A LEMMA ABOUT MEROMORPHIC FUNCTIONS SHARING A SMALL FUNCTION

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Abstract. We say that two meromorphic functions f and g *share a small function* α *counting multiplicities* if $f-\alpha$ and $g-\alpha$ admit the same zeros with the same multiplicities. Let Q be a polynomial of one variable. In [1, Theorem 1.1], we proved that if $(Q(f))^{(k)}$ and $(Q(g))^{(k)}$ share α counting multiplicities then, with suitable conditions on the degree of Q and on the number of zeros and the multiplicities of the zeros of Q', there are explicit relation between $Q(f)$ and $Q(g)$. Unfortunately, there is a gap at the beginning of the proof of Theorem [1, Theorem 1.1]. We will give a way to fix the gap. This proof also can be used to fix for the gaps of other authors's published literatures which were listed in [5].

Let $Q(z)$ be a polynomial of degree q in $\mathbb{C}$ and k be a positive integer. Denote by

$$Q'(z) = b\prod_{i=1}^{l}(z-\zeta_i)^{m_i}$$

with $b \in \mathbb{C}^*$, and denote by v and h the indexes such that $1 \le v \le h \le l$, and

$$m_1 \ge m_2 \ge \cdots \ge m_v > k \ge m_{v+1} \ge \cdots \ge m_l,$$
$$m_1 \ge m_2 \ge \cdots \ge m_h \ge k > m_{h+1} \ge \cdots \ge m_l.$$

Let f and g be meromorphic functions. A meromorphic function α is called *a small function respect to* f if it satisfies $T(r,\alpha) = o(T(r,f))$ as $r \to +\infty$ outside of a possible exceptional set with finite measure. Here, $T(r,f)$ denotes the Nevanlinna characteristic of f. In [1, Theorem 1.1], we proved that

Theorem 1. [1, Theorem 1.1] *Let f and g be nonconstant meromorphic functions, and let α be a non-zero small function with respect to f. Suppose $[Q(f)]^{(k)}$ and $[Q(g)]^{(k)}$ share α counting multiplicities. If $q > k + 6 + 2v(k+1) + 2\sum_{i=v+1}^{l} m_i$ then one of the following holds:*

(i) *$Q(f) = Q(g) + c$, for some constant c.*
(ii) *$[Q(f)]^{(k)}[Q(g)]^{(k)} = \alpha^2$.*

To prove Theorem [1, Theorem 1.1], similar to many other authors which were listed in [5], we used the following claim:

Claim 1. Denote by $F = [Q(f)]^{(k)}$ and $G = [Q(g)]^{(k)}$. If F and G share α counting multiplicities, then $\dfrac{F}{\alpha}$ and $\dfrac{G}{\alpha}$ share 1 counting multiplicities.

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Thank to Claim 1, the problem of meromorphic functions sharing a small functions can be converted to the problem of meromorphic functions sharing 1. Therefore, we can apply [6, Lemma 3] for the case $\dfrac{F}{\alpha}$ and $\dfrac{G}{\alpha}$ share 1. Claim 1 is used in the proofs of [3, Theorems 3,4, and 5], [4, Theorems 1.1 and 1.2], [7, Theorems 1.1, 1.2, and 1.5], and [8, Theorem 2] for meromorphic functions sharing a small functions.

Unfortunately, in [5] Schweizer pointed out Claim 1 is not correct and he gave counterexamples to show that F and G share α counting multiplicities, but $\dfrac{F}{\alpha}$ and $\dfrac{G}{\alpha}$ do not share 1.

To help Claim 1 can be hold, Schweizer gave some new definitions of "sharing a small function" by adding more conditions. The definitions, however, are not nature since they are not related with the generalizations of the Second Main Theorem that involve the counting function of the zeros of $F - \alpha$. On the other hands, from his definitions, the results have to be restricted on a subclass of meromorphic functions (see Definition 3, 4 and 5 in [5]). Some other authors tried to avoid the gap by adding the extra conditions on α, which do not have neither common poles nor common zeros with f and with g; or in [2, Definition 1.3], they consider a definition of weakly sharing.

In this paper, we will improve [6, Lemma 3] for a small function (see Lemma 2 as following). Thanks to Lemma 2, we do not need to use Claim 1 in the proof of Theorem 1 as well as in the proof of the theorems cited above. Therefore, the gap in all of above listed publications can be fixed if we replace [6, Lemma 3] by Lemma 2. Again, by Lemma 2, any additional restrictions for meromorphic functions as well as weakening the definition of sharing can be omitted.

First, we recall some notations and lemmas. Denote by $N_{p)}(r, f)$ the counting function of poles of f which have multiplicity at most p, each pole counted with its multiplicity, by $N_{(p}(r, f)$ the counting function of poles of f which have multiplicity at least p, each pole counted with its multiplicity, and the corresponding reduced counting functions are denoted by $\overline{N}_{p)}(r, f)$ and $\overline{N}_{(p}(r, f)$. Furthermore, we denote by $N_{1)}\big(r, \frac{1}{f-\alpha}|\alpha \neq 0, \alpha \neq \infty\big)$ the counting function of simple zeros of $f - \alpha$ which are not zeros or poles of α, by $N_{1)}\big(r, \frac{1}{f-\alpha}|\alpha = 0\big)$ the counting function of simple zeros of $f - \alpha$ which are zeros of α and by $N_{1)}\big(r, \frac{1}{f-\alpha}|\alpha = \infty\big)$ the counting function of simple zeros of $f - \alpha$ which are poles of α.

Lemma 1. [9, Lemma 2.4] *Let f be a non-constant meromorphic function, and let p and k be two positive integers. Then*

$$N_p\Big(r, \frac{1}{f^{(k)}}\Big) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}\Big(r, \frac{1}{f}\Big) + S(r, f),$$
$$N_p\Big(r, \frac{1}{f^{(k)}}\Big) \leq k\overline{N}(r, f) + N_{p+k}\Big(r, \frac{1}{f}\Big) + S(r, f).$$

Moreover, if $f^{(k)} \not\equiv 0$, then

$$N\Big(r, \frac{1}{f^{(k)}}\Big) \leq k\overline{N}(r, f) + N\Big(r, \frac{1}{f}\Big) + S(r, f).$$

Our result as following:

Lemma 2. *Let f and g be two non-constant meromorphic functions, and let α be a non-zero small function with respect to f and g. If f and g share α counting multiplicities, then precisely one of the following statements holds:*

(i) $T(r,f) \le N_2(r,f) + N_2(r,g) + N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + S(r,f) + S(r,g)$, *and* $T(r,g) \le N_2(r,f) + N_2(r,g) + N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + S(r,f) + S(r,g)$;
(ii) $f \equiv g$;
(iii) $fg \equiv \alpha^2$.

Proof. Set

$$F = \frac{f}{\alpha}, \qquad G = \frac{g}{\alpha}, \tag{1}$$

and

$$H = \frac{F''}{F'} - 2\frac{F'}{F-1} - \frac{G''}{G'} + 2\frac{G'}{G-1}. \tag{2}$$

Let $z_0 \notin \{z : \alpha(z) = 0\} \cup \{z : \alpha(z) = \infty\}$ be a common simple zero of $f - \alpha$ and $g - \alpha$. Then, it follows from (1) that z_0 is a common simple zero of $F - 1$ and $G - 1$. Hence, in some neighborhood of z_0, we have

$$\begin{aligned} F(z) &= 1 + a_1(z - z_0) + a_2(z - z_0)^2 + O((z - z_0)^3), \quad a_1 \ne 0, \\ G(z) &= 1 + b_1(z - z_0) + b_2(z - z_0)^2 + O((z - z_0)^3), \quad b_1 \ne 0. \end{aligned}$$

By (2) and a simple computation, we obtain

$$\begin{aligned} H(z) =& \frac{2a_2 + O(z - z_0)}{a_1 + 2a_2(z - z_0) + O((z - z_0)^2)} - \frac{2b_2 + O(z - z_0)}{b_1 + 2b_2(z - z_0) + O((z - z_0)^2)} \\ & - \frac{2(a_2b_1 - a_1b_2) + O(z - z_0)}{[a_1 + a_2(z - z_0) + O((z - z_0)^2)][b_1 + b_2(z - z_0) + O((z - z_0)^2)]}. \end{aligned}$$

Therefore, we have $H(z_0) = 0$.

Suppose that $H \not\equiv 0$. Since f and g share α counting multiplicities, we have

$$\begin{aligned} N_{1)}\big(r, \frac{1}{g - \alpha}\big) &= N_{1)}\big(r, \frac{1}{f - \alpha}\big) \\ &= N_{1)}\big(r, \frac{1}{f - \alpha}|\alpha \ne 0, \alpha \ne \infty\big) + N_{1)}\big(r, \frac{1}{f - \alpha}|\alpha = 0\big) \\ &\quad + N_{1)}\big(r, \frac{1}{f - \alpha}|\alpha = \infty\big) \\ &\le N_{1)}\big(r, \frac{1}{F - 1}\big) + N\big(r, \frac{1}{\alpha}\big) + N(r, \alpha) \\ &\le N\big(r, \frac{1}{H}\big) + S(r, f) \\ &\le T(r, H) + S(r, f) \le N(r, H) + S(r, f) + S(r, g). \end{aligned} \tag{3}$$

Let $z_1 \notin \{z : \alpha(z) = 0\} \cup \{z : \alpha(z) = \infty\}$ be a common multiple zero of $f - \alpha$ and $g - \alpha$. Then, by (1), z_1 is a common multiple zero of $F - 1$ and $G - 1$ and by calculating, we get $H(z_1) \ne \infty$. In addition, by a simple computation, we can easily follow from (2) that any simple pole of F and G is not a pole of H. Thus, it follows from (2) that the poles of H can only occur at the multiple zeros of F, or the multiple zeros of G, or the multiple poles of F, or the multiple poles of G, or zeros or poles of α, or zeros of F' that are not the zeros of $F(F - 1)$, or zeros of G' that are not the zeros of $G(G - 1)$. We denote by $N_0\big(r, \frac{1}{F'}\big)$ the counting function of zeros of F' that are not the zeros of $F(F - 1)$, and by $\overline{N}_0\big(r,\frac{1}{F'}\big)$ the corresponding reduced counting function. Similarly, we can define $N_0\big(r,\frac{1}{G'}\big)$ and $\overline{N}_0\big(r,\frac{1}{G'}\big)$. Hence, from (1), (2) and the above observations, we get

$$\begin{aligned} N_{1)}\big(r,\frac{1}{f-\alpha}\big) &\le \overline{N}_{(2}(r,F)+\overline{N}_{(2}(r,G)+\overline{N}_{(2}\big(r,\frac{1}{F}\big)+\overline{N}_{(2}\big(r,\frac{1}{G}\big)\\ &\quad+\overline{N}_0\big(r,\frac{1}{F'}\big)+\overline{N}_0\big(r,\frac{1}{G'}\big)+\overline{N}(r,\alpha)+\overline{N}\big(r,\frac{1}{\alpha}\big)\\ &\quad+S(r,f)+S(r,g)\\ &\le \overline{N}_{(2}(r,F)+\overline{N}_{(2}(r,G)+\overline{N}_{(2}\big(r,\frac{1}{F}\big)+\overline{N}_{(2}\big(r,\frac{1}{G}\big)\\ &\quad+\overline{N}_0\big(r,\frac{1}{F'}\big)+\overline{N}_0\big(r,\frac{1}{G'}\big)+S(r,f)+S(r,g). \end{aligned} \tag{4}$$

On the other hand, from the definition of $N_0\big(r,\frac{1}{G'}\big)$, we have

$$\overline{N}_0\big(r,\frac{1}{G'}\big)+\overline{N}_{(2}\big(r,\frac{1}{G-1}\big)+N_{(2}\big(r,\frac{1}{G}\big)-\overline{N}_{(2}\big(r,\frac{1}{G}\big)\le N\big(r,\frac{1}{G'}\big). \tag{5}$$

Combining (5) and Lemma 1, we get

$$\overline{N}_0\big(r,\frac{1}{G'}\big)+\overline{N}_{(2}\big(r,\frac{1}{G-1}\big)\le \overline{N}(r,G)+\overline{N}\big(r,\frac{1}{G}\big)+S(r,g).$$

Hence, we have

$$\begin{aligned} \overline{N}_0\big(r,\frac{1}{G'}\big)+\overline{N}_{(2}\big(r,\frac{1}{g-\alpha}\big) &= \overline{N}_0\big(r,\frac{1}{G'}\big)+\overline{N}_{(2}\big(r,\frac{1}{\alpha(G-1)}\big)\\ &\le \overline{N}_0\big(r,\frac{1}{G'}\big)+\overline{N}_{(2}\big(r,\frac{1}{G-1}\big)+\overline{N}_{(2}\big(r,\frac{1}{\alpha}\big)\\ &\le \overline{N}(r,G)+\overline{N}\big(r,\frac{1}{G}\big)+S(r,g). \end{aligned} \tag{6}$$

Since f and g share α counting multiplicities, we have

$$\overline{N}_{(2}\big(r,\frac{1}{f-\alpha}\big)=\overline{N}_{(2}\big(r,\frac{1}{g-\alpha}\big). \tag{7}$$

By combining (1), (4), (6) and (7), we get

$$\begin{aligned} \overline{N}\big(r,\frac{1}{F-1}\big) &\le \overline{N}\big(r,\frac{1}{f-\alpha}\big)+\overline{N}(r,\alpha)\\ &\le N_{1)}\big(r,\frac{1}{f-\alpha}\big)+\overline{N}_{(2}\big(r,\frac{1}{f-\alpha}\big)+S(r,f)\\ &= N_{1)}\big(r,\frac{1}{f-\alpha}\big)+\overline{N}_{(2}\big(r,\frac{1}{g-\alpha}\big)+S(r,f)\\ &\le \overline{N}_{(2}(r,F)+\overline{N}_{(2}(r,G)+\overline{N}_{(2}\big(r,\frac{1}{F}\big)+\overline{N}_{(2}\big(r,\frac{1}{G}\big)\\ &\quad+\overline{N}_0\big(r,\frac{1}{F'}\big)+\overline{N}_0\big(r,\frac{1}{G'}\big)+\overline{N}_{(2}\big(r,\frac{1}{g-\alpha}\big)+S(r,f)+S(r,g)\\ &\le \overline{N}_{(2}(r,F)+\overline{N}_{(2}\big(r,\frac{1}{F}\big)+N_2(r,G)+N_2\big(r,\frac{1}{G}\big)\\ &\quad+\overline{N}_0\big(r,\frac{1}{F'}\big)+S(r,f)+S(r,g). \end{aligned} \tag{8}$$

Applying the Second Main Theorem for F and $0, \infty$ and 1, we have

$$T(r,F) \le \overline{N}(r,F) + \overline{N}\big(r,\frac{1}{F}\big) + \overline{N}\big(r,\frac{1}{F-1}\big) - N_0\big(r,\frac{1}{F'}\big) + S(r,f). \tag{9}$$

Therefore, by combining (8) and (9), we have

$$T(r,F) \le N_2(r,F) + N_2(r,G) + N_2\big(r,\frac{1}{F}\big) + N_2\big(r,\frac{1}{G}\big) + S(r,f) + S(r,g). \tag{10}$$

It follows from (2) that

$$N_2(r,F) \le N_2(r,f) + N_2\big(r,\frac{1}{\alpha}\big) \le N_2(r,f) + S(r,f),$$

$$N_2\big(r,\frac{1}{F}\big) \le N_2\big(r,\frac{1}{f}\big) + N_2(r,\alpha) \le N_2\big(r,\frac{1}{f}\big) + S(r,f),$$

and similar inequalities

$$N_2(r,G) \le N_2(r,g) + S(r,g), \quad N_2\big(r,\frac{1}{G}\big) \le N_2\big(r,\frac{1}{g}\big) + S(r,g).$$

By combining these inequalities and (10), we obtain

$$T(r,F) \le N_2(r,f) + N_2(r,g) + N_2\big(r,\frac{1}{f}\big) + N_2\big(r,\frac{1}{g}\big) + S(r,f) + S(r,g). \tag{11}$$

On the other hand,

$$T(r,F) = T\big(r,\frac{f}{\alpha}\big) \ge T(r,f) - T(r,\alpha) + O(1). \tag{12}$$

The inequalities (10) and (12) imply (i).

Suppose that $H \equiv 0$. We deduce from (2) that

$$F = \frac{(b+1)G + a - b - 1}{bG + a - b},$$

where a, b are finite complex numbers and $a \ne 0$.

If $b \ne 0, -1$, then

$$F - \frac{b+1}{b} = -\frac{a}{b(bG + a - b)}.$$

Applying the Second Main Theorem for F and $0, \infty$ and $\frac{b+1}{b}$, we have

$$\begin{aligned} T(r,F) &\le \overline{N}(r,F) + \overline{N}\big(r,\frac{1}{F - \frac{b+1}{b}}\big) + \overline{N}\big(r,\frac{1}{F}\big) + S(r,f) \\ &\le \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}\big(r,\frac{1}{F}\big) + S(r,f) \end{aligned}$$

Hence, we get

$$\begin{aligned} T(r,f) &\le T(r,F) + S(r,f) \\ &\le \overline{N}(r,f) + \overline{N}(r,g) + \overline{N}\big(r,\frac{1}{f}\big) + S(r,f) + S(r,g), \end{aligned}$$

which implies (i).

If $b=0$, then $F=\dfrac{G+a-1}{a}$. If $a=1$, then $F=G$ which implies (ii). If $a\neq 1$, applying the Second Main Theorem for F and $0,\infty$ and $\dfrac{a-1}{a}$, we have

$$\begin{aligned}T(r,f) &\le T(r,F)+S(r,f)\\ &\le \overline{N}(r,F)+\overline{N}\big(r,\frac{1}{F-\frac{a-1}{a}}\big)+\overline{N}\big(r,\frac{1}{F}\big)+S(r,f)\\ &\le \overline{N}(r,F)+\overline{N}(r,\frac{1}{G})+\overline{N}\big(r,\frac{1}{F}\big)+S(r,f)\\ &\le \overline{N}(r,f)+\overline{N}(r,\frac{1}{g})+\overline{N}\big(r,\frac{1}{f}\big)+S(r,f)+S(r,g).\end{aligned}$$

We obtain (i).

If $b=-1$, then $F=\dfrac{a}{a+1-G}$. If $a\neq -1$, applying the Second Main Theorem for F and $0,\infty$ and $\dfrac{a}{a+1}$, we have

$$\begin{aligned}T(r,f) &\le T(r,F)+S(r,f)\\ &\le \overline{N}(r,F)+\overline{N}\big(r,\frac{1}{F-\frac{a}{a+1}}\big)+\overline{N}\big(r,\frac{1}{F}\big)+S(r,f)\\ &\le \overline{N}(r,F)+\overline{N}(r,\frac{1}{G})+\overline{N}\big(r,\frac{1}{F}\big)+S(r,f)\\ &\le \overline{N}(r,f)+\overline{N}(r,\frac{1}{g})+\overline{N}\big(r,\frac{1}{f}\big)+S(r,f)+S(r,g).\end{aligned}$$

Therefore, we get (i). If $a=-1$, then $FG\equiv 1$ which implies (iii). □

We now already for the proof of Theorem [1, Theorem 1.1] by using Lemma 2 replace to Lemma 3.3 in [1]. For a convenience of readers, we would like to explain as following.

Proof of Theorem 1. We denote

$$F:=[Q(f)]^{(k)},\qquad F_1:=Q(f),$$
$$G:=[Q(g)]^{(k)},\qquad G_1:=Q(g).$$

It is easy to see that

$$S(r,F)=S(r,f),\quad\text{and}\quad S(r,G)=S(r,g).$$

Since F and G share α counting multiplicities, applying Lemma 2, one of the following cases holds:

(i) $T(r,F)\le N_2\Big(r,\frac{1}{F}\Big)+N_2\Big(r,\frac{1}{G}\Big)+N_2(r,F)+N_2(r,G)+S(r,F)+S(r,G)$, and $T(r,G)\le N_2\Big(r,\frac{1}{F}\Big)+N_2\Big(r,\frac{1}{G}\Big)+N_2(r,F)+N_2(r,G)+S(r,F)+S(r,G)$,

(ii) $FG\equiv\alpha^2$,

(iii) $F\equiv G$.

If Case (i) holds, from Lemma 1 with $(G_1')^{(k-1)} = G$, we obtain

$$\begin{aligned} T(r,F) &\le N_2\Big(r,\frac{1}{F}\Big) + N_2\Big(r,\frac{1}{G}\Big) + N_2(r,F) + N_2(r,G) + S(r,f) + S(r,g) \\ &\le N_2\Big(r,\frac{1}{F}\Big) + (k-1)\overline{N}(r,G_1') + N_{k+1}\Big(r,\frac{1}{G_1'}\Big) + N_2(r,F) + N_2(r,G) \\ &\quad + S(r,f) + S(r,g). \end{aligned}$$

Then, the rest of proof can be continued as in Theorem [1, Theorem 1.1].

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References

[1] Ta Thi Hoai An, Nguyen Viet Phuong, *Uniqueness theorems for differential polynomials sharing a small function*, Computational Methods and Function Theory, 17 (2017), 613-634.

[2] Banerjee and S. Mukherjee, *Nonlinear differential polynomials sharing a small function,* Archivum Mathematicum (Brno) 44 (2008), 41-56.

[3] K. Boussaf, A. Escassut, J. Ojeda, *Complex meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function,* Indag. Math. 24 (2013), 15-41.

[4] A. Chen, X. Wang, G. Zhang, *Unicity of meromorphic function sharing one small function with its derivative*, Int. J. Math. Math. Sci. 2010 Art.ID507454, 11pp.

[5] Andreas Schweizer, *What is the definition of two meromorphic functions sharing a small function?*, arxiv:1705.05048v2.

[6] C.C. Yang, X. Hua, *Uniqueness and value sharing of meromorphic functions*, Ann. Acad. Sci. Fenn. Math. 22 (1997) 395-406.

[7] X.-B. Zhang and J.-F. Xu, *Uniqueness of meromorphic functions sharing a small function and its applications*, Comput. Math. Appl. 61 (2011), 722-730.

[8] J.-L. Zhang and L.-Z. Yang, *Some results related to a conjecture of R. Brück concerning meromorphic functions sharing one small function with their derivatives,* Ann. Acad. Sci. Fenn. Math. 32 (2007), 141-149.

[9] J.L. Zhang, L.Z. Yang, *Some results related to a conjecture of R. Brück,* J. Inequal. Pure Appl. Math. 8 (1) (2007) Art. 18.

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