THE BOUSSINESQ SYSTEMS ON THE BACKGROUND OF A LINE SOLITARY WAVE

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Abstract. This paper is motivated by the transverse stability properties of the line solitary wave solutions of the (a,b,c,d) class of Boussinesq systems introduced in [12]. We first review the known results on existence of one-dimensional solitary waves. Then we address the question of long time existence of the systems satisfied by a localized perturbation of a line solitary wave.

1. Introduction

The motivation of this paper is the study of transverse stability issues of line solitary waves to the (a,b,c,d) Boussinesq systems introduced in [12] to model surface water waves. They couple the elevation η and the horizontal velocity $\mathbf{u}$ of the wave:

$$(1.1)\qquad \begin{cases} \eta_t + \nabla\cdot\mathbf{u} + \varepsilon\nabla\cdot(\eta\mathbf{u}) + \mu[a\nabla\cdot\Delta\mathbf{u} - b\Delta\eta_t] = 0, \\ \mathbf{u}_t + \nabla\eta + \varepsilon\frac{1}{2}\nabla|\mathbf{u}|^2 + \mu[c\nabla\Delta\eta - d\Delta\mathbf{u}_t] = 0, \quad \text{in } \mathbb{R}^d\times\mathbb{R}, d=1,2. \end{cases}$$

Here μ and ε are the small parameters (shallowness and nonlinearity parameters respectively) defined as

$$\mu = \frac{h^2}{\lambda^2}, \quad \varepsilon = \frac{\alpha}{h}$$

where α is a typical amplitude of the wave, h a typical depth and λ a typical horizontal wavelength.

In the Boussinesq regime, ε and μ are supposed to be of same order, $\varepsilon \sim \mu \ll 1$, and we will take for simplicity $\varepsilon = \mu$, writing (1.1) as

$$(1.2)\qquad \begin{cases} \eta_t + \nabla\cdot\mathbf{u} + \varepsilon[\nabla\cdot(\eta\mathbf{u}) + a\nabla\cdot\Delta\mathbf{u} - b\Delta\eta_t] = 0 \\ \mathbf{u}_t + \nabla\eta + \varepsilon[\frac{1}{2}\nabla|\mathbf{u}|^2 + c\nabla\Delta\eta - d\Delta\mathbf{u}_t] = 0, \end{cases}$$

The class of systems (1.1), (1.2) models water waves on a flat bottom propagating in both directions in the aforementioned regime (see [12, 13, 11]).

One could also derive similar systems with a non trivial bathymetry (non flat bottom), see [18], and one has then to distinguish between the case when the bottom varies slowly and the case where it is strongly varying. In the former case, (1.2) has to be slightly modified and becomes

$$(1.3)\qquad \begin{cases} \eta_t + \nabla\cdot\mathbf{u} + \varepsilon[\nabla\cdot((\eta-\beta)\mathbf{u}) + a\nabla\cdot\Delta\mathbf{u} - b\Delta\eta_t] = 0, \\ \mathbf{u}_t + \nabla\eta + \varepsilon[\frac{1}{2}\nabla|\mathbf{u}|^2 + c\nabla\Delta\eta - d\Delta\mathbf{u}_t] = 0, \end{cases}$$

where β is a smooth function on $\mathbb{R}^d, d=1,2,$ bounded together with its derivatives measuring the bathymetry. In the second case one gets much more complicated systems [18, 47].

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Also, systems similar to (1.2) can be derived for internal waves, see for instance [14, 30, 35].

Recall (see [12, 31]) that the modeling parameters are constrained by the relation

$$a+b+c+d=\frac{1}{3}-\tau,$$

where $\tau \geq 0$ is the surface tension parameter (Bond number).

Recall also [12] that (1.2) is linearly well-posed when

$$a \leq 0, c \leq 0, b \geq 0, d \geq 0, \tag{1.4}$$

and when

$$a=c, b \geq 0, d \geq 0. \tag{1.5}$$

It has been established in [11] that the (a,b,c,d) systems are good approximations of the full water waves system, in the relevant regime, with an error of order $O(\varepsilon^2 t)$. The complete justification of the Boussinesq systems needs thus to establish the *long time existence* for the Cauchy problem, that is on time scales of order $O(1/\varepsilon)$. This has been achieved for all linearly well-posed Boussinesq systems in [48, 55, 16, 56]. Note that no global well-posedness result seems to be known for the Cauchy problem of (1.2) in the spatial two-dimensional case (however the global existence of solutions is established in a few one-dimensional cases, see [13, 2, 57]).

It has been proved by Zakharov [62] that the full water waves system is Hamiltonian. This is not the case in general for the (a,b,c,d) systems due to the transformations made to derive them starting from the following (ill-posed) "original" Boussinesq system obtained by expanding the Dirichlet-Neumann operator with respect to ε (see [12, 44]):

$$\begin{cases} \eta_t + \nabla\cdot\mathbf{u} + \frac{\varepsilon}{3}\Delta\nabla\cdot\mathbf{u} + \varepsilon\nabla\cdot(\eta\mathbf{u}) = 0 \\ \mathbf{u}_t + \nabla\eta + \frac{\varepsilon}{2}\nabla|\mathbf{u}|^2 = 0. \end{cases} \tag{1.6}$$

Nevertheless, an Hamiltonian structure is found when $b = d$. More precisely, denoting by J_ε the skew adjoint matrix operator

$$J_\varepsilon = \begin{pmatrix} 0 & \partial_x(I-\varepsilon b\Delta)^{-1} & \partial_y(I-\varepsilon b\Delta)^{-1} \\ \partial_x(I-\varepsilon b\Delta)^{-1} & 0 & 0 \\ \partial_y(I-\varepsilon b\Delta)^{-1} & 0 & 0 \end{pmatrix},$$

and

$$U = \begin{pmatrix} \eta \\ \mathbf{u} \end{pmatrix},$$

the Boussinesq systems write in this case

$$\partial_t U = -J_\varepsilon(\operatorname{grad}\mathcal{H}_\varepsilon)(U),$$

where $\mathcal{H}_\varepsilon(U)$ is the Hamiltonian given by

$$\mathcal{H}_\varepsilon(U) = \frac{1}{2}\int_{\mathbb{R}^2}\big(-c\varepsilon|\nabla\eta|^2 - a\varepsilon|\nabla\mathbf{u}|^2 + \eta^2 + |\mathbf{u}|^2 + \varepsilon\eta|\mathbf{u}|^2\big)dxdy,$$

so that $\mathcal{H}_\varepsilon(U)$ is (formally) conserved by the flow.

Note that when $d = 1$ the following quantity (momentum) is also formally conserved

$$\mathcal{I}_\varepsilon(U) = \int_{\mathbb{R}} (\eta u + \varepsilon b \eta_x u_x) dx.$$

We are interested here in *localized* solitary wave solutions to (1.2). No such solutions are known to exist in the two-dimensional case (see however [26] for the existence of asymmetric periodic two-dimensional wave patterns).

We will thus focus on one-dimensional solitary waves and more specifically to their transverse stability. To start with we review the known results on the existence of one dimensional traveling waves to (1.2).

We thus look for solutions of (1.2) of the form $\eta = \eta(x - \omega t), u = u(x - \omega t)$ where $x \in \mathbb{R}$ and $(\eta(x), u(x)) \to (0, 0)$ as $|x| \to \infty$ yielding the system

$$\left\{ \begin{array}{l} -\omega\eta + u + \varepsilon u\eta + \varepsilon a u_{xx} - \varepsilon b \eta_{xx} = 0 \\ -\omega u + \eta + \frac{\varepsilon}{2} u^2 + \varepsilon c u_{xx} - \varepsilon d \eta_{xx} = 0. \end{array} \right. \tag{1.7}$$

Min Chen [19, 20] has found in a few cases exact solutions when η and u are proportional but it turns out that the only case compatible with the well posedness conditions (1.4) and (1.5) are

$$a - b + 2d = 0, \ a = c, \ d > 0 \tag{1.8}$$

and

(1.9)
$$a - b + 2d \neq 0 \text{ and } p > 0, \ (p - \frac{1}{2})((b-a)p - b) > 0 \text{ where } p = (-b + c + 2d)/(a - b + 2d) > 0.$$

The exact solutions have a sech profile, namely

$$\eta(x, t) = \eta_0 \text{sech}^2\ (\lambda(x + x_0 - c_s t))$$

and

$$u(x, t) = \pm\sqrt{\frac{3}{\eta_0 + 3}}\eta_0 \text{sech}^2(\lambda(x + x_0 - c_s t)),$$

where λ and c_s are appropriate constants.

Note that conditions (1.8), (1.9) can hold in the non Hamiltonian case $b \neq d$.

On the other hand no general *non existence result* seems to be available. One has nevertheless the following remark which excludes the existence of solitary waves with small velocity when the well-posedness condition (1.5) holds true with moreover $b = d$.

Proposition 1.1. *Let $b = d \geq 0$. Then any H^1 solitary wave solution (η, u) satisfies the identity*

$$\int (u^2 + \eta^2 + 5\varepsilon(a u_x^2 + c\eta_x^2)) dx = 2\omega \int (3u\eta - 5\varepsilon b u_x \eta_x) dx$$

Proof. This is a standard Pohojaev type argument. The following computations can be justified by a regularizing procedure.

We multiply the first equation in (1.7) by xu_x, the second one by $x\eta_x$. Integrating the resulting equations and adding one gets

$$\frac{1}{2}\int (u^2+\eta^2+a\varepsilon u_x^2+c\varepsilon\eta_x^2)=\int(\omega u\eta-\varepsilon b\omega u_x\eta_x-\frac{\varepsilon}{2}u^2\eta). \tag{1.10}$$

We then multiply the first equation in (1.7) by u the second one by η. Integrating the resulting equations and adding one gets

$$\int (u^2+\eta^2-a\varepsilon u_x^2-c\varepsilon\eta_x^2)=\varepsilon\int(2\omega b u_x\eta_x-\frac{3}{2}\eta u^2). \tag{1.11}$$

The proposition results from substracting three times (1.10) from (1.11). □

As a consequence, one obtains that no non trivial solitary waves with ω small exist when

$$a\geq 0,\ c\geq 0,\ b=d=0,$$

or

$$a>0,\ c>0,\ b>0.$$

We recall that one need the condition $a=c$ to get a well-posed system when a, c are not negative.

The paper is organized as follows. In the next section we survey the known existing results on existence (and stability) of one-dimensional traveling waves to (1.2). In section 3 we prove the suitable long time existence result for the systems derived from (1.2) by a localized perturbation of a line solitary wave.

Notation:

1) The notation " $:=$" means the definition notation.
2) $|.|_p$ the $L^p(\mathbb{R}^n)$ norm.
3) The $L^2(\mathbb{R}^n)$ scalar product is denoted by $(u,v)_2 := \int_{\mathbb{R}^n} uvdx$.
4) For any $s\in\mathbb{R}$, we denote by $H^s(\mathbb{R}^n)$ the L^2 based Sobolev spaces with the norm $|.|_{H^s}$.
5) The notation $f\sim g$ means that there exists a constant C such that $\frac{1}{C}f\leq g\leq Cf$ and $f\lesssim g$ means that there exists a constant B such that $f\leq Bg$.
6) The condensed notation $A_s = B_s + \langle C_s\rangle_{s\geq\tilde{s}}$ means that $A_s = B_s$ if $s\leq\tilde{s}$ and $A_s = B_s + C_s$ if $s>\tilde{s}$.
7) Vectors is denoted in bold letters, e.g. $\mathbf{u}$. When $\mathscr{X}$ is a Banach space, $\mathbf{u}\in\mathscr{X}$ means that each component of $\mathbf{u}$ belongs to $\mathscr{X}$ and the norm of $\mathbf{u}$ is denoted by $|\mathbf{u}|_{\mathscr{X}}$. If the components of $\mathbf{u}$ belong to different spaces, we will precise the notations.
8) The Fourier transform of a tempered distribution $u\in\mathscr{S}'$ is denoted by $\widehat{u}$. If f and u are two functions defined on $\mathbb{R}^n$, the Fourier multiplier $f(D)u$ is defined in term of Fourier transforms, i.e.
$$\widehat{f(D)u}(\xi)=f(\xi)\widehat{u}(\xi).$$
9) If A,B are two operators, $[A,B]=AB-BA$ denotes their commutator.

10) For $\xi \in \mathbb{R}^n$ we denote: $\langle\xi\rangle = \sqrt{1+|\xi|^2}$ and define the operator Λ^s as a Fourier multiplier with the symbol $\langle\xi\rangle^{s/2}$ with $s \in \mathbb{R}$.
11) The notation $\mathbf{u}_j$ is the j^{th} component of the vector $\mathbf{u}$.

2. Known results on one-dimensional solitary waves

We now turn to the existence results for solitary waves. Except the aforementioned explicit cases aforementioned, most of the results concern the Hamiltonian case $b = d$.

When

$$b = d > 0, \; a < 0, \; c < 0,$$

the existence of solitary waves with small propagation speeds is obtained in [28] by using the concentration-compactness method to minimize in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ the functional (where $\mu > 0$ is fixed)

$$\mathcal{E}_\mu(\eta, u) = \frac{1}{2}\int_{\mathbb{R}} (-\varepsilon c u_x^2 - \varepsilon a \eta_x^2 + \eta^2 + u^2)dx - \mu \int_{\mathbb{R}} (\eta u + \varepsilon b \eta_x u_x)dx$$

under the constraint

$$P(\eta, u) = \frac{1}{2}\int_{\mathbb{R}} \eta u^2 dx = p$$

Assuming moreover that

$$ac > b^2, \tag{2.1}$$

the solitary waves are shown in [27] to be orbitally stable. Note that (2.1) implies that $a + b + c + d < 0$, that is $\tau > \frac{1}{3}$, corresponding to strong surface tension.

We now consider the case

$$a < 0, c < 0, b = d = 0.$$

Oliveira [49] has proven in this case the existence of a family of non-negative, radially decreasing, exponentially decaying, solitary waves with small velocity and having a uniform (in velocity) L^2 bound.

2.1. **Boussinesq and Euler-Korteweg.**

Of special interest is the case $a = b = d = 0, c < 0$ since it appears to be a particular case of Euler-Korteweg systems that write

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla(g(\rho)) = \nabla\left(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\right), \end{cases} \tag{2.2}$$

where $K(\rho) > 0$ is the capillarity coefficient. See Benzoni-Danchin-Descombes [9] for the LWP of the Cauchy problem for "general" $g(\rho)$ and $K(\rho)$ and also [10] where a long time existence result is obtained for Euler-Korteweg systems in a Boussinesq scaling.

Setting $K(\rho) = K > 0, g(\rho) = \rho, \rho = 1 + \eta, \mathbf{u} = \nabla\psi$, one gets

$$\begin{cases} \partial_t \eta + \nabla \cdot \mathbf{u} + \nabla \cdot (\eta \mathbf{u}) = 0 \\ \partial_t \mathbf{u} + \frac{1}{2}\nabla|\mathbf{u}|^2 + \nabla\eta = K\nabla\Delta\eta, \end{cases} \tag{2.3}$$

which is the $(0, 0, -K, 0)$ Boussinesq system.

To recover exactly an (abcd) Boussinesq system involving the small parameter ε, one has to consider for (2.2) the *Boussinesq regime* (see [10], that is

$$\eta(x,y,t)=\varepsilon\tilde{\eta}(\sqrt{\varepsilon}x,\sqrt{\varepsilon}y,\sqrt{\varepsilon}t),\ u(x,y,t)=\varepsilon\tilde{u}(\sqrt{\varepsilon}x,\sqrt{\varepsilon}y,\sqrt{\varepsilon}t).$$

When applied to (2.3) this yields theb (0,0,0-K) Boussinesq for $(\tilde{\eta},\tilde{u})$ in the variables $(X,Y,T)=(\sqrt{\varepsilon}x,\sqrt{\varepsilon}y,\sqrt{\varepsilon}t)$.

One -dimensional solitary waves for (2.2) have been studied in [5, 6, 8, 3, 50]. When applied to the Bousinesq system (2.3), they yield the existence of solitary waves of velocity $|\omega|<1$. The existence of solitary waves follows from a phase portrait analysis of the governing ODEs. They can also be viewed as critical points of the Hamiltonian

$$H(\eta,u)=\frac{1}{2}\int_{\mathbb{R}}(K\eta_x^2+\eta^2+u^2+u^2\eta)dx$$

under the constraint

$$\mathcal{Q}(\eta,u)=\int_{\mathbb{R}}\eta u dx=\text{const},$$

see[6].

The stability of solitary waves of speed ω is governed by the convexity of the Boussinesq momentum of instability

$$m(\omega)=\mathcal{H}_\varepsilon(\eta,u)-\omega\mathcal{Q}(\eta,u)$$

More precisely, the solitary wave (η_ω,u_ω) is orbitally stable when

$$\frac{\partial^2 m}{\partial\omega^2}(\omega)>0$$

and linearly unstable when

$$\frac{\partial^2 m}{\partial\omega^2}(\omega)<0.$$

The transverse stability of one-dimensional solitary waves of (2.2) is studied in [52, 50]. Again we restrict those results to the particular case (2.3). The linear instability sf solitary waves with velocity $|\omega|<1$ is proved in [6, 52]. The nonlinear instability (for localized perturbations in (x,y)) is established in [50].

2.2. **Further studies.** Other studies concern the weaker notion of spectral stability. Numerical investigations of the spectral stability (both for one-dimensional and transverse perturbations) are displayed in [29] for the cases

$$a=-\frac{1}{9},\ b=\frac{1}{3},\ c=-\frac{1}{9},\ d=\frac{2}{9}$$

and

$$a=-\frac{8}{9},\ c=-\frac{8}{9},\ b=\frac{10}{9},\ d=1,$$

for which the solitary waves are explicit ([19, 20].

Rigorous spectral stability results are proven in [37] for the explicit solitary waves known to exist in the cases

$$a = c = -b,\ b = d > 0$$

and

$$a = c < 0,\ b = d > 0.$$

On the other hand, localized solitary waves have been proved to exist for the Bona-Smith class of Boussinesq systems. The original Bona-Smith system derived in [15] corresponds to $a = 0, b = d = \frac{1}{3}, c = -\frac{1}{3}$. The existence of solitary waves in this case is due to Toland [58].

A general class of Bona-Smith systems corresponds to $a = 0, b = d = \frac{3\theta^2-1}{6}, c = \frac{2-3\theta^2}{3}, \frac{2}{3} \leq \theta^2 \leq 1$ where θ parametrizes the height of the fluid layer where the horizontal velocity is taken, see [15]. Then, the techniques of Toland ([59]) can be used to prove the existence of solitary waves of arbitrary velocity $c > 1$ (see [33, 32]). The uniqueness of such solitary waves is considered in [60].

Remark 2.1. *The Boussinesq systems of the Bona-Smith class are among the few for which one can get the global existence of solutions to the Cauchy problem. Actually (see* [15, 13]*), such a result is obtained for initial data satisfying the non cavitation condition* $1 + \varepsilon\eta_0 > 0$ *and having a sufficiently small Hamiltonian* $\mathcal{H}(\eta_0, u_0)$.

Remark 2.2. *The Boussinesq system corresponding to* $a = c = b = 0,\ d > 0$ *has a particular importance since it is the BBM version of the natural (ill-posed) system obtained by expanding the Dirichlet-Neumann operator with respect to* ε *in the full water wave system (see*[44]*).*

By considering it as a dispersive perturbation of the hyperbolic Saint-Venant system, Schonbek and Amick ([57, 2] *have proved that the global existence of the Cauchy problem for the one-dimensional* $a = c = d = 0,\ b > 0$ *system has global solutions for arbitrary large initial data. On the other hand this system is not Hamiltonian and to our knowledge no (positive or negative) existence result for solitary waves is known.*

The case $a = c = d = 0, b > 0$ *is also particular. It is linearly ill-posed but in the one-dimensional case this ill-posed system is also known as the Kaup-Kupperschmidt system and it is completely integrable (*[39, 41]*).*

3. Long time existence

We show here how to extend the long time existence results in [55, 56] to the systems obtained by perturbing a line soliton by a localized two-dimensional perturbation.

Remark 3.1. *Since it does not rely on dispersive techniques but rather on "hyperbolic" symmetrization arguments, the method used in* [55, 56] *works as well for periodic perturbations in* y*, that is for the problem posed on* $\mathbb{R} \times \mathbb{T}$*. It works also for the purely periodic Cauchy problem posed on* $\mathbb{T}^2$ *which is the right framework to investigate the transverse stability of cnoïdal waves which we do not address here.*

Since there are many cases where there exists the one dimensional solitary waves, we will focus on the following values of (a, b, c, d).

$$a<0, c<0, b>0, d>0, \tag{3.1}$$

(that is the *generic case* in the terminology of [55] and in fact it contains also the Hamiltonian case $b=d$.),
or

$$a=0, c<0, b=d>0 \tag{3.2}$$

(That corresponds to the Bona-Smith class),
or the case of Euler-Korteweg systems

$$a=b=d=0, c<0 \tag{3.3}$$

we will from now on assume that (3.1),(3.2) or (3.3) holds.

Let $\tilde{U}_\omega = (\tilde{\eta}_\omega, \tilde{u}_\omega)$ a one dimensional solitary wave of velocity ω. As aforementioned, studying the transverse stability of $\tilde{U}_\omega$ with respect to periodic transverse perturbations in y would necessitate to solve the Cauchy problem for (1.1) in a $H^s(\mathbb{R}\times\mathbb{T})$ setting and this can be done as in [55, 56] for the $H^s(\mathbb{R}^2)$ case. We consider now localized perturbations of $\tilde{U}_\omega$. They satisfy the system

$$\begin{cases} \eta_t + \nabla\cdot\mathbf{u} + \varepsilon[\nabla\cdot((\eta\mathbf{u}) + \tilde{\eta}_\omega\mathbf{u} + \eta\tilde{\mathbf{u}}_\omega) + a\nabla\cdot\Delta\mathbf{u} - b\Delta\eta_t] = 0 \\ \mathbf{u}_t + \nabla\eta + \varepsilon[\nabla(\mathbf{u}\cdot\tilde{\mathbf{u}}_\omega) + \frac{1}{2}\nabla|\mathbf{u}|^2 + c\nabla\Delta\eta - d\Delta\mathbf{u}_t] = 0, \end{cases} \tag{3.4}$$

with initial condition $(\eta_0, \mathbf{u}_0)$. Note that, $\tilde{\mathbf{u}}_\omega = (\tilde{u}_\omega, 0)$.

The solitary waves of the (a,b,c,d) Boussinesq system have either explicit form or implicit form. The explicit solitary waves have "Sech" profile and the implicit solitary wave is in H^1, so we need different treatment for each case.

3.1. H^1- solitary waves.

In this section we study the case: $b=d>0$, $a<0$, $c<0$ for which in [28], we know the existence of solitary waves in $H^1(\mathbb{R})\times H^1(\mathbb{R})$.

For the completeness, we follow the general lines in [64] to prove the Cauchy problem of (3.4).

We begin with following lemma (see [65])

Lemma 3.1. *(Grisvard) Let $s_1, s_2, s_3 \in \mathbb{R}$ such that $s_1 \geq s_3$, $s_2 \geq s_3, s_1+s_2 \geq 0$, $s_1+s_2-s_3 > n/2$. Then, $(f,g)\mapsto fg$ is bilinear continuous from $H^{s_1}(\mathbb{R}^n)\times H^{s_2}(\mathbb{R}^n)$ into $H^{s_3}(\mathbb{R}^n)$. The result is also valid for a bounded domain $\Omega\subset\mathbb{R}^n$ with Lipschitz boundary.*

Theorem 3.1. *Assume $b=d>0$, $a<0$, $c<0$. Let $(\eta_0,\mathbf{u}_0)\in H^2(\mathbb{R}^2)^3$. Then, there exist $T>0$ and a unique solution*

$$(\eta,\mathbf{u})\in C([0,T];H^1(\mathbb{R}^2))^3$$

of (3.4) with initial condition $(\eta_0,\mathbf{u}_0)$. The existence time scale is of order $O(1)$.

Proof. With the condition of (a,b,c,d) we rewrite (3.4) as

$$\begin{cases} \eta_t + (I-b\varepsilon\Delta)^{-1}\left[\nabla\cdot\mathbf{u} + \varepsilon\nabla\cdot(\eta u) + \varepsilon\nabla(\tilde{\eta}_\omega\mathbf{u}) + \varepsilon\nabla\cdot(\eta\tilde{\mathbf{u}}_\omega) + \mathbf{a}\varepsilon\nabla\cdot\boldsymbol{\Delta}\mathbf{u}\right] = 0 \\ \mathbf{u}_t + (I-d\varepsilon\Delta)^{-1}\left[\nabla\eta + \varepsilon\nabla(\mathbf{u}\cdot\tilde{\mathbf{u}}_\omega) + \dfrac{1}{2}\varepsilon\nabla\Delta\eta\right] = 0. \end{cases} \tag{3.5}$$

then take its Fourier transform

$$\frac{d}{dt}\begin{pmatrix} \widehat{\eta} \\ \widehat{u_1} \\ \widehat{u_2} \end{pmatrix} + i|\xi|\mathcal{A}(\xi)\begin{pmatrix} \widehat{\eta} \\ \widehat{u_1} \\ \widehat{u_2} \end{pmatrix} + i\begin{pmatrix} \frac{\varepsilon}{1+b\varepsilon|\xi|^2}\left(\xi_1\widehat{\eta u_1} + \xi_2\widehat{\eta u_2} + \xi_1\widehat{\tilde{\eta}_\omega u_1} + \xi_2\widehat{\tilde{\eta}_\omega u_2}\right) \\ \frac{\varepsilon}{1+d\varepsilon|\xi|^2}\left(\xi_1\widehat{u_1\tilde{u}_\omega} + \frac{1}{2}\xi_1\widehat{|\mathbf{u}|^2}\right) \\ \frac{\varepsilon}{1+d\varepsilon|\xi|^2}\frac{1}{2}\xi_2\widehat{|\mathbf{u}|^2} \end{pmatrix} = 0.$$

Where the dispersive matrix $\mathcal{A}(\xi)$ given by

$$\mathcal{A}(\xi) = \begin{pmatrix} 0 & \frac{\xi_1}{|\xi|}\left(\frac{1-a\varepsilon|\xi|^2}{1+b\varepsilon|\xi|^2}\right) & \frac{\xi_2}{|\xi|}\left(\frac{1-a\varepsilon|\xi|^2}{1+b\varepsilon|\xi|^2}\right) \\ \frac{\xi_1}{|\xi|}\left(\frac{1-c\varepsilon|\xi|^2}{1+d\varepsilon|\xi|^2}\right) & 0 & 0 \\ \frac{\xi_2}{|\xi|^2}\left(\frac{1-c\varepsilon|\xi|^2}{1+d\varepsilon|\xi|^2}\right) & 0 & 0 \end{pmatrix}$$

The eigenvalues of $\mathcal{A}(\xi)$ are $\{0, \pm\sigma(\xi)\}$ where

$$\sigma(\xi) = \left(\frac{(1-a\varepsilon|\xi|^2)(1-c\varepsilon|\xi|^2)}{(1+b\varepsilon|\xi|^2)(1+d\varepsilon|\xi|^2)}\right)^{1/2}.$$

Diagonalize the above system:

$$P^{-1}(\xi)\mathcal{A}(\xi)P(\xi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma(\xi) & 0 \\ 0 & 0 & -\sigma(\xi), \end{pmatrix}$$

with

$$P(\xi) = \begin{pmatrix} 0 & \alpha(\xi) & -\alpha(\xi) \\ -\frac{\xi_2}{|\xi|} & \frac{\xi_1}{|\xi|} & \frac{\xi_1}{|\xi|} \\ \frac{\xi_1}{|\xi|} & \frac{\xi_2}{|\xi|} & \frac{\xi_2}{|\xi|} \end{pmatrix}, P^{-1}(\xi) = \frac{1}{2\alpha(\xi)}\begin{pmatrix} 0 & -2\alpha(\xi)\frac{\xi_2}{|\xi|} & 2\alpha(\xi)\frac{\xi_1}{|\xi|} \\ 1 & \alpha(\xi)\frac{\xi_1}{|\xi|} & \alpha(\xi)\frac{\xi_2}{|\xi|} \\ -1 & \alpha(\xi)\frac{\xi_1}{|\xi|} & \alpha(\xi)\frac{\xi_2}{|\xi|}, \end{pmatrix},$$

where

$$\alpha(\xi) = \left(\frac{(1+d\varepsilon|\xi|^2)(1-a\varepsilon|\xi|^2)}{(1-c\varepsilon|\xi|^2)(1+b\varepsilon|\xi|^2)}\right)^{1/2}.$$

Performing the change of variables

$$\begin{pmatrix} \widehat{\mu} \\ \widehat{\mathbf{w}} \end{pmatrix} = P^{-1}\begin{pmatrix} \widehat{\eta} \\ \widehat{\mathbf{u}} \end{pmatrix},$$

where $\mathbf{w} = (w_1, w_2)$, we have

(3.6)

$$\frac{d}{dt}\begin{pmatrix} \widehat{\mu} \\ \widehat{w_1} \\ \widehat{w_2} \end{pmatrix} + i|\xi|\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma(\xi) & 0 \\ 0 & 0 & -\sigma(\xi) \end{pmatrix}\begin{pmatrix} \widehat{\mu} \\ \widehat{w_1} \\ \widehat{w_2} \end{pmatrix}$$

$$= -iP^{-1}(\xi)\begin{pmatrix} \frac{\varepsilon}{1+b\varepsilon|\xi|^2}\left(\xi_1\widehat{\eta u_1} + \xi_2\widehat{\eta u_2} + \xi_1\widehat{\tilde{\eta}_\omega u_1} + \xi_2\widehat{\tilde{\eta}_\omega u_2}\right) \\ \frac{\varepsilon}{1+d\varepsilon|\xi|^2}\left(\xi_1\widehat{u_1\tilde{u}_\omega} + \frac{1}{2}\xi_1\widehat{|\mathbf{u}|^2}\right) \\ \frac{\varepsilon}{1+d\varepsilon|\xi|^2}\frac{1}{2}\xi_2\widehat{|\mathbf{u}|^2} \end{pmatrix}.$$

We have

$$\begin{aligned}
\widehat{\mu}(\xi) &= -\frac{\xi_2}{|\xi|}\widehat{u_1}(\xi) + \frac{\xi_1}{|\xi|}\widehat{u_2}(\xi)\\
\widehat{w_1}(\xi) &= \frac{1}{2\alpha(\xi)}\widehat{\eta}(\xi) + \frac{1}{2}\frac{\xi_1}{|\xi|}\widehat{u_1}(\xi) + \frac{1}{2}\frac{\xi_2}{|\xi|}\widehat{u_2}(\xi)\\
\widehat{w_2}(\xi) &= -\frac{1}{2\alpha(\xi)}\widehat{\eta}(\xi) + \frac{1}{2}\frac{\xi_1}{|\xi|}\widehat{u_1}(\xi) + \frac{1}{2}\frac{\xi_2}{|\xi|}\widehat{u_2}(\xi).
\end{aligned}$$

Thus,

$$(\eta, u_1, u_2) \in (H^1(\mathbb{R}^2))^3 \Rightarrow (\mu, w_1, w_2) \in (H^1(\mathbb{R}^2))^3.$$

From (3.6)

$$\frac{d}{dt}\begin{pmatrix} \mu \\ w_1 \\ w_2 \end{pmatrix} + \mathcal{B}\begin{pmatrix} \mu \\ w_1 \\ w_2 \end{pmatrix} = \mathcal{N}\begin{pmatrix} \mu \\ w_1 \\ w_2 \end{pmatrix}, \tag{3.7}$$

where $\mathcal{B}$ is the skew-adjoint matrix operator with symbol

$$i|\xi|\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma(\xi) & 0 \\ 0 & 0 & -\sigma(\xi) \end{pmatrix}.$$

The nonlinear term's Fourier transform is

$$-iP^{-1}(\xi)\begin{pmatrix} \frac{\varepsilon}{1+b\varepsilon|\xi|^2}\left(\xi_1\widehat{\eta u_1} + \xi_2\widehat{\eta u_2} + \xi_1\widehat{\tilde{\eta}_\omega u_1} + \xi_2\widehat{\tilde{\eta}_\omega u_2}\right) \\ \frac{\varepsilon}{1+d\varepsilon|\xi|^2}\left(\xi_1\widehat{u_1\tilde{u}_\omega} + \frac{1}{2}\xi_1\widehat{|\mathbf{u}|^2}\right) \\ \frac{\varepsilon}{1+d\varepsilon|\xi|^2}\frac{1}{2}\xi_2\widehat{|\mathbf{u}|^2} \end{pmatrix}.$$

Denoting by $S(t)$ the group generated by $\mathcal{B}$ then $S(t)$ is a unitary group on $(H^1(\mathbb{R}^2))^3$. Let $W = (\mu, w_1, w_2)^T$ and W_0 be related to the initial data $(\eta_0, \mathbf{u}_0)$. By Duhamel's formula, (3.7) with initial data W_0 has integral form as follows

$$W(t) = S(t)W_0 + \int_0^t S(t-s)\mathcal{N}(W)\, ds. \tag{3.8}$$

We only estimate the new terms given by the line solitary wave which correspond to the nonlinear parts: $\widehat{\tilde{\eta}_\omega u_1}$, $\widehat{\tilde{\eta}_\omega u_2}$, $\widehat{u_1\tilde{u}_\omega}$.
Note that, $\alpha(\xi)$ is of order 0, all the pseudo-differential operators involved are of order 0, so is $P^{-1}(\xi)$. Then, the nonlinear term correspond to $\widehat{u_1\tilde{u}_\omega}$ takes the form (in Fourier space)

$$p(\xi)\frac{\varepsilon}{1+d\varepsilon|\xi|^2}\xi_1\widehat{u_1\tilde{u}_\omega},$$

where $p(\xi)$ is a symbol of order 0. We need to estimate

$$\begin{aligned}
&\left|\int_0^t e^{i|\xi|\sigma(\xi)(t-s)}\,\langle\xi\rangle\, p(\xi)\frac{\varepsilon}{1+d\varepsilon|\xi|^2}\xi_1\widehat{u_1\tilde{u}_\omega}\right|_{L^2(\mathbb{R}^2)}\\
&\leq t^{3/2}\left|\langle\xi\rangle\, p(\xi)\frac{\varepsilon}{1+d\varepsilon|\xi|^2}\xi_1\widehat{u_1\tilde{u}_\omega}\right|_{L^\infty([0,t],L^2(\mathbb{R}^2))}\\
&\leq t^{3/2}C\left|\langle\xi\rangle\,\widehat{u_1\tilde{u}_\omega}\right|_{L^\infty([0,t],L^2(\mathbb{R}^2))}\\
&\leq t^{3/2}C\left|(\langle\xi_1\rangle+\langle\xi_2\rangle)\widehat{u_1\tilde{u}_\omega}\right|_{L^\infty([0,t],L^2(\mathbb{R}^2))}\\
&\leq Ct^{3/2}\left|\tilde{u}_\omega\right|_{L^\infty([0,t],H^1(\mathbb{R}))}\left|u_1\right|_{L^\infty([0,t],H^1(\mathbb{R}^2))}.
\end{aligned}$$

Here C is a general constant independent of ε and we used Lemma (3.1) for $s_1 = s_2 = 1$, $s_3 = 0$ and $n = 0, 1$. Estimates for other terms are similar.
It is also clear that the existence time scale given by using this method is of order $O(1)$.

□

3.2. **Solitary waves with Sech profile.** Before stating the main result of this section we need to introduce the space (see [55]) :

Definition 3.1. *For any $s \in \mathbb{R}$, $k \in \mathbb{N}$, $\varepsilon \in (0,1)$, the Banach space $X^s_{\varepsilon^k}(\mathbb{R}^n)$ is defined as $H^{s+k}(\mathbb{R}^n)$ equipped with the norm:*

$$|u|^2_{X^s_{\varepsilon^k}} = |u|^2_{H^s} + \varepsilon^k|u|^2_{H^{s+k}}. \tag{3.9}$$

The solutions $(\eta, \mathbf{u})$ of system (3.4) will belong to some space $X^s_{\varepsilon^k}(\mathbb{R}^2)\times(X^s_{\varepsilon^{k'}}(\mathbb{R}^2))^2$ with k and k' determined by a, b, c, d as follows :

Definition 3.2. *For any a, b, c, d satisfying (3.1), (3.2), (3.3) we define a pair of numbers $(k, k') = (k(a, b, c, d), k'(a, b, c, d))$ according to the admissible sets of $(a, b, c, d)'s$ as follows:*

- $(k, k') = (3, 3)$ *for* $b \neq d, b, d > 0, a, c < 0$ *;*
- $(k, k') = (3, 4)$ *for* $b = 0, d > 0, a, c < 0$*;*
- $(k, k') = (1, 1)$ *for* $b = d = 0, a, c < 0$*;*

Theorem 3.2. *Let $t_0 > 1$, $s \geq t_0 + 2$ if $b \neq d$ or $b = d > 0$, $s \geq t_0 + 4$ if $b = d = 0$. Assume that $\eta_0 \in X^s_{\varepsilon^k}(\mathbb{R}^2), \mathbf{u}_0 \in X^s_{\varepsilon^{k'}}(\mathbb{R}^2)$ satisfy the (non-cavitation) condition*

$$1 - \varepsilon\eta_0 \geq H > 0, \quad H \in (0, 1), \tag{3.10}$$

where (k, k') is defined in Definition 3.2. Then there exists a constant $\varepsilon_0 = \varepsilon_0(H)$ such that for any $\varepsilon \leq \varepsilon_0$, there exists $T > 0$, such that (3.4) has a unique solution $(\eta, \mathbf{u})^T$ with $\eta \in C([0, T]; X^s_{\varepsilon^k}(\mathbb{R}^2))$ and $\mathbf{u} \in C([0, T]; (X^s_{\varepsilon^{k'}}(\mathbb{R}^2))^2)$. Moreover,

$$\max_{t\in[0,T]}(|\eta|_{X^s_{\varepsilon^k}} + |\mathbf{u}|_{X^s_{\varepsilon^{k'}}}) \leq \tilde{c}(|\eta_0|_{X^s_{\varepsilon^k}} + |\mathbf{u}_0|_{X^s_{\varepsilon^{k'}}}), \tag{3.11}$$

here $\tilde{c} = C(H^{-1})$ is nondecreasing functions of their argument.

More precisely, the existence time scale, $T = O(1/\sqrt{\varepsilon})$ if $b \neq d$, $b, d > 0$, $a, c < 0$. And $T = O(1/\varepsilon)$ if $b = d > 0$, $a = 0$, $c < 0$.

3.3. **Linearization of** (3.4)**.** We will follow the idea in [55]. Setting

$$\mathbf{U} = (\eta, \mathbf{u})^T,\ \mathbf{V} = (\zeta, \mathbf{v})^T = \varepsilon \mathbf{U},$$

we rewrite (1.2) as

$$\begin{cases} (1 - b\varepsilon\Delta)\partial_t \zeta + \nabla \cdot \mathbf{v} + \nabla \cdot (\zeta \mathbf{v}) + a\varepsilon \nabla \cdot \Delta \mathbf{v} = 0, \\ (1 - d\varepsilon)\partial_t \mathbf{v} + \nabla \zeta + \dfrac{1}{2}\nabla(|\mathbf{v}|^2) + c\varepsilon \nabla \Delta \zeta = \mathbf{0}. \end{cases} \tag{3.12}$$

If $b > 0$, $d \geq 0$ or $b = d = 0$, let $g(D) = (1 - b\varepsilon\Delta)(1 - d\varepsilon\Delta)^{-1}$, then (3.12) is equivalent after applying $g(D)$ to the second equation to the condensed system

$$(1 - b\varepsilon\Delta)\partial_t \mathbf{V} + M(\mathbf{V}, D)\mathbf{V} = \mathbf{0}, \tag{3.13}$$

where

$$M(\mathbf{V}, D) = \begin{pmatrix} \mathbf{v} \cdot \nabla & (1 + \zeta + a\varepsilon\Delta)\partial_{x_1} & (1 + \zeta + a\varepsilon\Delta)\partial_{x_2} \\ g(D)(1 + c\varepsilon\Delta)\partial_{x_1} & g(D)(v_1 \partial_{x_1}) & g(D)(v_2 \partial_{x_1}) \\ g(D)(1 + c\varepsilon\Delta)\partial_{x_2} & g(D)(v_1 \partial_{x_2}) & g(D)(v_2 \partial_{x_2}) \end{pmatrix} \tag{3.14}$$

We denote $\tilde{\mathbf{V}}_\omega = (\tilde{\zeta}_\omega, \tilde{\mathbf{v}}_\omega)^T = (\varepsilon \tilde{\eta}_\omega, \varepsilon \tilde{\mathbf{u}}_\omega)^T = \varepsilon \tilde{\mathbf{U}}_\omega$ the one dimensional solitary wave of (1.7) , then (3.4) is equivalent to the following perturbation of (3.13) replacing $\mathbf{V}$ by $\mathbf{V} + \tilde{\mathbf{V}}_\omega$

$$(1 - b\varepsilon\Delta)\partial_t(\mathbf{V} + \tilde{\mathbf{V}}_\omega) + M(\mathbf{V} + \tilde{\mathbf{V}}_\omega, D)(\mathbf{V} + \tilde{\mathbf{V}}_\omega) = 0,$$

or

$$(1 - b\varepsilon\Delta)\partial_t \mathbf{V} + M(\mathbf{V} + \tilde{\mathbf{V}}_\omega, D)\mathbf{V} = -(M(\mathbf{V} + \tilde{\mathbf{V}}_\omega, D) - M(\tilde{\mathbf{V}}_\omega, D))\tilde{\mathbf{V}}_\omega. \tag{3.15}$$

In order to solve (3.4), we consider the following linearized equation

$$(1 - b\varepsilon\Delta)\partial_t \mathbf{V} + M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)\mathbf{V} = -(M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) - M(\tilde{\mathbf{V}}_\omega, D))\tilde{\mathbf{V}}_\omega, \tag{3.16}$$

or

$$(1 - b\varepsilon\Delta)\partial_t \mathbf{V} + M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)\mathbf{V} = F \tag{3.17}$$

where $M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)$ and $M(\tilde{\underline{\mathbf{V}}}, D)$ are defined in (3.14), and note that the notation $\underline{V}$ is considered as a known function.

The idea is to treat the equation (3.17) as a symmetrizable hyperbolic system under some smallness assumption on $\underline{\mathbf{V}}$ and $\tilde{\mathbf{V}}_\omega$. It is known in [55] that there exists a symmetrizer $S_{\underline{\mathbf{V}}}(D)$ of $M(\underline{\mathbf{V}}, D)$ such that the principal part of $iS_{\underline{\mathbf{V}}}(\xi)M(\underline{\mathbf{V}}, \xi)$ is self-adjoint, and that of $S_{\underline{\mathbf{V}}}(\xi)$ is positive and self-adjoint under a smallness assumption on $\underline{\mathbf{V}}$. Fortunately, it is also true for the symmetrizer $S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D)$ of $M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)$, so that we can apply that method for solving (3.17) with some modifications.

We have

(i) If $b = d$, $g(D) = 1$, $S_{\underline{\mathbf{V}}}(D)$ is

$$\begin{pmatrix} 1 + c\varepsilon\Delta & \underline{v}_1 & \underline{v}_2 \\ \underline{v}_1 & 1 + \underline{\zeta} + a\varepsilon\Delta & 0 \\ \underline{v}_2 & 0 & 1 + \underline{\zeta} + a\varepsilon\Delta \end{pmatrix}; \tag{3.18}$$

then

$$S_{\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega}(D) = S_{\underline{\mathbf{V}}}(D) + S_1, \tag{3.19}$$

where

$$S_1 = \begin{pmatrix} 0 & \tilde{v}_{1\omega} & \tilde{v}_{2\omega} \\ \tilde{v}_{1\omega} & \tilde{\zeta}_\omega & 0 \\ \tilde{v}_{2\omega} & 0 & \tilde{\zeta}_\omega \end{pmatrix}.$$

(ii) If $b \neq d$, $S_{\underline{\mathbf{V}}}(D)$ is

(3.20)

$$\begin{pmatrix} (1+c\varepsilon\Delta)^2 g(D) & g(D)(\underline{v}_1(1+c\varepsilon\Delta)) & g(D)(\underline{v}_2(1+c\varepsilon\Delta)) \\ g(D)(\underline{v}_1(1+c\varepsilon\Delta)) & (1+\underline{\zeta}+a\varepsilon\Delta)(1+c\varepsilon\Delta) & 0 \\ g(D)(\underline{v}_2(1+c\varepsilon\Delta)) & 0 & (1+\underline{\zeta}+a\varepsilon\Delta)(1+c\varepsilon\Delta) \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \underline{v}_1\underline{v}_1 & \underline{v}_1\underline{v}_2 \\ 0 & \underline{v}_1\underline{v}_2 & \underline{v}_2\underline{v}_2 \end{pmatrix} (g(D)-1).$$

Then, we decompose

$$S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D) = S_{\underline{\mathbf{V}}} + S_1 + S_2, \tag{3.21}$$

where

$$S_1 = \begin{pmatrix} 0 & g(D)(\tilde{v}_{1\omega}(1+c\varepsilon\Delta)) & g(D)(\tilde{v}_{2\omega}(1+c\varepsilon\Delta)) \\ g(D)(\tilde{v}_{1\omega}(1+c\varepsilon\Delta)) & \tilde{\zeta}_\omega(1+c\varepsilon\Delta) & 0 \\ g(D)(\tilde{v}_{2\omega}(1+c\varepsilon\Delta)) & 0 & \tilde{\zeta}_\omega(1+c\varepsilon\Delta) \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{v}_{1\omega}^2 & \tilde{v}_{1\omega}\tilde{v}_{2\omega} \\ 0 & \tilde{v}_{1\omega}\tilde{v}_{2\omega} & \tilde{v}_{2\omega}^2 \end{pmatrix} (g(D)-1),$$

and

$$S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\underline{v}_1\tilde{v}_{1\omega} & \underline{v}_1\tilde{v}_{2\omega} + \underline{v}_2\tilde{v}_{1\omega} \\ 0 & \underline{v}_1\tilde{v}_{2\omega} + \underline{v}_2\tilde{v}_{1\omega} & 2\underline{v}_2\tilde{v}_{2\omega} \end{pmatrix} (g(D)-1).$$

The key of this decomposition is to separate the terms independent, dependent only and dependent partially on the 1D solitary wave. We also need the following expression

$$M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) = M(\tilde{\mathbf{V}}_\omega, D) + \tilde{M}_1, \tag{3.22}$$

where

$$\tilde{M}_1 = \begin{pmatrix} \underline{\mathbf{v}}\cdot\nabla & \underline{\zeta}\partial_{x_1} & \underline{\zeta}\partial_{x_2} \\ 0 & g(D)(\underline{v}_1\partial_{x_1}) & g(D)(\underline{v}_2\partial_{x_1}) \\ 0 & g(D)(\underline{v}_1\partial_{x_2}) & g(D)(\underline{v}_2\partial_{x_2}) \end{pmatrix}.$$

And

$$M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) = M(\underline{\mathbf{V}}, D) + M_1, \tag{3.23}$$

where

$$M_1 = \begin{pmatrix} \tilde{\mathbf{v}}_\omega\cdot\nabla & \tilde{\zeta}_\omega\partial_{x_1} & \tilde{\zeta}_\omega\partial_{x_2} \\ 0 & g(D)(\tilde{v}_{1\omega}\partial_{x_1}) & g(D)(\tilde{v}_{2\omega}\partial_{x_1}) \\ 0 & g(D)(\tilde{v}_{1\omega}\partial_{x_2}) & g(D)(\tilde{v}_{2\omega}\partial_{x_2}) \end{pmatrix}.$$

Remark 3.1. i) *Because the solitary wave has the "sech" profile and using the expression (3.22), we see that the term F on the right hand side of (3.17) is in $X^s_\varepsilon(\mathbb{R}^2)$ if $\underline{\mathbf{V}} \in X^s_{\varepsilon^k}(\mathbb{R}^2)$ for $s, k > 0, s \in \mathbb{N}$.*

ii) *The above remark is not true if our solitary wave is in $H^1(\mathbb{R})$ only. Furthermore, we need $s \in \mathbb{N}$ because we use sometimes the Leibnitz rule instead of usual commutator estimates.*

Next we define the energy functional associated to (3.17) as

$$E_s(\mathbf{V}) = \Big((1-b\varepsilon\Delta)\Lambda^s\mathbf{V}, S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)\Lambda^s\mathbf{V}\Big)_2. \tag{3.24}$$

We shall show that $E_s(\mathbf{V})$ defined in (3.24) is equivalent to some $X^s_{\varepsilon^k}(\mathbb{R}^2)$ norm.

Remark 3.2. *If $b=0$, $d>0$, (3.12) is equivalent after applying $(1-d\varepsilon\Delta)$ to the first equation to the condensed system*

$$(1-d\varepsilon\Delta)\partial_t\mathbf{V} + M(\mathbf{V},D)\mathbf{V} = 0,$$

with $M(\mathbf{V},D)$ defined by

$$\begin{pmatrix} (1-d\varepsilon\Delta)(\mathbf{v}\cdot\nabla) & (1-d\varepsilon\Delta)((1+\zeta+a\varepsilon\Delta)\partial_{x_1}) & (1-d\varepsilon\Delta)((1+\zeta+a\varepsilon\Delta)\partial_{x_2}) \\ (1+c\varepsilon\Delta)\partial_{x_1} & v_1\partial_{x_1} & v_2\partial_{x_1} \\ (1+c\varepsilon\Delta)\partial_{x_2} & v_1\partial_{x_2} & v_2\partial_{x_2} \end{pmatrix}.$$

Then the equation (3.4) is equivalent to

$$(1-d\varepsilon\Delta)\partial_t\mathbf{V} + M(\mathbf{V}+\tilde{\mathbf{V}}_\omega,D)\mathbf{V} = -\Big(M(\mathbf{V}+\tilde{\mathbf{V}}_\omega,D) - M(\tilde{\mathbf{V}}_\omega,D)\Big)\tilde{\mathbf{V}}_\omega.$$

The symmetrizer $S_{\underline{\mathbf{V}}}(D)$of $M(\underline{\mathbf{V}},D)$ is defined by

$$\begin{pmatrix} (1+c\varepsilon\Delta)^2 & \underline{v}_1(1+c\varepsilon\Delta) & \underline{v}_2(1+c\varepsilon\Delta) \\ \underline{v}_1(1+c\varepsilon\Delta) & (1+c\varepsilon\Delta)[(1+\underline{\zeta}+a\varepsilon\Delta)(1-d\varepsilon\Delta)] & 0 \\ \underline{v}_2(1+c\varepsilon\Delta) & 0 & (1+c\varepsilon\Delta)[(1+\underline{\zeta}+a\varepsilon\Delta)(1-d\varepsilon\Delta)] \end{pmatrix}$$
$$+\begin{pmatrix} 0 & 0 & 0 \\ 0 & d\varepsilon\underline{v}_1\underline{v}_1\Delta & d\varepsilon\underline{v}_1\underline{v}_2\Delta \\ 0 & d\varepsilon\underline{v}_1\underline{v}_2\Delta & d\varepsilon\underline{v}_2\underline{v}_2\Delta \end{pmatrix}.$$

Therefore, the symmetrizer $S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)$ of $M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega,D)$ is defined by

$$S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D) = S_{\underline{\mathbf{V}}}(D) + S_1 + S_2,$$

where

$$S_1 = \begin{pmatrix} 0 & \tilde{v}_{1\omega}(1+c\varepsilon\Delta) & \tilde{\underline{v}}_{2\omega}(1+c\varepsilon\Delta) \\ \tilde{v}_{1\omega}(1+c\varepsilon\Delta) & (1+c\varepsilon\Delta)[\tilde{\zeta}_\omega(1-d\varepsilon\Delta)] & 0 \\ \tilde{v}_{2\omega}(1+c\varepsilon\Delta) & 0 & (1+c\varepsilon\Delta)[\tilde{\zeta}_\omega(1-d\varepsilon\Delta)] \end{pmatrix}$$
$$+\begin{pmatrix} 0 & 0 & 0 \\ 0 & d\varepsilon\tilde{v}_{1\omega}\tilde{v}_{1\omega}\Delta & d\varepsilon\tilde{v}_{1\omega}\tilde{v}_{2\omega}\Delta \\ 0 & d\varepsilon\tilde{v}_{1\omega}\tilde{v}_{2\omega}\Delta & d\varepsilon\tilde{v}_{2\omega}\tilde{v}_{2\omega} \end{pmatrix},$$

and

$$S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2d\varepsilon\underline{v}_1\tilde{v}_{1\omega}\Delta & d\varepsilon(\underline{v}_1\tilde{v}_{2\omega}+\underline{v}_2\tilde{v}_{1\omega})\Delta \\ 0 & d\varepsilon(\underline{v}_1\tilde{v}_{2\omega}+\underline{v}_2\tilde{v}_{1\omega})\Delta & 2d\varepsilon\underline{v}_2\tilde{v}_{2\omega}\Delta \end{pmatrix}.$$

Before going to the main results of this section, we recall the following definitions and Lemma which can be found in [55].

Lemma 3.2. *If $t_0 > \frac{n}{2}$ $(n=1,2)$ and $s \geq 0$, one has*

$$|fg|_{H^s} \leq C\left(|f|_{H^{t_0}}|g|_{H^s} + \langle |f|_{H^s}|g|_{H^{t_0}}\rangle_{s>t_0}\right), \quad \forall f,g \in H^s\cap H^{t_0}(\mathbb{R}^n). \tag{3.25}$$

Lemma 3.3. *Let $\phi \in H^s(\mathbb{R}_{x_1})$ for every $s \geq 1$, $t_0 > 1$ and $f \in H^s \cap H^{t_0}(\mathbb{R}^2)$. Then*

$$|\phi f|_{H^s} \leq C \left(|\phi|_{H^{t_0}_{x_1}} |f|_{H^s} + \left\langle |\phi|_{H^s_{x_1}} . |f|_{H^{t_0}} \right\rangle_{s>t_0} \right) \tag{3.26}$$

Definition 3.3. *We say that a Fourier multiplier $\sigma(D)$ is of order $s\,(s \in \mathbb{R})$ and write $\sigma \in \mathcal{S}^s$ if $\xi \in \mathbb{R}^d \mapsto \sigma(\xi) \in \mathbb{C}$ is smooth and satisfies*

$$\forall \xi \in \mathbb{R}^d,\ \forall \beta \in \mathbb{N}^d, \quad \sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^{|\beta|-s} |\partial^\beta \sigma(\xi)| < \infty.$$

Lemma 3.4. *Let $t_0 > d/2$, $s \geq 0$ and $\sigma \in \mathcal{S}^s$. If $f \in H^s(\mathbb{R}^d) \cap H^{t_0+1}(\mathbb{R}^d)$ then, for all $g \in H^{s-1}(\mathbb{R}^d) \cap H^{t_0}(\mathbb{R}^d)$ then*

$$|[\sigma(D), f]g|_{L^2} \leq C(\sigma) \left(|\nabla f|_{L^\infty} |g|_{H^{s-1}} + |\nabla f|_{H^{s-1}} |g|_{L^\infty} \right), \tag{3.27}$$

where $C(\sigma)$ depends only on σ.

Lemma 3.5. *Let $b, d > 0$ and $b \neq d$, $s \in \mathbb{R}$, $\theta \geq 0$, then*

(i) *for all $f \in H^s(\mathbb{R}^n)$, there hold*

$$\min\left\{1, \left(\frac{b}{d}\right)^\theta\right\} |f|_{H^s} \leq \left|g(D)^\theta f\right|_{H^s} \leq \max\left\{1, \left(\frac{b}{d}\right)^\theta\right\} |f|_{H^s}, \tag{3.28}$$

$$|(g(D) - 1)f|_{H^s} \leq \frac{|b-d|}{d} |f|_{H^s}. \tag{3.29}$$

(ii) *Let $t_0 > \frac{n}{2}$, $-t_0 < r \leq t_0 + 1$, for all $f \in H^{t_0+1}(\mathbb{R}^n)$ and $u \in H^{r-1}(\mathbb{R}^n)$, there holds*

$$\left| \left[g(D)^\theta, f \right] u \right|_{H^r} \leq C |f|_{H^{t_0+1}} |u|_{H^{r-1}}, \tag{3.30}$$

where C is a constant independent of ε.

3.4. **Sovability of linearized equation.**

3.4.1. ***The case:*** $b \neq d$, $b, d > 0$, $a, c < 0$.

Proposition 3.1. *Let $t_0 > 1$, $s \geq t_0 + 1$, $s \in \mathbb{N}$, $T' > 0$. Assume that $F \in C\left([0, T']; X^s_\varepsilon(\mathbb{R}^2)\right)$ and $\underline{\mathbf{V}} \in C^1\left([0, T']; X^{s-1}_{\varepsilon^3}(\mathbb{R}^2)\right) \cap C\left([0, T']; X^s_{\varepsilon^3}(\mathbb{R}^2)\right)$ satisfy that*

$$1 + \underline{\zeta} + \tilde{\zeta}_\omega \geq H > 0,\ |\underline{\mathbf{V}}|_\infty + \left|\tilde{\mathbf{V}}_\omega\right|_\infty \leq \kappa_H,\ |\underline{\mathbf{V}}|_{H^s} + \left|\tilde{\mathbf{V}}_\omega\right|_{H^s} \leq 1,\ \forall t \in [0, T'], \tag{3.31}$$

with κ_H ensures the equivalence of $E_s(\mathbf{V})$ and $|\mathbf{V}|^2_{X^s_{\varepsilon^3}}$ (see the Appendix 4.1)

$$\frac{1}{c} |\mathbf{V}|^2_{X^s_{\varepsilon^3}} \leq E_s(\mathbf{V}) \leq C |\mathbf{V}|^2_{X^s_{\varepsilon^3}}. \tag{3.32}$$

Then for any $V_0 \in X^s_{\varepsilon^3}(\mathbb{R}^2)$, (3.17) has a unique solution

$$\mathbf{V} \in C^1([0, T']; X^{s-1}_{\varepsilon^3}(\mathbb{R}^2)) \cap C([0, T']; X^s_{\varepsilon^3}(\mathbb{R}^2)),$$

and one has for any $t \in (0, T')$.

$$\begin{aligned} \frac{d}{dt} E_s(\mathbf{V}) \leq \tilde{c} \Big(& \left(1 + |\partial_t \underline{\mathbf{V}}|_{X^{s-1}_{\varepsilon^3}} + |\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}} + |\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\right) |F|_{X^s_{\varepsilon^3}} E_s(\mathbf{V})^{1/2} \\ & + \varepsilon^{1/2} \left(1 + |\partial_t \underline{\mathbf{V}}|_{X^{s-1}_{\varepsilon^3}} + |\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}} + |\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\right) E_s(\mathbf{V}) \Big), \end{aligned} \tag{3.33}$$

(3.34)
$$|\mathbf{V}(t)|^2_{X^s_{\varepsilon^3}} \leq \tilde{c}\,|\mathbf{V}_0|^2_{X^s_{\varepsilon^3}} + \tilde{c}\int_0^t \Bigg(\Big(1+|\partial_t\underline{\mathbf{V}}|_{X^{s-1}_{\varepsilon^3}} + |\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}} + |\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\Big)|F|_{X^s_{\varepsilon^3}}|\mathbf{V}|_{X^s_{\varepsilon^3}} + \varepsilon^{1/2}\Big(1+|\partial_t\underline{\mathbf{V}}|_{X^{s-1}_{\varepsilon^3}} + |\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}} + |\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\Big)|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\Bigg),$$

$$|\partial_t\mathbf{V}|_{X^{s-1}_{\varepsilon^3}} \leq C\Big(|F|_{H^s} + |\mathbf{V}|_{X^s_{\varepsilon^3}}\Big). \tag{3.35}$$

Remark 3.3. *Since we defined $\underline{\mathbf{V}} = \varepsilon\underline{\mathbf{U}}$ and $\tilde{\mathbf{V}}_\omega = \varepsilon\tilde{\mathbf{U}}_\omega$, the assumption (3.31) is a smallness condition which holds if ε is small enough.*

Proof. We derive a priori estimate on (3.17). First, we have that
(3.36)
$$\begin{aligned}\frac{d}{dt}E_s(\mathbf{V}) &= \Big(\Lambda^s F,\, \Big(S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D) + S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D)^*\Big)\Lambda^s\mathbf{V}\Big)_2\\
&\quad - \Big(\Lambda^s\Big(M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D)\mathbf{V}\Big),\, \Big(S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D) + S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D)^*\Big)\Lambda^s\mathbf{V}\Big)_2\\
&\quad - b\varepsilon\Big(\Big[S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D)^*, \Delta\Big]\Lambda^s\mathbf{V},\, \Lambda^s\partial_t\mathbf{V}\Big)_2\\
&\quad + \Big((1-b\varepsilon\Delta)\Lambda^s\mathbf{V},\, \Big(\partial_t S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D)\Big)\Lambda^s\mathbf{V}\Big)_2\\
&= I + II + III + IV,\end{aligned}$$
where $S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D)^*$ is the adjoint operator of $S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D)$.

Estimate on I. By using integral by parts, we have
$$\begin{aligned}&\Big(\Lambda^s F,\, S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D)\Lambda^s\mathbf{V}\Big)_2\\
&\quad = \big(\Lambda^s F_1,\, (1+c\varepsilon\Delta)^2 g(D)\Lambda^s\zeta\big)_2 + (\Lambda^s F_1,\, g(D)((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)(1+c\varepsilon\Delta)\Lambda^s\mathbf{v}))_2\\
&\quad + \sum_{i=1}^2\big\{(\Lambda^s F_{i+1},\, g(D)((\underline{v}_i+\tilde{v}_{i\omega})(1+c\varepsilon\Delta)\Lambda^s\zeta))_2\\
&\quad + \Big(\Lambda^s F_{i+1},\, (1+\underline{\zeta}+\underline{\tilde{\zeta}})(1+c\varepsilon\Delta)\Lambda^s v_i\Big)_2 - a\varepsilon\,(\nabla\Lambda^s F_{i+1},\, \nabla(1+c\varepsilon\Delta)\Lambda^s v_i)_2\\
&\quad + (\Lambda^s F_{i+1},\, (\underline{v}_i+\tilde{v}_{i\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot(g(D)-1)\Lambda^s\mathbf{v})_2\big\},\end{aligned}$$
which together with (3.31), (3.28) and (3.29) implies that
$$\begin{aligned}&\Big|\Big(\Lambda^s F,\, S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D)\Lambda^s\mathbf{V}\Big)_2\Big|\\
&\qquad \lesssim \Big(|F|_{H^s} + \varepsilon^{1/2}|F|_{H^{s+1}}\Big)\Big(|\mathbf{V}|_{H^s} + \varepsilon|\mathbf{V}|_{H^{s+2}} + \varepsilon^{3/2}|\mathbf{V}|_{H^{s+3}}\Big).\end{aligned}$$
Similarly we have
$$\begin{aligned}&\Big|\Big(\Lambda^s F,\, S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D)^*\Lambda^s\mathbf{V}\Big)_2\Big|\\
&\qquad \lesssim \Big(|\mathbf{V}|_{H^s} + \varepsilon^{1/2}|\mathbf{V}|_{H^{s+1}}\Big)\Big(|F|_{H^s} + \varepsilon|F|_{H^{s+2}} + \varepsilon^{3/2}|F|_{H^{s+3}}\Big),\end{aligned}$$
Therefore,
$$|I| \lesssim |F|_{X^s_{\varepsilon^3}}|\mathbf{V}|_{X^s_{\varepsilon^3}}. \tag{3.37}$$

Estimate on II. We have

$$II = -\left(\left[\Lambda^s, M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D)\right]\mathbf{V},\ \left(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)+S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\right)\Lambda^s\mathbf{V}\right)_2$$
$$-\left(M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D)\Lambda^s\mathbf{V},\ \left(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)+S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\right)\Lambda^s\mathbf{V}\right)_2$$
$$= II_1 + II_2.$$

Using (3.23) and (3.21), we can rewrite

$$II_1 = -\left([\Lambda^s, M(\underline{\mathbf{V}}, D)]\mathbf{V},\ \left(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)+S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\right)\Lambda^s\mathbf{V}\right)_2$$
$$-\left([\Lambda^s, M_1]\mathbf{V},\ \left(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)+S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\right)\Lambda^s\mathbf{V}\right)_2$$
$$= II_{11} + II_{12}.$$

Estimate on II_{11}. Since $\tilde{\mathbf{V}}_\omega$ is sufficiently smooth and bounded, we can use the same argument as in [55] to get

$$|II_{11}| \lesssim |\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}|\mathbf{V}|^2_{X^s_{\varepsilon^3}}. \tag{3.38}$$

Estimate on II_{12}. M_1 depends only in $\tilde{\mathbf{V}}_\omega$ then there will be a problem if we apply the commutator estimate (3.27) directly to Λ^s. We use the expression $\Lambda^s = \Lambda_1^s + \Lambda_2^s$ and estimate II_{12} separately as follows

First, we have

$$\left([\Lambda_1^s, M_1]\mathbf{V},\ S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)\Lambda^s\mathbf{V}\right)_2$$
$$\lesssim |g(D)([\Lambda_1^s, M_1]\mathbf{V})|_2\left(|\underline{\mathbf{V}}|_\infty + \left|\tilde{\mathbf{V}}_\omega\right|_\infty\right)|(1+c\varepsilon\Delta)\Lambda^s\mathbf{V}|_2$$
$$+ |[\Lambda_1^s, M_1]\mathbf{V}|_2\left(|(1+c\varepsilon\Delta)g(D)\Lambda^s\mathbf{V}|_2 + |(g(D)-1)\Lambda^s\mathbf{V}|_2\right)$$
$$- c\varepsilon\left(\nabla([\Lambda_1^s, M_1]\mathbf{V})_1,\ \nabla(1+c\varepsilon\Delta)g(D)\Lambda^s\mathbf{V}\right)_2$$
$$- a\varepsilon\left(\nabla([\Lambda_1^s, M_1]\mathbf{V})_2,\ \nabla(1+c\varepsilon\Delta)\Lambda^s v_1\right)_2$$
$$- a\varepsilon\left(\nabla([\Lambda_1^s, M_1]\mathbf{V})_3,\ \nabla(1+c\varepsilon\Delta)\Lambda^s v_2\right)_2.$$

(*where* $(\cdot)_j$ *is the notation of* j^{th} *component of the vector*)

Using the expression of M_1 we have that

$$|[\Lambda_1^s, M_1]\mathbf{V}|_2 \lesssim \left|[\Lambda_1^s, \tilde{\mathbf{V}}_\omega]\mathbf{V}\right|_2 = \left|\left|[\Lambda_1^s, \tilde{\mathbf{V}}_\omega]\mathbf{V}\right|_{L^2_{x_1}}\right|_{L^2_{x_2}}$$

and

$$|\nabla([\Lambda_1^s, M_1]\mathbf{V})_j|_2 \lesssim \left|([\Lambda_1^s, \nabla\tilde{\mathbf{V}}_\omega]\mathbf{V})_j\right|_2 \lesssim \left|\left|[\Lambda_1^s, \nabla\tilde{\mathbf{V}}_\omega]\mathbf{V}\right|_{L^2_{x_1}}\right|_{L^2_{x_2}}.$$

Then, using (3.27) with $d = 1$, $\sigma(D) = \Lambda_1^s$ we obtain

$$\left|[\Lambda_1^s, \tilde{\mathbf{V}}_\omega]\mathbf{V}\right|_{L^2_{x_1}} \lesssim \left|\nabla\tilde{\mathbf{V}}_\omega\right|_{L^\infty_{x_1}}|\mathbf{V}|_{H^{s-1}_{x_1}} + \left|\nabla\tilde{\mathbf{V}}_\omega\right|_{H^{s-1}_{x_1}}|\mathbf{V}|_{L^\infty_{x_1}} \lesssim \varepsilon|\Lambda_1^s\mathbf{V}|_{L^2_{x_1}},$$

and, with Leibnitz rule

$$\left|[\Lambda_1^s, \nabla\tilde{\mathbf{V}}_\omega]\mathbf{V}\right|_{L^2_{x_1}} \lesssim \varepsilon|\Lambda_1^{s+1}\mathbf{V}|_{L^2_{x_1}}.$$

Remark: The coefficients on these above estimates depend on $\left|\tilde{\mathbf{V}}_\omega\right|_{W^{2,\infty}}$ and do not depend on ε since M_1 is independent of ε.

Therefore, by combining the above estimates and (3.28)-(3.29), we have

$$
\begin{aligned}
&\Big([\Lambda_1^s, M_1]\mathbf{V},\, S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)\Lambda^s\mathbf{V}\Big)_2 \\
&\qquad \lesssim \varepsilon\,|\Lambda_1^s\mathbf{V}|_2\,(|\mathbf{V}|_{H^s}+\varepsilon|V|_{H^{s+2}}) + |\Lambda_1^s\mathbf{V}|_2\,(|\mathbf{V}|_{H^s}+\varepsilon|\mathbf{V}|_{H^{s+2}}) \\
&\qquad\quad + \varepsilon^{3/2}|\Lambda_1^{s+1}\mathbf{V}|_2\left(\varepsilon^{1/2}|\mathbf{V}|_{H^{s+1}}+\varepsilon^{3/2}|\mathbf{V}|_{H^{s+3}}\right) \\
&\qquad \lesssim \varepsilon\left(|\mathbf{V}|_{H^s}+\varepsilon^{1/2}|\mathbf{V}|_{H^{s+1}}+\varepsilon|\mathbf{V}|_{H^{s+2}}+\varepsilon^{3/2}|\mathbf{V}|_{H^{s+3}}\right)^2 \\
&\qquad \lesssim \varepsilon\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,.
\end{aligned}
\tag{3.39}
$$

The estimate for $\Big([\Lambda_1^s, M_1]\mathbf{V},\, S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\Lambda^s\mathbf{V}\Big)_2$ follows similarly as (3.39).

The estimate for $\Big([\Lambda_2^s, M_1]\mathbf{V},\, \Big(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)+S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\Big)\,\Lambda^s\mathbf{V}\Big)_2$ is easier since $\tilde{\mathbf{V}}_\omega$ does not depend on x_2. Therefore we have

$$
\Big([\Lambda^s, M_1]\mathbf{V},\, \Big(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)+S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\Big)\,\Lambda^s\mathbf{V}\Big)_2 \lesssim \varepsilon\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,. \tag{3.40}
$$

Estimate on II_1: Thanks to (3.38) and (3.40) we get

$$
II_1 \lesssim \varepsilon\,|\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,. \tag{3.41}
$$

Estimate on II_2: In order to estimate II_2, we first calculate

$$
S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D) := A(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D) = (a_{ij})
$$

as follows:

$$
\begin{aligned}
a_{11} &= (1+c\varepsilon\Delta)^2 g(D)((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla) + g(D)\left((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla(1+c\varepsilon\Delta)^2 g(D)\right),\\
a_{12} &= (1+c\varepsilon\Delta)^2 g(D)\left((1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)\partial_{x_1}\right)\\
&\quad + g(D)\left((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot(1+c\varepsilon\Delta)g(D)((\underline{v}_1+\tilde{v}_{1\omega})\nabla)\right),\\
a_{13} &= (1+c\varepsilon\Delta)^2 g(D)\left((1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)\partial_{x_2}\right)\\
&\quad + g(D)\left((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot(1+c\varepsilon\Delta)g(D)((\underline{v}_2+\tilde{v}_{2\omega})\nabla)\right),\\
a_{21} &= g(D)\left((\underline{v}_1+\tilde{v}_{1\omega})(1+c\varepsilon\Delta)((\underline{\mathbf{v}}+\underline{\tilde{\mathbf{v}}}_\omega)\cdot\nabla)\right)\\
&\quad + (\underline{v}_1+\tilde{v}_{1\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla(1+c\varepsilon\Delta)g(D)(g(D)-1)\\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)(1+c\varepsilon\Delta)^2 g(D)\partial_{x_1},\\
a_{22} &= g(D)\left((\underline{v}_1+\tilde{v}_{1\omega})(1+c\varepsilon\Delta)[(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)\partial_{x_1}]\right)\\
&\quad + (\underline{v}_1+\tilde{v}_{1\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_1+\tilde{v}_{1\omega})\nabla)\\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)(1+c\varepsilon\Delta)g(D)((\underline{v}_1+\tilde{v}_{1\omega})\partial_{x_1}),\\
a_{23} &= g(D)\left((\underline{v}_1+\tilde{v}_{1\omega})(1+c\varepsilon\Delta)[(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)\partial_{x_2}]\right)\\
&\quad + (\underline{v}_1+\tilde{v}_{1\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_2+\tilde{v}_{2\omega})\nabla)\\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)(1+c\varepsilon\Delta)g(D)((\underline{v}_2+\tilde{v}_{2\omega})\partial_{x_1}),\\
a_{31} &= g(D)\left((\underline{v}_2+\tilde{v}_{2\omega})(1+c\varepsilon\Delta)((\underline{\mathbf{v}}+\underline{\tilde{\mathbf{v}}}_\omega)\cdot\nabla)\right)\\
&\quad + (\underline{v}_2+\tilde{v}_{2\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla(1+c\varepsilon\Delta)g(D)(g(D)-1)\\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)(1+c\varepsilon\Delta)^2 g(D)\partial_{x_2},\\
a_{32} &= g(D)\left((\underline{v}_2+\tilde{v}_{2\omega})(1+c\varepsilon\Delta)[(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)\partial_{x_1}]\right)\\
&\quad + (\underline{v}_2+\tilde{v}_{2\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_1+\tilde{v}_{1\omega})\nabla)\\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)(1+c\varepsilon\Delta)g(D)((\underline{v}_1+\tilde{v}_{1\omega})\partial_{x_2}),\\
a_{33} &= g(D)\left((\underline{v}_2+\tilde{v}_{2\omega})(1+c\varepsilon\Delta)[(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)\partial_{x_2}]\right)\\
&\quad + (\underline{v}_2+\tilde{v}_{2\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_2+\tilde{v}_{2\omega})\nabla)\\
&\quad + (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)(1+c\varepsilon\Delta)g(D)((\underline{v}_2+\tilde{v}_{2\omega})\partial_{x_2}),
\end{aligned}
$$

For a_{11},

(3.42)
$$
\begin{aligned}
&(a_{11}\Lambda^s\zeta,\,\Lambda^s\zeta)_2\\
&\quad= \left((1+c\varepsilon\Delta)^2 g(D)(\underline{\mathbf{v}}\cdot\nabla\Lambda^s\zeta) + g(D)(\underline{\mathbf{v}}\cdot\nabla(1+c\varepsilon\Delta)^2 g(D)\Lambda^s\zeta),\,\Lambda^s\zeta\right)_2\\
&\quad+ \left((1+c\varepsilon\Delta)^2 g(D)(\tilde{\mathbf{v}}_\omega\cdot\nabla\Lambda^s\zeta) + g(D)(\tilde{\mathbf{v}}_\omega\cdot\nabla(1+c\varepsilon\Delta)^2 g(D)\Lambda^s\zeta),\,\Lambda^s\zeta\right)_2\\
&\quad:= B_{11}+B_{12}.
\end{aligned}
$$

From [55] we have

$$B_{11} \lesssim |\underline{\mathbf{v}}|_{X^s_{\varepsilon^3}}\,|\zeta|^2_{X^s_{\varepsilon^3}}\,, \tag{3.43}$$

with B_{12}, we can not use the commutator estimate for $g(D)$ and $\tilde{\mathbf{v}}_\omega$. In order to get the relevant norm we will use that $\tilde{\mathbf{v}}_\omega = \varepsilon\tilde{\mathbf{u}}_\omega$. First, using integrating by part, we have

$$
\begin{aligned}
B_{12} &= ((1+c\varepsilon\Delta)(\underline{\tilde{\mathbf{v}}}_\omega\cdot\nabla\Lambda^s\zeta),\,(1+c\varepsilon\Delta)g(D)\Lambda^s\zeta)_2\\
&\quad-\Big((1+c\varepsilon\Delta)^2g(D)\Lambda^s\zeta,\,\nabla\cdot(\tilde{\mathbf{v}}_\omega g(D)\Lambda^s\zeta)\Big)_2\\
&=\varepsilon\,((1+c\varepsilon\Delta)(\tilde{\mathbf{u}}_\omega\cdot\nabla\Lambda^s\zeta),\,(1+c\varepsilon\Delta)g(D)\Lambda^s\zeta)_2\\
&\quad-\varepsilon\,((1+c\varepsilon\Delta)g(D)\Lambda^s\zeta,\,(1+c\varepsilon\Delta)\nabla\cdot(\tilde{\mathbf{u}}_\omega g(D)\Lambda^s\zeta))_2\,.
\end{aligned}
\tag{3.44}
$$

By using Leibniz rule, it is not hard to see that

$$
B_{12}\lesssim(\varepsilon^{1/2}+\varepsilon^2)\,|\tilde{\mathbf{u}}_\omega|_{W^{2,\infty}}\,|\zeta|^2_{X^s_{\varepsilon^3}}
\tag{3.45}
$$

Combining (3.45) and (3.43), we obtain

$$
\begin{aligned}
(a_{11}\Lambda^s\zeta,\,\Lambda^s\zeta)_2&\lesssim((\varepsilon^{1/2}+\varepsilon^2)\,|\tilde{\mathbf{u}}_\omega|_{W^{2,\infty}}+|\underline{\mathbf{v}}|_{X^s_{\varepsilon^3}})\,|\zeta|^2_{X^s_{\varepsilon^3}}\\
&\lesssim\varepsilon^{1/2}(1+\varepsilon^{1/2}\,|\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}})\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,.
\end{aligned}
\tag{3.46}
$$

For a_{12} and a_{21}, using integration by part, we get:

$$
\begin{aligned}
&(a_{12}\Lambda^s v_1,\,\Lambda^s\zeta)_2+(a_{21}\Lambda^s\zeta,\,\Lambda^s v_1)_2\\
&=-\Big((1+c\varepsilon\Delta)(\partial_{x_1}(\underline{\zeta}+\underline{\tilde{\zeta}}_\omega)\Lambda^s v_1),\,g(D)(1+c\varepsilon\Delta)\Lambda^s\zeta\Big)_2\\
&\quad+\Big\{(g(D)((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot(1+c\varepsilon\Delta)g(D)((\underline{v}_1+\tilde{v}_{1\omega})\nabla\Lambda^s v_1),\,\Lambda^s\zeta)_2\\
&\quad+(g(D)((\underline{v}_1+\tilde{v}_{1\omega})(1+c\varepsilon\Delta)((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla\Lambda^s\zeta),\,\Lambda^s v_1)_2\\
&\quad+((\underline{v}_1+\tilde{v}_{1\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla(1+c\varepsilon\Delta)g(D)(g(D)-1)\Lambda^s\zeta,\,\Lambda^s v_1)_2\Big\}\\
&=B_{21}+B_{22},
\end{aligned}
\tag{3.47}
$$

where B_{21} contains only the terms which depends only on $\mathbf{V}$ and

$$
\begin{aligned}
B_{22}&=-\Big((1+c\varepsilon\Delta)(\partial_{x_1}\tilde{\zeta}_\omega\Lambda^s v_1),\,g(D)(1+c\varepsilon\Delta)\Lambda^s\zeta\Big)_2\\
&\quad+((1+c\varepsilon\Delta)g(D)(\tilde{v}_{1\omega}\nabla\Lambda^s v_1),\,\underline{\mathbf{v}}(g(D)\Lambda^s\zeta))_2\\
&\quad+((1+c\varepsilon\Delta)g(D)(\tilde{v}_{1\omega}\nabla\Lambda^s v_1),\,\tilde{\mathbf{v}}_\omega(g(D)\Lambda^s\zeta))_2\\
&\quad+((1+c\varepsilon\Delta)g(D)(\underline{v}_1\nabla\Lambda^s v_1),\,\tilde{\mathbf{v}}_\omega(g(D)\Lambda^s\zeta))_2\\
&\quad+((1+c\varepsilon\Delta)(\tilde{\mathbf{v}}_\omega\cdot\nabla\Lambda^s\zeta),\,\underline{v}_1 g(D)\Lambda^s v_1)_2\\
&\quad+((1+c\varepsilon\Delta)(\tilde{\mathbf{v}}_\omega\cdot\nabla\Lambda^s\zeta),\,\tilde{v}_{1\omega}g(D)\Lambda^s v_1)_2\\
&\quad+((1+c\varepsilon\Delta)(\underline{\mathbf{v}}\cdot\nabla\Lambda^s\zeta),\,\tilde{v}_{1\omega}g(D)\Lambda^s v_1)_2\\
&\quad+(\underline{v}_1\tilde{\mathbf{v}}_\omega\cdot\nabla(1+c\varepsilon\Delta)g(D)(g(D)-1)\Lambda^s\zeta,\,\Lambda^s v_1)_2\\
&\quad+(\tilde{v}_{1\omega}\underline{\mathbf{v}}\cdot\nabla(1+c\varepsilon\Delta)g(D)(g(D)-1)\Lambda^s\zeta,\,\Lambda^s v_1)_2\\
&\quad+(\tilde{v}_{1\omega}\underline{\tilde{\mathbf{v}}}_\omega\cdot\nabla(1+c\varepsilon\Delta)g(D)(g(D)-1)\Lambda^s\zeta,\,\Lambda^s v_1)_2
\end{aligned}
\tag{3.48}
$$

By using Leibnitz rule, Hölder inequality and (3.28) we have

$$
\Big((1+c\varepsilon\Delta)((\partial_{x_1}\tilde{\zeta}_\omega)\Lambda^s v_1),\,g(D)(1+c\varepsilon\Delta)\Lambda^s\zeta\Big)_2\lesssim\Big|\tilde{\zeta}_\omega\Big|_{W^{3,\infty}}|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,,
$$

$$
((1+c\varepsilon\Delta)g(D)(\tilde{v}_{1\omega}\nabla\Lambda^s v_1),\,\underline{\mathbf{v}}(g(D)\Lambda^s\zeta))_2\lesssim(\varepsilon^{1/2}+\varepsilon^{3/2})\,|\underline{\tilde{u}}_{1\omega}|_{W^{2,\infty}}\,|\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,,
$$

$$((1+c\varepsilon\Delta)g(D)(\tilde{v}_{1\omega}\nabla\Lambda^s v_1),\, \tilde{\mathbf{v}}_\omega(g(D)\Lambda^s\zeta))_2 \lesssim (\varepsilon^{1/2}+\varepsilon^{3/2})\,|\underline{\tilde{u}}_{1\omega}|_{W^{2,\infty}}\,|\tilde{\mathbf{v}}_\omega|_\infty\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}},$$

$$\begin{aligned}&((1+c\varepsilon\Delta)g(D)(\underline{v}_1\nabla\Lambda^s v_1),\, \tilde{\mathbf{v}}_\omega(g(D)\Lambda^s\zeta))_2\\ &\quad= (g(D)(\underline{v}_1\nabla\Lambda^s v_1),\, (1+c\varepsilon\Delta)(\tilde{\mathbf{v}}_\omega(g(D)\Lambda^s\zeta)))_2\\ &\quad\lesssim (\varepsilon^{1/2}+\varepsilon^{3/2})\,|\tilde{\mathbf{u}}_\omega|_{W^{2,\infty}}\,|\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}},\end{aligned}$$

$$((1+c\varepsilon\Delta)(\tilde{\mathbf{v}}_\omega\cdot\nabla\Lambda^s\zeta),\, \underline{v}_1 g(D)\Lambda^s v_1)_2 \lesssim (\varepsilon^{1/2}+\varepsilon^{3/2})\,|\tilde{\mathbf{u}}_\omega|_{W^{2,\infty}}\,|\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}},$$

$$((1+c\varepsilon\Delta)(\tilde{\mathbf{v}}_\omega\cdot\nabla\Lambda^s\zeta),\, \tilde{v}_{1\omega} g(D)\Lambda^s v_1)_2 \lesssim (\varepsilon^{1/2}+\varepsilon^{3/2})\,|\tilde{\mathbf{u}}_\omega|_{W^{2,\infty}}\,|\tilde{v}_{1\omega}|_\infty\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}},$$

$$\begin{aligned}&((1+c\varepsilon\Delta)(\underline{\mathbf{v}}\cdot\nabla\Lambda^s\zeta),\, \tilde{v}_{1\omega} g(D)\Lambda^s v_1)_2\\ &\quad= (\underline{\mathbf{v}}\cdot\nabla\Lambda^s\zeta,\, (1+c\varepsilon\Delta)(\tilde{v}_{1\omega} g(D)\Lambda^s v_1))_2\\ &\quad\lesssim (\varepsilon^{1/2}+\varepsilon^{3/2})\,|\underline{\tilde{u}}_{1\omega}|_{W^{2,\infty}}\,|\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}},\end{aligned}$$

$$(\underline{v}_1\tilde{\mathbf{v}}_\omega\cdot\nabla(1+c\varepsilon\Delta)g(D)(g(D)-1)\Lambda^s\zeta,\, \Lambda^s v_1)_2 \lesssim (\varepsilon^{1/2}+\varepsilon^{3/2})\,|\tilde{\mathbf{u}}_\omega|_\infty\,|\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}},$$

$$(\tilde{v}_{1\omega}\underline{\mathbf{v}}\cdot\nabla(1+c\varepsilon\Delta)g(D)(g(D)-1)\Lambda^s\zeta,\, \Lambda^s v_1)_2 \lesssim (\varepsilon^{1/2}+\varepsilon^{3/2})\,|\underline{\tilde{u}}_{1\omega}|_\infty\,|\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}},$$

$$\begin{aligned}&(\tilde{v}_{1\omega}\underline{\tilde{\mathbf{v}}}_\omega\cdot\nabla(1+c\varepsilon\Delta)g(D)(g(D)-1)\Lambda^s\zeta,\, \Lambda^s v_1)_2\\ &\quad\lesssim (\varepsilon^{1/2}+\varepsilon^{3/2})\,|\underline{\tilde{u}}_{1\omega}|_\infty\,\left|\tilde{\mathbf{V}}_\omega\right|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}.\end{aligned}$$

Therefore,

$$B_{22} \lesssim (\varepsilon^{1/2}+\varepsilon^{3/2})\,|\tilde{\mathbf{u}}_\omega|_{W^{3,\infty}}\,(\varepsilon^{1/2}+|\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}})\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}. \tag{3.49}$$

From [55] we also have

$$B_{21} \lesssim |\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}. \tag{3.50}$$

Combining (3.49)-(3.50), it follows

$$(a_{12}\Lambda^s v_1,\, \Lambda^s\zeta)_2 + (a_{21}\Lambda^s\zeta,\, \Lambda^s v_1)_2 \lesssim |\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}. \tag{3.51}$$

□

The same estimate holds for $(a_{13}\Lambda^s v_1,\, \Lambda^s\zeta)_2 + (a_{31}\Lambda^s\zeta,\, \Lambda^s v_2)_2$.
For a_{22},

$$\begin{aligned}&(a_{22}\Lambda^s v_1,\, \Lambda^s v_1)_2\\ &\quad= \Big((\underline{v}_1+\tilde{v}_{1\omega})(1+c\varepsilon\Delta)(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)\partial_{x_1}\Lambda^s v_1,\, g(D)\Lambda^s v_1\Big)_2\\ &\quad+\Big((1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)(1+c\varepsilon\Delta)g(D)((\underline{v}_1+\tilde{v}_{1\omega})\partial_{x_1}\Lambda^s v_1),\, \Lambda^s v_1\Big)_2\\ &\quad+((\underline{v}_1+\tilde{v}_{1\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_1+\tilde{v}_{1\omega})\nabla\Lambda^s v_1),\, \Lambda^s v_1)_2\\ &\quad:= B_{31}+B_{32}.\end{aligned} \tag{3.52}$$

Where B_{31} contains only the terms which depends only on $\mathbf{V}$ and

$$B_{32} = \Big(\underline{v}_1(1+c\varepsilon\Delta)(\tilde{\zeta}_\omega\partial_{x_1}\Lambda^s v_1),\, g(D)\Lambda^s v_1\Big)_2 \\ + \Big(\tilde{v}_{1\omega}(1+c\varepsilon\Delta)(1+\underline{\zeta}+\underline{\tilde{\zeta}}_\omega + a\varepsilon\Delta)(\partial_{x_1}\Lambda^s v_1),\, g(D)\Lambda^s v_1\Big)_2 \\ + \Big(\tilde{\zeta}_\omega(1+c\varepsilon\Delta)g(D)((\underline{v}_1+\tilde{v}_{1\omega})\partial_{x_1}\Lambda^s v_1),\, \Lambda^s v_1\Big)_2 \\ + \left((1+\underline{\zeta}+a\varepsilon\Delta)(1+c\varepsilon\Delta)g(D)(\tilde{v}_{1\omega}\partial_{x_1}\Lambda^s v_1),\, \Lambda^s v_1\right)_2 \\ + \left(\tilde{v}_{1\omega}(\underline{\mathbf{v}}+\underline{\tilde{\mathbf{v}}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_1+\tilde{v}_{1\omega})\nabla\Lambda^s v_1),\, \Lambda^s v_1\right)_2 \\ + \left(\underline{v}_1\tilde{\mathbf{v}}_\omega\cdot g(D)(g(D)-1)((\underline{v}_1+\tilde{v}_{1\omega})\nabla\Lambda^s v_1),\, \Lambda^s v_1\right)_2 \\ + \left(\underline{v}_1\underline{\mathbf{v}}\cdot g(D)(g(D)-1)(\tilde{v}_{1\omega}\nabla\Lambda^s v_1),\, \Lambda^s v_1\right)_2. \tag{3.53}$$

Similarly as (3.50) we get

$$B_{32} \lesssim \varepsilon^{1/2}\,|\tilde{\mathbf{u}}_\omega|_{W^{2,\infty}}\,(1+|\underline{\mathbf{V}}|^2_{X^s_{\varepsilon^3}} + \left|\tilde{\mathbf{V}}_\omega\right|^2_\infty)\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}. \tag{3.54}$$

From [55] we have

$$B_{31} \lesssim |\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}, \tag{3.55}$$

it follows

$$(a_{22}\Lambda^s v_1,\, \Lambda^s v_1)_2 \lesssim \varepsilon^{1/2}(1+\varepsilon^{1/2}\,|\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}} + |\underline{V}|^2_{X^s_{\varepsilon^3}})\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}. \tag{3.56}$$

The same estimate holds for $(a_{33}\Lambda^s v_2,\, \Lambda^s v_2)_2$.
For a_{23} and a_{32}, we have

$$(a_{23}\Lambda^s v_2,\, \Lambda^s v_1)_2 + (a_{32}\Lambda^s v_1,\, \Lambda^s v_2)_2 \\ = \Big(g(D)((\underline{v}_1+\tilde{v}_{1\omega})(1+c\varepsilon\Delta)(1+\underline{\zeta}+\underline{\tilde{\zeta}}_\omega+a\varepsilon\Delta)\partial_{x_2}\Lambda^s v_2),\, \Lambda^s v_1\Big)_2 \\ + \left((\underline{v}_1+\tilde{v}_{1\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_2+\tilde{v}_{2\omega})\nabla\Lambda^s v_2),\, \Lambda^s v_1\right)_2 \\ + \Big((1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)(1+c\varepsilon\Delta)g(D)((\underline{v}_2+\tilde{v}_{2\omega})\partial_{x_1}\Lambda^s v_2),\, \Lambda^s v_1\Big)_2 \\ + \Big(g(D)((\underline{v}_2+\tilde{v}_{2\omega})(1+c\varepsilon\Delta)(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)\partial_{x_1}\Lambda^s v_1),\, \Lambda^s v_2\Big)_2 \\ + \left((\underline{v}_2+\tilde{v}_{2\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_1+\tilde{v}_{1\omega})\nabla\Lambda^s v_1),\, \Lambda^s v_2\right)_2 \\ + \Big((1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)(1+c\varepsilon\Delta)g(D)((\underline{v}_1+\tilde{v}_{1\omega})\partial_{x_2}\Lambda^s v_1),\, \Lambda^s v_2\Big)_2 \\ = B_{41}+B_{42}, \tag{3.57}$$

where B_{41} contains only the terms which depends only on $\mathbf{V}$ and

$$
\begin{aligned}
B_{42} = &\Big(g(D)(\tilde{v}_{1\omega}(1+c\varepsilon\Delta)(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)\partial_{x_2}\Lambda^s v_2),\, \Lambda^s v_1\Big)_2 \\
&+\Big(g(D)(\underline{v}_1(1+c\varepsilon\Delta)(\underline{\tilde{\zeta}}_\omega\partial_{x_2}\Lambda^s v_2)),\, \Lambda^s v_1\Big)_2 \\
&+(\tilde{v}_{1\omega}(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_2+\tilde{v}_{2\omega})\nabla\Lambda^s v_2),\, \Lambda^s v_1)_2 \\
&+(\underline{v}_1\tilde{\mathbf{v}}_\omega\cdot g(D)(g(D)-1)((\underline{v}_2+\tilde{v}_{2\omega})\nabla\Lambda^s v_2),\, \Lambda^s v_1)_2 \\
&+(\underline{v}_1\underline{\mathbf{v}}\cdot g(D)(g(D)-1)(\tilde{v}_{2\omega}\nabla\Lambda^s v_2),\, \Lambda^s v_1)_2 \\
&+\Big(\tilde{\zeta}_\omega(1+c\varepsilon\Delta)g(D)((\underline{v}_2+\tilde{v}_{2\omega})\partial_{x_1}\Lambda^s v_2),\, \Lambda^s v_1\Big)_2 \\
&+\big((1+\underline{\zeta}+a\varepsilon\Delta)(1+c\varepsilon\Delta)g(D)(\tilde{v}_{2\omega}\partial_{x_1}\Lambda^s v_2),\, \Lambda^s v_1\big)_2 \\
&+\Big(g(D)(\tilde{v}_{2\omega}(1+c\varepsilon\Delta)(1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)\partial_{x_1}\Lambda^s v_1),\, \Lambda^s v_2\Big)_2 \\
&+\Big(g(D)(\underline{v}_2(1+c\varepsilon\Delta)(\tilde{\zeta}_\omega\partial_{x_1}\Lambda^s v_1)),\, \Lambda^s v_2\Big)_2 \\
&+(\tilde{v}_{2\omega}(\underline{\mathbf{v}}+\underline{\tilde{\mathbf{v}}}_\omega)\cdot g(D)(g(D)-1)((\underline{v}_1+\tilde{v}_{1\omega})\nabla\Lambda^s v_1),\, \Lambda^s v_2)_2 \\
&+(\underline{v}_2\tilde{\mathbf{v}}_\omega\cdot g(D)(g(D)-1)((\underline{v}_1+\tilde{v}_{1\omega})\nabla\Lambda^s v_1),\, \Lambda^s v_2)_2 \\
&+(\underline{v}_2\underline{\mathbf{v}}\cdot g(D)(g(D)-1)(\tilde{v}_{1\omega}\nabla\Lambda^s v_1),\, \Lambda^s v_2)_2 \\
&+\Big(\tilde{\zeta}_\omega(1+c\varepsilon\Delta)g(D)((\underline{v}_1+\tilde{v}_{1\omega})\partial_{x_2}\Lambda^s v_1),\, \Lambda^s v_2\Big)_2 \\
&+\big((1+\underline{\zeta}+a\varepsilon\Delta)(1+c\varepsilon\Delta)g(D)(\tilde{v}_{1\omega}\partial_{x_2}\Lambda^s v_1),\, \Lambda^s v_2\big)_2 .
\end{aligned}
\tag{3.58}
$$

Similarly as (3.50) a v nd (3.54) we have

$$
B_{42} \lesssim \varepsilon^{1/2}\,|\tilde{\mathbf{u}}_\omega|_{W^{2,\infty}}\,(1+|\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}+\left|\underline{\tilde{V}}_\omega\right|_\infty)\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,,
\tag{3.59}
$$

from [55] we also have

$$
B_{41} \lesssim |\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}}\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,.
\tag{3.60}
$$

Therefore,

$$
(a_{23}\Lambda^s v_2,\, \Lambda^s v_1)_2+(a_{32}\Lambda^s v_1,\, \Lambda^s v_2)_2 \lesssim \varepsilon^{1/2}(1+\varepsilon^{1/2}\,|\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}}+|\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}})\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,.
\tag{3.61}
$$

Thanks to (3.46), (3.51), (3.56) and (3.61), we obtain

$$
\left|\Big(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega,D)\Lambda^s\mathbf{V},\,\Lambda^s\mathbf{V}\Big)_2\right| \lesssim \varepsilon^{1/2}(1+\varepsilon^{1/2}\,|\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}}+|\underline{\mathbf{V}}|^2_{X^s_{\varepsilon^3}})\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,.
$$

The same estimate holds for the term $\Big(M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega,D)\Lambda^s\mathbf{V},\,S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)\Lambda^s\mathbf{V}\Big)_2$, then

$$
|II_2| \lesssim \varepsilon^{1/2}(1+\varepsilon^{1/2}\,|\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}}+|\underline{\mathbf{V}}|^2_{X^s_{\varepsilon^3}})\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,.
\tag{3.62}
$$

Due to (3.41) and (3.62), we have

$$
|II| \lesssim \varepsilon^{1/2}(1+\varepsilon^{1/2}\,|\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}}+|\underline{\mathbf{V}}|^2_{X^s_{\varepsilon^3}})\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}}\,.
\tag{3.63}
$$

Estimate on III. Using (3.21) and the expression of $(S_1+S_2)^*$ we only need to estimate

$$\begin{aligned}&-b\varepsilon\left([(S_1+S_2)^*,\Delta]\Lambda^s\mathbf{V},\,\Lambda^s\partial_t\mathbf{V}\right)_2\\&\quad\lesssim\varepsilon\left|[\tilde{\mathbf{v}}_\omega,\Delta]g(D)\Lambda^s\mathbf{V}\right|_2\left|(1+c\varepsilon\Delta)\Lambda^s\partial_t\mathbf{V}\right|_2\\&\qquad+\varepsilon\sum_{i,j=1}^{2}\left|[\underline{v}_i\tilde{v}_{j\omega}+\tilde{v}_{i\omega}\underline{v}_j+\tilde{v}_{i\omega}\tilde{v}_{j\omega},\Delta]\Lambda^s\mathbf{V}\right|_2\left|(g(D)-1)\Lambda^s\partial_t\mathbf{V}\right|_2.\end{aligned}$$

In order to use the commutator estimate (3.27), we use the same argument in the estimate for II_{11}. It follows

$$\begin{aligned}&-b\varepsilon\left([(S_1+S_2)^*,\Delta]\Lambda^s\mathbf{V},\,\Lambda^s\partial_t\mathbf{V}\right)_2\\&\quad\lesssim\varepsilon^2\left|\mathbf{V}\right|_{H^{s+1}}\left(\left|\partial_t\mathbf{V}\right|_{H^s}+\varepsilon\left|\partial_t\mathbf{V}\right|_{H^{s+2}}\right)+\varepsilon^2(\left|\underline{\mathbf{V}}\right|+1)\left|\mathbf{V}\right|_{H^{s+1}}\left|\partial_t\mathbf{V}\right|_{H^s}\\&\quad\lesssim\varepsilon^{3/2}(\left|\underline{\mathbf{V}}\right|_{X^s_{\varepsilon^3}}+1)\left|\mathbf{V}\right|_{X^s_{\varepsilon^3}}\left|\partial_t\mathbf{V}\right|_{X^{s-1}_{\varepsilon^3}}\end{aligned}\tag{3.64}$$

From [55] we know that

$$-b\varepsilon\left([S_{\underline{\mathbf{V}}}(D)^*,\Delta]\Lambda^s\mathbf{V},\,\Lambda^s\partial_t\mathbf{V}\right)_2\lesssim\left|\underline{\mathbf{V}}\right|_{X^s_{\varepsilon^3}}\left|\mathbf{V}\right|_{X^s_{\varepsilon^3}}\left|\partial_t\mathbf{V}\right|_{X^{s-1}_{\varepsilon^3}},\tag{3.65}$$

Therefore,

$$III\lesssim(\varepsilon^{3/2}+\varepsilon^{3/2}\left|\underline{\mathbf{V}}\right|_{X^s_{\varepsilon^3}}+\left|\underline{\mathbf{V}}\right|_{X^s_{\varepsilon^3}})\left|\mathbf{V}\right|_{X^s_{\varepsilon^3}}\left|\partial_t\mathbf{V}\right|_{X^{s-1}_{\varepsilon^3}}.\tag{3.66}$$

Estimate on IV. Plugging the expression of $S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)$ into IV we get that

$$IV=IV_1+IV_2,\tag{3.67}$$

where $IV_1=\left((1-b\varepsilon\Delta)\Lambda^s\mathbf{V},\,(\partial_tS_{\underline{\mathbf{V}}}(D))\Lambda^s\mathbf{V}\right)_2$, and from [55] we know that

$$IV_1\lesssim\left|\partial_t\underline{\mathbf{V}}\right|_{H^{s-1}}\left|\mathbf{V}\right|^2_{X^s_{\varepsilon^3}}.\tag{3.68}$$

And

(3.69)

$$\begin{aligned}&IV_2\\&=((1-b\varepsilon\Delta)\Lambda^s\zeta,\,g(D)((\partial_t\tilde{v}_{1\omega})(1+c\varepsilon\Delta)\Lambda^sv_1))_2\\&+((1-b\varepsilon\Delta)\Lambda^s\zeta,\,g(D)(\partial_t(\tilde{v}_{2\omega})(1+c\varepsilon\Delta)\Lambda^sv_2))_2\\&+((1-b\varepsilon\Delta)\Lambda^sv_1,\,g(D)(\partial_t(\tilde{v}_{1\omega})(1+c\varepsilon\Delta)\Lambda^s\zeta))_2\\&+\Big((1-b\varepsilon\Delta)\Lambda^sv_1,\,(\partial_t\tilde{\zeta}_\omega)(1+c\varepsilon\Delta)\Lambda^sv_1+(\partial_t(2\underline{v}_1\tilde{v}_{1\omega}+\tilde{v}^2_{1\omega}))(g(D)-1)\Lambda^sv_1\Big)_2\\&+((1-b\varepsilon\Delta)\Lambda^sv_1,\,(\partial_t(\tilde{v}_{1\omega}\tilde{v}_{2\omega}+\underline{v}_1\tilde{v}_{2\omega}+\underline{v}_2\tilde{v}_{1\omega}))(g(D)-1)\Lambda^sv_2)_2\\&+((1-b\varepsilon\Delta)\Lambda^sv_2,\,g(D)((\partial_t\tilde{v}_{2\omega})(1+c\varepsilon\Delta)\Lambda^s\zeta))_2\\&+((1-b\varepsilon\Delta)\Lambda^sv_2,\,(\partial_t(\tilde{v}_{1\omega}\tilde{v}_{2\omega}+\underline{v}_1\tilde{v}_{2\omega}+\underline{v}_2\tilde{v}_{1\omega}))(g(D)-1)\Lambda^sv_1)_2\\&+\Big((1-b\varepsilon\Delta)\Lambda^sv_2,\,(\partial_t\tilde{\zeta}_\omega)(1+c\varepsilon\Delta)\Lambda^sv_2+(\partial_t(\tilde{v}^2_{2\omega}+2\underline{v}_2\tilde{v}_{2\omega}))(g(D)-1)\Lambda^sv_2\Big)_2.\end{aligned}$$

Therefore,

$$IV_2\lesssim\varepsilon(1+\left|\partial_t\underline{\mathbf{V}}\right|_{H^{s-1}})\left|\mathbf{V}\right|^2_{X^s_{\varepsilon^3}}.\tag{3.70}$$

By combining (3.68) and (3.70) we have

$$IV\lesssim\varepsilon(1+\left|\partial_t\underline{\mathbf{V}}\right|_{H^{s-1}})\left|\mathbf{V}\right|^2_{X^s_{\varepsilon^3}}.\tag{3.71}$$

Combining (3.37), (3.63), (3.66) and (3.71) we obtain

(3.72)
$$\frac{d}{dt}E_s(\mathbf{V}) \lesssim |F|_{X^s_{\varepsilon^3}} |\mathbf{V}|_{X^s_{\varepsilon^3}} + \varepsilon^{1/2}(1+\varepsilon^{1/2}|\partial_t \underline{\mathbf{V}}|_{H^{s-1}} + \varepsilon^{1/2}|\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}} + |\underline{\mathbf{V}}|^2_{X^s_{\varepsilon^3}})\,|\mathbf{V}|^2_{X^s_{\varepsilon^3}} + \varepsilon(1+\varepsilon^{1/2}|\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}} + |\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}})\,|\partial_t V|_{X^{s-1}_{\varepsilon^3}} |V|_{X^s_{\varepsilon^3}}\,.$$

We are going to estimate $|\partial_t \mathbf{V}|_{X^{s-1}_{\varepsilon^3}}$, (3.17) follows that

$$\partial_t \mathbf{V} = (1-b\varepsilon\Delta)^{-1}(F - M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D)\mathbf{V}) := (1-b\varepsilon\Delta)^{-1}G.$$

Since the solitary wave has the "sech" profile and using in addtional the estimate (3.26) we have the same estimate as in [55] as follows,

$$|\partial_t \mathbf{V}|_{X^{s-1}_{\varepsilon^3}} \lesssim |F|_{H^s} + |\mathbf{V}|_{X^s_{\varepsilon^3}}\,. \tag{3.73}$$

This is the estimate (3.35). Plugging (3.35) into (3.72), using (3.32), we obtain (3.33) and (3.34).

The *existence* and *uniqueness* of the solution of (3.17) are obtained similarly as in [55].

3.4.2. *The case $b = d > 0,\ a = 0, c < 0$.*

Proposition 3.2. *Let $t_0 > 1,\ s \geq t_0 + 2,\ T' > 0$. Assume that*

$$F \in C([0,T']; X^s_\varepsilon(\mathbb{R}^2))$$

and

$$\underline{\mathbf{V}} = (\underline{\zeta}, \underline{\mathbf{v}}) \in C^1\left([0,T']; (X^{s-1}_{\varepsilon^3}(\mathbb{R}^2))^3\right) \cap C([0,T']; X^s_{\varepsilon^2}(\mathbb{R}^2) \times (X^s_\varepsilon(\mathbb{R}^2))^2)$$

satisfies

$$1+\underline{\zeta}+\tilde{\zeta}_\omega \geq H > 0,\ |\underline{\mathbf{V}}|_\infty + \left|\tilde{\mathbf{V}}_\omega\right|_\infty \leq \kappa_H,\ |\underline{\mathbf{V}}|_{H^s} + \left|\tilde{\mathbf{V}}_\omega\right|_{H^s} \leq 1,\ \forall t \in [0,T'], \tag{3.74}$$

with κ_H ensures the equivalence of $E_s(\mathbf{V})$ and $|\zeta|^2_{X^s_{\varepsilon^2}} + |\mathbf{v}|^2_{X^s_\varepsilon}$. (see appendix 4.2)

Then for any $V_0 \in X^s_{\varepsilon^2}(\mathbb{R}^2) \times (X^s_\varepsilon(\mathbb{R}^2))^2)$, (3.17) has a unique solution

$$\mathbf{V} \in C([0,T']; X^s_{\varepsilon^2}(\mathbb{R}^2) \times (X^s_\varepsilon(\mathbb{R}^2))^2),$$

and one has for any $t \in (0,T')$

$$\begin{aligned}\frac{d}{dt}E_s(\mathbf{V}) \lesssim &\left(\varepsilon + |\underline{\mathbf{v}}|_{X^s_\varepsilon} + \left|\underline{\zeta}\right|_{X^s_{\varepsilon^2}} + |\partial_t \underline{\mathbf{V}}|_{X^{s-1}_{\varepsilon^2}}\right) E_s(\mathbf{V})^2 \\ &+ \left(\varepsilon + |\underline{\mathbf{v}}|_{X^s_\varepsilon} + \left|\underline{\zeta}\right|_{X^s_{\varepsilon^2}}\right) |\partial_t \mathbf{V}|_{X^{s-1}_{\varepsilon^2}} E_s(\mathbf{V}) \\ &+ |F|_{H^s} E_s(\mathbf{V}).\end{aligned} \tag{3.75}$$

$$
\begin{aligned}
&|\mathbf{v}(t)|^2_{X^s_\varepsilon} + |\zeta(t)|^2_{X^s_{\varepsilon^2}} \\
&\quad \le \tilde{c}\left(|\mathbf{v}(0)|^2_{X^s_\varepsilon} + |\zeta(0)|^2_{X^s_{\varepsilon^2}}\right) \\
&\qquad + \tilde{c}\int_0^t \Bigg(|F|_{H^s}\left(|\mathbf{v}(t')|_{X^s_\varepsilon} + |\zeta(t')|_{X^s_{\varepsilon^2}}\right) \\
&\qquad\quad + (\varepsilon + |\underline{\mathbf{v}}|_{X^s_\varepsilon} + \left|\underline{\zeta}\right|_{X^s_{\varepsilon^2}})(|\mathbf{v}(t')|_{X^s_\varepsilon} + |\zeta(t')|_{X^s_{\varepsilon^2}})\, |\partial_t \mathbf{V}|_{X^{s-1}_{\varepsilon^2}} \\
&\qquad\quad + (\varepsilon + |\underline{\mathbf{v}}|_{X^s_\varepsilon} + \left|\underline{\zeta}\right|_{X^s_{\varepsilon^2}} + |\partial_t \underline{\mathbf{V}}|_{X^{s-1}_{\varepsilon^2}})(|\mathbf{v}(t')|_{X^s_\varepsilon} + |\zeta(t')|_{X^s_{\varepsilon^2}})\Bigg)^2 dt'.
\end{aligned} \tag{3.76}
$$

$$
|\partial_t \mathbf{V}|_{X^{s-1}_{\varepsilon^2}} \lesssim |F|_{H^s} + |\zeta|_{X^s_{\varepsilon^2}} + |\mathbf{v}|_{H^s}\,. \tag{3.77}
$$

Proof. We start with the expression (3.36),

$$
\begin{aligned}
\frac{d}{dt}E_s(\mathbf{V}) &= \left(\Lambda^s F,\, \left(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\right)\Lambda^s \mathbf{V}\right)_2 \\
&\quad - \left(\Lambda^s\left(M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D)\mathbf{V}\right),\, \left(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\right)\Lambda^s \mathbf{V}\right)_2 \\
&\quad - b\varepsilon\left(\left[S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*, \Delta\right]\Lambda^s \mathbf{V},\, \Lambda^s \partial_t \mathbf{V}\right)_2 \\
&\quad + \left((1-b\varepsilon\Delta)\Lambda^s \mathbf{V},\, \left(\partial_t S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)\right)\Lambda^s \mathbf{V}\right)_2 \\
&= I + II + III + IV,
\end{aligned}
$$

□

Estimate on I. By using integral by part and the expression (3.18) of $S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)$, we obtain

$$
\begin{aligned}
&\left(\Lambda^s F,\, S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)\Lambda^s \mathbf{V}\right)_2 \\
&\quad = (\Lambda^s F_1,\, (1+c\varepsilon\Delta)\Lambda^s\zeta + (\underline{v}_1 + \tilde{v}_{1\omega})\Lambda^s v_1 + (\underline{v}_2 + \tilde{v}_{2\omega})\Lambda^s v_2)_2 \\
&\qquad + \left(\Lambda^s F_2,\, (\underline{v}_1 + \tilde{v}_{1\omega})\Lambda^s\zeta + (1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s v_1\right)_2 \\
&\qquad + \left(\Lambda^s F_3,\, (\underline{v}_2 + \tilde{v}_{2\omega})\Lambda^s\zeta + (1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s v_2\right)_2,
\end{aligned} \tag{3.78}
$$

with assumption (3.74), it is not hard to get that

$$
\left(\Lambda^s F,\, S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)\Lambda^s \mathbf{V}\right)_2 \lesssim |F|_{H^s}\,(|\zeta|_{X^s_{\varepsilon^2}} + |\mathbf{v}|_{H^s}). \tag{3.79}
$$

The same estimate holds for $\left(\Lambda^s F,\, S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\Lambda^s \mathbf{V}\right)_2$, then we have

$$
|I| \lesssim |F|_{H^s}\,(|\zeta|_{X^s_{\varepsilon^2}} + |\mathbf{v}|_{H^s}). \tag{3.80}
$$

Estimate on II. First we rewrite,

$$
\begin{aligned}
II &= -\left([\Lambda^s, M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D)]\mathbf{V},\, \left(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\right)\Lambda^s \mathbf{V}\right)_2 \\
&\quad - \left(M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D)\Lambda^s \mathbf{V},\, \left(S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D) + S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\right)\Lambda^s \mathbf{V}\right)_2 \\
&:= II_1 + II_2.
\end{aligned} \tag{3.81}
$$

Estimate on II_1. Using the expression of $M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D)$ and $S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)$, we can rewrite

$$
\begin{aligned}
&\Big([\Lambda^s, M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D)]\mathbf{V},\, S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)\Lambda^s\mathbf{V}\Big)_2\\
&= \big([\Lambda^s, \underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega]\cdot\nabla\zeta + [\Lambda^s, \underline{\zeta}+\tilde{\zeta}_\omega]\partial_{x_1}v_1 + [\Lambda^s, \underline{\zeta}+\tilde{\zeta}_\omega]\partial_{x_2}v_2, (1+c\varepsilon\Delta)\Lambda^s\zeta\\
&\qquad + (\underline{v}_1+\tilde{v}_{1\omega})\Lambda^s v_1 + (\underline{v}_2+\tilde{v}_{2\omega})\Lambda^s v_2\big)_2\\
&+ \Big([\Lambda^s, \underline{v}_1+\tilde{v}_{1\omega}]\partial_{x_1}v_1 + [\Lambda^s, \underline{v}_2+\tilde{v}_{2\omega}]\partial_{x_1}v_2, (\underline{v}_1+\tilde{v}_{1\omega})\Lambda^s\zeta + (1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s v_1\Big)_2\\
&+ \Big([\Lambda^s, \underline{v}_1+\tilde{v}_{1\omega}]\partial_{x_2}v_1 + [\Lambda^s, \underline{v}_2+\tilde{v}_{2\omega}]\partial_{x_2}v_2,\, (\underline{v}_2+\tilde{v}_{2\omega})\Lambda^s\zeta + (1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s v_2\Big)_2.
\end{aligned}
$$

By using the commutator estimate (3.27) with the trick of represent $\Lambda^s = \Lambda_1^s + \Lambda_2^s$, where $\Lambda_j^s = \mathcal{F}^{-1}(1+|\xi_j|^s)$, similarly as in the previous case, we can prove that

$$
\begin{aligned}
&\Big([\Lambda^s, M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D)]\mathbf{V},\, S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)\Lambda^s\mathbf{V}\Big)_2\\
&\quad\lesssim \left(\varepsilon + \left|\underline{\zeta}\right|_{X^s_{\varepsilon^2}} + |\underline{\mathbf{v}}|_{X^s_\varepsilon}\right)\left(|\mathbf{v}|_{X^s_\varepsilon} + |\zeta|_{X^s_{\varepsilon^2}}\right).
\end{aligned}
$$

The same estimate holds for $\Big([\Lambda^s, M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D)]\mathbf{V},\, S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*\Lambda^s\mathbf{V}\Big)_2$, then

$$
|II_1| \lesssim \left(\varepsilon + \left|\underline{\zeta}\right|_{X^s_{\varepsilon^2}} + |\underline{\mathbf{v}}|_{X^s_\varepsilon}\right)\left(|\mathbf{v}|_{X^s_\varepsilon} + |\zeta|_{X^s_{\varepsilon^2}}\right). \tag{3.82}
$$

Estimate on II_2. We need to calculate $S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega, D) := (a_{ij})$ as follows

$$
\begin{aligned}
a_{11} &= (1+c\varepsilon\Delta)((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla) + (\underline{v}_1+\tilde{v}_{1\omega})(1+c\varepsilon\Delta)\partial_{x_1} + (\underline{v}_2+\tilde{v}_{2\omega})(1+c\varepsilon\Delta)\partial_{x_2},\\
a_{12} &= (1+c\varepsilon\Delta)((1+\underline{\zeta}+\tilde{\zeta}_\omega)\partial_{x_1}) + (\underline{v}_1+\tilde{v}_{1\omega})^2\partial_{x_1} + (\underline{v}_2+\tilde{v}_{2\omega})(\underline{v}_1+\tilde{v}_{1\omega})\partial_{x_2},\\
a_{13} &= (1+c\varepsilon\Delta)((1+\underline{\zeta}+\tilde{\zeta}_\omega)\partial_{x_2}) + (\underline{v}_1+\tilde{\underline{\zeta}}_{1\omega})(\underline{v}_2+\tilde{v}_{2\omega})\partial_{x_1} + (\underline{v}_2+\tilde{v}_{2\omega})^2\partial_{x_2},\\
a_{21} &= (\underline{v}_1+\tilde{v}_{1\omega})((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla) + (1+\underline{\zeta}+\tilde{\zeta}_\omega)(1+c\varepsilon\Delta)\partial_{x_1},\\
a_{22} &= 2(1+\underline{\zeta}+\tilde{\zeta}_\omega)(\underline{v}_1+\tilde{v}_{1\omega})\partial_{x_1},\\
a_{23} &= (\underline{v}_1+\tilde{v}_{1\omega})(1+\underline{\zeta}+\tilde{\zeta}_\omega)\partial_{x_2} + (1+\underline{\zeta}+\tilde{\zeta}_\omega)(\underline{v}_2+\tilde{v}_{2\omega})\partial_{x_1},\\
a_{31} &= (\underline{v}_2+\tilde{v}_{2\omega})(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla + (1+\underline{\zeta}+\tilde{\zeta}_\omega)(1+c\varepsilon\Delta)\partial_{x_2},\\
a_{32} &= (\underline{v}_2+\tilde{v}_{2\omega})(1+\underline{\zeta}+\tilde{\zeta}_\omega)\partial_{x_1} + (1+\underline{\zeta}+\tilde{\zeta}_\omega)(\underline{v}_1+\tilde{v}_{1\omega})\partial_{x_2},\\
a_{33} &= 2(\underline{v}_2+\tilde{v}_{2\omega})(1+\underline{\zeta}+\tilde{\zeta}_\omega)\partial_{x_2} + (1+\underline{\zeta}.
\end{aligned}
$$

Remark: This case is much simpler than the previous case, since there is no appearance of $g(D)$. We will only treat here the terms that gave the order of $\varepsilon^{1/2}$ in the previous case.

For a_{11}, we have

$$
\begin{aligned}
(a_{11}\Lambda^s\zeta,\, \Lambda^s\zeta)_2 &= ((1+c\varepsilon\Delta)((\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\cdot\nabla\Lambda^s\zeta),\, \Lambda^s\zeta)_2\\
&\quad + ((\underline{v}_1+\tilde{v}_{1\omega})(1+c\varepsilon\Delta)\partial_{x_1}\Lambda^s\zeta,\, \Lambda^s\zeta)_2\\
&\quad + ((\underline{v}_2+\tilde{v}_{2\omega})(1+c\varepsilon\Delta)\partial_{x_2}\Lambda^s\zeta,\, \Lambda^s\zeta)_2\\
&:= C_{11} + C_{12} + C_{13}.
\end{aligned} \tag{3.83}
$$

Using the commutator estimate (3.27), integration by part, Sobolev embedding ($s>3$) and Hölder inequality we get

$$
\begin{aligned}
&((1+c\varepsilon\Delta)(\underline{\mathbf{v}}\cdot\nabla\Lambda^s\zeta),\,\Lambda^s\zeta)_2\\
&\quad=(\underline{\mathbf{v}}\cdot\nabla\Lambda^s\zeta,\,\Lambda^s\zeta)_2+c\varepsilon\,([\Delta,\underline{\mathbf{v}}]\cdot\nabla\Lambda^s\zeta,\,\Lambda^s\zeta)_2+c\varepsilon\,(\underline{\mathbf{v}}\cdot\Delta\nabla\Lambda^s\zeta,\,\Lambda^s\zeta)_2\\
&\quad=-\frac{1}{2}\,((\nabla\cdot\underline{\mathbf{v}})\Lambda^s\zeta,\,\Lambda^s\zeta)_2+c\varepsilon\,([\Delta,\underline{\mathbf{v}}]\cdot\nabla\Lambda^s\zeta,\,\Lambda^s\zeta)_2\\
&\qquad+c\varepsilon\left(\Lambda^s\zeta,\,\sum_{j=1}^{2}(\nabla\underline{v}_j\cdot\nabla\partial_{x_j}\Lambda^s\zeta)\right)_2-\frac{1}{2}c\varepsilon\,(\Delta\Lambda^s\zeta,\,(\nabla\cdot\underline{\mathbf{v}})\Lambda^s\zeta)_2\\
&\qquad-\frac{3}{2}c\varepsilon\,(\Lambda^s\zeta,\,\Delta\underline{\mathbf{v}}\cdot\nabla\Lambda^s\zeta)_2\\
&\quad\lesssim|\underline{\mathbf{v}}|_{H^s}\,|\zeta|^2_{X^s_{\varepsilon^2}}\,.
\end{aligned}
\tag{3.84}
$$

By using (3.27) with $d=1$ and the $\sigma(D)=\partial^2_{x_j}$, $j=1,2$ and similar argument as above, we get

$$
((1+c\varepsilon\Delta)(\tilde{\mathbf{v}}_\omega\cdot\nabla\Lambda^s\zeta),\,\Lambda^s\zeta)_2\lesssim|\tilde{\mathbf{v}}_\omega|_{W^{2,\infty}}\,|\zeta|_{X^s_{\varepsilon^2}}\,.
$$

Therefore,

$$
|C_{11}|\lesssim(|\underline{\mathbf{v}}|_{H^s}+|\tilde{\mathbf{v}}_\omega|_{W^{2,\infty}})\,|\zeta|^2_{X^s_{\varepsilon^2}}\,.
\tag{3.85}
$$

For C_{12},

$$
\begin{aligned}
C_{12}&=-\frac{1}{2}\,(\Lambda^s\zeta,\,(\partial_{x_1}(\underline{v}_1+\tilde{v}_{1\omega}))\Lambda^s\zeta)_2-\frac{1}{2}c\varepsilon\,(\Delta\Lambda^s\zeta,\,(\partial_{x_1}(\underline{v}_1+\tilde{v}_{1\omega}))\Lambda^s\zeta)_2\\
&\quad-\frac{1}{2}c\varepsilon\,(\Lambda^s\zeta,\,(\Delta(\underline{v}_1+\tilde{v}_{1\omega}))\partial_{x_1}\Lambda^s\zeta)_2-\frac{1}{2}c\varepsilon\,(\Lambda^s\zeta,\,\nabla(\underline{v}_1+\tilde{v}_{1\omega})\cdot\nabla\partial_{x_1}\Lambda^s\zeta)_2\\
&\lesssim(|\underline{v}_1|_{H^s}+|\tilde{v}_{1\omega}|_{W^{2,\infty}})\,|\zeta|^2_{X^s_{\varepsilon^2}}\,.
\end{aligned}
\tag{3.86}
$$

The same estimate holds for C_{13}, as long side with (3.85) and (3.86), it follows

$$
(a_{11}\Lambda^s\zeta,\,\Lambda^s\zeta)_2\lesssim(|\underline{\mathbf{v}}|_{H^s}+|\tilde{\mathbf{v}}_\omega|_{W^{2,\infty}})\,|\zeta|^2_{X^s_{\varepsilon^2}}\,.
\tag{3.87}
$$

The same estimate holds for $(a_{22}\Lambda^s v_1,\,\Lambda^s v_1)_2$ and $(a_{33}\Lambda^s v_2,\,\Lambda^s v_2)_2$.

For a_{23} and a_{32},

$$
\begin{aligned}
&(a_{23}\Lambda^s v_2,\,\Lambda^s v_1)_2+(a_{32}\Lambda^s v_1,\,\Lambda^s v_2)_2\\
&\quad=-\Big(\Lambda^s v_2,\,\partial_{x_2}((\underline{v}_1+\tilde{v}_{1\omega})(1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s v_1)\Big)_2\\
&\qquad-\Big(\Lambda^s v_2,\,\partial_{x_1}((\underline{v}_2+\tilde{v}_{2\omega})(1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s v_2)\Big)_2\\
&\quad\lesssim(|\underline{\mathbf{v}}|_{H^s}+|\tilde{\mathbf{v}}_\omega|_{W^{1,\infty}})\,|\mathbf{v}|^2_{X^s_\varepsilon}\,.
\end{aligned}
\tag{3.88}
$$

By doing similarly as previous case for remaining terms, it is not hard to get that

$$
|II|\lesssim(\varepsilon+|\underline{\mathbf{v}}|_{X^s_\varepsilon}+\left|\underline{\zeta}\right|_{X^s_{\varepsilon^2}})(|\mathbf{v}|^2_{X^s_\varepsilon}+|\zeta|^2_{X^s_{\varepsilon^2}}),
\tag{3.89}
$$

Estimate on III. By using the expression of $S_{\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega}(D)^*$ and similar argument as before, we get

$$\begin{aligned} III = &-b\varepsilon\left([\underline{v}_1+\tilde{v}_{1\omega},\Delta]\Lambda^s v_1+[\underline{v}_2+\tilde{v}_{2\omega},\Delta]\Lambda^s v_2,\ \Lambda^s\partial_t\zeta\right)_2\\ &-b\varepsilon\Big([\underline{\zeta}+\tilde{\zeta}_\omega,\Delta]\Lambda^s v_1+[\underline{v}_1+\tilde{v}_{1\omega},\Delta]\Lambda^s\zeta,\ \Lambda^s\partial_t v_1\Big)_2\\ &-b\varepsilon\Big([\underline{v}_2+\tilde{v}_{2\omega},\Delta]\Lambda^s\zeta+[\underline{\zeta}+\tilde{\zeta}_\omega,\Delta]\Lambda^s v_2,\ \Lambda^s\partial_t v_2\Big)_2, \end{aligned}$$

then,

$$|III|\lesssim(\varepsilon+\left|\underline{\zeta}\right|_{X^s_{\varepsilon^2}}+|\underline{\mathbf{v}}|_{X^s_\varepsilon})(|\zeta|_{X^s_{\varepsilon^2}}+|\mathbf{v}|_{X^s_\varepsilon})(|\partial_t\zeta|_{H^s}+|\partial_t\mathbf{v}|_{H^s}). \tag{3.90}$$

Estimate on IV. Similarly as (3.69), (3.70) and (3.71), we have

$$\begin{aligned} IV = &\left((1-b\varepsilon\Delta)\Lambda^s\zeta,\ (\partial_t\underline{v}_1+\partial_t\tilde{v}_{1\omega})\Lambda^s v_1+(\partial_t\underline{v}_2+\partial_t\tilde{v}_{2\omega})\Lambda^s v_2\right)_2\\ &+\Big((1-b\varepsilon\Delta)\Lambda^s v_1,\ (\partial_t\underline{v}_1+\partial_t\tilde{v}_{1\omega})\Lambda^s\zeta+(\partial_t\underline{\zeta}+\partial_t\tilde{\zeta}_\omega)\Lambda^s v_1\Big)_2\\ &+\Big((1-b\varepsilon\Delta)\Lambda^s v_2,\ (\partial_t\underline{v}_2+\partial_t\tilde{v}_{2\omega})\Lambda^s\zeta+(\partial_t\underline{\zeta}+\partial_t\tilde{\zeta}_\omega)\Lambda^s v_2\Big)_2. \end{aligned}$$

Then,

$$|IV|\lesssim(\varepsilon+\left|\partial_t\underline{\zeta}\right|_{H^s}+|\partial_t\underline{\mathbf{v}}|_{H^s})(|\zeta|^2_{X^s_{\varepsilon^2}}+|\mathbf{v}|^2_{X^s_\varepsilon}). \tag{3.91}$$

Thanks to (3.80), (3.89), (3.90) and (3.91), we obtain

$$\begin{aligned} \frac{d}{dt}E_s(\mathbf{V})\lesssim\ &(\varepsilon+|\underline{\mathbf{v}}|_{X^s_\varepsilon}+\left|\underline{\zeta}\right|_{X^s_{\varepsilon^2}}+|\partial_t\underline{\mathbf{V}}|_{X^{s-1}_{\varepsilon^2}})(|\zeta|^2_{X^s_{\varepsilon^2}}+|\mathbf{v}|^2_{X^s_\varepsilon})\\ &+|F|_{H^s}(|\zeta|_{X^s_{\varepsilon^2}}+|\mathbf{v}|_{X^s_\varepsilon})\\ &+(\varepsilon+|\underline{\mathbf{v}}|_{X^s_\varepsilon}+\left|\underline{\zeta}\right|_{X^s_{\varepsilon^2}})(|\zeta|_{X^s_{\varepsilon^2}}+|\mathbf{v}|_{X^s_\varepsilon})\,|\partial_t\mathbf{V}|_{X^{s-1}_{\varepsilon^2}}. \end{aligned} \tag{3.92}$$

That is (3.75) and then (3.76).

Returning to (3.17) we set $G=F-M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega,D)\mathbf{V}$ then $\partial_t\mathbf{V}=(1-b\varepsilon\Delta)^{-1}G$. By using the expression of $M(\underline{\mathbf{V}}+\tilde{\mathbf{V}}_\omega,D)$ in this case we have that

$$|G|_{H^{s-1}}\lesssim|F|_{H^{s-1}}+|\zeta|_{X^s_{\varepsilon^2}}+|\mathbf{v}|_{H^s}. \tag{3.93}$$

Beside that we also have

$$\begin{aligned} |\partial_t\mathbf{V}|_{X^{s-1}_{\varepsilon^2}}&\lesssim|(1-b\varepsilon\Delta)\partial_t\mathbf{V}|_{H^{s-1}}\\ &=\left|(1-b\varepsilon\Delta)(1-b\varepsilon\Delta)^{-1}G\right|_{H^{s-1}}\\ &=|G|_{H^{s-1}}. \end{aligned}$$

Therefore, we obtain (3.77).

The *existence* and *uniqueness* of the solution of (3.17) are obtained similarly as in [55].

3.5. **Proof of Theorem (3.2).** Using the compactness method, the proof of Theorem (3.2) follows exactly 4 standard steps of the proof of Theorem 1.1 in [55], the crucial part is the linear solvability which we obtained previously. Therefore, we will only precise here the existence time scale for each case of (a,b,c,d).

Since the existence time is obtained by using the Gronwall's inequality, we need to precise the order of ε in the estimates (3.33) and (3.75)

If $b \neq d,\ b, d > 0,\ a, c < 0$.
Note that

$$F = \Big(M(\underline{\mathbf{V}} + \tilde{\mathbf{V}}_\omega, D) - M(\tilde{\mathbf{V}}_\omega, D) \Big) \tilde{\mathbf{V}}_\omega$$
$$= - \begin{pmatrix} \underline{\mathbf{v}} & \underline{\zeta}\partial_{x_1} & \underline{\mathbf{v}}\partial_{x_2} \\ 0 & g(D)(\underline{v_1}\partial_{x_1}) & g(D)(\underline{v_2}\partial_{x_1}) \\ 0 & g(D)(\underline{v_1}\partial_{x_2}) & g(D)(\underline{v_2}\partial_{x_2}) \end{pmatrix} \tilde{\mathbf{V}}_\omega,$$

then the order of ε in F is 2. So the order of ε in :

$$\Big(1 + |\partial_t \underline{\mathbf{V}}|_{X^{s-1}_{\varepsilon^3}} + |\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}} + |\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}} \Big) |F|_{X^s_{\varepsilon^3}} \text{ is } 2,$$

$$\varepsilon^{1/2} \Big(1 + |\partial_t \underline{\mathbf{V}}|_{X^{s-1}_{\varepsilon^3}} + |\underline{\mathbf{U}}|_{X^s_{\varepsilon^3}} + |\underline{\mathbf{V}}|_{X^s_{\varepsilon^3}} \Big) \text{ is } 1/2.$$

Therefore the existence time scale in this case is of order $O(\varepsilon^{-1/2})$.
If $b = d > 0,\ a = 0,\ c < 0$.
The order of ε in :

$$\varepsilon + |\underline{\mathbf{v}}|_{X^s_\varepsilon} + \big|\underline{\zeta}\big|_{X^s_{\varepsilon^2}} + |\partial_t \underline{\mathbf{V}}|_{X^{s-1}_{\varepsilon^2}} \text{ is } 1,$$

$$\Big(\varepsilon + |\underline{\mathbf{v}}|_{X^s_\varepsilon} + \big|\underline{\zeta}\big|_{X^s_{\varepsilon^2}} \Big) |\partial_t \mathbf{V}|_{X^{s-1}_{\varepsilon^2}} \text{ is } 2,$$

$$|F|_{H^s} \text{ is } 2.$$

Therefore, the existence time scale in this case is of order $O(1/\varepsilon)$.

3.6. **The case** $a = b = d = 0, c < 0$**.** We recall that this case corresponds to a particular version of Euler-Korteweg systems (2.2), that is re-intoducing the small parameter ε and taking without loss of generality $c = -1$

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \varepsilon[\nabla \cdot (\eta \mathbf{u})] = 0 \\ \mathbf{u}_t + \nabla \eta + \varepsilon[\frac{1}{2}\nabla|\mathbf{u}|^2 - \nabla\Delta\eta] = 0. \end{cases} \tag{3.94}$$

Well-posedness on time scales of order $O(1/\varepsilon)$ has been established in [56, 10] (actually [10] considers long time existence issues for a general class of Euler-Korteweg systems).

We now consider the system system satisfied by a localized perturbation on a line solitary wave $\tilde{U_\omega} = (\tilde{\eta}, \tilde{\mathbf{u}})$ of velocity ω
that is

$$\begin{cases} \eta_t + \nabla \cdot \mathbf{u} + \varepsilon[\nabla \cdot ((\eta \mathbf{u}) + \tilde{\eta}_\omega \mathbf{u} + \tilde{\mathbf{u}}_\omega \eta) +] = 0 \\ \mathbf{u}_t + \nabla \eta + \varepsilon[\nabla(\mathbf{u} \cdot \tilde{\mathbf{u}}_\omega) + \frac{1}{2}\nabla|\mathbf{u}|^2 - \nabla\Delta\eta] = 0. \end{cases} \tag{3.95}$$

The proof of longtime existence will follow that of [56] and it differs somewhat of the corresponding proof for the other cases.

4. Appendix

4.1. **Appendix 1.** (The case $b \neq d$, $b, d > 0$, $a, c < 0$.)

We are going to prove there exists a constant κ_H such that (3.32) holds under the assumption (3.31), plugging $S_{\underline{\mathbf{V}}+\tilde{\mathbf{v}}_\omega}(D)$ into (3.24) we have

$$
\begin{aligned}
E_s(\mathbf{V}) &= \big((1-b\varepsilon\Delta)\Lambda^s\zeta,\, (1+c\varepsilon\Delta)^2 g(D)\Lambda^s\zeta\big)_2 \\
&\quad + ((1-b\varepsilon\Delta)\Lambda^s\zeta,\, g(D)[(\underline{\mathbf{v}}+\underline{\tilde{\mathbf{v}}}_\omega)(1+c\varepsilon\Delta)\Lambda^s\mathbf{v}])_2 \\
&\quad + ((1-b\varepsilon\Delta)\Lambda^s\mathbf{v},\, g(D)[(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)(1+c\varepsilon\Delta)\Lambda^s\zeta])_2 \\
&\quad + \Big((1-b\varepsilon\Delta)\Lambda^s\mathbf{v},\, (1+\underline{\zeta}+\tilde{\zeta}_\omega+a\varepsilon\Delta)(1+c\varepsilon\Delta)\Lambda^s\mathbf{v}\Big)_2 \\
&\quad + \sum_{i,j=1}^{2}\big((1-b\varepsilon\Delta)\Lambda^s v_i,\, (\underline{v}_i+\tilde{v}_{i\omega})(\underline{v}_j+\tilde{v}_{j\omega})(g(D)-1)\Lambda^s v_j\big)_2 \\
&= A_1+A_2+A_3+A_4+A_5.
\end{aligned} \tag{A.1}
$$

For A_1 we have the following result from [55]

$$
\begin{aligned}
A_1 \geq \min\left\{1, \frac{b}{d}\right\}\bigg(&|\Lambda^s\zeta|_2^2 + c^2\varepsilon^2|\Delta\Lambda^s\zeta|_2^2 + (b-2c)\varepsilon|\nabla\Lambda^s\zeta|_2^2 \\
&+ bc^2\varepsilon^3|\Delta\nabla\Lambda^s\zeta|_2^2 + (-2bc)\varepsilon^2|\nabla\nabla\Lambda^s\zeta|_2^2\bigg),
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
A_1 \leq \max\left\{1, \frac{b}{d}\right\}\bigg(&|\Lambda^s\zeta|_2^2 + c^2\varepsilon^2|\Delta\Lambda^s\zeta|_2^2 + (b-2c)\varepsilon|\nabla\Lambda^s\zeta|_2^2 \\
&+ bc^2\varepsilon^3|\Delta\nabla\Lambda^s\zeta|_2^2 + (-2bc)\varepsilon^2|\nabla\nabla\Lambda^s\zeta|_2^2\bigg).
\end{aligned} \tag{A.3}
$$

For A_2 and A_3, using (3.28), we have

$$
\begin{aligned}
|A_2|+|A_3| &\leq (|\underline{\mathbf{v}}|_\infty + |\tilde{\mathbf{v}}_\omega|_\infty)\,|(1-b\varepsilon\Delta)g(D)\Lambda^s\zeta|_2\,|(1+c\varepsilon\Delta)\Lambda^s\mathbf{v}|_2 \\
&\quad + (|\underline{\mathbf{v}}|_\infty + |\tilde{\mathbf{v}}_\omega|_\infty)\,|(1-b\varepsilon\Delta)g(D)\Lambda^s\mathbf{v}|_2\,|(1+c\varepsilon\Delta)\Lambda^s\zeta|_2 \\
&\leq \max\left\{1, \frac{b}{d}\right\}(|\underline{\mathbf{v}}|_\infty + |\tilde{\mathbf{v}}_\omega|_\infty)\Big(2(|\Lambda^s\zeta|_2^2 + |\Lambda^s\mathbf{v}|_2^2) \\
&\qquad + (b^2+c^2)\varepsilon^2(|\Delta\Lambda^s\zeta|_2^2 + |\Delta\Lambda^s\mathbf{v}|_2^2)\Big).
\end{aligned} \tag{A.4}
$$

For A_4, we have

$$
\begin{aligned}
A_4 &= \Big(\Lambda^s\mathbf{v},\, (1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s\mathbf{v}\Big)_2 + abc\varepsilon^3|\nabla\Delta\Lambda^s\mathbf{v}|_2^2 + ac\varepsilon^2|\Delta\Lambda^s\mathbf{v}|_2^2 \\
&\quad + b\varepsilon^2\Big(\Delta\Lambda^s\mathbf{v},\, -[a+c(1+\underline{\zeta}+\tilde{\zeta}_\omega)]\Delta\Lambda^s\mathbf{v}\Big)_2 \\
&\quad + \varepsilon\Big(\Lambda^s\mathbf{v},\, [a+(c-b)(1+\underline{\zeta}+\tilde{\zeta}_\omega)]\Delta\Lambda^s\mathbf{v}\Big)_2
\end{aligned}
$$

which along with the assumption $1+\underline{\zeta}+\tilde{\zeta}_\omega \geq H>0$ implies that

$$\begin{aligned}
A_4 &\geq H\left|\Lambda^s \mathbf{v}\right|_2^2+abc\varepsilon^3\left|\nabla\Delta\Lambda^s \mathbf{v}\right|_2^2+(ac-b(a+cH))\varepsilon^2\left|\Delta\Lambda^s \mathbf{v}\right|_2^2 \\
&\quad +(-a-c+b)\varepsilon\left|\nabla\Lambda^s \mathbf{v}\right|_2^2+(c-b)\varepsilon\left(|\zeta|_\infty+\left|\tilde{\zeta}_\omega\right|_\infty\right)\left|\Lambda^s \mathbf{v}\right|_2\left|\Delta\Lambda^s \mathbf{v}\right|_2 \\
&\geq \frac{H}{2}\left|\Lambda^s \mathbf{v}\right|_2^2+abc\varepsilon^3\left|\nabla\Delta\Lambda^s \mathbf{v}\right|_2^2+(-a-c+b)\varepsilon\left|\nabla\Lambda^s \mathbf{v}\right|_2^2 \\
&\quad +\left(ac-b(a+cH)-\frac{(c-b)^2(\left|\underline{\zeta}\right|_\infty+\left|\tilde{\zeta}_\omega\right|_\infty)^2}{2H}\right)\varepsilon^2\left|\Delta\Lambda^s \mathbf{v}\right|_2^2 .
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
A_4 &\leq \left(1+\left|\underline{\zeta}\right|_\infty+\left|\tilde{\zeta}_\omega\right|_\infty\right)\left|\Lambda^s \mathbf{v}\right|_2^2+abc\varepsilon^3\left|\nabla\Delta\Lambda^s \mathbf{v}\right|_2^2 \\
&\quad +\left(ac-ab-bc\left(1+\left|\underline{\zeta}\right|_\infty+\left|\tilde{\zeta}_\omega\right|_\infty\right)\right)\varepsilon^2\left|\Delta\Lambda^s \mathbf{v}\right|_2^2 \\
&\quad +\left(-a+(-c+b)\left(1+\left|\underline{\zeta}\right|_\infty+\left|\tilde{\zeta}_\omega\right|_\infty\right)\right)\varepsilon\left|\Lambda^s \mathbf{v}\right|_2\left|\Delta\Lambda^s \mathbf{v}\right|_2 .
\end{aligned} \tag{A.6}$$

For A_5, using (3.29) we have

$$\begin{aligned}
|A_5| &\leq 2\left(|\underline{\mathbf{v}}|_\infty+|\tilde{\mathbf{v}}_\omega|_\infty\right)^2\left|(1-b\varepsilon\Delta)\Lambda^s \mathbf{v}\right|_2\left|(g(D)-1)\Lambda^s \mathbf{v}\right| \\
&\leq \frac{2|b-d|}{d}\left(|\underline{\mathbf{v}}|_\infty+|\tilde{\mathbf{v}}_\omega|_\infty\right)^2\left(2\left|\Lambda^s \mathbf{v}\right|_2^2+\frac{b^2}{4}\varepsilon^2\left|\Delta\Lambda^s \mathbf{v}\right|_2^2\right).
\end{aligned} \tag{A.7}$$

Combining (A.1), (A.2), (A.3), (A.4), (A.5), (A.6) and (A.7) we obtain that

$$\begin{aligned}
&E_s(\mathbf{V}) \\
&\geq \left(\min\left\{1,\frac{b}{d}\right\}-2\max\left\{1,\frac{b}{d}\right\}\left(|\underline{\mathbf{v}}|_\infty+|\tilde{\mathbf{v}}_\omega|_\infty\right)\right)\left|\Lambda^s\zeta\right|_2^2 \\
&\quad +\min\left\{1,\frac{b}{d}\right\}bc^2\varepsilon^3\left|\Delta\nabla\Lambda^s\zeta\right|_2^2 \\
&\quad +\left(\min\left\{1,\frac{b}{d}\right\}(c^2-2bc)-\max\left\{1,\frac{b}{d}\right\}\left(|\underline{\mathbf{v}}|_\infty+|\tilde{\mathbf{v}}_\omega|_\infty\right)(b^2+c^2)\right)\varepsilon^2\left|\Delta\Lambda^s\zeta\right|_2^2 \\
&\quad +\min\left\{1,\frac{b}{d}\right\}(b-2c)\varepsilon\left|\nabla\Lambda^s\zeta\right|_2^2 \\
&\quad +\left(-2\max\left\{1,\frac{b}{d}\right\}\left(|\underline{\mathbf{v}}|_\infty+|\tilde{\mathbf{v}}_\omega|_\infty\right)+\frac{H}{2}-\frac{4|b-d|}{d}\left(|\underline{\mathbf{v}}|_\infty+|\tilde{\mathbf{v}}_\omega|_\infty\right)^2\right)\left|\Lambda^s \mathbf{v}\right|_2^2 \\
&\quad +\Bigg(-\max\left\{1,\frac{b}{d}\right\}\left(|\underline{\mathbf{v}}|_\infty+|\tilde{\mathbf{v}}_\omega|_\infty\right)(b^2+c^2)+ac-b(a+cH) \\
&\qquad -\frac{(c-b)^2\left(\left|\underline{\zeta}\right|_\infty+\left|\tilde{\zeta}_\omega\right|_\infty\right)^2}{2H}-\frac{b^2|b-d|}{2d}\left(|\underline{\mathbf{v}}|_\infty+|\tilde{\mathbf{v}}_\omega|_\infty\right)^2\Bigg)\varepsilon^2\left|\Delta\Lambda^s \mathbf{v}\right|_2^2 ,
\end{aligned}$$

which along with a suitable lower-bound of $|\underline{\mathbf{v}}|_\infty+|\tilde{\mathbf{v}}_\omega|_\infty$ implies the left-hand side inequality of (3.32). Similarly, we get the right hand side inequality of (3.32) and therefore we can choose a suitable constant κ_H.

4.2. **Appendix 2.** (The case $b=d>0$, $a=0$, $c<0$).

We are going to prove the existence of κ_H which ensures the equivalence of $E_s(\mathbf{V})$ and $|\zeta|^2_{X^s_{\varepsilon^2}}+|\mathbf{v}|^2_{X^s_\varepsilon}$ in the assumption of Proposition (3.2). By doing similarly as previous case, we get

$$\begin{aligned}
E_s(\mathbf{V}) &= ((1-b\varepsilon\Delta)\Lambda^s\zeta,\ (1+c\varepsilon\Delta)\Lambda^s\zeta + (\underline{v}_1+\tilde{v}_{1\omega})\Lambda^s v_1 + (\underline{v}_2+\tilde{v}_{2\omega})\Lambda^s v_2)_2 \\
&\quad + \Big((1-b\varepsilon\Delta)\Lambda^s v_1,\ (\underline{v}_1+\tilde{v}_{1\omega})\Lambda^s\zeta + (1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s v_1\Big)_2 \\
&\quad + \Big((1-b\varepsilon\Delta)\Lambda^s v_2,\ (\underline{v}_2+\tilde{v}_{2\omega})\Lambda^s\zeta + (1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s v_2\Big)_2 \\
&= ((1-b\varepsilon\Delta)\Lambda^s\zeta,\ (1+c\varepsilon\Delta)\Lambda^s\zeta)_2 \\
&\quad + ((1-b\varepsilon\Delta)\Lambda^s\zeta,\ (\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\Lambda^s\mathbf{v})_2 \\
&\quad + \Big((1-b\varepsilon\Delta)\Lambda^s\mathbf{v},\ (1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s\mathbf{v}\Big)_2 \\
&\quad + ((1-b\varepsilon\Delta)\Lambda^s\mathbf{v},\ (\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\Lambda^s\zeta)_2 \\
&= B_1+B_2+B_3+B_4.
\end{aligned}$$

For B_1, we have

$$B_1 = |\Lambda^s\zeta|_2^2 + (b-c)\varepsilon\,|\nabla\Lambda^s\zeta|_2^2 - bc\varepsilon^2\,|\Delta\Lambda^s\zeta|_2^2 \tag{B.1}$$

For B_2 , we have

$$|B_2| \le (|\underline{\mathbf{v}}|_\infty + |\tilde{\mathbf{v}}_\omega|_\infty)\,|\Lambda^s\mathbf{v}|_2\,(|\Lambda^s\zeta|_2 + b\varepsilon\,|\Delta\Lambda^s\zeta|_2). \tag{B.2}$$

Then,

$$\begin{aligned}
B_2 &\ge -4(|\underline{\mathbf{v}}|_{W^{2,\infty}} + |\tilde{\mathbf{v}}_\omega|_{W^{2,\infty}})\,|\Lambda^s\mathbf{v}|_2^2 - \frac{1}{4}(|\underline{\mathbf{v}}|_{W^{2,\infty}} + |\tilde{\mathbf{v}}_\omega|_{W^{2,\infty}})\,|\Lambda^s\zeta|_2^2 \\
&\quad - 4b(|\underline{\mathbf{v}}|_{W^{2,\infty}} + |\tilde{\mathbf{v}}_\omega|_{W^{2,\infty}})\,|\Lambda\mathbf{v}|_2^2 - \frac{b}{4}\varepsilon^2\,|\Delta\Lambda^s\zeta|_2^2.
\end{aligned} \tag{B.2.1}$$

And

$$\begin{aligned}
B_2 &\ge 4(|\underline{\mathbf{v}}|_{W^{2,\infty}} + |\tilde{\mathbf{v}}_\omega|_{W^{2,\infty}})\,|\Lambda^s\mathbf{v}|_2^2 + \frac{1}{4}(|\underline{\mathbf{v}}|_{W^{2,\infty}} + |\tilde{\mathbf{v}}_\omega|_{W^{2,\infty}})\,|\Lambda^s\zeta|_2^2 \\
&\quad + 4b(|\underline{\mathbf{v}}|_{W^{2,\infty}} + |\tilde{\mathbf{v}}_\omega|_{W^{2,\infty}})\,|\Lambda\mathbf{v}|_2^2 + \frac{b}{4}\varepsilon^2\,|\Delta\Lambda^s\zeta|_2^2.
\end{aligned} \tag{B.2.2}$$

For B_3, by integrating by parts, we have

$$\begin{aligned}
B_3 &= \Big(\Lambda^s\mathbf{v},\ (1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s\mathbf{v}\Big)_2 - b\varepsilon\sum_{j=1}^{2}\Big(\Delta\Lambda^s v_j,\ (1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s v_j\Big)_2 \\
&= \Big(\Lambda^s\mathbf{v},\ (1+\underline{\zeta}+\tilde{\zeta}_\omega)\Lambda^s\mathbf{v}\Big)_2 + b\varepsilon\sum_{j=1}^{2}\Bigg[\Big(\nabla\Lambda^s v_j,\ (1+\underline{\zeta}+\tilde{\zeta}_\omega)\nabla\Lambda^s v_j\Big)_2 \\
&\quad - \frac{1}{2}\sum_{k=1}^{2}\Big(\Lambda^s v_j,\ (\partial_{x_k}^2\underline{\zeta}+\partial_{x_k}^2\tilde{\zeta}_\omega)\Lambda^s v_j\Big)_2\Bigg].
\end{aligned} \tag{B.3}$$

Which alongside with (3.2) give that

$$\begin{aligned}
B_3 &\ge \left(H - \frac{b\varepsilon}{2}(|\underline{\mathbf{v}}|_{W^{2,\infty}} + |\tilde{\mathbf{v}}_\omega|_{W^{2,\infty}})\right)|\Lambda^s\mathbf{v}|_2^2 \\
&\quad + b\varepsilon\sum_{j=1}^{2} H\,|\nabla\Lambda^s v_j|_2^2,
\end{aligned} \tag{B.3.1}$$

and

$$
\text{(B.3.2)}\quad
\begin{aligned}
B_3 \leq & \left(1+(1+\frac{b\varepsilon}{2})(\left|\underline{\zeta}\right|_{W^{2,\infty}}+\left|\tilde{\zeta}_\omega\right|_{W^{2,\infty}})\right)\left|\Lambda^s\mathbf{v}\right|_2^2 \\
& + b\varepsilon(1+\left|\underline{\zeta}\right|_\infty+\left|\tilde{\zeta}_\omega\right|_\infty)\sum_{j=1}^{2}\left|\nabla\Lambda^s v_j\right|_2^2.
\end{aligned}
$$

For B_4, we have

$$
\begin{aligned}
B_4 = & (\Lambda^s\mathbf{v},\,(\underline{\mathbf{v}}+\tilde{\mathbf{v}}_\omega)\Lambda^s\zeta)_2 - b\varepsilon\sum_{j=1}^{2}\Big[\left(\Lambda^s v_j,\,(\underline{v}_j+\tilde{v}_{j\omega})\Delta\Lambda^s\zeta\right)_2 \\
& + \left(\Lambda^s v_j,\,(\Delta\underline{v}_j+\Delta\tilde{v}_{j\omega})\Lambda^s\zeta\right)_2 + 2\left(\Lambda^s v_j,\,\nabla(\underline{v}_j+\tilde{v}_{j\omega})\cdot\nabla\Lambda^s\zeta\right)_2\Big].
\end{aligned}
$$

Then

$$
\text{(B.4)}\quad
\begin{aligned}
|B_4| \leq & (1+b\varepsilon)(\left|\underline{\mathbf{v}}\right|_{W^{2,\infty}}+\left|\tilde{\mathbf{v}}_\omega\right|_{W^{2,\infty}})\left|\Lambda^s\mathbf{v}\right|_2\left|\Lambda^s\zeta\right|_2 \\
& + 2b\varepsilon(\left|\underline{\mathbf{v}}\right|_{W^{2,\infty}}+\left|\tilde{\mathbf{v}}_\omega\right|_{W^{2,\infty}})\left|\Lambda^s\mathbf{v}\right|_2\left|\nabla\Lambda^s\zeta\right|_2.
\end{aligned}
$$

That follows

$$
\text{(B.4.1)}\quad
\begin{aligned}
B_4 \geq & -(1+b\varepsilon)(\left|\underline{\mathbf{v}}\right|_{W^{2,\infty}}+\left|\tilde{\mathbf{v}}_\omega\right|_{W^{2,\infty}})(4\left|\Lambda^s\mathbf{v}\right|_2^2+\frac{1}{4}\left|\Lambda^s\zeta\right|_2^2) \\
& - 2b(\left|\underline{\mathbf{v}}\right|_{W^{2,\infty}}+\left|\tilde{\mathbf{v}}_\omega\right|_{W^{2,\infty}})(4\left|\Lambda^s\mathbf{v}\right|_2^2+\frac{\varepsilon^2}{4}\left|\nabla\Lambda^s\zeta\right|_2^2),
\end{aligned}
$$

and

$$
\text{(B.4.2)}\quad
\begin{aligned}
B_4 \leq & (1+b\varepsilon)(\left|\underline{\mathbf{v}}\right|_{W^{2,\infty}}+\left|\tilde{\mathbf{v}}_\omega\right|_{W^{2,\infty}})(4\left|\Lambda^s\mathbf{v}\right|_2^2+\frac{1}{4}\left|\Lambda^s\zeta\right|_2^2) \\
& + 2b(\left|\underline{\mathbf{v}}\right|_{W^{2,\infty}}+\left|\tilde{\mathbf{v}}_\omega\right|_{W^{2,\infty}})(4\left|\Lambda^s\mathbf{v}\right|_2^2+\frac{\varepsilon^2}{4}\left|\nabla\Lambda^s\zeta\right|_2^2).
\end{aligned}
$$

Thanks to (B.1), (B.2.1), (B.3.1) and (B.4.1), we get

$$
\begin{aligned}
E_s(\mathbf{V}) \geq & \frac{1}{2}\left|\Lambda^s\zeta\right|_2^2+(\frac{b}{2}-c)\varepsilon\left|\nabla\Lambda^s\zeta\right|_2^2 - b\varepsilon(c+\frac{\left|\underline{\mathbf{v}}\right|_{W^{2,\infty}}+\left|\tilde{\mathbf{v}}_\omega\right|_{W^{2,\infty}}}{4})\left|\Delta\Lambda^s\zeta\right|_2^2 \\
& + (H-(\left|\underline{\mathbf{v}}\right|_{W^{2,\infty}}+\left|\tilde{\mathbf{v}}_\omega\right|_{W^{2,\infty}}))(8+12b+\frac{9}{2}b\varepsilon)\left|\Lambda^s\mathbf{v}\right|_2^2.
\end{aligned}
$$

Note that in this case we have $b > 0$ and $c < 0$, so similarly as the case in the appendix 4.1 we can choose a suitable constant κ_H.

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