Positivity and stability of mixed fractional-order systems with unbounded delays: Necessary and sufficient conditions

H.T. Tuan, H. Trinh and J. Lam

***Abstract*—This paper provides a comprehensive study on the quantitative properties of linear mixed fractional-order systems with multiple time-varying delays. The delays can be bounded or unbounded. We first obtain a result on the existence and uniqueness of solutions to these systems. Then, we prove a necessary and sufficient condition for their positivity. Finally, we provide a necessary and sufficient criterion to characterize the asymptotic stability of positive linear mixed fractional-order systems with multiple time-varying delays.**

Key words: Fractional differential equations, Linear mixed fractional-order systems, Time-varying delays, Positive systems, Asymptotic stability.

I. Introduction

Fractional differential equations are widely used to describe memory and hereditary properties of materials and processes. For details, we refer the reader to some monographs [1], [2], [3] and the references therein. On the other hand, time-delay systems have received considerable attention due to the fact that many processes include after-effect phenomena in their inner dynamics, see, e.g., [4], [5], [6]. While positive systems play a key role in understanding many processes in biological and medical sciences, see e.g., [7], [8]. As such, the qualitative theory of positive fractional-order systems with delays is an important and interesting research topic, which is the main focus of this paper.

One of the important problems in the dynamical system theory of time-delay fractional-order systems is stability analysis. Using the characteristic polynomial, in [9], [10], the authors obtained conditions depending on the magnitude of the delay for the asymptotic stability of fractional-order systems with the linear part comprises a pure delay. Recently, some results on stability of fractional-order systems with delays by using Lyapunov-candidate-functions were proposed in [11], [12]. In [13], the author discussed linear fractional systems with multiple delays in control input. An analytical approach based on the Laplace transform and 'inf-sup' method for studying the finite-time stability of singular fractional-order switched systems with delay was presented in [14]. By using the Lyapunov method combined with the concept of uniformly positive definite matrix functions and Hamilton–Jacobi–Riccati inequalities, the robust stability of the almost periodic solution to uncertain impulsive functional differential systems of fractional order was investigated in [15].

Up to now, in our view, an important contribution to the study of the asymptotic behavior of solutions to positive mixed fractional-order systems with delays is the paper by Shen and Lam [16]. In that paper, the authors reported a criterion for the positivity of linear mixed fractional-order systems with a time-varying delay. They also obtained a result on the asymptotic stability of positive linear mixed fractional-order system with a bounded time-varying delay.

Let $d \in \mathbb{N}$, $\hat{\alpha} = (\alpha_1, \ldots, \alpha_d)^{\mathrm{T}} \in (0,1] \times \cdots \times (0,1]$, $r > 0$, $m \in \mathbb{N}$. Motivated by [16], in this paper, we consider the following linear mixed fractional-order systems with multiple *unbounded* time-varying delays

$$^{C}D^{\hat{\alpha}}_{0+}x(t) = Ax(t) + \sum_{1}^{m} B_k x(t - h_k(t)),\ t > 0, \quad (1)$$

with the initial condition $x(\cdot) = \phi(\cdot) \in C([-r,0];\mathbb{R}^d)$ on $[-r,0]$, where $^{C}D^{\hat{\alpha}}_{0+}x(t) = (^{C}D^{\alpha_1}_{0+}x_1(t), \ldots, ^{C}D^{\alpha_k}_{0+}x_k(t), \ldots, ^{C}D^{\alpha_d}_{0+}x_d(t))^{\mathrm{T}}$ in which $^{C}D^{\alpha_k}_{0+}$ is the Caputo derivative operator of the order α_k, $A = (a_{ij})_{1\leq i,j\leq d}$, $B_k = (b^k_{ij})_{1\leq i,j \leq d}$, $h_k : [0,\infty) \to \mathbb{R}_{\geq 0}$ is continuous and satisfies the growth rate as in [17]. Our main aim is to study the asymptotic stability of the system (1) for the case it is positive. It is worth noting that the approaches as in [9], [10] (based on the eigenvalues of the characteristic polynomials) and [16] (based on comparing the trajectory of the time-varying delay system with that of the constant delay system) cannot be applied for (1) where the delays $h_k(\cdot)$ $(1 \leq k \leq m)$ are time-varying and *unbounded*.

This paper is organized as follows. In Section II, we first introduce a result on the existence and uniqueness of global solutions to linear mixed fractional-order with multiple time-varying delays. Then, we give a necessary and sufficient condition to characterize the positivity of these systems. The main result of the paper is given in Section III. In particular, in Theorem III.2, we provide a necessary and sufficient criterion to ensure the asymptotic stability of positive linear mixed fractional-order systems with bounded and unbounded time-varying delays.

Before concluding this section, we introduce some notations which are used throughout this paper. Let $\mathbb{N}$ be the set of

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natural numbers, $Z_{\geq 0}$ be the set of non-negative integers, $\mathbb{R}$ ($\mathbb{R}_{\geq 0}$) be the set of real numbers (nonnegative real numbers, respectively), and $\mathbb{R}^d$ be the d-dimensional Euclidean space endowed with a norm $\|\cdot\|$. Without loss of generality, in this paper we use the symbol $\|\cdot\|$ to denote the max norm of Euclidean spaces. For any $[a,b]\subset\mathbb{R}$, let $C([a,b];\mathbb{R}^d)$ be the space of continuous functions $\xi:[a,b]\to\mathbb{R}^d$. A matrix $A=(a_{ij})_{1\leq i,j\leq d}\in\mathbb{R}^{d\times d}$ is called Metzler if $a_{ij}\geq 0$ for all $1\leq i\neq j\leq d$. A matrix $A\in\mathbb{R}^{d\times d}$ is said to be Hurwitz if its spectrum $\sigma(A)$ satisfies

$$\sigma(A)\subset\{\lambda\in\mathbb{C}:\text{Re}\,\lambda<0\}.$$

Let $n,m\in\mathbb{N}$ and $A=(a_{ij})_{1\leq i\leq n}^{1\leq j\leq m}, B=(B_{ij})_{1\leq i\leq n}^{1\leq j\leq m}\in\mathbb{R}^{n\times m}$. We write $A\succ B$ ($A\succeq B$) if $a_{ij}>b_{ij}$ ($a_{ij}\geq b_{ij}$, respectively) for all $1\leq i\leq n,\ 1\leq j\leq m$. The matrix A is said to be nonnegative if $a_{ij}\geq 0$ for all $1\leq i\leq n$, $1\leq j\leq m$. For $\alpha\in(0,1)$ and an integrable function $x:[a,b]\to\mathbb{R}$, the Riemann–Liouville integral operator of $x(\cdot)$ with the order α is defined by

$$(I^\alpha_{a+}x)(t):=\frac{1}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}x(\tau)\,d\tau,\quad t\in(a,b],$$

where $\Gamma(\cdot)$ is the Gamma function. The *Caputo fractional derivative* ${}^{C}D^\alpha_{a+}x$ of a function $x\in AC([a,b];\mathbb{R})$ is defined by

$$({}^{C}D^\alpha_{a+}x)(t):=(I^{1-\alpha}_{a+}Dx)(t),\quad t\in(a,b],$$

where $AC([a,b];\mathbb{R})$ denotes the space of absolutely continuous functions and D is the classical derivative.

II. Positivity of linear mixed-order fractional systems with time-varying delays

Let $\hat{\alpha}=(\alpha_1,\dots,\alpha_d)^{\mathrm{T}}\in(0,1]\times\cdots\times(0,1]\subset\mathbb{R}^d$, $T,r>0$, $m\in\mathbb{N}$. Consider the following system on $(0,T]$

$${}^{C}D^{\hat{\alpha}}_{0+}x(t)=Ax(t)+\sum_1^m B_kx(t-h_k(t))+Uw(t),\quad (2)$$

and $x(\cdot)=\phi(\cdot)\in C([-r,0];\mathbb{R}^d)$ on $[-r,0]$, where $A=(a_{ij})_{1\leq i,j\leq d}$, $B_k=(b^k_{ij})_{1\leq i,j\leq d}$ $(1\leq k\leq m)$, $U=(u_{ij})_{1\leq i,j\leq d}\in\mathbb{R}^{d\times d}$ and $w(\cdot)\in C([0,T];\mathbb{R}^d)$. Assume that $h_k:[0,T]\to\mathbb{R}_{\geq 0}$ $(1\leq k\leq m)$ is continuous such that

(F1) $h_k(0)>0$;
(F2) $t-h_k(t)\geq -r$ for all $t\in[0,T]$;
(F3) $h_k(0)\neq h_l(0)$ for any $1\leq k\neq l\leq m$.

Using the same arguments as in the proof of [18, Lemma 6.2, pp. 86], we see that a vector valued function $\varphi(\cdot,\phi)\in C([-r,T];\mathbb{R}^d)$ is a solution of (2) with $x(\cdot)=\phi(\cdot)$ on $[-r,T]$ if and only if it satisfies the time-delay integral system on $(0,T]$,

$$x_i(t)=\phi_i(0)+\frac{1}{\Gamma(\alpha_i)}\int_0^t(t-s)^{\alpha_i-1}\sum_{1\leq j\leq d}\Big(a_{ij}x_j(s)+\sum_1^m b^k_{ij}x_j(s-h_k(s))+u_{ij}w_j(s)\Big)ds,\ 1\leq i\leq d,$$

and $x(\cdot)=\phi(\cdot)$ on $[-r,0]$.

Surprisingly, up to now, there has been no result reported in the literature on the existence and uniqueness of solutions to mixed fractional-order systems with multiple time-varying delays. Hence, we first introduce here a rigorous proof for the existence and uniqueness of global solutions to the system in (2).

Lemma II.1 (Existence and uniqueness of linear mixed fractional-order with time-varying delays)**.** Assume that $h_k:[0,T]\to\mathbb{R}_{\geq 0}$ $(1\leq k\leq m)$ is continuous such that condition (F2) holds. Then, for any $\phi(\cdot)\in C([-r,0];\mathbb{R}^d)$ and $w(\cdot)\in C([0,T];\mathbb{R}^d)$, system (2) with initial condition $x(t)=\phi(t),\ t\in[-r,0]$ has a unique solution $\varphi(\cdot,\phi)$ on $[-r,T]$.

Proof. Let

$$C_\phi:=\big\{\xi\in C([-r,T];\mathbb{R}^d):\xi(t)=\phi(t),\ t\in[-r,0]\big\}$$

and define a functional $\|\cdot\|_\gamma$ on C_ϕ by

$$\|\xi\|_\gamma=\max_{t\in[0,T]}\frac{\xi^*(t)}{\exp(\gamma t)},$$

where $\gamma>0$ is fixed and chosen later and $\xi^*(t)=\max_{-r\leq\theta\leq t}\|\xi(\theta)\|$. Notice that $\|\cdot\|_\gamma$ is a norm and $(C_\phi,\|\cdot\|_\gamma)$ is a Banach space. On this space, we establish an operator $\mathcal{T}_\phi:C_\phi\to C_\phi$ as follows.

$$(\mathcal{T}_\phi\xi)^i(t)=\phi_i(0)+\frac{1}{\Gamma(\alpha_i)}\int_0^t(t-s)^{\alpha_i-1}\Big(\sum_{1\leq j\leq d}a_{ij}\xi_j(s)+\sum_1^m b^k_{ij}\xi_j(s-h_k(s))+u_{ij}w_j(s)\Big)d\tau,$$

for $t\in(0,T]$, $1\leq i\leq d$, and $(\mathcal{T}_\phi\xi)(t)=\phi(t)$ on $[-r,0]$. To complete the proof of this lemma, we only have to show that $\mathcal{T}_\phi$ is contractive. For that, for any $\xi(\cdot),\hat{\xi}(\cdot)\in C_\phi$, $t\in[0,T]$, $1\leq i\leq d$, we have

$$\begin{aligned}I(t)&=|(\mathcal{T}_\phi\xi)^i(t)-(\mathcal{T}_\phi\hat{\xi})^i(t)|\\&\leq\frac{1}{\Gamma(\alpha_i)}\int_0^t(t-s)^{\alpha_i-1}\sum_{1\leq j\leq d}\big(|a_{ij}||\xi_j(s)-\hat{\xi}_j(s)|\\&\quad+\sum_{1\leq k\leq m}|b^k_{ij}||\xi_j(s-h_k(s))-\hat{\xi}_j(s-h_k(s))|\big)ds\\&\leq\frac{\max_{i=1}^d\big(\sum_{j=1}^d(|a_{ij}|+\sum_{k=1}^m|b^k_{ij}|)\big)\exp(\gamma t)}{\Gamma(\alpha_i)}\\&\quad\times\int_0^t(t-s)^{\alpha_i-1}\exp(-\gamma(t-s))\frac{(\xi-\hat{\xi})^*(s)}{\exp(\gamma s)}ds.\end{aligned}$$

Hence,

$$I(t) \le \frac{\max_{i=1}^d \left(\sum_{j=1}^d (|a_{ij}| + \sum_{k=1}^m |b_{ij}^k|)\right) \exp(\gamma t)}{\Gamma(\alpha_i)} \times \int_0^t v^{\alpha_i - 1} \exp(-\gamma v) dv \|\xi - \hat{\xi}\|_\gamma$$
$$\le \frac{\max_{i=1}^d \left(\sum_{j=1}^d (|a_{ij}| + \sum_{k=1}^m |b_{ij}^k|)\right) \exp(\gamma t)}{\Gamma(\alpha_i)\lambda^{\alpha_i}} \times \int_0^{\gamma t} u^{\alpha_i - 1} \exp(-u) du \|\xi - \hat{\xi}\|_\gamma,$$

which implies

$$\frac{|(\mathcal{T}_\phi \xi)^i(t) - (\mathcal{T}_\phi \hat{\xi})^i(t)|}{\exp(\gamma t)} \le \max_{1 \le i \le d} \frac{\sum_{j=1}^d (|a_{ij}| + \sum_{k=1}^m |b_{ij}^k|)}{\gamma^{\alpha_i}} \|\xi - \hat{\xi}\|_\gamma, \qquad (3)$$

where we used the fact that

$$\int_0^\infty u^{\alpha_i - 1} \exp(-u) du = \Gamma(\alpha_i)$$

and the estimates

$$\|\xi(s) - \hat{\xi}(s)\|,\ \|\xi(s - h_k(s)) - \hat{\xi}(s - h_k(s))\| \le (\xi - \hat{\xi})^*(s)$$

for $s \in [0, T]$, $1 \le k \le m$. From (3), we obtain

$$\frac{(\mathcal{T}_\phi \xi - \mathcal{T}_\phi \hat{\xi})^*(t)}{\exp(\gamma t)} \le \max_{l=1}^d \frac{\max_{i=1}^d \sum_{j=1}^d (|a_{ij}| + \sum_{k=1}^m |b_{ij}^k|)}{\gamma^{\alpha_l}} \|\xi - \hat{\xi}\|_\gamma$$

for all $t \in [0, T]$. Thus,

$$\|\mathcal{T}_\phi \xi - \mathcal{T}_\phi \hat{\xi}\|_\gamma \le \|\xi - \hat{\xi}\|_\gamma \times \max_{1 \le l \le d} \frac{\max_{1 \le i \le d} \sum_{1 \le j \le d} (|a_{ij}| + \sum_{1 \le k \le m} |b_{ij}^k|)}{\gamma^{\alpha_l}}.$$

By choosing $\gamma > 0$ such that

$$\max_{1 \le l \le d} \frac{\max_{i=1}^d \sum_{j=1}^d (|a_{ij}| + \sum_{k=1}^m |b_{ij}^k|)}{\gamma^{\alpha_l}} < 1,$$

then $\mathcal{T}_\phi$ is contractive. Banach fixed point theorem implies that this operator has a fixed point in $(C_\phi, \|\cdot\|_\gamma)$ which is also the unique solution to initial value problem (2) with initial condition $x(t) = \phi(t)$, $t \in [-r, 0]$. The proof is complete. □

Our main aim in this section is to introduce a criterion to characterize the positivity of linear mixed-order fractional systems with time-varying delays.

Definition II.2. System (2) is positive if for any $\phi(t) \succeq 0$ on $[-r, 0]$ and $w(t) \succeq 0$ on $[0, T]$, its solution $\varphi(\cdot, \phi)$ satisfies $\varphi(t, \phi) \succeq 0$ on $[0, T]$.

The main result in this section is the following proposition.

Proposition II.3 (A necessary and sufficient condition for the positivity of linear mixed fractional-order systems with time-varying delays)**.** Let $h_k : [0, T] \to \mathbb{R}_{\ge 0}$ $(1 \le k \le m)$ be continuous such that conditions (F1), (F2) and (F3) hold. Then, system (2) is positive if and only if A is Metzler, B_k $(1 \le k \le m)$ and U are nonnegative.

Proof. **Necessity:** Let system (2) be positive. We first show that $U = (u_{ij})_{1 \le i, j \le d}$ is nonnegative. To do this, assume that there is an element $u_{i_0 j_0} < 0$. By choosing $\phi(t) = 0$ on $[-r, 0]$ and $w(t) = e_{j_0}$ on $[0, T]$, we have the representation of the i_0-component of $\varphi(\cdot, \phi)$ as

$$\varphi_{i_0}(t, \phi) = \frac{1}{\Gamma(\alpha_{i_0})} \int_0^t (t - s)^{i_0 - 1} \sum_{1 \le j \le d} a_{i_0 j} \varphi_j(s, \phi) ds + \frac{1}{\Gamma(\alpha_{i_0})} \int_0^t (t - s)^{i_0 - 1} \sum_{1 \le k \le m} \sum_{1 \le j \le d} b_{i_0 j}^k \varphi_j(s - h_k(s), \phi) ds + \frac{1}{\Gamma(\alpha_{i_0})} \int_0^t (t - s)^{i_0 - 1} u_{i_0 j_0} ds, \quad t \in [0, T],$$

where $e_{j_0} = (0, \ldots, 1, \ldots, 0)^{\mathrm{T}}$ denotes the unit vector in $\mathbb{R}^d$ with the j_0-coordinate equals to 1. Hence, for $t_0 > 0$ small enough, for example, for all $t \in [0, t_0]$,

$$t - h_k(t) < -\max_{1 \le k \le m} h_k(0)/2, \quad \sum_{1 \le j \le d} a_{i_0 j} \varphi_j(s, \phi) < |u_{i_0 j_0}|,$$

then

$$\varphi_{i_0}(t_0, \phi) = \frac{1}{\Gamma(\alpha_{i_0})} \int_0^{t_0} (t_0 - s)^{i_0 - 1} \sum_{1 \le j \le d} a_{i_0 j} \varphi_j(s, \phi) ds + \frac{1}{\Gamma(\alpha_{i_0})} \int_0^{t_0} (t_0 - s)^{i_0 - 1} u_{i_0 j_0} ds < 0,$$

a contradiction. Next, assume, ad absurdum, $A = (a_{ij})_{1 \le i, j \le d}$ is not Metzler, that is, there exist indexes $1 \le i_0 \ne j_0 \le d$ such that $a_{i_0 j_0} < 0$. Let $\phi(\cdot) \in C([-r, 0]; \mathbb{R}^d)$ be a vector valued function with

$$\phi(t) = \begin{cases} e_{j_0}, & \text{if} \quad t = 0, \\ 0, & \text{if} \quad t \in [-r, -\max_{1 \le k \le m} h_k(0)/2], \end{cases}$$

and $w(t) = 0$ on $[0, T]$. Due to the continuity of solutions and the delay $h_k(\cdot)$ and that $h_k(0) > 0$ $(1 \le k \le m)$, we can find $t_0 > 0$ (small enough) such that $t - h_k(t) \le -\max_{1 \le k \le m} h_k(0)/2$, $\varphi_{j_0}(t, \phi) > 1/2$, and $\sum_{1 \le j \le d, j \ne j_0} a_{i_0 j} \varphi_j(t, \phi) < \frac{|a_{i_0 j_0}|}{2}$ for all $t \in [0, t_0]$. Then, the i_0-component of $\varphi(t_0, \phi)$ satisfies

$$\varphi_{i_0}(t_0, \phi) = \frac{1}{\Gamma(\alpha_{i_0})} \int_0^{t_0} (t_0 - s)^{i_0 - 1} a_{i_0 j_0} \varphi_{j_0}(s, \phi) ds + \frac{1}{\Gamma(\alpha_{i_0})} \int_0^{t_0} (t_0 - s)^{i_0 - 1} \sum_{1 \le j \le d, j \ne j_0} a_{i_0 j} \varphi_j(s, \phi) ds < 0,$$

a contradiction. We now prove that B_k is nonnegative for any $1 \le k \le m$. From (F1) and (F3), without loss of generality, let $0 < h_1(0) < \cdots < h_m(0)$. First, we show that B_1 is nonnegative. Suppose, ad absurdum,

$B_1 = (b^1_{ij})_{1\le i,j\le d}$ is not nonnegative. That is, there is $b^1_{i_0j_0} < 0$. Choose $\phi(\cdot) \in C([-r,0];\mathbb{R}^d)$ such that

$$\phi(t) = \begin{cases} 0, & \text{if} \quad t=0, \\ e_{j_0}, & \text{if} \quad t \in [\frac{-2h_1(0)-h_2(0)}{3}, \frac{-h_1(0)}{2}], \\ 0, & \text{if} \quad t \in [-r, \frac{-2h_2(0)-h_1(0)}{3}], \end{cases}$$

and $w(t) = 0$ on $[0,T]$. Then, for $t_0 > 0$ small enough so that on the interval $[0,t_0]$:

- $\frac{-2h_1(0)-h_2(0)}{3} \le t - h_1(t) \le \frac{-h_1(0)}{2}$;
- $-r \le t - h_k(t) \le \frac{-2h_2(0)-h_1(0)}{3}$, $2 \le k \le m$;
- $\sum_{1\le j\le d} a_{i_0 j}\varphi_j(t,\phi) < |b^1_{i_0j_0}|$.

Then, the i_0-component of the solution $\varphi(\cdot,\phi)$ at $t = t_0$ verifies

$$\begin{aligned}\varphi_{i_0}(t_0,\phi) &= \frac{1}{\Gamma(\alpha_{i_0})}\int_0^{t_0}(t_0-s)^{i_0-1}b^1_{i_0j_0}ds \\ &+ \frac{1}{\Gamma(\alpha_{i_0})}\int_0^{t_0}(t_0-s)^{i_0-1}\sum_{1\le j\le d} a_{i_0j}\varphi_j(s,\phi)ds \\ &< 0,\end{aligned}$$

which implies a contradiction. By similar arguments, we also see B_k, $2 \le k \le m$, is nonnegative. Thus, B_k ($1 \le k \le m$) are nonnegative.

Sufficiency: Let $A = (a_{ij})_{1\le i,j\le d}$ be Metzler and $B_k = (b^k_{ij})_{1\le i,j\le d}$, $U = (u_{ij})_{1\le i,j\le d}$ be nonnegative. We first show that if $\phi(t) \succ 0$ on $[-r,0]$ and $w(t) \succeq 0$ on $[0,T]$, then $\varphi(t,\phi) \succeq 0$ on $[0,T]$. Indeed, due to the fact that A is Metzler, there exists a positive constant $\rho > 0$ such that

$$A = -\rho I_d + (\rho I_d + A),$$

where $\rho I_d + A$ is nonnegative. Then, system (2) is rewritten as

$$\begin{aligned}{}^C D^{\hat\alpha}_{0+}x(t) &= \rho I_d x(t) + (\rho I_d + A)x(t) \\ &+ \sum_{1\le k\le m} B_k x(t-h_k(t)) + Uw(t), \quad t \in (0,T].\end{aligned}$$

By virtue of the variation of constants formula (see, e.g., [19, Lemma 3.1]), the solution $\varphi(\cdot,\phi) = (\varphi_1(\cdot,\phi),\dots,\varphi_d(\cdot,\phi))^{\mathrm{T}}$ of (2) with $\varphi(\cdot,\phi) = \phi(\cdot)$ on $[-r,0]$ has the following form:

$$\begin{aligned}&\varphi_i(t,\phi) = E_{\alpha_i}(-\rho t^{\alpha_i})\phi_i(0) \\ &+\int_0^t (t-s)^{\alpha_i-1}E_{\alpha_i,\alpha_i}(-\rho(t-s)^{\alpha_i})\sum_{1\le j\le d}\big((a_{ij}+\rho\delta_{ij})\varphi_j(s,\phi) \\ &\quad + \sum_{1\le k\le m} b^k_{ij}\varphi_j(s-h_k(s),\phi) + u_{ij}w_j(s)\big)ds \qquad (4)\end{aligned}$$

for $t \in [0,T]$, $1 \le i \le d$, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and

$$E_{\alpha_i}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha_i+1)}, \quad E_{\alpha_i,\alpha_i}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha_i+\alpha_i)}$$

are Mittag-Leffler functions. Suppose that there exists $t_0 > 0$ so that $\varphi(t_0,\phi) \not\succeq 0$. From this, we can find an index $i_0 \in \{1,\dots,d\}$ satisfying $\varphi_{i_0}(t_0,\phi) = 0$. Take

$$t^* = \inf\{t \in [0,T] : \varphi_{i_0}(t,\phi) = 0\}.$$

Then $t^* > 0$, $\varphi_{i_0}(t^*,\phi) = 0$ and $\varphi_{i_0}(t,\phi) > 0$ for all $t \in [0,t^*)$. However, from (4),

$$\begin{aligned}&\varphi_{i_0}(t^*,\phi) = E_{\alpha_{i_0}}(-\rho t^{*\alpha_{i_0}})\phi_{i_0}(0) \\ &+ \int_0^{t^*}(t^*-s)^{\alpha_{i_0}-1}E_{\alpha_{i_0},\alpha_{i_0}}(-\rho(t^*-s)^{\alpha_{i_0}}) \\ &\times \sum_{1\le j\le d}\big((a_{i_0j}+\rho\delta_{i_0j})\varphi_j(s,\phi) + \sum_{1\le k\le m} b^k_{i_0j}\varphi_j(s-h_k(s),\phi) \\ &+ u_{i_0j}w_j(s)\big)ds \\ &\qquad \ge E_{\alpha_{i_0}}(-\rho t^{*\alpha_{i_0}})\phi_{i_0}(0) > 0,\end{aligned}$$

a contradiction. Thus, $\varphi(t,\phi) \succ 0$ on $[0,T]$. We now consider the case where the inputs $\phi(t) \succeq 0$ on $[-r,0]$ and $w(t) \succeq 0$ on $[0,T]$. Using the arguments as in [20, Proposition 1], we get the initial conditions $\phi^n(\cdot) = \phi(\cdot) + \frac{1}{n}\mathbf{1}$ on $[-r,0]$ with $n \in \mathbb{N}$ and $\mathbf{1} = (1,\dots,1)^{\mathrm{T}}$. It is obvious to see that $\{\varphi(\cdot,\phi^n)\}_{n=1}^{\infty}$ is a decreasing sequence of continuous positive functions on $[-r,T]$. Define $\varphi^*(t) := \lim_{n\to\infty}\varphi(t,\phi_n)$ for each $t \in [-r,T]$. By Dini's theorem (see, e.g., [21, Theorem 7.13, pp. 150]), the sequence $\{\varphi(\cdot,\phi^n)\}_{n=1}^{\infty}$ converges uniformly to $\varphi^*(\cdot)$ and this function is also continuous and nonnegative on $[-r,T]$. Notice that for each $n \in \mathbb{N}$, $\varphi(\cdot,\phi^n)$ verifies

$$\begin{aligned}\varphi_i(t,\phi^n) =& E_{\alpha_i}(-\rho t^{\alpha_i})(\phi^n(0))_i \\ &+ \int_0^t (t-s)^{\alpha_i-1}E_{\alpha_i,\alpha_i}(-\rho(t-s)^{\alpha_i}) \\ &\times \sum_{1\le j\le d}\big((a_{ij}+\rho\delta_{ij})\varphi_j(s,\phi^n) \\ &+ \sum_{1\le k\le m} b^k_{ij}\varphi_j(s-h_k(s),\phi^n) + u_{ij}w_j(s)\big)ds,\end{aligned}$$

for $1 \le i \le d$, $t \in [0,T]$ and $\varphi(t,\phi^n) = \phi^n(t)$ on $[-r,0]$. Let $n \to \infty$, we obtain

$$\begin{aligned}\varphi^*_i(t) &= E_{\alpha_i}(-\rho t^{\alpha_i})\phi_i(0) \\ &+ \int_0^t (t-s)^{\alpha_i-1}E_{\alpha_i,\alpha_i}(-\rho(t-s)^{\alpha_i}) \\ &\times \sum_{1\le j\le d}\big((a_{ij}+\rho\delta_{ij})\varphi^*_j(s) \\ &+ \sum_{1\le k\le m} b^k_{ij}\varphi^*_j(s-h_k(s)) + u_{ij}w_j(s)\big)ds,\end{aligned}$$

for $1 \le i \le d$, $t \in [0,T]$ and $\varphi^*(t) = \phi(t)$ on $[-r,0]$. Since the original system has a unique solution (see Lemma II.1) and it has the form as in (4), $\varphi^*(\cdot)$ is the unique solution of this system. On the other hand, as shown above, $\varphi^*(t) \succeq 0$ on $[-r,T]$, which implies that $\varphi(\cdot,\phi)$ is nonnegative on the existence interval $[0,T]$. The proof is complete. □

Remark II.4. In the classical case, to prove the positivity of the time-delay system

$$\begin{cases}\frac{dx(t)}{dt} = Ax(t) + \sum_{1\le k\le m} B_k x(t-h_k), \quad t\ge 0,\\ x(t)=\phi(t)\in\mathbb{R}^d, \quad t\in[-r,0],\end{cases}$$

one usually adopts the following representation for its solution on $[0,\infty)$

$$x(t) = \exp(tA)\phi(0) + \int_0^t \exp((t-s)A)\sum_1^m B_k x(s-h_k))ds,$$

see, e.g., [23, Proposition 3.1]. In our opinion, this approach is also true for time-delay systems with a non-integer derivative. However, it does not work for mixed fractional-order systems because there is not a similar variation of constants formula for the solution to these systems.

III. Asymptotic stability of positive linear mixed-order fractional systems with time-varying delays

Let $\hat\alpha = (\alpha_1,\dots,\alpha_d)^{\mathrm{T}} \in (0,1]\times\cdots\times(0,1] \subset \mathbb{R}^d\times\cdots\times\mathbb{R}^d$, $r>0$, $m\in\mathbb{N}$. In this section, we consider the following linear mixed-order fractional system on $(0,\infty)$

$$^{C}D^{\hat\alpha}_{0+}x(t) = Ax(t) + \sum_{1\le k\le m} B_k x(t-h_k(t)) \tag{5}$$

with $x(\cdot)=\phi(\cdot)\in C([-r,0];\mathbb{R}^d)$ on $[-r,0]$, where $A, B_k\in\mathbb{R}^{d\times d}$, $h_k:[0,\infty)\to\mathbb{R}_+$ $(1\le k\le m)$ is continuous and satisfies the following conditions

(G1) $h_k(0)>0$;
(G2) $t-h_k(t)\ge -r$ for all $t\in[0,\infty)$;
(G3) $h_k(0)\ne h_l(0)$ for any $1\le k\ne l\le m$
(G4) $\lim_{t\to\infty} t-h_k(t)=\infty$ $(1\le k\le m)$.

For linear systems, the asymptotic stability in the Lyapunov sense and the attractivity are equivalent, see e.g., [24, Theorem 6]. Hence, in this paper, we use the following definition for the asymptotic stability of system (5).

Definition III.1. System (5) is said to be asymptotically stable if for any $\phi(\cdot)\in C([-r,0];\mathbb{R}^d)$, its solution $\varphi(\cdot,\phi)$ converges to the origin as $t\to\infty$.

Based on Proposition II.3 about the positivity of time-delay linear fractional-order systems, we obtain a necessary and sufficient condition for the asymptotic stability of positive linear mixed-order fractional systems with unbounded time-varying delays in the following theorem.

Theorem III.2 (A characterization of the asymptotic stability of linear mixed fractional-order systems with unbounded time-varying delays)**.** Assume that system (5) is positive. Then, it is asymptotically stable if and only if $A+\sum_{1\le k\le m}B_k$ is Hurwitz.

Proof. **Necessity:** Let the positive system (5) be asymptotically stable. Suppose, ad absurdum, $A+\sum_{1\le k\le m}B_k$ is not Hurwitz. Notice that A is Metzler and B_k $(1\le k\le m)$ is nonnegative and thus $A+\sum_{1\le k\le m}B_k$ is also Metzler. From [25, Theorem 2.5.3, p. 114], we have $(A+\sum_{1\le k\le m}B_k)\lambda\succeq 0$ for any $\lambda\succ 0$. Choose and fix such a positive vector $\lambda\in\mathbb{R}^d$, and put $e_0(t):=\varphi(t,\lambda)-\lambda$ for all $t\in[-r,\infty)$. Then, $e_0(\cdot)$ is the unique solution to the system

$$\begin{aligned}^{C}D^{\hat\alpha}_{0+}x(t) &= Ax(t) + \sum_{1\le k\le m} B_k x(t-h_k(t))\\ &\quad + (A+\sum_{1\le k\le m}B_k)\lambda, \quad t>0, \\ x(\cdot) &= 0 \quad \text{on } [-r,0].\end{aligned} \tag{6}$$

On the other hand, by virtue Proposition II.3, system (6) is positive. Hence, $e_0(t)\succeq 0$ on $[0,\infty)$. This implies that $\varphi(t,\lambda)\succeq\lambda\succ 0$, $\forall t\in[0,\infty)$. It is a contradiction because from the original assumption, $\lim_{t\to\infty}\varphi(t,\lambda)=0$.

Sufficiency: Let $A+\sum_{1\le k\le m}B_k$ be Hurwitz. By virtue [25, Theorem 2.5.3, p. 114], we can find a vector $\lambda\succ 0$ such that

$$(A+\sum_{1\le k\le m}B_k)\lambda\prec 0. \tag{7}$$

First step: In this step, we will prove that there exists $t_1>0$ and $\nu\in(0,1)$ such that

$$\varphi(t,\lambda)\prec\nu\lambda, \quad \forall t\ge t_1. \tag{8}$$

For that, at first, let $u_0(t)=\lambda-\varphi(t,\lambda)$, $t\ge -r$. Then, $u_0(\cdot)$ is the unique solution of the system

$$\begin{aligned}^{C}D^{\hat\alpha}_{0+}u_0(t) &= Au_0(t) + \sum_{1\le k\le m} B_k u_0(t-h_k(t)) -\\ &\quad (A+\sum_{1\le k\le m}B_k)\lambda, \quad t>0,\\ u_0(t) &= 0, \quad t\in[-r,0].\end{aligned}$$

This system is positive, hence, $u_0(t)\succeq 0$ on $[0,\infty)$, which implies that $\varphi(t,\lambda)\preceq\lambda$ for all $t\ge 0$. Next, let $y(\cdot)$ is the unique solution of the system

$$\begin{cases}^{C}D^{\hat\alpha}_{0+}y(t) = Ay(t) + \sum_{1\le k\le m}B_k\lambda, \quad t>0,\\ y(0)=\lambda.\end{cases} \tag{9}$$

Using the same arguments as above, we see that $0\prec y(t)\preceq\lambda$ for all $t\ge 0$. Moreover,

$$0\preceq\varphi(t,\lambda)\preceq y(t), \quad t\ge 0. \tag{10}$$

Now, for any $c>0$, define $u_1(t)=y(t)-y(t+c)$, $t\ge 0$. This vector valued function satisfies the system

$$\begin{cases}^{C}D^{\hat\alpha}_{0+}u_1(t) = Au_1(t), \quad t>0,\\ u_1(0)\succeq 0.\end{cases} \tag{11}$$

Due to the fact that system (11) is positive, $u_1(t)\succeq 0$ for all $t\ge 0$, that is, $y(t)\succeq y(t+c)$, for all $t\ge 0$. In particular,

(S1) $0\prec y(t)\preceq\lambda$ for all $t\ge 0$;

(S2) $y(\cdot)$ is decreasing on $[0,\infty)$.

From (S1) and (S2), the limit $\lim_{t\to\infty} y(t)$ exists. Put $y^* = \lim_{t\to\infty} y(t)$ and denote by $\mathcal{L}$ the Laplace transform. In light of the Final value theorem (see, e.g., [18, Theorem D13]), we obtain

$$\begin{aligned}\lim_{s\to+0} s\mathcal{L}\{{}^C D^{\hat\alpha}_{0+}y(\cdot)\} &= \lim_{t\to\infty} {}^C D^{\hat\alpha}_{0+}y(t)\\ &= \lim_{t\to\infty}(Ay(t) + \sum_{1\le k\le m} B_k\lambda)\\ &= Ay^* + \sum_{1\le k\le m} B_k\lambda.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\lim_{s\to+0} s\mathcal{L}\{{}^C D^{\hat\alpha}_{0+}y(\cdot)\}\\ &= \lim_{s\to+0} s[s^{\alpha_1}\mathcal{L}\{y_1(\cdot)\}(s) - s^{\alpha_1-1}\lambda_1,\\ &\quad \dots, s^{\alpha_d}\mathcal{L}\{y_d(\cdot)\}(s) - s^{\alpha_d-1}\lambda_d]\\ &= \lim_{s\to+0}[s^{\alpha_1}(s\mathcal{L}\{y_1(\cdot)\}(s) - \lambda_1),\\ &\quad \dots, s^{\alpha_d}(s\mathcal{L}\{y_d(\cdot)\}(s) - \lambda_d)]\\ &= 0\end{aligned}$$

due to the fact that, for all $1 \le j \le d$,

$$\lim_{s\to+0} s\mathcal{L}\{y_j(\cdot)\}(s) = \lim_{t\to\infty} y_j(t) = y_j^*.$$

This leads to that $y^* = \lim_{t\to\infty} y(t) = -A^{-1}\sum_{1\le k\le m} B_k\lambda$. Note that A is Metzler and Hurwitz. From [25, Theorem 2.5.3, p. 114], $-A^{-1} \succeq 0$ which together with (7) implies that

$$\lim_{t\to\infty} y(t) = -A^{-1}\sum_{1\le k\le m} B_k\lambda \prec \lambda. \tag{12}$$

By combining (10) and (12), we can find $t_1 > 0$ and $\nu \in (0,1)$ such that the estimate (8) holds.

<u>Second step:</u> In this step, we will show that there exists an increasing sequence $\{T_n\}_{n=0}^\infty$ with $T_0 = 0$ and $\lim_{n\to\infty} T_n = \infty$ such that for any $n \in \mathbb{Z}_{\ge 0}$,

$$\varphi(t,\lambda) \preceq \nu^n\lambda, \quad \forall t \in [T_n, T_{n+1}]. \tag{13}$$

To do this, we use a proof by induction. From (G4), there exists $\hat t_1 > t_1$ such that $t - h_k(t) \ge t_1$ for all $t \ge \hat t_1$, $1 \le k \le m$. Put $T_1 := \hat t_1$. Then, (13) holds for $n = 0$ and $\varphi(t,\lambda) \preceq \nu\lambda$ for all $t \ge T_1$.

Next, define $y_1(t) = \varphi(t+T_1,\lambda)$, $t \ge 0$. Then, $y_1(\cdot)$ satisfies the system

$$\begin{cases}{}^C D^{\hat\alpha}_{0+}y_1(t) = Ay_1(t) + \sum_1^m B_k f_k(t),\ t > 0,\\ y_1(0) = \varphi(T_1,\lambda),\end{cases} \tag{14}$$

where $f_k(t) = \varphi(t + T_1 - h_k(t+T_1),\lambda)$, $t \ge 0$. Thus, $0 \preceq f_k(t) \preceq \nu\lambda$ for all $t \ge 0$. Now, consider the system

$$\begin{cases}{}^C D^{\hat\alpha}_{0+}z_1(t) = Az_1(t) + \sum_1^m B_k\nu\lambda,\ t > 0,\\ z_1(0) = \nu\lambda.\end{cases} \tag{15}$$

By the comparison principle for solutions of (14) and (15) and the similar arguments as shown above, we obtain

- $0 \preceq y_1(t) \preceq z_1(t) \preceq \nu\lambda$ for all $t \ge 0$;
- $\lim_{t\to\infty} z_1(t) = -A^{-1}\sum_{1\le k\le m} B_k\nu\lambda$.

Hence, there exists $t_2 > 0$ such that $\varphi(t+T_1,\lambda) = y_1(t) \preceq \nu^2\lambda$ for all $t \ge t_2$. Take $\hat t_2 = T_1 + t_2$, then $\varphi(t,\lambda) \preceq \nu^2\lambda$ for all $t \ge \hat t_2$. Using (G4) again, we have $T_2 > \hat t_2$ so that $t - h_k(t) \ge \hat t_2$ for all $t \ge T_2$, $1 \le k \le m$. Thus, (13) holds for $n = 1$ and $\varphi(t,\lambda) \preceq \nu^2\lambda$ for all $t \ge T_2$. By a similar procedure, we also see that (13) holds for $n = 2, 3, \dots,$ and thus the proof of *Second step* is complete.

<u>Third step:</u> From (13), we see that $\lim_{t\to\infty}\varphi(t,\lambda) = 0$. Let $\phi(\cdot) \in C([-r,0];\mathbb{R}^d_+)$ be arbitrary. There is a positive constant γ such that

$$\phi(t) \preceq \gamma\lambda, \quad t \in [-r,0].$$

Due to the positivity, the linearity and the existence and uniqueness of solutions of system (5), we have

$$\varphi(t,\phi) \preceq \varphi(t,\gamma\lambda) = \gamma\varphi(t,\lambda), \quad t \ge 0.$$

Thus,

$$0 \preceq \lim_{t\to\infty}\varphi(t,\phi) \preceq \gamma\lim_{t\to\infty}\varphi(t,\lambda) = 0.$$

This shows that system (5) is asymptotically stable. □

Remark III.3. In [16, Theorem 2], the authors studied the asymptotic stability of linear mixed fractional-order with a bounded time-varying delay

$$\begin{cases}{}^C D^{\hat\alpha}_{0+}x(t) = Ax(t) + Bx(t-\tau(t)), \quad t \ge 0,\\ x(t) = \varphi(t) \in \mathbb{R}^d, \quad t \in [-r,0],\end{cases} \tag{16}$$

where $0 \le \tau(t) \le r$ for all $t \ge 0$. Assume that $\lambda \succ 0$ satisfying $(A+B)\lambda \prec 0$. Their approach is to compare the solution $\varphi(\cdot,\lambda)$ of the system (16) with the solution of the following system

$$\begin{cases}{}^C D^{\hat\alpha}_{0+}x(t) = Ax(t) + Bx(t-r), \quad t \ge 0,\\ x(t) = \lambda, \quad t \in [-r,0].\end{cases}$$

It is easy to see that this approach cannot be applied for the case where the delay $\tau(\cdot)$ is not bounded which is the main objective in our research.

Finally, we give an example to illustrate the effectiveness of the proposed result.

Example III.4. Let $\hat\alpha = (\alpha_1,\alpha_2,\alpha_3)^{\mathrm{T}} \in (0,1]\times(0,1]\times(0,1]$, and continuous function $h : [0,\infty) \to \mathbb{R}_{\ge 0}$ be defined by $h(t) = \frac{t\sin^2 t}{2} + 1$ for all $t \ge 0$. Consider the following positive linear mixed-order fractional system with an unbounded time-varying delay

$${}^C D^{\hat\alpha}_{0+}x(t) = Ax(t) + Bx(t-h(t)), \quad t \ge 0, \tag{17}$$

where

$$A = \begin{pmatrix}-5 & 1 & 0\\ 0.5 & -4 & 0.5\\ 1 & 0 & -6\end{pmatrix}, \quad B = \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}.$$

We see that the delay $h(t)$ satisfies assumptions (G1), (G2) and (G4), A is Metzler, B is nonnegative and $A+B$ is

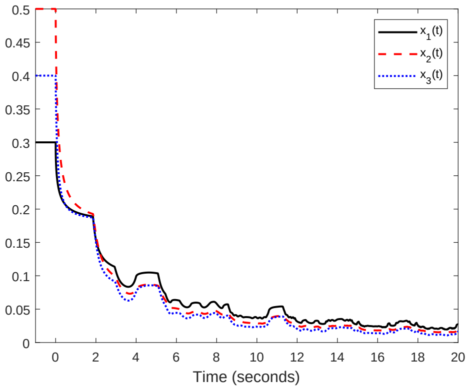

Fig. 1. Trajectories of the solution $\varphi(\cdot,\phi)$ to system (17) when $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\alpha_3 = 0.8$.

Hurwitz. Thus, by Theorem III.2, system (17) is asymptotically stable, that is, for any $\phi(\cdot) \in C([-1,0];\mathbb{R}^3)$, the solution $\varphi(t,\phi) \to 0$ as $t \to \infty$. In Figure 1, we simulate the trajectories of the solution $\varphi(\cdot,\phi)$ to system (17) when $\alpha_1 = 0.5$, $\alpha_2 = 0.7$, $\alpha_3 = 0.8$ and the initial condition as $\phi(t) = (0.3, 0.5, 0.4)^{\rm T}$ on the interval $[-1,0]$.

IV. Conclusion

In this paper, by using a new weighted type norm which is adaptive with time-delay systems, we have obtained a result on the existence and uniqueness of solutions to linear mixed fractional-order systems with time-varying delays. Then, by using the integral representation of solutions, we have derived a necessary and sufficient condition for the positivity of these systems. Finally, by comparing the trajectories of solutions of the time-delay system with that of inhomogeneous systems having the inhomogeneous parts constant and decreasing on time and the inductive principle, we have established a necessary and sufficient criterion to guarantee the asymptotic stability of positive linear mixed fractional-order systems with both multiple bounded and unbounded time-varying delays.

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