bf}(t,s) = \exp\left(\int_s^t (a(u) + b(u)f(u))\, du\right). \tag{23}$$

In particular, for all $k \in \mathbb{Z}_{\geq 0}$ we have

$$X_{a+bf}((k+1)\vartheta, k\vartheta) = \exp\left(\int_{k\vartheta}^{(k+1)\vartheta} (a(s) + b(s)f(s))\ ds\right).$$

Then, by definition of f we arrive at

$$X_{a+bf}((k+1)\vartheta, k\vartheta) = \begin{cases} e^{\alpha\vartheta} & \text{if } k \in \mathcal{I}, \\ e^{\beta\vartheta} & \text{if } k \notin \mathcal{I} \end{cases}. \tag{24}$$

Thus, for arbitrary but fixed $\gamma \in [\alpha, \beta]$ we have

$$X_{a+bf}((k+1)\vartheta, k\vartheta) \leq e^{\gamma\vartheta} \qquad \text{for } k \in \mathcal{I}$$

and

$$X_{a+bf}((k+1)\vartheta, k\vartheta) \geq e^{\gamma\vartheta} \qquad \text{for } k \notin \mathcal{I}.$$

By the structure of $\mathcal{I}$, we arrive at

$$X_{a+bf}((2^{2n+1}-1)\vartheta, 2^{2n}\vartheta) \leq e^{\gamma\vartheta 2^{2n}} \qquad \text{for } n \in \mathbb{Z}_{\geq 0}$$

and

$$X_{a+bf}((2^{2n+2}-1)\vartheta, 2^{2n+1}\vartheta) \geq e^{\gamma\vartheta 2^{2n+1}} \text{ for } n \in \mathbb{Z}_{\geq 0}.$$

Consequently, the shifted system

$$\dot{x} = (a(t) + b(t)f(t) - \gamma)x(t)$$

does not admit an exponential dichotomy. Since γ can be chosen arbitrarily in $[\alpha, \beta]$ it follows that $[\alpha, \beta] \subset \Sigma_{\mathrm{ED}}^{\mathbb{R}_+}(a+bf)$. To conclude the proof, we need to verify that for all $\gamma \notin [\alpha, \beta]$ the shifted system $\dot{x} = (a(t)+b(t)f(t)-\gamma)x(t)$ exhibits an exponential dichotomy. Since $a + bf \in \mathcal{K}_{1,1}(\mathbb{R}_+)$ there exists $K \geq 1$ such that

$$\sup_{|t-s|\leq\vartheta} |X_{a+bf}(t, s)| \leq K,$$

which implies that

$$\sup_{|t-s|\leq\vartheta} |X_{a+bf}(t, s)e^{-\beta(t-s)}| \leq \widehat{K}, \text{where } \widehat{K} := Ke^{|\beta|\vartheta}.$$

Thus, by virtue of (23), for all $t \geq s \geq 0$ we have

$$\begin{aligned}
&\frac{|X_{a+bf}(t,s)|}{\left|X_{a+bf}\left(\left[\frac{t}{\vartheta}\right]\vartheta, \left[\frac{s}{\vartheta}\right]\vartheta\right)\right|}\\
&= \frac{\exp\left(\int_s^t (a(u) + b(u)f(u))\, du\right)}{\exp\left(\int_{\left[\frac{s}{\vartheta}\right]\vartheta}^{\left[\frac{t}{\vartheta}\right]\vartheta} (a(u) + b(u)f(u))\, du\right)}\\
&= \frac{\exp\left(\int_{\left[\frac{t}{\vartheta}\right]\vartheta}^{t} (a(u) + b(u)f(u))\, du\right)}{\exp\left(\int_{\left[\frac{s}{\vartheta}\right]\vartheta}^{s} (a(u) + b(u)f(u))\, du\right)}\\
&= \left|X_{a+bf}\left(t, \left[\frac{t}{\vartheta}\right]\vartheta\right)\right| \left|X_{a+bf}\left(\left[\frac{s}{\vartheta}\right]\vartheta, s\right)\right|\\
&\leq \widehat{K}^2 e^{\beta(t-s-(\left[\frac{t}{\vartheta}\right]\vartheta-\left[\frac{s}{\vartheta}\right]\vartheta))},
\end{aligned}$$

where $[x]$ denotes the largest integer smaller or equal x. On the other hand, by (24) we have for all $t \geq s \geq 0$

$$\begin{aligned}
\left|X_{a+bf}\left(\left[\frac{t}{\vartheta}\right]\vartheta, \left[\frac{s}{\vartheta}\right]\vartheta\right)\right| &= \prod_{k=\left[\frac{s}{\vartheta}\right]}^{\left[\frac{t}{\vartheta}\right]-1} |X_{a+bf}((k+1)\vartheta, k\vartheta)|\\
&\leq e^{\beta(\left[\frac{t}{\vartheta}\right]\vartheta-\left[\frac{s}{\vartheta}\right]\vartheta)}.
\end{aligned}$$

Consequently, we have

$$|X_{a+bf}(t,s)| \le \widehat{K}^2 e^{\beta(t-s)} \quad \text{for all } t \ge s.$$

Thus, for any $\gamma > \beta$ the shifted system $\dot{x} = (a(t) + b(t)f(t) - \gamma)x(t)$ exhibits an exponential dichotomy. Similarly, for $\gamma < \alpha$ the corresponding shifted system also exhibits an exponential dichotomy. The proof is complete. ■

We end this paragraph with an example of a system that has assignable Lyapunov spectrum but its dichotomy spectrum is not assignable.

Example 2 *Let us define a sequence $(t_k)_{k\in\mathbb{N}}$ by the recurrent formula*

$$t_1 = 1, \quad t_{2m} = mt_{2m-1}, \quad t_{2m+1} = m + t_{2m}$$

for all $m \in \mathbb{N}$. The sequence $(t_k)_{k\in\mathbb{N}}$ is strictly increasing for $k \ge 2$ and tends to $+\infty$.
Put

$$b(t) = \begin{cases} 1 & \text{if either } t \in (-\infty, 2) \text{ or } t \in [t_{2m-1}, t_{2m}), \\ 0 & \text{if } t \in [t_{2m}, t_{2m+1}), \end{cases}$$

for $m = 2, 3, \dots,$ and consider the scalar linear control system

$$\dot{x} = b(t)u. \tag{25}$$

This example has been considered in [21], where it has been shown that system (25) is not uniformly completely controllable neither on $\mathbb{R}$ nor on $\mathbb{R}_+$ but it has assignable Lyapunov spectrum. From Theorem 2 it follows that system (25) does not have the dichotomy spectrum assignable. Consider this system on $\mathbb{R}_+$. We will show that only intervals of the form $[\alpha, \beta]$, where $0 \in [\alpha, \beta]$ may be a dichotomy spectrum of system

$$\dot{x} = b(t)f(t)x, \tag{26}$$

where $f \in \mathcal{KC}_{1,1}(\mathbb{R}_+)$ and that each such interval is a dichotomy spectrum of system (26) for certain linear piecewise continuous and bounded feedback. To prove this we will use Lemma 1. Suppose that for certain $f \in \mathcal{KC}_{1,1}(\mathbb{R}_+)$ an interval $[x, y]$ is the dichotomy spectrum of (26). Since $t_{2m+1} - t_{2m} \to \infty$ when $m \to \infty$, then

$$y = \limsup_{t-s\to\infty} \frac{1}{t-s} \int_s^t b(\tau)f(\tau)d\tau \ge$$

$$\limsup_{m\to\infty} \frac{1}{t_{2m+1} - t_{2m}} \int_{t_{2m}}^{t_{2m+1}} b(\tau)f(\tau)d\tau = 0$$

and

$$x = \liminf_{t-s\to\infty} \frac{1}{t-s} \int_{\tau}^{t} b(\tau)f(\tau)d\tau \leq$$

$$\liminf_{m\to\infty} \frac{1}{t_{2m+1} - t_{2m}} \int_{t_{2m}}^{t_{2m+1}} b(\tau)f(\tau)d\tau = 0.$$

Therefore, $0 \in [x, y]$.

Now let us fix an interval $[\alpha, \beta]$ *such that* $0 \in [\alpha, \beta]$ *and define the linear feedback gain* f *as follows*

$$f(t) = \begin{cases} \beta & \text{for } t \in \left[t_{2m-1}, \frac{t_{2m}+t_{2m-1}}{2}\right), \\ \alpha & \text{for } t \in \left[\frac{t_{2m}+t_{2m-1}}{2}, t_{2m}\right), \\ 0 & \text{otherwise,} \end{cases}$$

for $m \in \mathbb{N}$. *To calculate the dichotomy spectrum* $[x, y]$ *of this system we will apply Lemma 1. Since* $t_{2m} - t_{2m-1} \to \infty$ *when* $m \to \infty$, *then we have*

$$x = \liminf_{t-s\to\infty} \frac{1}{t-s} \int_{\tau}^{t} b(\tau)f(\tau)d\tau \leq$$

$$\liminf_{m\to\infty} \frac{2}{t_{2m} - t_{2m-1}} \int_{\frac{t_{2m}+t_{2m-1}}{2}}^{t_{2m}} b(\tau)f(\tau)d\tau = \alpha.$$

On the other hand, since $b(\tau)f(\tau) \geq \alpha$, *then*

$$x = \liminf_{t-s\to\infty} \frac{1}{t-s} \int_{s}^{t} b(\tau)f(\tau)d\tau \geq \alpha$$

and therefore $x = \alpha$. *In the same way we may show that* $y = \beta$.

6 Conclusions

In this paper we investigated a problem of assignability of dichotomy spectrum by time-varying bounded linear feedback for continuous time-varying control system. We have shown that the dichotomy spectrum is assignable if and only if the system is uniformly completely controllable. We proved this by using the concept of uniform complete stabilizability which is equivalent to uniform complete controllability.

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