LIMITS OF TANGENT SPACES TO DEFINABLE SETS

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Abstract. We study the set of tangent limits at a given point to a set definable in any o-minimal structure by characterizing the set of "exceptional rays" in the tangent cone to the set at that point and investigating the set of tangent limits along these rays. Several criteria for determining "exceptional rays" will be given. The main results of the paper generalize, to the o-minimal setting and to arbitrary dimension, the main results of [12] which deals with algebraic surfaces in $\mathbb{R}^3$.

1. Introduction

One of the ways to study singular varieties is to investigate their tangent cones and limits of tangent spaces, which was initialized by Whitney in the 1960s [15, 16]. Given a variety $X \subset \mathbb{K}^n$ of (pure) dimension d ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, varieties nature will be specified), one constructs the *Nash blow-up* $\mathcal{N}(X)$ of X, made of the closure of the tangent bundle of its regular part X_{reg} (we say that $x \in X$ is regular if X is C^1 at x) and studies the *Nash fiber* $\mathcal{N}(X)_x$ of the bundle $\mathcal{N}(X)$ over $x \in X$. Namely, if $\mathbb{G}(d, n)$ denotes the Grassmannian of the d-dimensional linear subspaces of $\mathbb{K}^n$ and $T_x X$ is the space tangent to X at x, then

$$\mathcal{N}(X) = \overline{\{(x, P) \in X_{reg} \times \mathbb{G}(d, n) :\ P = T_x X_{reg}\}},$$

so (x, P) belongs to $\mathcal{N}(X)$ if there exists a sequence $x^k \in X_{reg}$ approaching x with $T_{x^k} X \to P$. If x is a regular point, the Nash fiber $\mathcal{N}(X)_x$ is reduced to the tangent space to X at x, but for singular x, this Nash fiber contains all limits at x of the spaces tangent to X_{reg}, then carries informations on the singularity germ.

Nash fibers are better analyzed together with an additional data which keeps track of the direction along which limits are taken. For this, set

$$\mathcal{N}'(X) = \overline{\{(x, t, v, P) \in X_{reg} \times \mathbb{K}_+ \times \mathbb{K}^n \times \mathbb{G}(d, n) :\ x + tv \in X_{reg},\ P = T_{x+tv} X_{reg}\}},$$

where $\mathbb{K}_+ = \mathbb{R}_+ := (0, +\infty)$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{K}_+ = \mathbb{C} \setminus \{0\}$ if $\mathbb{K} = \mathbb{C}$. The fiber $\mathcal{N}'(X)_{(x,0)}$ of $\mathcal{N}'(X)$ for $x \in X$ and $t = 0$ gives again the Nash fiber $\mathcal{N}(X)_x$ when projected on the

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P coordinate, while its projection on the v coordinate is the tangent (semi)cone to X at x. Most importantly, $\mathcal{N}'(X)_{(x,0)}$ connects a plane in the Nash fiber to the direction of the tangent (semi)cone it comes from. Indeed, $(v, P) \in \mathcal{N}'(X)_{(x,0)}$ if there exists a sequence $x^k \in X$ approaching x with simultaneously $T_{x^k}X \to P$ and $t_k(x^k - x) \to v$ for some $t_k \in \mathbb{K}_+$. For fixed x and for each $v \in \mathbb{K}^n \setminus \{0\}$, the fiber of $\mathcal{N}'(X)$ over $(x, 0, u)$ does not depend on the point u in the ray $\ell := \mathbb{R}_+ v$ if $\mathbb{K} = \mathbb{R}$ or the line $\ell := \mathbb{C}v$ if $\mathbb{K} = \mathbb{C}$, and is denoted by $\mathcal{N}_\ell$ if X and x are clear from the context. We call $\mathcal{N}_\ell$, the *Nash fiber (over x) along ℓ* or *the set of tangent limits to X (at x) along ℓ*. Studying a Nash fiber with respect to the tangent cone consists of describing $\mathcal{N}_\ell$ with respect to ℓ.

For complex varieties, these studies were carried out by Henry, Lê and Teissier [4, 7, 8]. The case of complex algebraic surfaces is well described by [4] for isolated singularities and by [7] in general, and the results can be stated as follows.

Theorem Henry–Lê ([4, 7])**.** *Let $X \subset \mathbb{C}^3$ be a complex algebraic surface containing the origin $0 \in \mathbb{C}^3$. Denote by $\mathcal{C}$ the tangent cone to X at 0. Then there exists a cone $\mathcal{E} \subset \mathcal{C}$ consisting of finitely many lines $\ell_1, \dots, \ell_r$ (called exceptional lines) such that:*

- *For $\ell \subset \mathcal{C}$ such that $\ell \not\subset \mathcal{E}$, the set $\mathcal{N}_\ell$ is reduced to one plane, which is the (common) tangent plane to $\mathcal{C}$ at a non zero point of ℓ.*
- *For $\ell \subset \mathcal{E}$, we have $\mathcal{N}_\ell = \{P \in \mathbb{G}(2,3) : \ \ell \subset P\}$, i.e., $\mathcal{N}_\ell$ contains the whole pencil of planes containing ℓ.*
- *Singular lines of $\mathcal{C}$ are exceptional, i.e., $\mathcal{C}_{sing} \subset \mathcal{E}$.*

In the real setting, to the best of our knowledge, only algebraic surfaces in $\mathbb{R}^3$ have already been considered elaborately, by O'Shea and Wilson in [12]. Compared to the complex case, the structure of the set of tangent limits at a singular point of a real algebraic surface is more flexible, and characterizations of exceptional rays that coincide for complex varieties become inequivalent. In [12], a ray ℓ is said to be *exceptional* if $\mathcal{N}_\ell$ has positive dimension. The main results in [12] can be summarized as follows.

Theorem O'Shea–Wilson ([12])**.** *Let $X \subset \mathbb{R}^3$ be a real algebraic surface containing the origin $0 \in \mathbb{R}^3$. Let $\mathcal{C}$ be the tangent semicone to X at 0, and $\mathcal{C}'$ be the tangent semicone to the singular part X_{sing} of X at 0. Then there exists a semicone $\mathcal{E} \subset \mathcal{C}$ not containing rays in $\mathcal{C}'$ and consisting of finitely many rays $\ell_1, \dots, \ell_r$ (called exceptional rays), such that:*

- *If $\ell \subset \mathcal{C} \setminus (\mathcal{E} \cup \mathcal{C}')$, then $\mathcal{N}_\ell$ is reduced to one plane, which is the (common) tangent plane to $\mathcal{C}$ at a non zero point of ℓ.*
- *If $\ell \subset \mathcal{E}$, then $\mathcal{N}_\ell$ is closed, connected and has dimension 1.*
- *If $\ell \subset \mathcal{C}_{sing} \setminus \mathcal{C}'$ is a singular ray of $\mathcal{C}$, then ℓ is exceptional, except, possibly, if the tangent semicone to $\mathcal{C}$ at non zero points of ℓ is a plane. In particular, if ℓ is an isolated ray or a boundary ray in $\mathcal{C}$, then ℓ is exceptional.*

The authors of [12] remark that their results should have generalizations to higher dimension and/or codimension, but, as they fairly recognized, their methods are merely specific to algebraic surfaces in $\mathbb{R}^3$. This article aims to make this extension for arbitrary dimension and codimension. It happens moreover that the proofs can be made very general: we show that our results hold in the setting of an arbitrary o-minimal structure; in particular, the given description of Nash fibers is indeed not of an algebraic nature, but steams from the tame topology of the considered sets.

For the remainder of the paper, abusing of terminology, a "semicone" is called briefly a "cone" for short. If not mentioned otherwise, the term "ray" means "open ray emanating from the origin $0 \in \mathbb{R}^n$", i.e., we consider only "rays" with the endpoint 0 but 0 is not included.

Given an o-minimal expansion $\mathcal{R}$ of the field of real numbers, we call a set *definable* if it is definable in $\mathcal{R}$ with real parameters. We include in Section 2 a short introduction to o-minimality and refer to [2] or [14] for further general references. Let $X \subset \mathbb{R}^n$ be a definable set with $x \in \overline{X}$. The *geometric tangent semicone*, which we will call "*tangent cone*" for short from now on, of X at x is defined by

$$C_xX := \left\{ \begin{array}{ll} v \in \mathbb{R}^n : & \text{there are sequences } x^k \in X \text{ and } t_k \in (0, +\infty) \text{ such that} \\ & x^k \to x \text{ and } t_k(x^k - x) \to v \text{ as } k \to +\infty \end{array} \right\}.$$

With no loss of generality, suppose that $x = 0$ for the remainder of the paper. For short we set

$$\mathcal{C} := C_0X \text{ and } \mathcal{C}' := C_0(\overline{X})_{sing},$$

where $(\overline{X})_{sing}$ is the singular part of the closure $\overline{X}$ of X. As mentioned previously, different subsets of $\mathcal{C} \setminus \mathcal{C}'$ might be considered as *exceptional* (like in Theorem O'Shea–Wilson, we exclude rays in $\mathcal{C}'$, both because such rays are certainly not ordinary, and because Nash fibers along these rays seem to be wilder as they contain the degeneracy at 0 of singular Nash fibers at $x \neq 0$); namely:

(a) non singular rays ℓ in $\mathcal{C}$ whose Nash fiber is not the tangent space $T_v\mathcal{C}$ to the tangent cone $\mathcal{C}$ at an arbitrary point v in ℓ: $\mathcal{N}_\ell \neq \{T_v\mathcal{C}\}$;

(b) rays ℓ whose Nash fiber is not a unique plane: $\#(\mathcal{N}_\ell) > 1$;

(c) rays ℓ whose Nash fiber has positive dimension: $\dim \mathcal{N}_\ell \geqslant 1$;

(d) rays ℓ that contains non zero critical values of the canonical projection

$$pr\colon \mathcal{N}'(X)_{(x=0,t=0)} \subset \mathcal{C} \times \mathcal{N}(X)_{x=0} \to \mathcal{C}, \ (v, P) \mapsto v.$$

($\mathcal{N}_\ell$ is precisely the (common) preimage of $v \in \ell$, by this projection).

We will focus mainly on the criterion (b) in this article, so we define the following set:

$$\mathcal{E} = \{v \in \mathbb{R}^n \setminus \{0\} \colon \ \ell := \mathbb{R}_+ v \subset \mathcal{C} \setminus \mathcal{C}', \ \#(\mathcal{N}_\ell) > 1\}. \tag{1}$$

The choice of criterion (b) as principal interest can be explained in light of our results as follows. We show that criteria (a) and (b) coincide for rays $\ell \not\subset \mathcal{C}_{sing} \cup \mathcal{C}'$ with $\dim_0 X = \dim_v \mathcal{C}$, where $v \in \ell$ and $\dim_0 X$ is the dimension of X at 0 (Theorem 1.1). For hypersurfaces, we show that (b) and (c) coincide (Theorem 1.2); they do not in full generality (Example 5.7), and we do not know if they coincide for any ray $\ell \not\subset \mathcal{C}_{sing}$. Criterion (d) is not studied here since we follow [12], while in view of Singularity Theory, it is a natural candidate. The projection pr and its critical values emerge however here and there during the proofs, and we believe that (d) deserves its own study.

It is noticeable that the criteria (a), (b) and (c) collapse when $\dim_0 X > \dim \mathcal{C}$. In order to deal with this situation, we also introduce the following cone of rays ℓ whose Nash fiber has a tangent limit not containing the plane tangent to $\mathcal{C}$ along ℓ:

$$\mathcal{E}' = \{v \in \mathbb{R}^n \setminus \{0\} : \ \ell := \mathbb{R}_+ v \subset \mathcal{C} \setminus \mathcal{C}_{sing}, \text{ there exists } P \in \mathcal{N}_\ell \text{ such that } T_v\mathcal{C} \not\subset P\}, \quad (2)$$

so $\mathcal{E}'$ contains rays that are exceptional for a criterion derived from (a) in the case $\dim_0 X > \dim \mathcal{C}$.

We now state our results. The first one shows that the rays we call exceptional are rare. For surfaces in $\mathbb{R}^3$, it recovers the finiteness of the number of exceptional rays and the first item in Theorem O'Shea–Wilson.

Theorem 1.1 (Nowhere dense - Dimension)**.** *Let $X \subset \mathbb{R}^n$ be a definable set of pure dimension $d > 0$ at the origin $0 \in \mathbb{R}^n$, $\mathcal{C}$ be its tangent cone at 0, $\mathcal{C}'$ be the tangent cone to $(\overline{X})_{sing}$ at 0, and $\mathcal{E}, \mathcal{E}'$ be given by (1) and (2) respectively. Then the following statements hold:*

(i) *The set $\mathcal{E}' \cap \mathbb{S}^{n-1}$ is nowhere dense in $\mathcal{C} \cap \mathbb{S}^{n-1}$. In particular $\dim \mathcal{E}' < \dim \mathcal{C} \leqslant d$.*

(ii) *Let ℓ be a ray in $\mathcal{C}$ and $v \in \ell$. Assume that $\dim_v \mathcal{C} = d$. Then $\ell \subset \mathcal{E} \setminus \mathcal{C}_{sing}$ if and only if $\ell \subset \mathcal{E}' \setminus \mathcal{C}'$. Furthermore, if $\ell \subset \mathcal{C} \setminus (\mathcal{E} \cup \mathcal{C}_{sing} \cup \mathcal{C}')$, then $\mathcal{N}_\ell = \{T_v\mathcal{C}\}$. In particular $\dim \mathcal{E} < d$.*

No analogue of the second item in Theorem O'Shea–Wilson can be reached in full generality, according to Example 5.7. We however are able to generalize it for hypersurfaces, as follows.

Theorem 1.2 (Connected exceptional Nash fibers)**.** *Let $X \subset \mathbb{R}^n$, with $n \geqslant 2$, be a definable set of pure dimension $n-1$ at the origin $0 \in \mathbb{R}^n$ and $\mathcal{E}$ be given by (1). Then for each ray $\ell \subset \mathcal{E}$, the Nash fiber $\mathcal{N}_\ell$ is a closed, connected and definable set of positive dimension.*

It remains to get an analogue of the last item in Theorem O'Shea–Wilson. For this, we study the rays ℓ for which the tangent cone to $\mathcal{C}$ along ℓ is not a plane of dimension d. In fact, the following result recovers the last item in Theorem O'Shea–Wilson.

Theorem 1.3 (Singular cone)**.** *Let $X \subset \mathbb{R}^n$ be a definable set of pure dimension $d > 0$ at the origin $0 \in \mathbb{R}^n$, $\mathcal{C}$ be its tangent cone at 0, $\mathcal{C}'$ be the tangent cone to $(\overline{X})_{sing}$ at 0, and $\mathcal{E}$*

be given by (1)*. If* $v \in \mathcal{C} \setminus \mathcal{C}'$ *is a non zero point such that* $C_v\mathcal{C}$ *is not a plane of dimension* d*, then* $v \in \mathcal{E}$*.*

The paper is organized as follows. We fix the notation which will be used throughout the paper in Section 2. This section also contains some basic properties of o-minimal structures needed in the paper. In Section 3, we give some elementary properties of tangent cones and tangent limits. The main results of the paper will be proved in Section 4. In the last section 5, we give some remarks and examples.

2. Preliminaries

2.1. **Notation.** For the remainder of the paper, we denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^n$ with respect to the Euclidean inner product $\langle\cdot,\cdot\rangle$. The closed ball, the open ball and the sphere centered at $x \in \mathbb{R}^n$ and of radius r are denoted respectively by $\mathbb{B}^n_r(x)$, $\mathring{\mathbb{B}}^n_r(x)$ and $\mathbb{S}^{n-1}_r(x)$. If $x = 0$, we write $\mathbb{B}^n_r$, $\mathring{\mathbb{B}}^n_r$ and $\mathbb{S}^{n-1}_r$. If in addition $r = 1$, then we write $\mathbb{B}^n$, $\mathring{\mathbb{B}}^n$ and $\mathbb{S}^{n-1}$. For $X \subset \mathbb{R}^n$, the sets $\overline{X}$ and ∂X designate respectively the closure and the boundary of X. The cardinality of X is denoted by $\#(X)$.

Let $\operatorname{dist}(X, Y)$ stand for the Euclidean distance between two subsets X and Y of $\mathbb{R}^n$, i.e.,

$$\operatorname{dist}(X, Y) := \inf\{\|x - y\| : \ x \in X, \ y \in Y\}.$$

By convention, set $\operatorname{dist}(X, Y) = 0$ if $X = \emptyset$ or $Y = \emptyset$. Furthermore, the Hausdorff distance between X and Y is given by

$$\operatorname{dist}_{\mathcal{H}}(X, Y) := \max\Big\{\sup_{x\in X} \operatorname{dist}(x, Y), \sup_{y\in Y} \operatorname{dist}(y, X)\Big\}.$$

The set $Y \subset \mathbb{R}^n$ is called the limit of a sequence of subsets Y_k of $\mathbb{R}^n$, i.e., $Y = \lim\limits_{k\to+\infty} Y_k$, if and only if $\operatorname{dist}_{\mathcal{H}}(Y_k, Y) \to 0$ as $k \to +\infty$.

For a non empty definable set $X \subset \mathbb{R}^n$, let X_{reg} be the set of regular points of X, which is the set of points where X is a C^1-manifold. The complement of X_{reg} in X is denoted by X_{sing}, the set of singular points of X. Note that X_{sing} is nowhere dense in X.

If X is non empty, for a number $t \in \mathbb{R}$, let

$$tX := \{tx : \ x \in X\}.$$

Let $v, w \in \mathbb{R}^n$ be not equal to 0 simultaneously, denote by $\widehat{v, w}$ the angle between v and w. For convenience, if either $v = 0$ or $w = 0$, set $\widehat{v, w} := \dfrac{\pi}{2}$. So $0 \leqslant \widehat{v, w} = \widehat{w, v} \leqslant \pi$. The angle between two rays ℓ_1 and ℓ_2, denoted by $\widehat{\ell_1, \ell_2}$, is defined to be the angle between the unit directions in each ray. If $V \neq \{0\}$ is a linear subspace of $\mathbb{R}^n$, let π_V be the orthogonal projection on V and the angle between a non zero vector v and V is given by

$$\angle(v, V) = \widehat{v, \pi_V(v)}.$$

For two linear subspaces $V_1 \neq \{0\}$ and $V_2 \neq \{0\}$ of $\mathbb{R}^n$, let π_{V_i} be the orthogonal projection on V_i $(i = 1, 2)$. We define the angle between V_1 and V_2 by

$$\begin{aligned} \angle(V_1, V_2) \ &:= \ \begin{cases} \sup\{\widehat{v, \pi_{V_2}(v)} : \ v \in V_1 \setminus \{0\}\} & \text{if } \dim V_1 \leqslant \dim V_2 \\ \sup\{\widehat{v, \pi_{V_1}(v)} : \ v \in V_2 \setminus \{0\}\} & \text{if } \dim V_1 \geqslant \dim V_2 \end{cases} \\ &= \ \begin{cases} \max\{\widehat{v, \pi_{V_2}(v)} : \ v \in V_1 \cap \mathbb{S}^{n-1}\} & \text{if } \dim V_1 \leqslant \dim V_2 \\ \max\{\widehat{v, \pi_{V_1}(v)} : \ v \in V_2 \cap \mathbb{S}^{n-1}\} & \text{if } \dim V_1 \geqslant \dim V_2. \end{cases} \end{aligned}$$

Observe that if $\dim V_1 = \dim V_2$, then

$$\sup\{\widehat{v, \pi_{V_2}(v)} : \ v \in V_1 \setminus \{0\}\} = \sup\{\widehat{v, \pi_{V_1}(v)} : \ v \in V_2 \setminus \{0\}\},$$

so the definition of angle between linear subspaces makes sense. By definition,

$$0 \leqslant \angle(V_1, V_2) \leqslant \frac{\pi}{2}.$$

Furthermore, the equality $\angle(V_1, V_2) = 0$ implies that $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$. If V_1 and V_2 are affine subspaces of $\mathbb{R}^n$, then the angle between V_1 and V_2 are determined by the angle between the corresponding parallel linear subspaces. It is not hard to verify that $\angle(\cdot, \cdot)$ defines a metric on the Grassmannian of d-dimensional linear subspaces of $\mathbb{R}^n$, for $1 \leqslant d \leqslant n$.

2.2. **O-minimal structures.** The notion of o-minimality was developed in the late 1980s after it was noticed that many proofs of analytic and geometric properties of semi-algebraic sets and mappings can be carried over verbatim for sub-analytic sets and mappings. We refer the reader to [2, 13, 14] for the basic properties of o-minimal structures used in this paper.

Definition 2.1. A *structure* expanding the field of real numbers $\mathbb{R}$ is a collection

$$\mathcal{R} = (\mathcal{R}_n)_{n \in \mathbb{N}},$$

where each $\mathcal{R}_n$ is a set of subsets of the affine space $\mathbb{R}^n$, satisfying the following axioms:

(a) All algebraic subsets of $\mathbb{R}^n$ are in $\mathcal{R}_n$.
(b) For every n, $\mathcal{R}_n$ is a Boolean subalgebra of the powerset of $\mathbb{R}^n$.
(c) If $A \in \mathcal{R}_m$ and $B \in \mathcal{R}_n$, then $A \times B \in \mathcal{R}_{m+n}$.
(d) If $p \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates and $A \in \mathcal{R}^{n+1}$, then $p(A) \in \mathcal{R}_n$.

The elements of $\mathcal{R}_n$ are called the *definable subsets* of $\mathbb{R}^n$. A mapping whose graph is a definable set is called a *definable mapping*. The structure $\mathcal{R}$ is said to be *o-minimal* if, moreover, it satisfies the following axiom:

(e) Each element of $\mathcal{R}_1$ is a finite union of points and intervals.

Examples of o-minimal structures are:

- the semi-algebraic sets (by the Tarski–Seidenberg theorem),
- the globally sub-analytic sets, i.e., the sub-analytic sets of $\mathbb{R}^n$ whose (compact) closures in $\mathbb{RP}^n$ are sub-analytic (using Gabrielov's complement theorem).

From now on, we fix an arbitrary o-minimal structure expanding $\mathbb{R}$. The term "definable" means definable in this structure. In the sequel, we will make use of the following Curve Selection lemma, Morse–Sard theorem and Hardt's definable triviality theorem.

Lemma 2.2 (Curve Selection)**.** [11, Lemma 3.1], [14, 1.17] *Let $X \subset \mathbb{R}^n$ be a definable set and $x \in \overline{X} \setminus X$. Then there is a C^1 definable curve $\gamma\colon (0, \varepsilon) \to X \setminus \{x\}$, for some $\varepsilon > 0$, such that $\lim_{t\to 0^+} \gamma(t) = x$.*

Theorem 2.3 (Morse–Sard's Theorem)**.** [10, Theorem 1.4], [17, Theorem 2.7] *Let N and M be C^1 definable manifolds of dimensions respectively n and m with $n \geqslant m \geqslant 1$, and $f\colon N \to M$ be a C^1 definable mapping. Let*

$$\Sigma(f) := \{x \in N : \ \text{rank } d_x f < m\}.$$

Then $f(\Sigma(f))$ is a definable set of dimension less than m.

Theorem 2.4 (Hardt's triviality theorem)**.** [2, Theorem 5.22], [3], [13, Theorem 1.2, p. 142] *Let X and Y be definable sets and $f\colon X \to Y$ be a continuous definable mapping. Then there exists a finite partition*

$$Y = Y_1 \sqcup \cdots \sqcup Y_p$$

into definable subsets Y_i, $i = 1, \dots p$, such that f is definably trivial over each Y_i, i.e., $f^{-1}(Y_i)$ is definably homeomorphic to $f^{-1}(y_i) \times Y_i$ for each i and any $y_i \in Y_i$.

Let $X \subset \mathbb{R}^n$ be a definable set. We define the *dimension* of X by

$$\dim X := \max\{\dim Y : \ Y \text{ is a } C^1\text{-manifold contained in } X\}.$$

For $x \in \overline{X}$, the *dimension of X at x* is defined by

$$\dim_x X := \min\{\dim(X \cap U) : \ U \text{ is an open neighborhood of } x \text{ in } \mathbb{R}^n\}.$$

Moreover, we say that X is of *pure dimension d at x* if there exists an open neighborhood U of x in $\mathbb{R}^n$ such that $\dim_y X = d$ for any $y \in X \cap U$. Finally, we say that X has *pure dimension d* if $\dim_y X = d$ for any $y \in X$.

Lemma 2.5. [14, 1.16(3)] *Let X be a definable set in $\mathbb{R}^n$. Then the following statements hold.*

(i) *If* $X \neq \emptyset$ *then* $\dim(\overline{X} \setminus X) < \dim X$. *In particular,* $\dim \overline{X} = \dim X$.

(ii) *For any* $x \in \overline{X}$, *we have* $\dim_x X = \dim_x \overline{X}$. *Moreover* X *is of pure dimension* d *at* x *if and only if* $\overline{X}$ *is of pure dimension* d *at* x.

3. Tangent cones and tangent limits

In this section, some elementary properties of tangent cones and tangent limits will be given. First of all, we state the following simple lemma whose proof is left to the reader.

Lemma 3.1. *Let* $C \subset \mathbb{R}^n$ *be a definable cone at the origin* 0 *and let* $D := C \cap \mathbb{S}_R^{n-1}$, *where* $R \in (0, +\infty)$. *The following statements hold true:*

(i) D *is definable.*

(ii) *A non zero point* $v \in C$ *is a singular point of* C *if and only if the ray through* v *contains only singular points of* C. *In particular,* C_{sing} *is also a cone and we have*

$$C_{sing} \cap \mathbb{S}_R^{n-1} = D_{sing}.$$

(iii) *If* $v \in D \setminus D_{sing}$, *then for all* $t > 0$ *we have* $tv \in C \setminus C_{sing}$ *and* $T_{tv}C \cong T_v D \oplus \mathbb{R}v$.

(iv) $C \setminus \{0\}$ *is homeomorphic to* $D \times (0, +\infty)$. *In particular,* $\dim_v C = \dim_v D + 1$ *for all* $v \in D$ *and so* $\dim C = \dim D + 1$.

Remark 3.2. In view of Lemma 3.1(iv), the dimension of C at any point v in a ray ℓ of C is constant. If ℓ is a non singular ray in C, for all $v \in \ell$, the tangent spaces T_vC define the same plane in the Grassmannian $\mathbb{G}(\dim_v C, n)$. Moreover, it is not hard to check that the Nash fiber of C along ℓ contains only one element which is T_vC.

For the remainder of the section, let X be a definable set with $0 \in \overline{X}$. Recall that $\mathcal{C} := C_0X$ is the tangent cone to X at 0. The following lemma is a definable version of [6, Lemma 1.2].

Lemma 3.3. *The set* $\mathcal{C}$ *is a nonempty closed definable cone of dimension at most* $\dim_0 X$. *In addition, if* $\dim_0 X > 0$, *then* $\mathcal{C}$ *has positive dimension.*

Proof. The last statement is clear so it remains to prove the first one. By definition, it is easy to check that $\mathcal{C}$ is a nonempty closed definable cone. Let us show that $\dim \mathcal{C} \leqslant \dim_0 X$. To do this, for each $r > 0$, define

$$A_r := \left\{ \left(\frac{x}{t}, t \right) \in \mathbb{R}^n \times (0, +\infty) : \ x \in X \cap \mathring{\mathbb{B}}_r^n \right\}.$$

Obviously, A_r is a nonempty definable set, which is homeomorphic to $(X \cap \mathring{\mathbb{B}}_r^n) \times (0, +\infty)$. Hence, for all r sufficiently small, we have

$$\dim A_r = \dim_0 X + 1.$$

Observe that

$$\mathcal{C} \times \{0\} \subset \overline{A}_r \cap (\mathbb{R}^n \times \{0\}) \subset \overline{A}_r \setminus A_r.$$

Hence

$$\dim \mathcal{C} \leqslant \dim(\overline{A}_r \setminus A_r) < \dim A_r = \dim_0 X + 1,$$

where the second inequality follows from Lemma 2.5. This ends the proof of the lemma. $\square$

Recall that the term "ray" means "open ray emanating from the origin $0 \in \mathbb{R}^n$". The following lemma shows that tangent cones and Nash fibers are invariant by taking closure.

Lemma 3.4. *Suppose that X is of pure dimension $d > 0$ at the origin $0 \in \mathbb{R}^n$. Then $C_0X = C_0\overline{X}$. Moreover, $\mathcal{N}_\ell(X) = \mathcal{N}_\ell(\overline{X}) \neq \emptyset$ for any ray $\ell \subset C_0X$, where $\mathcal{N}_\ell(X)$ and $\mathcal{N}_\ell(\overline{X})$ are respectively the Nash fibers of X and $\overline{X}$ along ℓ.*

Proof. In view of Lemma 3.3, we have $\dim C_0X > 0$. So $\dim C_0\overline{X} > 0$ as $C_0X \subset C_0\overline{X}$. Let ℓ be a ray in $C_0\overline{X}$ and $x^k \in \overline{X} \setminus \{0\}$ be a sequence tending to 0 such that $\dfrac{x^k}{\|x^k\|} \to v \in \ell$ as $k \to +\infty$. Since $\overline{X \setminus X_{sing}} = \overline{X}$ and since X is of pure dimension $d > 0$ at 0, for k large enough, there is $y^k \in X \setminus X_{sing}$ such that

$$\dim T_{y^k}X = d, \ \|y^k - x^k\| < \frac{\|x^k\|}{k}, \quad \text{and} \quad \angle(T_{y^k}X, T_{x^k}\overline{X}) < \frac{1}{k} \text{ if } x^k \notin (\overline{X})_{sing}. \tag{3}$$

Hence

$$\|y^k\| \leqslant \|x^k\| + \|y^k - x^k\| \leqslant \|x^k\| + \frac{\|x^k\|}{k} \to 0 \text{ as } k \to +\infty. \tag{4}$$

On the other hand,

$$\|y^k\| \geqslant \|x^k\| - \|y^k - x^k\| \geqslant \|x^k\| - \frac{\|x^k\|}{k} > \frac{\|x^k\|}{2} > 0,$$

for k large enough. Thus

$$\frac{\|y^k - x^k\|}{\|y^k\|} < \frac{\|x^k\|}{k\|y^k\|} < \frac{2}{k} \to 0 \quad \text{as} \quad k \to +\infty.$$

Therefore

$$\begin{aligned}
\left\| \frac{y^k}{\|y^k\|} - v \right\| &\leqslant \left\| \frac{y^k - x^k}{\|y^k\|} \right\| + \left\| \frac{x^k}{\|y^k\|} - \frac{x^k}{\|x^k\|} \right\| + \left\| \frac{x^k}{\|x^k\|} - v \right\| \\
&< \frac{2}{k} + \|x^k\| \left| \frac{1}{\|y^k\|} - \frac{1}{\|x^k\|} \right| + \left\| \frac{x^k}{\|x^k\|} - v \right\| \\
&= \frac{2}{k} + \left| \frac{\|x^k\| - \|y^k\|}{\|y^k\|} \right| + \left\| \frac{x^k}{\|x^k\|} - v \right\| \\
&\leqslant \frac{2}{k} + \frac{\|x^k - y^k\|}{\|y^k\|} + \left\| \frac{x^k}{\|x^k\|} - v \right\| < \frac{4}{k} + \left\| \frac{x^k}{\|x^k\|} - v \right\| \to 0 \text{ as } k \to +\infty.
\end{aligned}$$

Combining this with (4) yields $v \in C_0X$ and so $\ell \subset C_0X$. Hence $C_0X = C_0\overline{X}$.

Now taking a subsequence if necessary, we can suppose that the limit $\lim_{k\to+\infty} T_{y^k}X$ exists. Clearly, this limit belongs to $\mathcal{N}_\ell(X)$, i.e., $\mathcal{N}_\ell(X) \neq \emptyset$.

It remains to show that $\mathcal{N}_\ell(X) = \mathcal{N}_\ell(\overline{X})$. For this, assume that

$$x^k \notin (\overline{X})_{sing},\ \dim_{x^k} \overline{X} = d \text{ and the limit } P := \lim_{k\to+\infty} T_{x^k}\overline{X} \text{ exists,}$$

so $P \in \mathcal{N}_\ell(\overline{X})$. Then from (3), it follows that $P \in \mathcal{N}_\ell(X)$. Therefore $\mathcal{N}_\ell(\overline{X}) \subset \mathcal{N}_\ell(X)$ and so $\mathcal{N}_\ell(X) = \mathcal{N}_\ell(\overline{X})$ as the inclusion $\mathcal{N}_\ell(X) \subset \mathcal{N}_\ell(\overline{X})$ is clear. This ends the proof of the lemma. □

Lemma 3.5. *Suppose that X is of pure dimension $d > 0$ at the origin $0 \in \mathbb{R}^n$. Let ℓ be a ray in $\mathcal{C}$ and let $\ell_k \subset \mathcal{C}$ be a sequence of rays such that $\ell_k \to \ell$. Assume that, for each k, P_k is a tangent limit to X at 0 along ℓ_k such that $P_k \to P$, then $P \in \mathcal{N}_\ell$.*

Proof. By definition and by the assumption, for each k, there is a sequence $x^{kl} \in X \setminus X_{sing}$ such that

$$x^{kl} \to 0,\ \frac{x^{kl}}{\|x^{kl}\|} \to v_k \in \ell_k,\ \dim T_{x^{kl}}X = d \ \text{ and } \ T_{x^{kl}}X \to P_k \in \mathcal{N}_{\ell_k} \ \text{ as } \ l \to +\infty.$$

We can find a subsequence x^{kl_k} such that

$$\|x^{kl_k}\| < \frac{1}{k},\ \left\|\frac{x^{kl_k}}{\|x^{kl_k}\|} - v_k\right\| < \frac{1}{k} \ \text{ and } \ \angle(T_{x^{kl_k}}X, P_k) < \frac{1}{k}.$$

Since $\ell_k \to \ell$, it follows that $v^k \to v$, where v is the unit direction in ℓ, as $k \to +\infty$. Then it is clear that

$$x^{kl_k} \to 0,\ \frac{x^{kl_k}}{\|x^{kl_k}\|} \to v \text{ and } T_{x^{kl_k}}X \to P \text{ as } k \to +\infty.$$

Hence $P \in \mathcal{N}_\ell$. □

The following lemma shows that any tangent limit along a ray always contains that ray.

Lemma 3.6 (see also [12, Lemma 4], [16, Theorem 11.8, Theorem 22.1])**.** *Suppose that X is of pure dimension $d > 0$ at the origin $0 \in \mathbb{R}^n$. Let P be a tangent limit along a ray $\ell \subset \mathcal{C}$ and $v \in \ell$ be the unit direction in ℓ. Then there exists a C^1 definable curve $\gamma\colon (0,\varepsilon) \to X \setminus X_{sing}$ such that*

$$\|\gamma(t)\| = t \textit{ for } t \in (0,\varepsilon),\ \lim_{t\to0^+} \frac{\gamma(t)}{\|\gamma(t)\|} = v \ \textit{ and } \ \lim_{t\to0^+} T_{\gamma(t)}X = P.$$

Moreover $\ell \subset P$.

Proof. In view of Lemma 2.2, there is a C^1 definable curve $x\colon (0,\epsilon) \to X \setminus X_{sing}$ such that

$$\lim_{r\to0^+} x(r) = 0,\ \lim_{r\to0^+} \frac{x(r)}{\|x(r)\|} = v \ \text{ and } \ \lim_{r\to0^+} T_{x(r)}X = P.$$

Let $t(r) := \|x(r)\|$ which is a definable function. By shrinking ϵ, we may assume that $t(r)$ is of class C^1 and strictly increasing on $(0, \epsilon)$; moreover, the inverse function $r(t)$ is also of class C^1. Set $\gamma(t) := x(r(t))$. Then clearly

$$\|\gamma(t)\| = \|x(r(t))\| = t(r(t)) = t.$$

It is not hard to check that $\lim_{t \to 0^+} \frac{\gamma(t)}{\|\gamma(t)\|} = v$ and $\lim_{t \to 0^+} T_{\gamma(t)} X = P$. Hence the first statement follows.

Now for all t, the point $\gamma(t) \in X$ so $\gamma'(t) \in T_{\gamma(t)} X$. Hence, when t tends to 0^+, L'Hospital's Rule gives

$$v = \lim_{t \to 0^+} \frac{\gamma(t)}{\|\gamma(t)\|} = \lim_{t \to 0^+} \frac{\gamma(t)}{t} = \lim_{t \to 0^+} \gamma'(t) \in \lim_{t \to 0^+} T_{\gamma(t)} X = P.$$

Therefore $\ell \subset P$. This ends the proof of the lemma. $\square$

4. Proofs of the main results

Proof of Theorem 1.1. (i) Observe that the second statement follows from the first one and Lemma 3.3, so it remains to prove the first statement. Consider the definable function

$$\rho \colon X \setminus X_{sing} \to [0, +\infty), \quad x \mapsto \|x\|.$$

Observe that the set of critical values of ρ is finite in view of Theorem 2.3. Thus there exists a constant $\varepsilon > 0$ such that ρ has no critical values in the interval $(0, \varepsilon)$. Let

$$B := \{(u, r) \in \mathbb{S}^{n-1} \times (0, \varepsilon) : \ ru \in X \setminus X_{sing}\}.$$

Clearly B is definable and nonempty. Moreover, B is non singular as it is the inverse image of the non singular set $(X \setminus X_{sing}) \cap \mathring{\mathbb{B}}^n_\varepsilon$ by the diffeomorphism

$$\phi \colon \mathbb{S}^{n-1} \times (0, +\infty) \to \mathbb{R}^n \setminus \{0\}, \quad (u, r) \mapsto ru.$$

Let $D := (\mathcal{C} \setminus \mathcal{C}_{sing}) \cap \mathbb{S}^{n-1}$. Clearly D is a non singular definable set. By Lemma 3.4, $\dim \mathcal{C} > 0$, so D is nonempty. Set $\widetilde{B} := D \times \{0\}$ and consider the definable function

$$f \colon \widetilde{B} \cup B \to \mathbb{R}, \quad (u, r) \mapsto r.$$

Note that $\widetilde{B} \subset \overline{B} \setminus B$. Furthermore, we have

$$\operatorname{rank}(f|_{\widetilde{B}}) = 0 \quad \text{and} \quad \operatorname{rank}(f|_B) = 1.$$

Indeed, the first equation is clear. To prove the second one, suppose for contradiction that $d_{(u,r)} f = 0$ for some $(u, r) \in B$. We have $\|ru\| = r < \varepsilon$ and so ru is not a critical point of ρ. Moreover,

$$T_{(u,r)} B = \ker d_{(u,r)} f \subset T_u \mathbb{S}^{n-1} \times \{0\}.$$

Therefore

$$T_{ru}(X \setminus X_{sing}) = d_{(u,r)}\phi(T_{(u,r)}B) \subset d_{(u,r)}\phi(T_u\mathbb{S}^{n-1} \times \{0\}) = T_{ru}\mathbb{S}^{n-1}_r,$$

and so ru is a critical point of ρ, which is a contradiction.

Let

$$Z := \left\{ \begin{array}{ll} v \in D: & \text{the pair } (B, \widetilde{B}) \text{ does not satisfy} \\ & \text{the Thom's } a_f \text{ condition at } (v, 0) \end{array} \right\}. \tag{5}$$

Recall that the pair $(B, \widetilde{B})$ satisfies the *Thom's a_f condition* at $(v, 0) \in \widetilde{B}$ if and only if for any sequence $w^k = (u^k, r_k) \in B$ converging to $(v, 0) \in \widetilde{B}$, we have

$$\angle(T_{w^k} f^{-1}(r_k), T_v D \times \{0\}) \to 0 \quad \text{as} \quad k \to +\infty.$$

Let $D_1, \ldots, D_q$ be the connected components of D. By [9, Lemma 2] (see also [1, Lemma 15] and [5, Lemma 1.2]), we have $\dim(Z \cap D_i) < \dim D_i$ for each $i = 1, \ldots, q$. Hence $Z \cap D_i$ is nowhere dense in D_i, and so it is not hard to check that Z is nowhere dense in D.

Now to prove (i), it is enough to show that $\mathcal{E}' \cap \mathbb{S}^{n-1} \subset Z$, or, equivalently,

$$D \setminus Z \subset D \setminus \mathcal{E}'.$$

To do this, take arbitrarily $v \in D \setminus Z$. We need to show that $v \notin \mathcal{E}'$. In fact, let P be an arbitrary tangent limit to X at 0 along $\ell := \mathbb{R}_+ v$. By definition, there is a sequence $x^k \in X \setminus X_{sing}$ such that

$$x^k \to 0, \ \frac{x^k}{\|x^k\|} \to v \quad \text{and} \quad T_{x^k} X \to P \quad \text{as} \quad k \to +\infty.$$

Set $r_k := \|x^k\|$ and $u^k := \dfrac{x^k}{\|x^k\|}$. Then the sequence $w^k := (u^k, r_k) \in B$ tends to $(v, 0) \in \widetilde{B}$. Since $v \notin (Z \cup \mathcal{C}_{sing})$, we have

$$\angle(T_{w^k}(f^{-1}(r_k)), T_v D \times \{0\}) \to 0 \quad \text{as} \quad k \to +\infty.$$

Equivalently,

$$\angle\left(T_{w^k}\left[\left(\frac{1}{r_k}(X \setminus X_{sing}) \cap \mathbb{S}^{n-1}\right) \times \{r_k\}\right], T_v D \times \{0\}\right) \to 0 \quad \text{as} \quad k \to +\infty,$$

where

$$\frac{1}{r_k}(X \setminus X_{sing}) = \left\{\frac{x}{r_k} : \ x \in X \setminus X_{sing}\right\}.$$

Consequently,

$$\angle\left(T_{u^k}\left[\frac{1}{r_k}(X \setminus X_{sing}) \cap \mathbb{S}^{n-1}\right], T_v D\right) \to 0 \text{ as } \ k \to +\infty. \tag{6}$$

On the other hand, $\|x^k\| < \varepsilon$ for all large k. For all such k, x^k is a regular point of the function ρ and so it is a non singular point of $X \cap \mathbb{S}^{n-1}_{r_k}$. Combining this with the assumption that X is of pure dimension d at 0, Lemma 3.1(iv) and Lemma 3.3 yields

$$\begin{aligned}\dim_{x^k}(X \cap \mathbb{S}^{n-1}_{r_k}) = \dim_{x^k} X - 1 = \dim_0 X - 1 &\geqslant \dim \mathcal{C} - 1 = \dim D \\ &\geqslant \dim_v D = \dim T_v D.\end{aligned}$$

Then, by definition, we obtain

$$\begin{aligned}\angle(T_{x^k}X, T_vD) &= \sup\{\widehat{z, \pi_{T_{x^k}X}(z)} : z \in T_vD \setminus \{0\}\} \\ &\leqslant \sup\{\widehat{z, \pi_{T_{x^k}(X\cap\mathbb{S}^{n-1}_{r_k})}(z)} : z \in T_vD \setminus \{0\}\} \\ &= \angle(T_{x^k}(X \cap \mathbb{S}^{n-1}_{r_k}), T_vD) \\ &= \angle(T_{x^k}((X \setminus X_{sing}) \cap \mathbb{S}^{n-1}_{r_k}), T_vD),\end{aligned}$$

where $\pi_{T_{x^k}X}$ and $\pi_{T_{x^k}(X\cap\mathbb{S}^{n-1}_{r_k})}$ are the orthogonal projections on $T_{x^k}X$ and $T_{x^k}(X \cap \mathbb{S}^{n-1}_{r_k})$ respectively. Observe that the homothety

$$(X \setminus X_{sing}) \cap \mathbb{S}^{n-1}_{r_k} \to \frac{1}{r_k}(X \setminus X_{sing}) \cap \mathbb{S}^{n-1}, \quad x \mapsto \frac{x}{r_k},$$

is a diffeomorphism. In particular, $T_{x^k}\left[(X \setminus X_{sing}) \cap \mathbb{S}^{n-1}_{r_k}\right]$ and $T_{u^k}\left[\frac{1}{r_k}(X \setminus X_{sing}) \cap \mathbb{S}^{n-1}\right]$ determine the same plane in the Grassmannian $\mathbb{G}(d-1, n)$. Therefore

$$\angle(T_{x^k}X, T_vD) \leqslant \angle\left(T_{u^k}\left[\frac{1}{r_k}(X \setminus X_{sing}) \cap \mathbb{S}^{n-1}\right], T_vD\right).$$

This, together with (6), gives us $\angle(P, T_vD) = 0$. Note that $T_v\mathcal{C} = T_vD \oplus \mathbb{R}v$ (in light of Lemma 3.1(iii)) and $v \in P$ (by Lemma 3.6). Hence, $\angle(P, T_v\mathcal{C}) = 0$. This yields $v \notin \mathcal{E}'$ and so Item (i) follows.

Before proving Item (ii), we need the following lemma.

Lemma 4.1. *Let $X \subset \mathbb{R}^n$ be a definable set of pure dimension $d > 0$ at the origin $0 \in \mathbb{R}^n$. For each ray $\ell \subset \mathcal{C} \setminus \mathcal{C}_{sing}$, there exists at least one tangent limit along ℓ containing $T_v\mathcal{C}$ for some $v \in \ell$.*

Proof. In view of Lemma 3.4, we have $\mathcal{N}_\ell \neq \emptyset$. From the definition of $\mathcal{E}'$, it is enough to prove the statement for any ray $\ell \subset \mathcal{E}'$.

First of all, consider the case when ℓ is an isolated ray in $\mathcal{C}$. In view of Lemma 3.6, we have $\ell \subset P$ for any $P \in \mathcal{N}_\ell$. Observe that $T_v\mathcal{C}$ is the line containing ℓ. Hence $T_v\mathcal{C} \subset P$ for any $P \in \mathcal{N}_\ell$.

Now assume that ℓ is not an isolated ray in $\mathcal{C}$. As $\mathcal{E}'$ is nowhere dense in $\mathcal{C}$ by Item (i), it holds that $\mathcal{C}_{sing} \cup \mathcal{E}'$ is nowhere dense in $\mathcal{C}$. Hence there exists a sequence of rays $\ell_k \subset \mathcal{C} \setminus (\mathcal{C}_{sing} \cup \mathcal{E}')$ converging to ℓ. For each k, let v^k be a direction in ℓ_k such that the sequence v^k tends to v and let P_k be a tangent limit of X along ℓ_k. Then $T_{v^k}\mathcal{C} \subseteq P_k$ by definition. Taking a subsequence if necessary, we may suppose that P_k converges to P. By Lemma 3.5, $P \in \mathcal{N}_\ell$. Since $T_{v^k}\mathcal{C} \to T_v\mathcal{C}$, we deduce that $T_v\mathcal{C} \subseteq P$. The lemma is proved. $\square$

(ii) Let ℓ be a ray in $\mathcal{C}$ and let $v \in \ell$. Suppose that $\dim_v \mathcal{C} = d$.

Let us prove the first statement. Assume first that $\ell \subset \mathcal{E} \setminus \mathcal{C}_{sing}$. Then ℓ is not a ray in $\mathcal{C}'$ and by Lemma 4.1, there is $P \in \mathcal{N}_\ell$ containing $T_v\mathcal{C}$. By the assumption, $\dim_v \mathcal{C} = \dim P$. Hence $P = T_v\mathcal{C}$. As $\#(\mathcal{N}_\ell) > 1$, there is $P' \in \mathcal{N}_\ell$ such that $P' \neq P$. Hence $T_v\mathcal{C} \not\subset P'$, i.e., $\ell \subset \mathcal{E}'$. Now suppose that $\ell \subset \mathcal{E}' \setminus \mathcal{C}'$, so ℓ is not a ray in $\mathcal{C}_{sing}$ and there is $P \in \mathcal{N}_\ell$ not containing $T_v\mathcal{C}$. On the other hand, by Lemma 4.1, there is $P' \in \mathcal{N}_\ell$ such that $T_v\mathcal{C} \subset P'$. Clearly $P \neq P'$, so $\ell \subset \mathcal{E}$. The first statement follows. In fact, we have proved that

$$((\mathcal{E} \setminus \mathcal{C}_{sing}) \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} = d\}) = ((\mathcal{E}' \setminus \mathcal{C}') \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} = d\}) \tag{7}$$

Suppose that $\ell \subset \mathcal{C} \setminus (\mathcal{E} \cup \mathcal{C}_{sing} \cup \mathcal{C}')$. Then $\mathcal{N}_\ell$ contains only one element, say P. In view of Lemma 4.1, we must have $T_v\mathcal{C} \subseteq P$. As $\dim_v \mathcal{C} = \dim P$ by the assumption, we get $T_v\mathcal{C} = P$, i.e., $\mathcal{N}_\ell = \{T_v\mathcal{C}\}$.

Now it remains to show that $\dim \mathcal{E} < d$. We have

$$\begin{aligned}
\mathcal{E} &= (\mathcal{E} \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} = d\}) \cup (\mathcal{E} \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} < d\}) \\
&= ((\mathcal{E} \setminus \mathcal{C}_{sing}) \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} = d\}) \cup ((\mathcal{E} \cap \mathcal{C}_{sing} \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} = d\}) \\
&\quad \cup (\mathcal{E} \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} < d\}) \\
&\subset ((\mathcal{E} \setminus \mathcal{C}_{sing}) \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} = d\}) \cup \mathcal{C}_{sing} \cup (\mathcal{E} \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} < d\}) \\
&= ((\mathcal{E}' \setminus \mathcal{C}') \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} = d\}) \cup \mathcal{C}_{sing} \cup (\mathcal{E} \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} < d\}) \\
&\subset \mathcal{E}' \cup \mathcal{C}_{sing} \cup (\mathcal{E} \cap \{u \in \mathcal{C} :\ \dim_u \mathcal{C} < d\}),
\end{aligned}$$

where the last equality follows from (7). Observe that $\dim \mathcal{E}' < d$ by Item (i) and the dimension of the third set is clearly smaller than d. Moreover, in view of Lemma 3.3, we get $\dim \mathcal{C}_{sing} < \dim \mathcal{C} \leqslant \dim_0 X = d$. Hence $\dim \mathcal{E} < d$. This ends the proof of the theorem. $\square$

Remark 4.2. It is not hard to check that $\mathcal{E}' \cap \mathbb{S}^{n-1} = Z$ where Z is defined by (5).

By Lemma 3.5 and Lemma 4.1, the following corollary is straightforward.

Corollary 4.3. *Let $X \subset \mathbb{R}^n$ be a definable set of pure dimension $d > 0$ at the origin $0 \in \mathbb{R}^n$. Let ℓ be a ray in $\mathcal{C}$ and let $\ell_k \subset \mathcal{C} \setminus \mathcal{C}_{sing}$ be a sequence of rays such that $\ell_k \to \ell$. Assume that $T_{v^k}\mathcal{C} \to \mathcal{P}$ for $v^k \in \ell_k$, then there exists $P \in \mathcal{N}_\ell$ such that $\mathcal{P} \subset P$.*

Before proving Theorem 1.2, we need some results of preparation. For each $x \in \mathbb{R}^n \setminus \{0\}$, let ℓ_x denote the open ray emanating from the origin through x, namely, $\ell_x := \{rx :\ r > 0\}$. Given a real number $\delta \in [0, 1]$ and a ray ℓ, let $N_\delta(\ell)$ denote the *δ-conical neighborhood* of ℓ, i.e.,

$$N_\delta(\ell) := \{x \in \mathbb{R}^n \setminus \{0\} :\ \widehat{\ell, \ell_x} \leqslant \frac{\pi}{2} \text{ and } \sin \widehat{\ell, \ell_x} \leqslant \delta\},$$

recall that $\widehat{\ell, \ell_x}$ denotes the angle between ℓ and ℓ_x.

Lemma 4.4. *Assume that X is a definable set of positive pure dimension at 0. Let ℓ be a ray in $\mathcal{C}$. Then there exists a constant $\delta_0 > 0$ and a continuous definable function $h\colon (0, \delta_0) \to \mathbb{R}$ with $h(\delta) \in (0, \delta]$ such that for $\delta \in (0, \delta_0)$, we have:*

(i) *The number of connected components of $(X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}$ is constant.*

(ii) *For each connected component Y of $(X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}$, the intersection $Y \cap N_{\delta'}(\ell) \cap \mathbb{S}^{n-1}_{r'}$ is nonempty and connected for $\delta' \in (0, \delta]$ and $r' \in (0, h(\delta'))$.*

(iii) *If ℓ is not in $\mathcal{C}'$, then $((\overline{X})_{sing} \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)} = \emptyset$.*

(iv) *The Nash fiber of X along ℓ is equal to the Nash fiber of $(X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}$ along ℓ, i.e.,*

$$\mathcal{N}_\ell(X) = \mathcal{N}_\ell((X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}).$$

Proof. Consider the set

$$A := \{(x, \delta, r) \in \mathbb{R}^n \times [0, 1] \times (0, +\infty) : \ x \in (X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathbb{S}^{n-1}_r\}$$

and the projection

$$\mathrm{pr}\colon A \to [0, 1] \times (0, +\infty), \ (x, \delta, r) \mapsto (\delta, r).$$

Let v be an arbitrary point in ℓ. Note that A is a definable set since it can be written in the following form

$$A = \left\{ \begin{array}{ll} (x, \delta, r) \in \mathbb{R}^n \times [0, 1] \times (0, +\infty) : & x \in (X \setminus \{0\}) \cap \mathbb{S}^{n-1}_r, \\ & \mathrm{dist}(x, \ell) \leqslant \delta \|x\|, \\ & \langle x, v \rangle \geqslant 0 \end{array} \right\}$$

where $\mathrm{dist}(\cdot, \cdot)$ still denotes the Euclidean distance. In light of Theorem 2.4, there is a partition of $[0, 1] \times (0, +\infty)$ into disjoint definable sets $B_1, \ldots, B_k$ such that pr is definably trivial over each B_i. Obviously, there is a unique set $B \in \{B_1, \ldots, B_k\}$ such that:

(a) $\dim B = 2$; and

(b) there exists a constant $\delta_0 > 0$ such that $[0, \delta_0] \times \{0\} \subset \overline{B}$.

For $\delta \in (0, \delta_0)$, if there is $r > 0$ such that $\{\delta\} \times (0, r) \cap B = \emptyset$, then set $h(\delta) = 0$; otherwise, set

$$h(\delta) := \sup\{r \in (0, \delta) : \ \{\delta\} \times (0, r) \subset B\}.$$

Clearly h is a definable function. Shrinking δ_0, if necessary, so that either $h(\delta) > 0$ for all $\delta \in (0, \delta_0)$ or $h \equiv 0$ on $(0, \delta_0)$. Observe that the latter can not happen since otherwise, we shall have $[0, \delta_0] \times \{0\} \not\subset \overline{B}$, which is a contradiction.

Now by triviality, Items (i) and (ii) follows.

As $\mathcal{C}'$ is closed and ℓ is not a ray in $\mathcal{C}'$, by shrinking δ_0, we have $N_{\delta_0}(\ell) \cap \mathcal{C}' = \{0\}$. Hence, it is not hard to see that there is a constant $r_0 > 0$ such that

$$((\overline{X})_{sing} \setminus \{0\}) \cap N_{\delta_0}(\ell) \cap \mathring{\mathbb{B}}^n_{r_0} = \emptyset.$$

By shrinking δ_0, if necessary, so that $\delta_0 < r_0$, Item (iii) follows.

Let $P \in \mathcal{N}_\ell(X)$. By Lemma 3.6, there exists a C^1 definable curve $\gamma\colon (0, \varepsilon) \to X \setminus X_{sing}$, such that

$$\|\gamma(t)\| = t \text{ for } t \in (0, \varepsilon), \ \lim_{t \to 0^+} \frac{\gamma(t)}{\|\gamma(t)\|} = v \in \ell \ \text{ and } \ \lim_{t \to 0^+} T_{\gamma(t)} X = P.$$

Shrinking ε so that $\varepsilon \in (0, h(\delta))$ and $\dfrac{\gamma(t)}{\|\gamma(t)\|} \in N_\delta(\ell)$. Clearly $\gamma(t) \in (X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}$ for all $t \in (0, \varepsilon)$. Therefore $P \in \mathcal{N}_\ell((X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)})$. Consequently

$$\mathcal{N}_\ell(X) \subset \mathcal{N}_\ell((X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}).$$

Now Item (iv) follows from observing that the inclusion $\mathcal{N}_\ell((X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}) \subset \mathcal{N}_\ell(X)$ is trivial. $\square$

Lemma 4.5. *With the notation in Lemma 4.4, let ℓ be a ray in $\mathcal{C} \setminus \mathcal{C}'$ and let Y be a connected component of $(X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}$, where $\delta \in (0, \delta_0)$. Then $\mathcal{N}_\ell(Y)$ is connected.*

Proof. Without loss of generality, we may suppose that $(X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}$ is connected, so

$$Y = (X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}.$$

Furthermore, by Lemma 4.4(iv), $\mathcal{N}_\ell(Y) = \mathcal{N}_\ell(X) \subset \mathbb{G}(n-1, n)$. Assume for contradiction that $\mathcal{N}_\ell(Y)$ is not connected. Let M_1 and M_2 be two distinct connected components of $\mathcal{N}_\ell(X)$ which are clearly closed sets in $\mathbb{G}(n-1, n)$. Let $\eta > 0$ be such that

$$U_\eta(M_1) \cap (\mathcal{N}_\ell \setminus M_1) = \emptyset, \tag{8}$$

where $U_\eta(M_1)$ is the closed neighborhood of radius η of M_1, namely,

$$U_\eta(M_1) = \{P \in \mathbb{G}(n-1, n) : \text{ there is } Q \in M_1 \text{ such that } \angle(P, Q) \leqslant \eta\}.$$

Let $P_1 \in M_1$ and $P_2 \in M_2$. By Lemma 3.6, there exist two C^1 definable curves

$$\gamma_1, \gamma_2\colon (0, \varepsilon) \to Y$$

such that:

(a) $\|\gamma_1(t)\| = \|\gamma_2(t)\| = t$ for $t \in (0, \varepsilon)$;
(b) $\gamma_1(t) \cap X_{sing} = \emptyset$ and $\gamma_2(t) \cap X_{sing} = \emptyset$ for $t \in (0, \varepsilon)$;
(c) $\displaystyle\lim_{t \to 0^+} \frac{\gamma_1(t)}{\|\gamma_1(t)\|} = \lim_{t \to 0^+} \frac{\gamma_2(t)}{\|\gamma_2(t)\|} = v \in \ell$;
(d) $\displaystyle\lim_{t \to 0^+} T_{\gamma_1(t)} X = P_1$ and $\displaystyle\lim_{t \to 0^+} T_{\gamma_2(t)} X = P_2$.

Shrinking ε so that $\varepsilon \leqslant h(\delta)$ and let

$$g(t) := \max \left\{ \sup_{0<t'\leqslant t} \sin \widehat{\ell, \ell_{\gamma_1(t')}}, \sup_{0<t'\leqslant t} \sin \widehat{\ell, \ell_{\gamma_2(t')}} \right\}.$$

Clearly

$$g(t) \to 0 \text{ as } t \to 0^+ \quad \text{and} \quad g(t) \leqslant \delta \text{ for } t \in (0, h(\delta)). \tag{9}$$

The latter, together with the assumption that $Y = (X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}$ is connected and Lemma 4.4(ii), implies that, for $t \in (0, \varepsilon)$, the set $(X \setminus \{0\}) \cap N_{g(t)}(\ell) \cap \mathbb{S}^{n-1}_{b(t)}$ is also connected, where

$$b(t) = \frac{\min\{h(g(t)), h(\delta)\}}{2}. \tag{10}$$

Hence there exists a continuous curve

$$\alpha_t \colon [0,1] \to (X \setminus \{0\}) \cap N_{g(t)}(\ell) \cap \mathbb{S}^{n-1}_{b(t)}, \; s \mapsto \alpha_t(s),$$

such that $\alpha_t(0) = \gamma_1(b(t))$ and $\alpha_t(1) = \gamma_2(b(t))$. Obviously, the mapping

$$s \mapsto \beta_t(s) := T_{\alpha_t(s)} X$$

is continuous. By Item (d), for $t > 0$ small enough, we have

$$\beta_t(0) \in U_\eta(M_1) \quad \text{and} \quad \beta_t(1) \notin U_\eta(M_1).$$

In particular, these hold for $t = \frac{1}{k}$ with $k \in \mathbb{N}$ large enough. By continuity, for all such k, there is $s_k \in (0,1)$ such that $\beta_{\frac{1}{k}}(s_k) \in \partial U_\eta(M_1)$. By the compactness of $\partial U_\eta(M_1)$, which follows from the closedness of $\partial U_\eta(M_1)$ and the compactness of $\mathbb{G}(n-1,n)$, the sequence $T_{\alpha_{\frac{1}{k}}(s_k)} X = \beta_{\frac{1}{k}}(s_k)$ has an accumulation point in $\partial U_\eta(M_1)$, say P. From the definition of $g(t)$, α_t, (9) and (10), we have

$$\sin \widehat{\ell, \ell_{\alpha_{\frac{1}{k}}(s_k)}} \leqslant g\left(\frac{1}{k}\right) \to 0 \text{ and } \|\alpha_{\frac{1}{k}}(s_k)\| = b\left(\frac{1}{k}\right) \leqslant \frac{h(g(\frac{1}{k}))}{2} \leqslant \frac{g(\frac{1}{k})}{2} \to 0 \text{ as } k \to +\infty.$$

Thus, $P \in \mathcal{N}_\ell$. Therefore

$$\emptyset \neq \partial U_\eta(M_1) \cap \mathcal{N}_\ell = \partial U_\eta(M_1) \cap (\mathcal{N}_\ell \setminus M_1) \subset U_\eta(M_1) \cap (\mathcal{N}_\ell \setminus M_1).$$

This contradicts (8) and so ends the proof of the lemma. $\square$

We also need the following key lemma which relates the differential of the distance function between two disjoint non singular hypersurfaces with the angle between the corresponding tangent hyperplanes.

Lemma 4.6. *Let $Y, Z \subset \mathbb{R}^n$, with $n \geqslant 2$, be two disjoint non singular hypersurfaces. Define the function*

$$\widetilde{\rho} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \; (y,z) \mapsto \|y - z\| = \sqrt{\sum_{i=1}^n (y_i - z_i)^2}.$$

For any $y \in Y$ and $z \in Z$, the following statements hold:

(i) *If (y, z) is a critical point of the restriction $\widetilde{\rho}|_{Y\times Z}$, then $\angle(T_yY, T_zZ) = 0$.*

(ii) *For each $(y, x) \in Y \times Z$, we have*

$$\|\nabla(\widetilde{\rho}|_{Y\times Z})(y, z)\| \geqslant \frac{\sin\angle(T_yY, T_zZ)}{\sqrt{2}},$$

where $\nabla(\widetilde{\rho}|_{Y\times Z})$ denotes the gradient of $\widetilde{\rho}|_{Y\times Z}$.

Proof. (i) Assume that (y, z) is a critical point of $\widetilde{\rho}|_{Y\times Z}$, i.e.,

$$d_{(y,z)}\widetilde{\rho}(T_yY \times T_zZ) = 0.$$

Hence, for any $(a, 0) \in T_yY \times \{0\} \subset T_yY \times T_zZ$, we have

$$0 = d_{(y,z)}\widetilde{\rho}(a, 0) = \left\langle \frac{(y - z, z - y)}{\|y - z\|}, (a, 0) \right\rangle = \frac{\langle y - z, a\rangle}{\|y - z\|}.$$

Consequently, $y - z \perp T_yY$. Similarly, we have also $y - z \perp T_zZ$. Therefore, Item (i) follows from remarking that $\dim T_yY = \dim T_zZ = n - 1$ by assumption.

(ii) Since the statement is clear if $\angle(T_yY, T_zZ) = 0$, let us suppose that $\angle(T_yY, T_zZ) \neq 0$. Set

$$W = (\{y\} + T_yY) \cap (\{z\} + T_zZ).$$

Note that T_yY and T_zZ are of dimension $n - 1$, so $W \neq \emptyset$. We consider two cases:

Case 1: $y, z \in W$. We have

$$\nabla\widetilde{\rho}(y, z) = \frac{(y - z, z - y)}{\|y - z\|} \in T_yY \times T_zZ.$$

This, together with the fact that $\nabla(\widetilde{\rho}|_{Y\times Z})(y, z)$ is the orthogonal projection of $\nabla\widetilde{\rho}(y, z)$ on $T_yY \times T_zZ$, implies

$$\|\nabla(\widetilde{\rho}|_{Y\times Z})(y, z)\| = \|\nabla\widetilde{\rho}(y, z)\| = \left\| \frac{(y - z, z - y)}{\|y - z\|} \right\| = \sqrt{2} > \frac{\sin\angle(T_yY, T_zZ)}{\sqrt{2}}.$$

Case 2: $y \notin W$ or $z \notin W$. Assume with no loss of generality that $y \notin W$. Let w be the orthogonal projection of y on W. We denote by L_y the line through w and y, and by L_z the line through w and z if $w \neq z$. If $w = z$, we let L_z be any line in W through z. It is clear that

$$\angle(L_y, L_z) \geqslant \angle(T_yY, T_zZ).$$

Furthermore, since $\widetilde{L}_y \times \widetilde{L}_z$ is a subspace of $T_yY \times T_zZ$, where

$$\widetilde{L}_y := \{-y\} + L_y \quad \text{and} \quad \widetilde{L}_z := \{-z\} + L_y,$$

we get

$$\|\nabla(\widetilde{\rho}|_{Y\times Z})(y, z)\| = \|\mathrm{pr}_{T_yY\times T_zZ}\nabla\widetilde{\rho}(y, z)\| \geqslant \|\mathrm{pr}_{\widetilde{L}_y\times\widetilde{L}_z}\nabla\widetilde{\rho}(y, z)\|,$$

where $\mathrm{pr}_{T_yY\times T_zZ}$ and $\mathrm{pr}_{\widetilde{L}_y\times\widetilde{L}_z}$ are respectively the orthogonal projections on $T_yY\times T_zZ$ and $\widetilde{L}_y\times\widetilde{L}_z$. So to prove the statement in this case, it is enough to show that

$$\|\mathrm{pr}_{\widetilde{L}_y\times\widetilde{L}_z}\nabla\widetilde{\rho}(y,z)\| \geqslant \frac{\sin\angle(L_y,L_z)}{\sqrt{2}}. \tag{11}$$

Let y' and z' be the orthogonal projection of y and z on L_z and L_y, respectively. Then it is clear that

$$\begin{aligned}\|\mathrm{pr}_{\widetilde{L}_y\times\widetilde{L}_z}\nabla\widetilde{\rho}(y,z)\| &= \left\|\mathrm{pr}_{\widetilde{L}_y\times\widetilde{L}_z}\frac{(y-z,z-y)}{\|y-z\|}\right\| \\ &= \frac{\left\|(\mathrm{pr}_{\widetilde{L}_y}(y-z),\mathrm{pr}_{\widetilde{L}_z}(z-y))\right\|}{\|y-z\|} \\ &= \frac{\sqrt{\|y-z'\|^2+\|z-y'\|^2}}{\|y-z\|}\ \text{(see Figure 1)}.\end{aligned} \tag{12}$$

If $z=z'$, then we have

$$\|\mathrm{pr}_{\widetilde{L}_y\times\widetilde{L}_z}\nabla\widetilde{\rho}(y,z)\| \geqslant \frac{\|y-z'\|}{\|y-z\|} = \frac{\|y-z\|}{\|y-z\|} = 1,$$

and (11) follows. So suppose that $z\neq z'$. Remark that $y\neq y'$ as $y\notin W$ and $y'\in W$. Denote by x the intersection point of the line through y,y' and the line through z,z'. Then it is not hard to check that (see Figure 1)

$$\|y-z'\| = \|y-x\|\sin\angle(L_y,L_z) \quad \text{and} \quad \|z-y'\| = \|z-x\|\sin\angle(L_y,L_z).$$

This and (12) together yield

$$\begin{aligned}\|\mathrm{pr}_{\widetilde{L}_y\times\widetilde{L}_z}\nabla\widetilde{\rho}(y,z)\| &= \frac{\sqrt{\|y-x\|^2\sin^2\angle(L_y,L_z)+\|z-x\|^2\sin^2\angle(L_y,L_z)}}{\|y-z\|} \\ &= \frac{\sqrt{\|y-x\|^2+\|z-x\|^2}\sin\angle(L_y,L_z)}{\|y-z\|} \\ &\geqslant \frac{\sqrt{(\|y-x\|+\|z-x\|)^2}\sin\angle(L_y,L_z)}{\sqrt{2}\|y-z\|} \\ &\geqslant \frac{\|y-z\|\sin\angle(L_y,L_z)}{\sqrt{2}\|y-z\|} = \frac{\sin\angle(L_y,L_z)}{\sqrt{2}},\end{aligned}$$

which proves (11) and so the lemma follows. $\square$

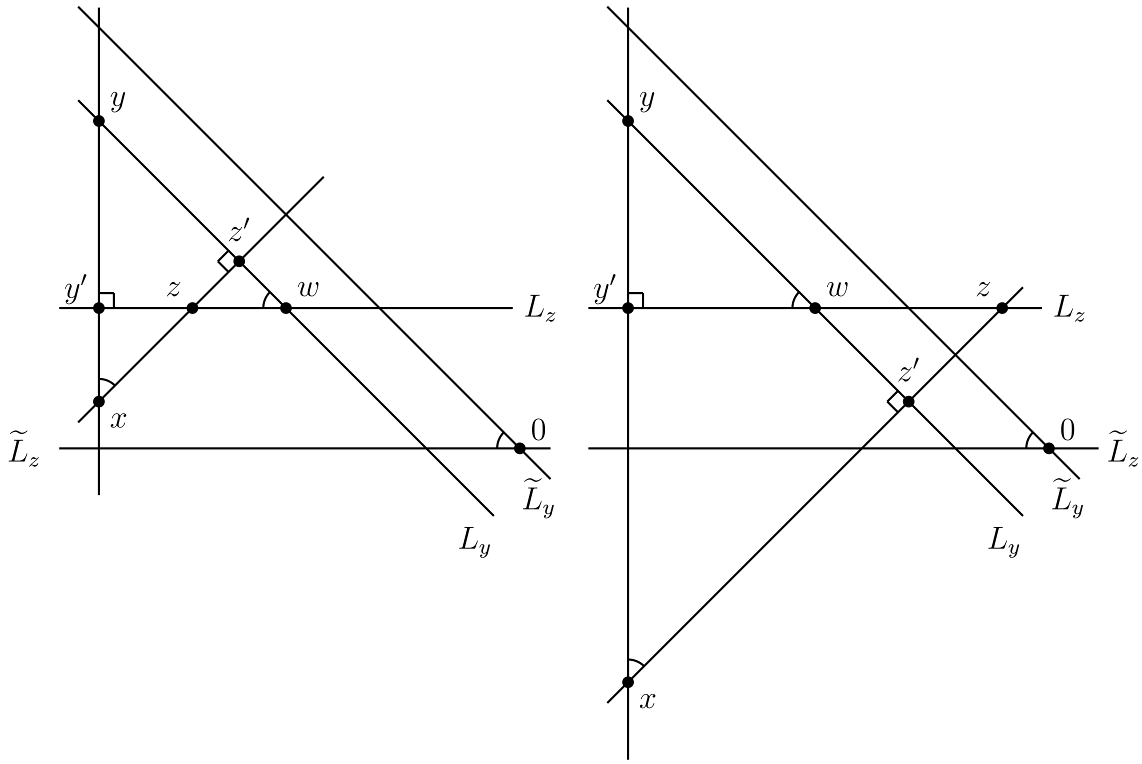


Figure 1.

Now we are in position to prove Theorem 1.2.

Proof of Theorem 1.2. In view of Lemma 3.4, we may assume that X is closed. Let ℓ be any ray in $\mathcal{E}$ with the unit direction v. It is not hard to show that $\mathcal{N}_\ell$ is a closed definable set, so we leave it to the reader. Note that if $\mathcal{N}_\ell$ is connected, then it is path connected and so is of positive dimension as $\#(\mathcal{N}_\ell) > 1$, so it remains to prove the connectedness. Let $X_i = X_i(\delta)$ with $i = 1, \ldots, m,\ \delta \in (0, \delta_0)$, be the connected components of $(X \setminus \{0\}) \cap N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)}$, where δ_0 and $h(\delta)$ are the constants in Lemma 4.4. Then it is not hard to see that $\ell \subset C_0(X_i)$ for any i. Let $\mathcal{N}_\ell(X_i)$ be the Nash fiber of X_i over 0 along ℓ. Obviously

$$\mathcal{N}_\ell = \bigcup_{i=1}^{m} \mathcal{N}_\ell(X_i)$$

in light of Lemma 4.4(iv). Furthermore, $\mathcal{N}_\ell(X_i)$ is connected in view of Lemma 4.5. Hence, to prove that $\mathcal{N}_\ell$ is connected, it is enough to show that

$$\mathcal{N}_\ell(X_i) \cap \mathcal{N}_\ell(X_j) \neq \emptyset \quad \text{for} \quad i \neq j.$$

In order to do this, let $Y, Z \in \{X_i,\ i = 1, \ldots, m\}$ with $Y \neq Z$. Shrinking δ if necessary so that $\delta < 1$. By the construction, for $0 < r < \frac{h(\delta)}{2}$, we have $\mathring{\mathbb{B}}^n_{r\delta}(rv) \subset N_\delta(\ell) \cap \mathring{\mathbb{B}}^n_{h(\delta)} \setminus \{0\}$, recall that $\mathring{\mathbb{B}}^n_{r\delta}(rv)$ is the open ball of radius $r\delta$ centered at rv. Set

$$Y_r = Y \cap \mathring{\mathbb{B}}^n_{r\delta}(rv) \quad \text{and} \quad Z_r = Z \cap \mathring{\mathbb{B}}^n_{r\delta}(rv).$$

Clearly Y_r and Z_r are non singular hypersurfaces in view of Lemma 4.4(iii). Hence we can define the following function θ given by

$$\theta(r) := \inf_{y\in Y_r, z\in Z_r} \sin\angle(T_yY_r, T_zZ_r) = \max\{\|u - \pi_{T_yY_r}(u)\| :\ u \in T_zZ_r \cap \mathbb{S}^{n-1}\},$$

recall that $\pi_{T_yY_r}$ is the orthogonal projection on T_yY_r. Then θ is a non negative definable function. Assume that we have proved

$$\lim_{r\to 0^+} \theta(r) = 0. \tag{13}$$

This, of course, implies that $\mathcal{N}_\ell(Y) \cap \mathcal{N}_\ell(Z) \neq \emptyset$ by shrinking δ to 0. So it remains to prove (13). For contradiction, suppose that $\theta(r) \not\to 0$. Then there are some positive constants $\varepsilon < \frac{h(\delta)}{2}$ and $\theta_0 < \frac{\pi}{2}$ such that $\theta(r) \geqslant \theta_0$ for $r \in (0, \varepsilon)$.

On the other hand, since ℓ is a ray in $C_0(Y)$ and $C_0(Z)$, by Lemma 3.6, shrinking ε if necessary, there are two C^1 definable curves $y\colon (0,\varepsilon) \to Y$ and $z\colon (0,\varepsilon) \to Z$ such that

$$\|y(r)\| = \|z(r)\| = r, \quad \left\|\frac{y(r)}{\|y(r)\|} - v\right\| \to 0 \ \text{ and } \ \left\|\frac{z(r)}{\|z(r)\|} - v\right\| \to 0 \ \text{ as } \ r \to 0^+.$$

So

$$\frac{\|y(r) - rv\|}{r} = \left\|\frac{y(r)}{\|y(r)\|} - v\right\| \to 0 \ \text{ and } \ \frac{\|z(r) - rv\|}{r} = \left\|\frac{z(r)}{\|z(r)\|} - v\right\| \to 0 \ \text{ as } \ r \to 0^+.$$

Consequently, $y(r) \in Y_r$ and $z(r) \in Z_r$ for any $r > 0$ small enough. Fix $r > 0$ such that

$$\frac{\|y(r) - rv\|}{r} < \frac{\delta\theta_0}{4\sqrt{2}} \quad \text{and} \quad \frac{\|z(r) - rv\|}{r} < \frac{\delta\theta_0}{4\sqrt{2}}. \tag{14}$$

Let

$$\gamma\colon [0, t_0) \to Y_r \times Z_r,\ t \mapsto (\alpha(t), \beta(t))$$

be the maximal trajectory of the vector field $-\frac{\nabla\rho}{\|\nabla\rho\|^2}$ on $Y_r \times Z_r$ with the initial condition $\gamma(0) = (y(r), z(r))$, where ρ is the restriction of the function $\widetilde{\rho}$, defined in Lemma 4.6, on $Y_r \times Z_r$. The following facts are easy to verify:

(a) $\rho(\gamma(t)) = \|y(r) - z(r)\| - t$ for $t \in [0, t_0)$ and so $t_0 < \|y(r) - z(r)\|$; and

(b) $\|\nabla\rho(\gamma(s))\| \geqslant \frac{\theta_0}{\sqrt{2}}$ for $t \in [0, t_0)$. (This follows from Lemma 4.6(ii) and the assumption $\theta(r) \geqslant \theta_0 > 0$.)

Moreover we have

$$\begin{array}{rcl} \text{length}(\gamma) &=& \displaystyle\int_0^{t_0} \|\dot\gamma(t)\| dt &=& \displaystyle\int_0^{t_0} \frac{1}{\|\nabla\rho(\gamma(t))\|} dt \\ &\leqslant& \displaystyle\int_0^{t_0} \frac{\sqrt{2}}{\theta_0} dt &=& \displaystyle\frac{\sqrt{2}t_0}{\theta_0} \\ &<& \displaystyle\frac{\sqrt{2}\|y(r) - z(r)\|}{\theta_0} &\leqslant& \displaystyle\sqrt{2}\frac{\|y(r) - rv\| + \|z(r) - rv\|}{\theta_0}. \end{array}$$

This, together with (14), yields $\operatorname{length}(\gamma) < \frac{r\delta}{2}$, which implies that the limit

$$(\alpha_{t_0}, \beta_{t_0}) = \gamma_{t_0} := \lim_{t \to t_0} \gamma(t)$$

exists. Clearly $\gamma_{t_0} \in \overline{Y}_r \times \overline{Z}_r$. In addition, we have

$$\begin{aligned} \|\alpha_{t_0} - rv\| \leqslant \|\alpha_{t_0} - y(r)\| + \|y(r) - rv\| &\leqslant \operatorname{length}(\gamma) + \|y(r) - rv\| \\ &< \frac{r\delta}{2} + \frac{r\delta\theta_0}{4\sqrt{2}} < r\delta. \end{aligned}$$

Similarly, we also have $\|\beta_{t_0} - rv\| < r\delta$. Furthermore, as X is closed, it follows that

$$\overline{Y}_r \setminus Y_r \subset \mathbb{S}^{n-1}_{r\delta}(rv) \ \text{ and } \ \overline{Z}_r \setminus Z_r \subset \mathbb{S}^{n-1}_{r\delta}(rv).$$

Therefore $\gamma_{t_0} \in Y_r \times Z_r$. Recall that γ is maximal, so γ_{t_0} is a critical point of ρ. In light of Lemma 4.6, $\angle(T_{\alpha_{t_0}} Y_r, T_{\beta_{t_0}} Z_r) = 0$. Consequently, $\theta(r) = 0$ which contradicts the assumption. Therefore, (13) must hold and the theorem follows. $\square$

Let us make some preparation before proving Theorem 1.3. First of all, we need the following technical lemma.

Lemma 4.7. *Let $Y \subset \mathbb{R}^n$ be a definable set and $v \in Y$. Assume that ℓ^* is a ray not in C_vY. Then, for any $R > 0$, there exists a positive constant $\delta_R \leqslant \frac{1}{R}$ such that:*

(i) *for any $w \in C_vY$ with $\|w\| \leqslant \delta_R$, we have* $\operatorname{dist}(v + w, Y) < \frac{\|w\|}{R}$;

(ii) *for any $x \in Y$ with $\|x - v\| \leqslant \delta_R$, we have* $\operatorname{dist}(x - v, C_vY) < \frac{\|x - v\|}{R}$;

(iii) *for all $t \in (0, \delta_R)$, we have*

$$t\left(\sin\theta - \frac{1}{R}\right) < \operatorname{dist}(v + tp, Y) < t\left(\sin\theta + \frac{1}{R}\right),$$

where p is the unit direction in ℓ^* and

$$\theta := \min\left\{\frac{\pi}{2}, \min_{\ell' \subset C_vY} \widehat{\ell^*, \ell'}\right\} > 0, \tag{15}$$

recall that $\widehat{\ell^*, \ell'}$ is the angle between the rays ℓ^* and ℓ'.

Proof. (i) Suppose for contradiction that there is $R > 0$ such that for any integer $k > 0$, there is $w^k \in C_vY$ with $\|w^k\| \leqslant \frac{1}{k}$ such that $\operatorname{dist}(v + w^k, Y) \geqslant \frac{\|w^k\|}{R}$. Taking a subsequence if necessary, we may suppose that the sequence $\frac{w^k}{\|w^k\|}$ converges to a limit w. By Lemma 3.6, there is a C^1 definable curve $\gamma : (0, \varepsilon) \to C_vY \setminus \{0\}$ such that:

(a) $\|\gamma(t)\| = t$ for $t \in (0, \varepsilon)$;

(b) $\operatorname{dist}(v + \gamma(t), Y) \geqslant \frac{\|\gamma(t)\|}{R} = \frac{t}{R}$ for $t \in (0, \varepsilon)$; and

(c) $\lim_{t\to 0^+} \frac{\gamma(t)}{\|\gamma(t)\|} = w.$

Evidently $w \in C_vY$. So by the definition of tangent cone and by Lemma 3.6, there is a C^1 definable curve $\alpha : (0, \varepsilon') \to Y \setminus \{v\}$ such that:

(d) $\|\alpha(t) - v\| = t$ for $t \in (0, \varepsilon')$; and

(e) $\lim_{t\to 0^+} \frac{\alpha(t) - v}{\|\alpha(t) - v\|} = w.$

Now we have

$$\begin{aligned} \operatorname{dist}(v + \gamma(t), Y) &\leqslant \|v + \gamma(t) - \alpha(t)\| \\ &\leqslant \|\gamma(t) - tw\| + \|v - \alpha(t) + tw\| \\ &= t\left(\left\|\frac{\gamma(t)}{\|\gamma(t)\|} - w\right\| + \left\|\frac{\alpha(t) - v}{\|\alpha(t) - v\|} - w\right\|\right). \end{aligned}$$

This, together with Item (c) and Item (e), implies that $\lim_{t\to 0^+} \frac{\operatorname{dist}(v + \gamma(t), Y)}{t} = 0$ which contradicts Item (b). Hence, Item (i) follows.

(ii) By contradiction, suppose that there is $R > 0$ such that for any integer $k > 0$, there is $x^k \in Y$ with $\|x^k - v\| \leqslant \frac{1}{k}$ such that $\operatorname{dist}(x^k - v, C_vY) \geqslant \frac{\|x^k - v\|}{R}$, i.e.,

$$\operatorname{dist}\left(\frac{x^k - v}{\|x^k - v\|}, C_vY\right) \geqslant \frac{1}{R}.$$

Taking a subsequence if necessary, we may suppose that the sequence $\frac{x^k - v}{\|x^k - v\|}$ converges to a limit w. Then, clearly, $w \in C_vY$ on one hand and we have $\operatorname{dist}(w, C_vY) \geqslant \frac{1}{R}$ on the other hand. This is a contradiction and Item (ii) follows.

(iii) By Item (i), for each R, there exists $\delta_R > 0$ such that for any $w \in C_vY$ with $\|w\| \leqslant \delta_R$, we have $\operatorname{dist}(v + w, Y) < \frac{\|w\|}{R}$. Since C_vY is closed, there is $\widetilde{w} \in C_vY$ such that the distance function $\operatorname{dist}(tp, C_vY)$ is attained, i.e.,

$$\operatorname{dist}(tp, C_vY) = \|tp - \widetilde{w}\| = t\sin\theta.$$

Then clearly $\|\widetilde{w}\| = t\cos\theta < t < \delta_R$. In addition,

$$\|tp - \widetilde{w}\| - \operatorname{dist}(v + \widetilde{w}, Y) \leqslant \operatorname{dist}(v + tp, Y) \leqslant \|tp - \widetilde{w}\| + \operatorname{dist}(v + \widetilde{w}, Y).$$

Thus, in view of Item (i), we get

$$t\sin\theta - \frac{t\cos\theta}{R} \leqslant \operatorname{dist}(v + tp, Y) \leqslant t\sin\theta + \frac{t\cos\theta}{R},$$

which implies Item (iii). $\square$

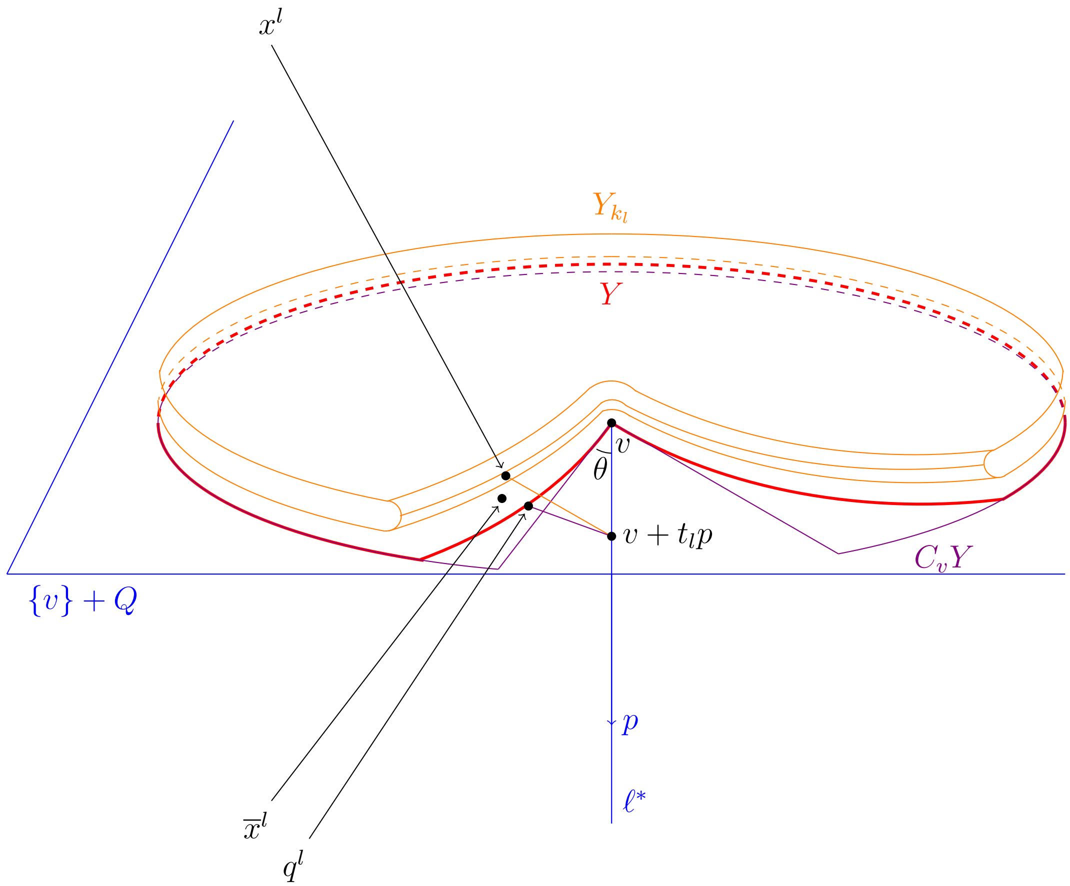
The following lemma is the key to prove Theorem 1.3.

Lemma 4.8. *Let $Y \subset \mathbb{R}^n$ be a closed definable set and $v \in Y$. Let Y_k be a sequence of closed definable sets of pure dimension $d > 0$ for all integer $k > 0$ such that*

$$Y = \lim_{k \to +\infty} Y_k \quad \textit{and} \quad v \notin \lim_{k \to +\infty} (Y_k)_{sing}.$$

Assume that $y^k \in Y_k \setminus (Y_k)_{sing}$ is a sequence tending to v such that the limit $Q := \lim_{k \to +\infty} T_{y^k} Y_k$ exists and $C_v Y \subsetneq Q$. Then there is a subsequence $\{k_1, k_2, k_3 \dots\}$ of the sequence $\{1, 2, 3, \dots\}$ and a sequence $x^l \in Y_{k_l} \setminus (Y_{k_l})_{sing}$ tending to v such that the limit $P := \lim_{l \to +\infty} T_{x^l} Y_{k_l}$ exists and $\angle(P, Q) = \dfrac{\pi}{2}$.

Proof. The construction in the proof is described in Figure 2 below.



Y is the region with red boundary
$C_v Y$ is the region with violet boundary

Figure 2.

By the assumption, there exists a ray ℓ^* in Q which is not a ray in $C_v Y$. Let $\theta > 0$ be the constant given by (15). For each integer $l > 0$, let $\delta_l \in (0, \frac{1}{l}]$ be the constant given by Lemma 4.7. By the assumption $\lim_{k \to +\infty} Y_k = Y$, for each integer $l > 0$, there is an integer

$k_l > 0$ such that

$$\mathrm{dist}_{\mathcal{H}}(Y_{k_l}, Y) < \frac{\delta_l}{l}, \tag{16}$$

where $\mathrm{dist}_{\mathcal{H}}(\cdot, \cdot)$ still denotes the Hausdorff distance. Let $t_l = \dfrac{\delta_l}{3}$. Then for l large enough, we have

$$\begin{array}{rcl} \mathrm{dist}(v + t_l p, Y_{k_l}) & \geqslant & \mathrm{dist}(v + t_l p, Y) - \mathrm{dist}(Y, Y_{k_l}) \\ & > & t_l \left(\sin\theta - \dfrac{1}{l}\right) - \dfrac{\delta_l}{l} \\ & = & t_l \left(\sin\theta - \dfrac{1}{l}\right) - \dfrac{3t_l}{l} = t_l \left(\sin\theta - \dfrac{4}{l}\right) > 0, \end{array} \tag{17}$$

where the second inequality follows from Lemma 4.7(iii) and (16). Hence $v + t_l p \notin Y_{k_l}$. By the closedness of Y and Y_{k_l}, let $x^l \in Y_{k_l}$ and $q^l \in Y$ be respectively points where the distance functions $\mathrm{dist}(v + t_l p, Y_{k_l})$ and $\mathrm{dist}(v + t_l p, Y)$ are attained. Then for l large enough, we have

$$\begin{array}{rcl} \|x^l - v\| & \leqslant & \|x^l - v - t_l p\| + t_l \\ & = & \mathrm{dist}(v + t_l p, Y_{k_l}) + t_l \\ & \leqslant & \mathrm{dist}(v + t_l p, q^l) + \mathrm{dist}(q^l, Y_{k_l}) + t_l \\ & \leqslant & \mathrm{dist}(v + t_l p, Y) + \sup\limits_{x \in Y} \mathrm{dist}(x, Y_{k_l}) + t_l \\ & \leqslant & \mathrm{dist}(v + t_l p, Y) + \mathrm{dist}_{\mathcal{H}}(Y, Y_{k_l}) + t_l \\ & < & \dfrac{\delta_l}{3} \left(\sin\theta + \dfrac{1}{l}\right) + \dfrac{\delta_l}{l} + \dfrac{\delta_l}{3} \leqslant \delta_l \left(\dfrac{2}{3} + \dfrac{4}{3l}\right) < \delta_l \leqslant \dfrac{1}{l}, \end{array} \tag{18}$$

where the fifth inequality follows from Lemma 4.7(iii) and (16). Consequently

$$\lim_{l \to +\infty} x^l = v. \tag{19}$$

Let $z^l \in Y$ be such that

$$\mathrm{dist}(x^l, Y) = \|x^l - z^l\|.$$

Obviously $\mathrm{dist}(x^l, Y) \leqslant \|x^l - v\|$, so in view of (18), we have

$$\|z^l - v\| \leqslant \|z^l - x^l\| + \|x^l - v\| \leqslant 2\|x^l - v\| < 2\delta_l. \tag{20}$$

Then

$$\begin{array}{rcl} \mathrm{dist}(x^l, \{v\} + Q) & \leqslant & \mathrm{dist}(x^l, \{v\} + C_v Y) \\ & \leqslant & \mathrm{dist}(x^l, Y) + \mathrm{dist}(z^l, \{v\} + C_v Y) \\ & = & \mathrm{dist}(x^l, Y) + \mathrm{dist}(z^l - v, C_v Y) \\ & \leqslant & \mathrm{dist}_{\mathcal{H}}(Y_{k_l}, Y) + \mathrm{dist}(z^l - v, C_v Y) \\ & \leqslant & \dfrac{\delta_l}{l} + \dfrac{\|z^l - v\|}{l} < \dfrac{3\delta_l}{l}, \end{array} \tag{21}$$

where the first inequality follows from the assumption $C_v Y \subset Q$, the forth inequality follows from Lemma 4.7(ii) and (16) while the fifth one follows from (20). Let $\overline{x}^l$ be the orthogonal projection of x^l on $\{v\}+Q$, then by (17) and (21), we have

$$\frac{\|x^l-\overline{x}^l\|}{\|v+t_lp-x^l\|}<\frac{3\delta_l}{lt_l\Big(\sin\theta-\frac{4}{l}\Big)}=\frac{9}{l\Big(\sin\theta-\frac{4}{l}\Big)}\to 0\quad\text{as}\quad l\to+\infty. \tag{22}$$

By taking a subsequence if necessary, we may assume that the sequence $\dfrac{v+t_lp-x^l}{\|v+t_lp-x^l\|}$ converges to a limit w. Then from (22), it is not hard to check that

$$\lim_{l\to+\infty}\frac{v+t_lp-\overline{x}^l}{\|v+t_lp-\overline{x}^l\|}=w.$$

Since $v+t_lp,\overline{x}^l\in\{v\}+Q$, we have

$$\angle(v+t_lp-\overline{x}^l,Q)=\angle(v+t_lp-\overline{x}^l,\{v\}+Q)=0.$$

Hence $\angle(w,Q)=0$. By the assumption $v\notin\lim\limits_{l\to+\infty}(Y_{k_l})_{sing}$ and (19), it is clear that x^l is not a singular point of Y_{k_l} for l large enough. Taking a subsequence if necessary, we can assume that there exists the limit

$$P:=\lim_{l\to+\infty}T_{x^l}Y_{k_l}.$$

Observe that $v+t_lp-x^l$ is perpendicular to $T_{x^l}Y_{k_l}$ as x^l is a point where the distance function $\operatorname{dist}(v+t_lp,Y_{k_l})$ is attained, so by taking limit as $l\to+\infty$, we get $\angle(w,P)=\dfrac{\pi}{2}$. Hence $\angle(P,Q)=\dfrac{\pi}{2}$ and the lemma follows. □

We finish the section by giving the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $Q\in\mathbb{G}(d,n)$ be a tangent limit of X along ℓ. There are two cases to be considered.

Case 1*: $C_v\mathcal{C}\not\subset Q$, i.e., $C_v\mathcal{C}\setminus Q\neq\emptyset$.* Let $\widetilde{\ell}$ be a ray in $C_v\mathcal{C}\setminus Q$. As $\mathcal{C}_{sing}$ is nowhere dense in $\mathcal{C}$, it is not hard to see that there is a sequence $v^k\in\mathcal{C}\setminus\mathcal{C}_{sing}$ such that

$$v^k\to v,\ \frac{v^k-v}{\|v^k-v\|}\to u\in\widetilde{\ell}\ \text{ and }\ T_{v^k}\mathcal{C}\to\mathcal{P}\text{ as }k\to+\infty.$$

In light of Lemma 3.6, $\widetilde{\ell}\subset\mathcal{P}$. Since $\widetilde{\ell}\not\subset Q$, obviously $\mathcal{P}\not\subset Q$. Denote by ℓ_k the ray in $\mathcal{C}$ through v^k. It is clear that

$$\ell_k\subset\mathcal{C}\setminus\mathcal{C}_{sing}\text{ for any }k\ \text{ and }\ \ell_k\to\ell:=\mathbb{R}_+v\text{ as }k\to+\infty.$$

By Corollary 4.3, there exists $P\in\mathcal{N}_\ell$ such that $\mathcal{P}\subset P$. As $\mathcal{P}\not\subset Q$, it follows that $P\neq Q$ and so $\#(\mathcal{N}_\ell)>1$.

Case 2: $C_v\mathcal{C} \subset Q$. By the assumption, we have $C_v\mathcal{C} \neq Q$. We will show that there is a tangent limit P of X along ℓ such that $\angle(P,Q) = \frac{\pi}{2}$, which yields the theorem. By the definition, there is a sequence $z^k \in X \setminus X_{sing}$ and a sequence $t_k \in (0,+\infty)$ such that

$$\lim_{k\to+\infty} z^k = 0, \ \lim_{k\to+\infty} t_k z^k = v \ \text{ and } \ Q = \lim_{k\to+\infty} T_{z^k}X.$$

Since X is of pure dimension d at 0 and since $t_k \to +\infty$ as $k \to +\infty$, there is $\delta \in (0,1)$ such that $X \cap \mathring{\mathbb{B}}^n_{\frac{\delta}{t_k}}(\frac{v}{t_k})$ is of pure dimension d for k large enough. For such k, set

$$X_k := t_k X = \{t_k x : \ x \in X\}, \ \ Y_k := \overline{X_k \cap \mathring{\mathbb{B}}^n_\delta(v)} \ \text{ and } \ Y := \lim_{k\to+\infty} Y_k.$$

We are going to apply Lemma 4.8, so we need to verify the conditions required by this lemma. Set $y^k = t_k x^k$. It is clear that Y_k and Y are closed, $v \in Y$, $y^k \in Y_k \setminus (Y_k)_{sing}$ and $Q = \lim_{k\to+\infty} T_{y^k}Y_k$.

We will show that $v \notin \lim_{k\to+\infty} (Y_k)_{sing}$. By contradiction, suppose that $v \in \lim_{k\to+\infty} (Y_k)_{sing}$. Then there is a sequence $w^k \in (Y_k)_{sing}$ tending to v. Clearly $w^k \in \mathring{\mathbb{B}}^n_\delta(v)$ for k large enough. This and the condition $w^k \in (Y_k)_{sing}$ implies that $w^k \in (\overline{X}_k)_{sing}$, i.e., $u^k := \dfrac{w^k}{t_k} \in (\overline{X})_{sing}$. As $t_k \to +\infty$, we have $u^k \to 0$ as $k \to +\infty$. Moreover, it is clear that $t_k u^k \to v$. Therefore $v \in \mathcal{C}'$. This contradiction implies that $v \notin \lim_{k\to+\infty} (Y_k)_{sing}$.

Next, for k large enough, we must have that $X \cap \mathring{\mathbb{B}}^n_{\frac{\delta}{t_k}}(\frac{v}{t_k})$ is of pure dimension d. Therefore $X_k \cap \mathring{\mathbb{B}}^n_\delta(v)$ is also of pure dimension d as it is the image of $X \cap \mathring{\mathbb{B}}^n_{\frac{\delta}{t_k}}(\frac{v}{t_k})$ by the linear isomorphism

$$\mathbb{R}^n \to \mathbb{R}^n, \ x \mapsto t_k x.$$

Consequently $Y_k = \overline{X_k \cap \mathring{\mathbb{B}}^n_\delta(v)}$ is of pure dimension d.

In order to apply Lemma 4.8, it remains to prove that $C_vY \subsetneq Q$. For this, it is sufficient to show that $C_v\mathcal{C} = C_vY$. Let $u \in \mathcal{C} \cap \mathring{\mathbb{B}}^n_\delta(v)$. Clearly $u \neq 0$ by the choice of δ. In view of Lemma 3.6, there is a C^1 definable curve $\gamma : (0,\varepsilon) \to X \setminus X_{sing}$ such that

$$\|\gamma(r)\| = r \text{ for } r \in (0,\varepsilon) \ \text{ and } \ \lim_{r\to 0^+} \frac{\gamma(r)}{r} = u.$$

Thus it is clear that, for $k > 0$ large enough, $t_k\gamma(\frac{1}{t_k}) \in \mathring{\mathbb{B}}^n_\delta(v)$ and so $t_k\gamma(\frac{1}{t_k}) \in Y_k$. Consequently $u \in Y$ and we get $\mathcal{C} \cap \mathring{\mathbb{B}}^n_\delta(v) \subset Y$. On the other hand, it is easy to see that $Y \subset \mathcal{C} \cap \mathbb{B}^n_\delta(v)$. Therefore $\mathcal{C} \cap \mathring{\mathbb{B}}^n_\delta(v) = Y \cap \mathring{\mathbb{B}}^n_\delta(v)$, which implies that $C_v\mathcal{C} = C_vY$.

Now in light of Lemma 4.8, there is a subsequence k_l of $\{1,2,\dots\}$ and a sequence $x^l \in Y_{k_l} \setminus (Y_{k_l})_{sing}$ tending to v as $l \to +\infty$ such that the limit $P := \lim_{l\to+\infty} T_{x^l}Y_{k_l}$ exists and $\angle(P,Q) = \frac{\pi}{2}$. Let $\widetilde{x}^l := \dfrac{x^l}{t_{k_l}} \in \overline{X}$. Clearly $\widetilde{x}^l \to 0$, $t_{k_l}\widetilde{x}^l = x^l \to v$ as $l \to +\infty$ and $\widetilde{x}^l$ is a non singular point of $\overline{X}$. In addition, since $T_{\widetilde{x}^l}\overline{X}$ and $T_{x^l}Y_{k_l}$ determine the same plane in the

Grassmannian $\mathbb{G}(d,n)$, we get $\lim_{l\to+\infty} T_{\widetilde{x}^l}\overline{X} = P$, i.e., P belongs to the Nash fiber of $\overline{X}$ along ℓ. In view of Lemma 3.4, we also have $P \in \mathcal{N}_\ell$. The theorem follows. $\square$

The following corollary follows immediately from the proof of Theorem 1.3.

Corollary 4.9. *Let $X \subset \mathbb{R}^n$ be a definable set of pure dimension d at* 0. *If* $\dim \mathcal{C} < d$, *then* $\mathcal{E} = \mathcal{C} \setminus \mathcal{C}'$.

5. Remarks and examples

In this section we give some remarks and examples concerning the results presented in the paper.

Remark 5.1.
(i) It is possible that $\dim \mathcal{E} = \dim \mathcal{C}$. In addition, under the assumptions of Theorem 1.3, it does not necessarily hold that $v \in \mathcal{E}'$. These will be seen in Example 5.3.
(ii) Theorem 1.2 does not necessarily hold if we replace $\mathcal{E}$ by $\mathcal{E}'$. In fact, for a ray ℓ in $\mathcal{C}'$, the Nash fiber $\mathcal{N}_\ell$ along ℓ is not necessary connected as shown in Example 5.4.
(iii) If $\dim \mathcal{C} = \dim X$, a ray $\ell \subset \mathcal{C}_{sing}$ does not necessary belong to $\mathcal{E}$. This is illustrated in Example 5.5.
(iv) If X is not closed, it is worth noting that we need to remove from $\mathcal{E}$ the rays in $\mathcal{C}' = C_0(\overline{X})_{sing}$, not only the rays in C_0X_{sing}. Precisely, Theorem 1.2 may not hold if we set
$$\mathcal{E} = \{\ell \subset \mathcal{C} \setminus C_0X_{sing} : \ \#(\mathcal{N}_\ell) > 1\}.$$
An illustration is given in Example 5.6.
(v) Theorem 1.2 does not hold, in general, for definable sets of codimension greater than 1 as shown in Example 5.7.

Example 5.2. Consider the Whitney umbrella
$$X := \{(x, y, x) \in \mathbb{R}^3 : \ x^2 - y^2 z = 0\}.$$
It is not hard to see that
$$X_{sing} = \{x = y = 0\} \ \text{ and } \ \mathcal{C} = \{x = 0, z \geqslant 0\}.$$
We will show that the rays $\mathbb{R}_+(0,1,0)$ and $\mathbb{R}_+(0,-1,0)$ belong to $\mathcal{E}$ by computing the Nash fibers along these rays (in fact, this is straightforward in view of Theorem O'Shea–Wilson or Theorem 1.3); moreover,
$$\begin{aligned} \mathcal{N}_{\mathbb{R}_+(0,1,0)} &= \mathcal{N}_{\mathbb{R}_+(0,-1,0)} \\ &= \{P \in \mathbb{G}(2,3) : \ P \text{ contains the axis } Oy\} \\ &= \left\{ P \in \mathbb{G}(2,3) : \ P = \left\{ \begin{array}{ll} (w_1, w_2, w_3) : & aw_1 + bw_3 = 0, \\ & a^2 + b^2 \neq 0, \ b \geqslant 0 \end{array} \right\} \right\}. \end{aligned} \tag{23}$$

Assume that $\ell = \mathbb{R}_+(0,1,0)$. The case $\ell = \mathbb{R}_+(0,-1,0)$ is similar. Set

$$f := x^2 - y^2 z.$$

For $(x,y,z) \in X$, we have

$$\nabla f(x,y,z) = (2x, -2yz, -y^2).$$

Set

$$A(x,y,z) := \frac{\nabla f(x,y,z)}{\|\nabla f(x,y,z)\|}.$$

Let $\gamma\colon (0,\epsilon) \to X \setminus X_{sing}$ be an analytic curve such that $\gamma(t) \to 0$ and $\dfrac{\gamma(t)}{\|\gamma(t)\|} \to (0,1,0)$ as $t \to 0$.

If the curve γ lies in the axis Oy, then $A(\gamma(t)) = (0,0,-1)$. Thus

$$\mathcal{N}_\ell \ni (0,0,-1)^\perp = \{w_1, w_2, w_3) \in \mathbb{R}^3 : w_3 = 0\}. \tag{24}$$

Now assume that γ does not intersects the axis Oy. Write

$$\gamma(t) = (x(t), y(t), z(t)) = (x_0 t^\alpha + \cdots, y_0 t^\beta + \cdots, z_0 t^\gamma + \cdots).$$

Clearly $y_0 > 0$ and $\beta > 0$. Moreover, as γ does not intersect the axis Oy, it follows that $z(t) > 0$ for all t. Hence $z_0 > 0$ and $\gamma > 0$. Consequently $x_0 \neq 0$ and $\alpha > 0$. As

$$z(t) = \frac{x^2(t)}{y^2(t)} = \frac{x_0^2}{y_0^2} t^{2\alpha - 2\beta} + \ldots,$$

we get

$$\gamma(t) = \left(x_0 t^\alpha, y_0 t^\beta, \frac{x_0^2}{y_0^2} t^{2\alpha-2\beta} \right) + \cdots \text{ and}$$
$$\nabla(\gamma(t)) = \left(2x_0 t^\alpha, -2\frac{x_0^2}{y_0} t^{2\alpha-\beta}, -y_0^2 t^{2\beta} \right) + \cdots$$

Observe that $\gamma = 2\alpha - 2\beta$, so $\alpha > \beta$. Assume that $\alpha = 2\beta$, then it is not hard to verify that

$$\lim_{t\to 0} \frac{\gamma(t)}{\|\gamma(t)\|} = (0,1,0) \quad \text{and} \quad \lim_{t \to 0} A(\gamma(t)) = \frac{(2x_0, 0, -y_0^2)}{\sqrt{4x_0^2 + y_0^4}}.$$

For $a \neq 0$ and $b > 0$, set $x_0 = -\frac{a}{2}$ and $y_0 = \sqrt{b}$. Then

$$\begin{aligned} \mathcal{N}_\ell \ni \left(\frac{(2x_0, 0, -y_0^2)}{\sqrt{4x_0^2 + y_0^4}} \right)^\perp &= (2x_0, 0, -y_0^2)^\perp \\ &= (-a, 0, -b)^\perp = (a, 0, b)^\perp \\ &= \{(w_1, w_2, w_3) : aw_1 + bw_3 = 0\}. \end{aligned} \tag{25}$$

Now suppose that $\frac{3\beta}{2} < \alpha < 2\beta$. By simple computations, we have

$$\lim_{t \to 0} \frac{\gamma(t)}{\|\gamma(t)\|} = (0,1,0) \quad \text{and} \quad \lim_{t\to 0} A(\gamma(t)) = (1,0,0),$$

i.e., $\mathcal{N}_\ell \ni (1,0,0)^\perp = \{(w_1, w_2, w_3) :\ w_1 = 0\}$. Combining this with (24) and (25) yields (23).

For any ray ℓ different from $\mathbb{R}_+(0,1,0)$ and $\mathbb{R}_+(0,-1,0)$, it can be verified that $\mathcal{N}_\ell$ contains only one element given by $\{(w_1,w_2,w_3):\ w_1=0\}$. So $\ell\notin\mathcal{E}$.

Example 5.3. Let $X:=\{(x,y,x)\in\mathbb{R}^3:\ x^2+y^2=z^3\}$. Then $\mathcal{C}=\{x=y=0,\ z\geqslant 0\}$. Let $\ell:=\mathbb{R}_+(0,0,1)$. We have $\mathcal{C}=\ell\cup\{0\}$ and $\mathcal{N}_\ell=\{P\in\mathbb{G}(2,3):\ \ell\subset P\}$. Consequently $\dim\mathcal{E}=\dim\mathcal{C}$.

Example 5.4. Let $X:=X_1\cup X_2$, where

$$X_1:=\{(x,y,x)\in\mathbb{R}^3:\ z^3\geqslant x^2,\ y=0\},$$

$$X_2:=\{(x,y,x)\in\mathbb{R}^3:\ x=0\}.$$

It is not hard to check that $\mathcal{C}=\{x=0\}$ and $\mathcal{C}'=\{x=y=0\}$. So $\mathcal{C}_{sing}=\emptyset$. Consider the ray $\ell=\mathbb{R}_+(0,0,1)\subset\mathcal{C}'$. Clearly $\mathcal{N}_\ell$ is disconnected since it contains two elements which are determined respectively by $\{(w_1,w_2,w_3):\ w_1=0\}$ and $\{(w_1,w_2,w_3):\ w_2=0\}$.

Example 5.5. Let $X:=X_1\cup X_2$, where

$$X_1:=\{(x,y,x)\in\mathbb{R}^3:\ f_1(x,y,z):=x^2+(y-z)^2+z^4-z^2=0,z\geqslant 0\},$$

$$X_2:=\{(x,y,x)\in\mathbb{R}^3:\ f_2(x,y,z):=x^2+(y+z)^2+z^4-z^2=0,z\geqslant 0\}.$$

Clearly,

$$\mathcal{C}=\{(x^2+(y-z)^2-z^2)(x^2+(y+z)^2-z^2)=0,z\geqslant 0\},$$

which is the union of two cones tangent to each other along the ray $\ell=\mathbb{R}_+(0,0,1)$. Therefore, $\ell\subset\mathcal{C}_{sing}$. On the other hand, let $a^k:=(x_k,y_k,z_k)\in X\setminus\{(0,0,0)\}$ be any sequence such that $a^k\to 0$ and $\dfrac{a^k}{\|a^k\|}\to(0,0,1)$, so $\dfrac{x_k}{z_k}\to 0$ and $\dfrac{y_k}{z_k}\to 0$. Without loss of generality, suppose that $a^k\in X_1$ for all k. We have

$$\nabla f_1(a^k)=(2x_k,2y_k-2z_k,-2y_k+4z_k^3)=2z_k\left(\frac{x_k}{z_k},\frac{y_k}{z_k}-1,-\frac{y_k}{z_k}+2z_k^2\right).$$

Consequently, $\frac{\nabla f_1(a^k)}{\|\nabla f_1(a^k)\|}\to(0,-1,0)$, which implies that $T_{a^k}X$ tends to the plane Oxz. Hence $\mathcal{N}_\ell$ contains only one element which is the plane Oxy, so $\ell\not\subset\mathcal{E}$.

Example 5.6. Set

$$X:=\{(x,y,z)\in\mathbb{R}^3:\ x=0,\ y\neq 0\}\cup\{(x,y,z)\in\mathbb{R}^3:\ y=0,\ x\neq 0\}.$$

Note that $X_{sing}=\emptyset$, so $C_0X_{sing}=\emptyset$. Obviously,

$$\mathcal{C}=\{x=0\}\cup\{y=0\},$$

Moreover,

$$\ell_1:=\mathbb{R}_+(0,0,1)\subset\mathcal{C}\setminus C_0X_{sing},\ \ell_2:=\mathbb{R}_+(0,0,-1)\subset\mathcal{C}\setminus C_0X_{sing}$$

and

$$\mathcal{N}_{\ell_1} = \mathcal{N}_{\ell_2} = \{P_1, P_2\},$$

where

$$P_1 = \{(w_1, w_2, w_3) : \; w_1 = 0\} \;\text{ and }\; P_2 = \{(w_1, w_2, w_3) : \; w_2 = 0\}.$$

So $\mathcal{N}_{\ell_1}$ and $\mathcal{N}_{\ell_2}$ are disconnected.

Example 5.7. Let $X := X_1 \cup X_2$, where

$$\begin{aligned} X_1 &:= \{(x, y, z, t) \in \mathbb{R}^4 : \; x = y = 0\} \\ X_2 &:= \{(x, y, z, t) \in \mathbb{R}^4 : \; z = 0, \; x^2 + y^2 = t^3\}. \end{aligned}$$

It is clear that $X \setminus \{0\}$ is not singular, $\dim X = 2$, and that

$$C_0(X_1) = X_1 \;\text{ and }\; C_0(X_2) = \{x = y = z = 0\}.$$

So $C_0(X) = \{x = y = 0\}$. Let

$$\ell := \mathbb{R}_+(0, 0, 0, 1) \subset C_0(X_1) \cap C_0(X_2) \subset \mathcal{C}.$$

Denote by $\mathcal{N}_\ell(X_i)$ the set of tangent limits of X_i $(i = 1, 2)$ along ℓ. It is clear that $\mathcal{N}_\ell(X_1)$ has only one element given by

$$P = \{(w_1, w_2, w_3, w_4) : \; w_1 = w_2 = 0\}.$$

On the other hand, for any $Q \in \mathcal{N}_\ell(X_2)$, its equation is given by

$$Q = \{(w_1, w_2, w_3, w_4) : \; aw_1 + bw_2 = 0, w_3 = 0\}$$

with $a^2 + b^2 \neq 0$. So we have $\angle(P, Q) = \dfrac{\pi}{2}$. Consequently, the set of tangent limits of X along ℓ, given by $\mathcal{N}_\ell(X) = \mathcal{N}_\ell(X_1) \cup \mathcal{N}_\ell(X_2)$, is disconnected. Obviously, ℓ is a ray in $\mathcal{E}$ and $\dim \mathcal{N}_\ell = 1$.

Now let

$$X_3 := \{(x, y, z, t) \in \mathbb{R}^4 : \; z = 0, \; x^2 = t^3\}$$

and let $X' := X_1 \cup X_3$. Then $(X')_{sing} = (X_3)_{sing} = \{x = z = t = 0\}$, $\dim X' = 2$ and we have

$$C_0(X') = \{x = y = 0\} \cup \{x = z = 0, t \geqslant 0\}.$$

Let $\ell := \mathbb{R}_+(0, 0, 0, 1) \subset (C_0(X'))_{sing}$. Clearly, ℓ is not tangent to $(X')_{sing}$ at 0. Note that $\mathcal{N}_\ell(X_3)$ contains only one element given by

$$R = \{(w_1, w_2, w_3, w_4) : \; w_1 = w_3 = 0\},$$

so $R \neq P$. Therefore $\#(\mathcal{N}_\ell) = 2$ and ℓ is a ray in $\mathcal{E}$.

Question. If X is a definable set of pure dimension $d > 0$ at the origin $0 \in \mathbb{R}^n$ and $\ell \subset \mathcal{E} \setminus (\mathcal{C}' \cup \mathcal{C}_{sing})$, then $\dim \mathcal{N}_\ell \geqslant 1$?

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