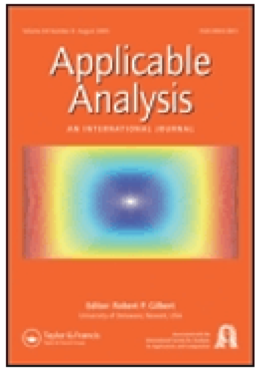

Optimal Economic Growth Problems with High Values of Total Factor Productivity

Journal:	*Applicable Analysis*
Manuscript ID	GAPA-2019-0901.R1
Manuscript Type:	Original Paper
Date Submitted by the Author:	n/a
Complete List of Authors:	Huong, Vu Thi; Vietnam Academy of Science and Technology,
Keywords:	optimal economic growth, optimal control, maximum principle, finite horizon, total factor productivity
2010 Mathematics Subject Classification:	91B62, 49J15, 37N40, 46N10, 91B55

Optimal Economic Growth Problems with High Values of Total Factor Productivity

Vu Thi Huong*

March 30, 2020

Abstract. This paper solves a question raised in the paper of Huong, Yao, and Yen [1] about optimal economic growth problems with production functions and utility functions being all in the form of AK functions. By using a solution existence theorem from the paper of Huong [2] and a maximum principle from the book of Vinter [3], we prove that the problem in question has a unique solution and give a comprehensive synthesis of the optimal processes. Our results show that if the value of total factor productivity is enough high and the planning time is short, then expanding the production facility does not lead to a higher total consumption satisfaction of the society. Meanwhile, if the value of total factor productivity is enough high and the planning time is relatively long, then the highest total consumption satisfaction of the society is attained only if the largest expansion of the production facility is made until a special time.

Keywords: optimal economic growth, optimal control, maximum principle, finite horizon, total factor productivity

2010 Mathematics Subject Classification: 91B62, 49J15, 37N40, 46N10, 91B55.

1 Introduction

Models of economic growth have played an essential role in economic and mathematical studies since the 30s of the twentieth century. Namely, they allow ones to analyze, plan, and predict relations between global factors, which include capital, labor force, production

*Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi 10307, Vietnam; email: vthuong@math.ac.vn; huong263@gmail.com.

technology, and national product, of a particular economy in a given planning interval of time. Ramsey [4], Harrod [5], Domar [6], Solow [7], Swan [8], and other authors have studied principal models and their basic properties. We refer to Barro and Sala-i-Martin [9] and Acemoglu [10] for details about the development of the economic growth theory. If one wants to define the consumption/saving curve to maximize a certain target of consumption satisfaction, then one has deal with the *optimal economic growth problem*. The problem was considered firstly by Ramsey [4] and extended by Cass [11] and Koopmans [12] afterwards.

In the recent paper Huong [2], we have studied the solution existence of finite horizon *optimal economic growth problems of an aggregative economy* (see, e.g., Takayama [13, Sections C and D in Chapter 5]). The results therein on the solution existence were obtained under some mild conditions on the *utility function* and the *per capita production function*, which are two major inputs of the model in question. In addition, the results for general problems were specified for typical ones with the utility function and the production function being either the *AK function* or the *Cobb–Douglas one* (see, e.g., Barro and Sala-i-Martin [9] and Takayama [13]). The just mentioned typical problems were classified into four classes. The question of finding optimal processes for these classes of problems is raised in Huong, Yao and Yen [1] where, under an additional assumption, the problems with production functions and utility functions being all in the form of AK functions have been solved. Namely, using a maximum principle in Vinter [3], Huong, Yao and Yen [1] have proved that the problem has a unique local solution, which is also a global one, provided that the data triple (A, σ, λ) playing as three (among six) parameters of the problem satisfies the strict linear inequality $A < \sigma + \lambda$. The meanings of the parameters are as follows: A expresses the *total factor productivity* (which is a measure of economic efficiency appearing in the description of the production function and depends on some intangible factors such as technological change, education, research and development, etc.), σ is the *rate of labor force* (closely related to the population growth), and λ is *the real interest rate* (which indicates the rate of the decrease along time of the satisfaction level of the society w.r.t. the same amount of consumption). From the obtained results it follows that *if the value of total factor productivity is relatively small, then an expansion of the production facility does not lead to a higher total consumption satisfaction of the society.*

As a continuation of Huong, Yao and Yen [1], the present paper aims at answering the question "*What happens if $A > \sigma + \lambda$?*", which was raised in [1]. The last inequality means that the total factor productivity A is relatively large. It was conjectured in [1, Sect. 4] that *the optimal strategy requires to make the maximum saving until a special time $\bar{t} \in (t_0, T)$, which depends on the data tube (A, σ, λ), then switch the saving to minimum.* Our main result (see Theorem 2.2 in next section) shows that the conjecture is just partly true, since the optimal strategy depends not only on the data triple (A, σ, λ), but also on the length of the planning interval. Namely, the following rules are valid:

(a) *If the total factor productivity A is enough high but the planning time is short, then an expansion of the production facility does not lead to a higher total consumption satisfaction of the society.* (This strategy is a must if $A < \sigma + \lambda$; see Huong, Yao and Yen [1, Remark 4.13].)

(b) *If the total factor productivity A is enough high and the planning time is relatively long, then the highest total consumption satisfaction of the society is attained only if the largest expansion of the production facility is made until a special time.*

It is worthy to note that the approach adopted herein has the origin in the preceding papers of Huong, Yao and Yen [1, 14, 15]. This means that we will combine a result on the solution existence and an intensive treatment of the system of necessary optimality conditions given by the maximum principle with some specific properties of the problem to obtain a comprehensive synthesis of optimal processes. In comparison with Huong, Yao and Yen [1], where the same problem was solved under the assumption $A < \sigma + \lambda$, to deal with the reserve inequality, herein we have to use more sophisticated techniques to determine the minimizer among several candidates for minimizers obtained from the maximum principle.

The organization of the paper is as follows. In Section 2, we recall the model of finite horizon optimal economic growth considered in Huong, Yao and Yen [1, Section 2], and formulate our main result. We also give a remark on economic interpretations of the latter. Section 3 contains three subsections. The first one is devoted to a maximum principle from Vinter [3], which is our main tool. Subsection 3.2 presents our preliminary analysis of the application of the maximum principle to the optimal economic growth problem in question. The main result is proved in Subsection 3.3.

2 Problem Formulation and the Result

By $\mathbb{R}$ (resp., $\mathbb{R}_+$) we denote the set of real numbers (resp., the set of nonnegative real numbers). The Euclidean norm in the n-dimensional space $\mathbb{R}^n$ is denoted by $\|.\|$. The *Sobolev space* $W^{1,1}([t_0, T], \mathbb{R}^n)$ (see, e.g., Ioffe and Tihomirov [16, p. 21]) is the linear space of the *absolutely continuous functions* $x : [t_0, T] \to \mathbb{R}^n$ equipped with the norm

$$\|x\|_{W^{1,1}} = \|x(t_0)\| + \int_{t_0}^{T} \|\dot{x}(t)\| dt.$$

For systematical expositions of optimal economic growth models, the interested reader is referred to Takayama [13, Chapter 5], Pierre [17, Chapters 5, 7, 10, and 11], Chiang and Wainwright [18, Chapter 20], and Acemoglu [10, Chapters 7 and 8]. The problem of *optimal growth of an aggregative economy* with all the related economic concepts was briefly presented in Huong [2, Subsection 2.1] and recalled in Huong, Yao and Yen [1, Section 2]. It is as follows:

$$\text{Maximize } \ I(k,s) := \int_{t_0}^{T} \omega[(1-s(t))\phi(k(t))]e^{-\lambda t}dt \tag{2.1}$$

over $k \in W^{1,1}([t_0,T],\mathbb{R})$ and measurable functions $s : [t_0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{k}(t) = s(t)\phi(k(t)) - \sigma k(t), & \text{a.e. } t \in [t_0,T] \\ s(t) \in [0,1], & \text{a.e. } t \in [t_0,T] \\ k(t_0) = k_0 \\ k(t) \geq 0, & \forall t \in [t_0,T], \end{cases} \tag{2.2}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\omega : \mathbb{R}_+ \to \mathbb{R}$ are given functions, $\sigma > 0$, $\lambda \geq 0$, $k_0 > 0$, and $T > t_0 \geq 0$ are given as five parameters. The problem (2.1)–(2.2) is denoted by (GP).

Recall that $k(t)$ and $s(t)$ respectively are the *capital-to-labor ratio* and the *propensity to save* at a time moment t in the *planning interval* $[t_0,T]$. The values $\phi(k)$, $k \geq 0$, of the *per capita production function* $\phi(\cdot)$ express the *outputs per capita*. The *utility function* $\omega(\cdot)$ depends on the variable c, which is the *per capita consumption*. The integral $\int_{t_0}^{T} \omega(c(t))e^{-\lambda t}dt$, where $\lambda \geq 0$ is the *real interest rate*, represents the total amount of the utility gained by the society on the time period $[t_0,T]$. Since $c(t) = (1-s(t))\phi(k(t))$ for all $t \geq 0$, the just-mentioned integral equals to the value $I(k,s)$ defined in (2.1). The parameters σ and k_0 in (2.2) respectively stand for the *rate of labor force* and the *initial capital-to-labor ratio*.

Note that (GP) is a *finite horizon optimal control problem of the Lagrange type*, where $s(\cdot)$ and $k(\cdot)$ play as the role of *control variable* and *state variable*, respectively. Besides, due to the appearance of the *state constraint* $k(t) \geq 0$ for $t \in [t_0,T]$, (GP) belong to the class of *optimal control problems with state constraints*.

For each $\alpha, \beta \in (0,1]$ and $A > 0$, let $\phi(k) = Ak^\alpha$ for all $k \geq 0$ and $\omega(c) = c^\beta$ for every $c \geq 0$ in (GP). Then, we have the optimal control problem

$$\text{Maximize } \ \int_{t_0}^{T} [1-s(t)]^\beta k^{\alpha\beta}(t)e^{-\lambda t}dt \tag{2.3}$$

over $k \in W^{1,1}([t_0,T],\mathbb{R})$ and measurable functions $s : [t_0,T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{k}(t) = Ak^\alpha(t)s(t) - \sigma k(t), & \text{a.e. } t \in [t_0,T] \\ s(t) \in [0,1], & \text{a.e. } t \in [t_0,T] \\ k(t_0) = k_0 \\ k(t) \geq 0, & \forall\, t \in [t_0,T] \end{cases} \tag{2.4}$$

with $\alpha \in (0,1]$, $\beta \in (0,1]$, $A > 0$, $\sigma > 0$, $\lambda \geq 0$, $k_0 > 0$, T and t_0, where $T > t_0 \geq 0$, being given as eight parameters. We will denote the problem in (2.3)–(2.4) by (GP_1).

Thus, apart from the five parameters σ, λ, k_0, t_0, T in (GP), in the description of (GP_1) there are three more parameters α, β, and A. Each choice of the pair (α, β) with α and β either in $(0, 1)$ or equal to 1 yields a typical problem with the production function and the utility function being either the *Cobb–Douglas function* or the *AK function* (see, e.g., [9] and [13]). Moreover, the production function w.r.t. the per capita production function in (GP_1) is given by $F(K, L) = AK^{\alpha}L^{1-\alpha}$, $K \geq 0, L \geq 0$ (see, e.g., [2, Section 4]). In the last formula, the exponent α (resp., $1 - \alpha$) refers to the *output elasticity of capital* (resp., the *output elasticity of labor*), which represents the share of the contribution of the capital K (resp., of the labor L) to the total product $F(K, L)$. Meanwhile, the coefficient A expresses the *total factor productivity*[1] (TFP). This measure of economic efficiency is calculated by dividing output by the weighted average of labour and capital input. TFP represents the increase in total production which is in excess of the increase that results from increase in inputs and depends on some intangible factors such as technological change, education, research and development, etc.

It was shown in Huong, Yao and Yen [1] that, if $\alpha = \beta = 1$ and the data triple (A, σ, λ) satisfies a certain strict linear inequality, then (GP_1) admits an explicit simple solution.

Theorem 2.1. ([1, Theorem 4.12]) *Consider (GP_1) with $\alpha = \beta = 1$ and suppose that the positive parameters A, σ, λ satisfy the inequality $A < \sigma + \lambda$. Then, (GP_1) possesses a unique solution $(\bar{k}, \bar{s})$, where $\bar{s}(t) = 0$ for almost every $t \in [t_0, T]$ and $\bar{k}(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$.*

It follows from Theorem 2.1 that if the total factor productivity A is *smaller* than the sum of the rate of labor force σ and the real interest rate λ, then keeping the saving equal 0 is the optimal strategy. In other words, *if the value of total factor productivity is relatively small, then an expansion of the production facility does not lead to a higher total consumption satisfaction of the society.*

In connection with Theorem 2.1, the question "*What happens if $A > \sigma + \lambda$?*" was asked in [1]. The last inequality means that the value of total factor productivity is relatively high. It was conjectured [1, Sect. 4] that *the optimal strategy requires to make the maximum saving until a special time $\bar{t} \in (t_0, T)$, which depends on the data tube (A, σ, λ), then switch the saving to minimum.* The forthcoming theorem, which is the main result in this paper, shows that the conjecture is just partly true, since the optimal strategy will be depended on not only the data triple (A, σ, λ) but also the length $T - t_0$ of the planning interval $[t_0, T]$.

Theorem 2.2. *Consider the problem (GP_1) with $\alpha = \beta = 1$ and suppose that the positive parameters A, σ, λ satisfy the inequality*

$$A > \sigma + \lambda. \tag{2.5}$$

[1] See, e.g., https://en.wikipedia.org/wiki/Total_factor_productivity.

Let

$$\rho = \frac{1}{\sigma+\lambda}\ln\frac{A}{A-(\sigma+\lambda)} \quad \textit{and} \quad \bar{t} = T-\rho. \tag{2.6}$$

Then, (GP_1) has a unique global solution $(\bar{k},\bar{s})$. Moreover, the following assertions hold:

(a) *If $T-t_0 \le \rho$ (i.e., $t_0 \ge \bar{t}$), then $(\bar{k},\bar{s})$ is given by*

$$\bar{s}(t) = 0, \quad a.e.\, t\in[t_0,T], \quad \textit{and} \quad \bar{k}(t) = k_0 e^{-\sigma(t-t_0)}, \quad \forall t\in[t_0,T].$$

(b) *If $T-t_0 > \rho$ (i.e., $t_0 < \bar{t}$), then $(\bar{k},\bar{s})$ is given by*

$$\bar{s}(t) = \begin{cases} 1, & a.e. \quad t\in[t_0,\bar{t}] \\ 0, & a.e. \quad t\in(\bar{t},T] \end{cases} \quad \textit{and} \quad \bar{k}(t) = \begin{cases} k_0 e^{(A-\sigma)(t-t_0)}, & t\in[t_0,\bar{t}] \\ k(\bar{t})e^{-\sigma(t-\bar{t})}, & t\in(\bar{t},T]. \end{cases}$$

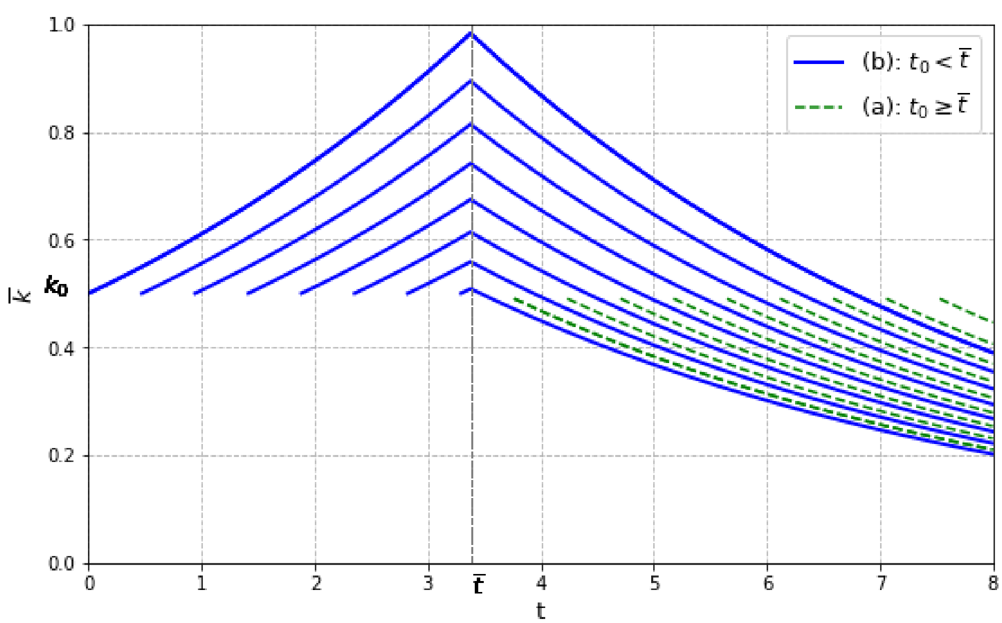

Figure 1: Optimal trajectories $\bar{k}(\cdot)$ of (GP_1) corresponding to parameters $\alpha = 1, \beta = 1$, $A = 0.4$, $\sigma = 0.1$, $\lambda = 0.2, k_0 = 0.5$, $T = 8$, and t_0 varying in $[0,T]$

The economic interpretation of Theorem 2.2 is as follows.

Remark 2.1. According to Theorem 2.2, if the total factor productivity A is higher than the sum of the rate of labor force σ and the real interest rate λ, then optimal strategy depends on the length of the planning time $[t_0,T]$. Namely, if the situation $T-t_0 \le \rho$ occurs, then keeping the saving equal to 0 is the optimal strategy. In the opposite situation where $T-t_0 > \rho$, the best strategy is to implement the maximum saving until the special time instance $\bar{t} = T-\rho$

in (t_0, T), where ρ is defined in (2.6), and switch the saving to minimum afterwards. So, *if the total factor productivity A is enough high but the planning time is short, then an expansion of the production facility does not lead to a higher total consumption satisfaction of the society.* (This strategy is a must if $A < \sigma + \lambda$; see [1, Remark 4.13].) Meanwhile, *if the total factor productivity A is enough high and the planning time is relatively long, then the highest total consumption satisfaction of the society is attained only if the largest expansion of the production facility is made until the special time instance $\bar{t}$.*

In the terminology of [2], a global solution $(\bar{k}, \bar{s})$ of (GP_1) is said to be *regular* if the propensity to save function $\bar{s}(\cdot)$ only has finitely many discontinuities of first type on $[t_0, T]$. By Lemma 4.11 in Huong, Yao and Yen [1], if $A \neq \sigma + \lambda$ is fulfilled, then any solution $(\bar{k}, \bar{s})$ of (GP_1) with $\alpha = \beta = 1$ is a regular one. Moreover, the control function $\bar{s}$ can have at most one switching point. This result is clearly illustrated by the explicit descriptions of the solution $(\bar{k}, \bar{s})$ of (GP_1) provided by Theorem 2.1 for the case where $A < \sigma + \lambda$ and by Theorem 2.2 for the case where $A > \sigma + \lambda$. We conclude this section by noting that problem (GP_1) with $\alpha = \beta = 1$ has not been solved in the case where $A = \sigma + \lambda$. In this case, it may happen that the problem can admit some irregular global solutions.

3 Proof of the Result

3.1 A Maximum Principle

As in Vinter [3, p. 321], we consider the following *finite horizon optimal control problem of the Mayer type*, denoted by $\mathcal{M}$,

$$\text{Minimize } \ g(x(t_0), x(T)), \tag{3.7}$$

over $x \in W^{1,1}([t_0, T], \mathbb{R}^n)$ and measurable functions $u : [t_0, T] \to \mathbb{R}^m$ satisfying

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & \text{a.e. } t \in [t_0, T] \\ u(t) \in U(t), & \text{a.e. } t \in [t_0, T] \\ (x(t_0), x(T)) \in C \\ h(t, x(t)) \leq 0, & \forall t \in [t_0, T], \end{cases} \tag{3.8}$$

where $[t_0, T]$ is a given interval, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $h : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}$ are given functions, $C \subset \mathbb{R}^n \times \mathbb{R}^n$ is a closed set, and $U : [t_0, T] \rightrightarrows \mathbb{R}^m$ is a set-valued map.

A measurable function $u : [t_0, T] \to \mathbb{R}^m$ satisfying $u(t) \in U(t)$ a.e. $t \in [t_0, T]$ is called a *control function*. A *process* (x, u) consists of a control function u and an arc

$x \in W^{1,1}([t_0,T];\mathbb{R}^n)$ that is a solution to the differential equation in (3.8). A *state trajectory* x is the first component of some process (x,u). A process (x,u) is called *feasible* if the state trajectory satisfies the *endpoint constraint* $(x(t_0),x(T)) \in C$ and the *state constraint* $h(t,x(t)) \le 0$ for all $t \in [t_0,T]$. Due to the appearance of the state constraint, the problem $\mathcal{M}$ in (3.7)–(3.8) is said to be an *optimal control problem with state constraints*.

Definition 3.1. A feasible process $(\bar{x},\bar{u})$ is called a $W^{1,1}$ *local minimizer* for $\mathcal{M}$ if there exists $\delta > 0$ such that

$$g(\bar{x}(t_0),\bar{x}(T)) \le g(x(t_0),x(T)) \tag{3.9}$$

for any feasible process (x,u) satisfying $\|\bar{x}-x\|_{W^{1,1}} \le \delta$. If (3.9) holds true for every feasible process (x,u), then $(\bar{x},\bar{u})$ is called a *global minimizer* for $\mathcal{M}$.

Definition 3.2. The *Hamiltonian* $\mathcal{H} : [t_0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ of (3.8) is defined by

$$\mathcal{H}(t,x,p,u) := p.f(t,x,u) = \sum_{i=1}^{n} p_i f_i(t,x,u). \tag{3.10}$$

The *partial hybrid subdifferential* (see [3, p. 329]) $\partial_x^> h(t,x)$ of $h(t,x)$ w.r.t. x is given by

$$\begin{aligned}\partial_x^> h(t,x) := \operatorname{co}\big\{\xi \,:\, & \text{there exists } (t_i,x_i) \xrightarrow{h} (t,x) \text{ such that} \\ & h(t_k,x_k) > 0 \text{ for all } k \text{ and } \nabla_x h(t_k,x_k) \to \xi\big\},\end{aligned} \tag{3.11}$$

where $(t_k,x_k) \xrightarrow{h} (t,x)$ means that $(t_k,x_k) \to (t,x)$ and $h(t_k,x_k) \to h(t,x)$ as $k \to \infty$.

In order to introduce the necessary condition for a process $(\bar{x},\bar{u})$ be a $W^{1,1}$ local minimizer for $\mathcal{M}$, we now present some related materials.

The *convex hull* of a subset $C \subset \mathbb{R}^n$ is denoted by $\operatorname{co} C$. The *graph* of a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is defined by $\operatorname{gph} F := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \,:\, y \in F(x)\}$. For a given segment $[t_0,T]$ of the real line, we denote the σ-algebra of its Lebesgue measurable subsets (resp., the σ-algebra of its Borel measurable subsets) by $\mathcal{L}$ (resp., $\mathcal{B}$).

Let $\Omega \subset \mathbb{R}^n$ be a closed set and $\bar{v} \in \Omega$. The *Fréchet normal cone* to Ω at $\bar{v}$ is given by

$$\widehat{N}_\Omega(\bar{v}) = \left\{ v' \in \mathbb{R}^n \,:\, \limsup_{v \xrightarrow{\Omega} \bar{v}} \frac{\langle v', v-\bar{v}\rangle}{\|v-\bar{v}\|} \le 0 \right\},$$

where $v \xrightarrow{\Omega} \bar{v}$ means $v \to \bar{v}$ with $v \in \Omega$. The *limiting normal cone* to Ω at $\bar{v}$ is defined by

$$N_\Omega(\bar{v}) = \big\{ v' \in \mathbb{R}^n : \exists \text{ sequences } v_k \to \bar{v},\ v'_k \to v' \text{ with } v'_k \in \widehat{N}_\Omega(v_k), \forall k \in \mathbb{N} \big\}.$$

Given an extended real-valued function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty,+\infty\}$, one defines the *epigraph* of φ by $\operatorname{epi}\varphi = \{(x,\mu) \in \mathbb{R}^n \times \mathbb{R} \,:\, \mu \ge \varphi(x)\}$. The *limiting subdifferential* of φ at $\bar{x} \in \mathbb{R}^n$ with $|\varphi(\bar{x})| < \infty$ is defined by

$$\partial\varphi(\bar{x}) = \big\{ x^* \in \mathbb{R}^n \,:\, (x^*,-1) \in N_{\operatorname{epi}\varphi}\big((\bar{x},\varphi(\bar{x}))\big) \big\}.$$

If $|\varphi(x)| = \infty$, then one puts $\partial\varphi(\bar{x}) = \emptyset$. The reader is referred to [20, 21, 22] for comprehensive treatments of the Fréchet normal cone, the limiting normal cone, the limiting subdifferential, and the related calculus rules.

To deal with the state constraint $h(t, x(t)) \leq 0$ in $\mathcal{M}$, one introduces a multiplier that is an element in the topological dual $C^*([t_0, T]; \mathbb{R})$ of the space of continuous functions $C([t_0, T]; \mathbb{R})$ with the supremum norm. By the Riesz Representation Theorem (see, e.g., [23, Theorem 6, p. 374] and [24, Theorem 1, pp. 113–115]), any bounded linear functional f on $C([t_0, T]; \mathbb{R})$ can be uniquely represented in the form $f(x) = \int_{[t_0,T]} x(t)dv(t)$, where v is *a function of bounded variation* on $[t_0, T]$ which vanishes at t_0 and which are continuous from the right at every point $\tau \in (t_0, T)$, and $\int_{[t_0,T]} x(t)dv(t)$ is the Riemann-Stieltjes integral of x with respect to v (see, e.g., [23, p. 364]). The set of the elements of $C^*([t_0, T]; \mathbb{R})$ which are given by nondecreasing functions v is denoted by $C^{\oplus}(t_0, T)$. The integrals $\int_{[t_0,t)} \nu(s)d\mu(s)$ and $\int_{[t_0,T]} \nu(s)d\mu(s)$ of a Borel measurable function ν in next theorem are understood in the sense of the Lebesgue-Stieltjes integration [23, p. 364].

Theorem 3.1 (See [3, Theorem 9.3.1])**.** *Let $(\bar{x}, \bar{u})$ be a $W^{1,1}$ local minimizer for $\mathcal{M}$. Assume that for some $\delta > 0$, the following hypotheses are satisfied:*

(H1) *$f(., x, .)$ is $\mathcal{L} \times \mathcal{B}^m$ measurable, for fixed x. There exists a Borel measurable function $k(., .) : [t_0, T] \times \mathbb{R}^m \to \mathbb{R}$ such that $t \mapsto k(t, \bar{u}(t))$ is integrable and*

$$\|f(t, x, u) - f(t, x', u)\| \leq k(t, u)\|x - x'\|, \quad \forall x, x' \in \bar{x}(t) + \delta\bar{B},\ \forall u \in U(t)$$

for almost every $t \in [t_0, T]$;

(H2) $\operatorname{gph} U$ *is a Borel set in $[t_0, T] \times \mathbb{R}^m$;*

(H3) *g is Lipschitz continuous on the ball $(\bar{x}(t_0), \bar{x}(T)) + \delta\bar{B}$;*

(H4) *h is upper semicontinuous and there exists $K > 0$ such that*

$$\|h(t, x) - h(t, x')\| \leq K\|x - x'\|, \quad \forall x, x' \in \bar{x}(t) + \delta\bar{B},\ \forall t \in [t_0, T].$$

Then there exist $p \in W^{1,1}([t_0, T]; \mathbb{R}^n)$, $\gamma \geq 0$, $\mu \in C^{\oplus}(t_0, T)$, and a Borel measurable function $\nu : [t_0, T] \to \mathbb{R}^n$ such that $(p, \mu, \gamma) \neq (0, 0, 0)$, and for $q(t) := p(t) + \eta(t)$ with

$$\eta(t) := \int_{[t_0,t)} \nu(s)d\mu(s), \text{ for } t \in [t_0, T), \quad \text{and} \quad \eta(T) := \int_{[t_0,T]} \nu(s)d\mu(s),$$

the following holds true:

(i) $\nu(t) \in \partial_x^{>} h(t, \bar{x}(t))\ \mu - a.e.$;

(ii) $-\dot{p}(t) \in \operatorname{co} \partial_x \mathcal{H}(t, \bar{x}(t), q(t), \bar{u}(t))$ *a.e.;*

(iii) $(p(t_0), -q(T)) \in \gamma \partial g(\bar{x}(t_0), \bar{x}(T)) + N_C(\bar{x}(t_0), \bar{x}(T))$*;*

(iv) $\mathcal{H}(t, \bar{x}(t), q(t), \bar{u}(t)) = \max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), q(t), u)$ *a.e.*

3.2 Preliminary Analysis

Consider (GP_1) with $\alpha = \beta = 1$. To apply Theorem 3.1 for finding optimal processes for (GP_1), we have to interpret (GP_1) in the form of $\mathcal{M}$. For doing so, we set $x(t) = (x_1(t), x_2(t))$, where $x_1(t)$ plays the role of $k(t)$ in (2.3)–(2.4) and

$$x_2(t) := \int_{t_0}^{t} (s(\tau) - 1) x_1(\tau) e^{-\lambda \tau} d\tau, \quad t \in [0, T]. \tag{3.12}$$

Thus, (GP_1) is equivalent to the following problem:

$$\text{Minimize} \;\; x_2(T) \tag{3.13}$$

over $x = (x_1, x_2) \in W^{1,1}([t_0, T], \mathbb{R}^2)$ and measurable functions $s : [t_0, T] \to \mathbb{R}$ satisfying

$$\begin{cases} \dot{x}_1(t) = (As(t) - \sigma) x_1(t), & \text{a.e. } t \in [t_0, T] \\ \dot{x}_2(t) = (s(t) - 1) x_1(t) e^{-\lambda t}, & \text{a.e. } t \in [t_0, T] \\ s(t) \in [0, 1], & \text{a.e. } t \in [t_0, T] \\ (x(t_0), x(T)) \in \{(k_0, 0)\} \times \mathbb{R}^2 \\ x_1(t) \geq 0, & \forall t \in [t_0, T]. \end{cases} \tag{3.14}$$

The optimal control problem with state constraints in (3.13)–(3.14) is denoted by (GP_{1a}).

To see (GP_{1a}) in the form of $\mathcal{M}$, we choose $n = m = 1$, $C = \{(k_0, 0)\} \times \mathbb{R}^2$, $U(t) = [0, 1]$ for all $t \in [t_0, T]$, $g(x, y) = y_2$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, $y = (y_1, y_2) \in \mathbb{R}^2$, $h(t, x) = -x_1$ for all $(t, x) \in [t_0, T] \times \mathbb{R}^2$, and

$$f(t, x, s) = \big((As - \sigma) x_1, (s - 1) x_1 e^{-\lambda t}\big), \quad \forall (t, x, s) \in [t_0, T] \times \mathbb{R}^2 \times \mathbb{R}. \tag{3.15}$$

In accordance with (3.10) and (3.15), the Hamiltonian of (GP_{1a}) is given by

$$\mathcal{H}(t, x, p, s) = (As - \sigma) x_1 p_1 + (s - 1) x_1 e^{-\lambda t} p_2, \quad \forall (t, x, p, s) \in [t_0, T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}.$$

Clearly, $\mathcal{H}$ is continuously differentiable in x and

$$\partial_x \mathcal{H}(t, x, p, u) = \big\{\big((As - \sigma) p_1 + (s - 1) e^{-\lambda t} p_2\big)\big\}, \quad \forall (t, x, p, s) \in [t_0, T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}.$$

By (3.11), the partial hybrid subdifferential of h at $(t,x) \in [t_0,T] \times \mathbb{R}^2$ is given by

$$\partial_x^> h(t,x) = \begin{cases} \emptyset, & \text{if } x_1 > 0 \\ \{(-1,0)\}, & \text{if } x_1 \le 0. \end{cases}$$

The following lemma is a known result in Huong [1]. It descibes the relationships between a control function $s(\cdot)$ and the corresponding trajectory $x(\cdot)$ of (3.14).

Lemma 3.1. ([1, Lemma 4.1]) *For each measurable function $s : [t_0,T] \to \mathbb{R}$ with $s(t) \in [0,1]$, there exists a unique trajectory $x = (x_1,x_2) \in W^{1,1}([t_0,T],\mathbb{R}^2)$ such that (x,s) is a feasible process of (3.14). Moreover, for every $\tau \in [t_0,T]$, one has*

$$x_1(t) = x_1(\tau)e^{\int_\tau^t (As(z)-\sigma)dz}, \quad \forall t \in [t_0,T].$$

In particular, $x_1(t) > 0$ for all $t \in [t_0,T]$.

The next two remarks, which were mentioned in [1], are aimed at clarifying the tool used to solve (GP_{1a}).

Remark 3.3. By Lemma 3.1, any process satisfying the first four conditions in (3.14) automatically satisfies the state constraint $x_1(t) \ge 0$ for all t in $[t_0,T]$. Thus, the latter can be omitted in the problem formulation. This means that, for the case $\alpha = 1$, instead of the maximum principle in Theorem 3.1 one can apply the maximum principle for optimal control problems without state constraints in [3, Theorem 6.2.1]. Note that both theorems yield the same necessary optimality conditions (see, e.g., [14, Subsection 3.2]).

Remark 3.4. For the case $\alpha \in (0,1)$, one cannot claim that any process satisfying the first four conditions in (3.14) automatically satisfies the state constraint $x_1(t) \ge 0$ for all $t \in [t_0,T]$. Thus, we have to rely on Theorem 3.1. Referring to the classification of optimal economic growth models given in [2, Sect. 4], we can say that models of the types “nonlinear-linear” and “nonlinear-nonlinear” may require the use of Theorem 3.1. For this reason, we prefer to present the latter in this paper to prepare a suitable framework for dealing with (GP_{1a}) under different sets of assumptions.

Let $(\bar{x},\bar{s})$ be a $W^{1,1}$ local minimizer for (GP_{1a}). As it was shown in [1], the hypotheses (H1)–(H4) in Theorem 3.1 are satisfied. By that theorem, we can find $p \in W^{1,1}([t_0,T];\mathbb{R}^2)$, $\gamma \ge 0$, $\mu \in C^\oplus(t_0,T)$, and a Borel measurable function $\nu : [t_0,T] \to \mathbb{R}^2$ such that $(p,\mu,\gamma) \ne (0,0,0)$, and for $q(t) := p(t) + \eta(t)$ with

$$\eta(t) := \int_{[t_0,t)} \nu(\tau)d\mu(\tau), \text{ for } t \in [t_0,T), \quad \text{and} \quad \eta(T) := \int_{[t_0,T]} \nu(\tau)d\mu(\tau), \tag{3.16}$$

conditions (i)–(iv) in Theorem 3.1 hold true.

As $\bar{x}_1(t) > 0$ for all $t \in [t_0, T]$ by Lemma 3.1, condition (i) implies that $\mu([t_0, T]) = 0$, i.e., $\mu = 0$. So, we have $\eta(t) = 0$ for all $t \in [t_0, T]$. Thus, the relation $q(t) = p(t) + \eta(t)$ allows us to have $q(t) = p(t)$ for every $t \in [t_0, T]$. Analyzing conditions (ii)–(iv) with $p(t)$ playing the role of $q(t)$ therein, we get $p_2(t) = -\gamma$ for every $t \in [t_0, T]$, $p_1(T) = 0$,

$$\dot{p}_1(t) = -(A\bar{s}(t) - \sigma)p_1(t) + \gamma(\bar{s}(t) - 1)e^{-\lambda t}, \quad \text{a.e. } t \in [t_0, T] \tag{3.17}$$

and $\big(Ap_1(t) - \gamma e^{-\lambda t}\big)\bar{x}_1(t)\bar{s}(t) = \max_{s\in[0,1]} \big\{\big(Ap_1(t) - \gamma e^{-\lambda t}\big)\bar{x}_1(t)s\big\}$ for a.e. $t \in [t_0, T]$. Since $\bar{x}_1(t) > 0$ for all $t \in [t_0, T]$, it follows from the last relation that

$$\big(Ap_1(t) - \gamma e^{-\lambda t}\big)\bar{s}(t) = \max_{s\in[0,1]} \big\{\big(Ap_1(t) - \gamma e^{-\lambda t}\big)s\big\}, \quad \text{a.e. } t \in [t_0, T]. \tag{3.18}$$

Describing the adjoint trajectory p corresponding to $(\bar{x}, \bar{s})$ in (3.17), the next lemma is an analogue of Lemma 3.1.

Lemma 3.2. ([1, Lemma 4.2]) *The Cauchy problem defined by the differential equation* (3.17) *and the condition that* $p_1(T) = 0$ *possesses a unique solution* $p_1(\cdot) : [t_0, T] \to \mathbb{R}$,

$$p_1(t) = -\int_t^T c(z)\bar{\Omega}(z, t)dz, \quad \forall t \in [t_0, T], \tag{3.19}$$

where $\bar{\Omega}(t, \tau)$ *is defined by*

$$\bar{\Omega}(t, \tau) := e^{\int_\tau^t (A\bar{s}(z) - \sigma)dz}, \quad \forall t, \tau \in [t_0, T], \tag{3.20}$$

and

$$c(t) := \gamma(\bar{s}(t) - 1)e^{-\lambda t}, \quad \forall t \in [t_0, T]. \tag{3.21}$$

Looking back to the maximum principle in Theorem 3.1, we see that the objective function g is taken into a full account in condition (iii) only if $\gamma > 0$. In such a situation, the maximum principle is said to be *normal*. The reader is referred to [25, 26, 27] for investigations on the normality of maximum principles for optimal control problems. For the problem (GP_{1a}), thanks to formulas (3.19)–(3.21) and the property that $(p, \mu, \gamma) \neq (0, 0, 0)$, in [1] it was shown that the situation $\gamma = 0$ can be excluded.

Lemma 3.3. ([1, Lemma 4.4]) *One must have* $\gamma > 0$.

In accordance with (3.18), to define the control value $\bar{s}(t)$, it is important to know the sign of the real-valued function

$$\psi(t) := Ap_1(t) - \gamma e^{-\lambda t}, \quad t \in [t_0, T]. \tag{3.22}$$

By (3.18), one has $\bar{s}(t) = 1$ when $\psi(t) > 0$ and $\bar{s}(t) = 0$ when $\psi(t) < 0$. Hence $\bar{s}(\cdot)$ is a constant function on each segment where $\psi(\cdot)$ has a fixed sign. The following lemma gives formulas for $\bar{x}_1(\cdot)$ and $p_1(\cdot)$ on such a segment. The proof of this lemma (see [1]) was based on Lemmas 3.1 and 3.2.

Lemma 3.4. ([1, Lemma 4.3]) *Let $[t_1, t_2] \subset [t_0, T]$ and $\tau \in [t_1, t_2]$ be given arbitrarily.*

(a) *If $\bar{s}(t) = 1$ for a.e. $t \in [t_1, t_2]$, then*

$$\bar{x}_1(t) = \bar{x}_1(\tau)e^{(A-\sigma)(t-\tau)}, \quad \forall t \in [t_1, t_2]$$

and

$$p_1(t) = p_1(\tau)e^{-(A-\sigma)(t-\tau)}, \quad \forall t \in [t_1, t_2].$$

(b) *If $\bar{s}(t) = 0$ for a.e. $t \in [t_1, t_2]$, then*

$$\bar{x}_1(t) = \bar{x}_1(\tau)e^{-\sigma(t-\tau)}, \quad \forall t \in [t_1, t_2]$$

and

$$p_1(t) = p_1(\tau)e^{\sigma(t-\tau)} + \frac{\gamma}{\sigma+\lambda}e^{\sigma t}\left[e^{-(\sigma+\lambda)t} - e^{-(\sigma+\lambda)\tau}\right], \quad \forall t \in [t_1, t_2].$$

For any $t \in [t_0, T]$, if $\psi(t) = 0$, then (3.18) holds automatically no matter what $\bar{s}(t)$ is. Thus, by (3.18) we can assert nothing about the control function $\bar{s}(\cdot)$ at this t. Motivated by this observation, we consider the set

$$\Gamma = \{t \in [t_0, T] : \psi(t) = 0\}. \tag{3.23}$$

Since the functions $p_1(\cdot)$ is absolutely continuous on $[t_0, T]$, so is $\psi(\cdot)$. It follows that Γ is a compact set. Besides, as $p_1(T) = 0$ and $\gamma > 0$, the equality $\psi(T) = Ap_1(T) - \gamma e^{-\lambda T}$ implies that $\psi(T) < 0$. Thus, $T \notin \Gamma$.

We are now in a position to prove Theorem 2.2, the main result of this paper.

3.3 Proof of Theorem 2.2

Consider (GP_1) with $\alpha = \beta = 1$. Suppose that the positive parameters A, σ, λ satisfy the inequality (2.5).

Let $(\bar{x}, \bar{s})$ be a $W^{1,1}$ local minimizer of (GP_{1a}). As it has been explained in the previous subsection, by Theorem 3.1 we can find $p \in W^{1,1}([t_0, T]; \mathbb{R}^2)$, $\gamma \geq 0$, $\mu \in C^{\oplus}(t_0, T)$, and a Borel measurable function $\nu : [t_0, T] \to \mathbb{R}^2$ such that $(p, \mu, \gamma) \neq (0, 0, 0)$ and the conditions (i)–(iv) in Theorem 3.1 hold true for $q(t) := p(t) + \eta(t)$ with $\eta(t)$ being given in (3.16) for all $t \in [t_0, T]$. In what follows, we will keep all the notations of Subsection 3.2. In particular, $\psi(t)$ is given by (3.22) and the set Γ is defined by (3.23). Recall that $T \notin \Gamma$ as $\psi(T) < 0$.

First, consider the situation where $\Gamma = \emptyset$. Then we have $\psi(t) < 0$ on the whole segment $[t_0, T]$. Indeed, otherwise we would find a point $\tau \in [t_0, T)$ such that $\psi(\tau) > 0$, as $\psi(T) < 0$. Since $\psi(\tau)\psi(T) < 0$, by the continuity of $\psi(\cdot)$ on $[t_0, T]$ we can assert that $\Gamma \cap (\tau, T) \neq \emptyset$. This contradicts our supposition that $\Gamma = \emptyset$. Now, as $\psi(t) < 0$ for all $t \in [t_0, T]$, from (3.18) we have $\bar{s}(t) = 0$ for a.e. $t \in [t_0, T]$. Applying Lemma 3.4 for $t_1 = t_0$, $t_2 = T$, and $\tau = t_0$, we get $\bar{x}_1(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$.

Now, consider the situation where $\Gamma \neq \emptyset$. According to [1, Lemma 4.11], the set Γ contains at most one element when the data triple (A, σ, λ) satisfies the inequality $A \neq \sigma + \lambda$. Since (2.5) implies the latter inequality and $\Gamma \neq \emptyset$, we can assert that Γ is a singleton, say, $\Gamma = \{t^*\}$. By the continuity of $\psi(\cdot)$, and the fact that $\psi(T) < 0$, we have $\psi(t) < 0$ for every $t \in (t^*, T]$. This and (3.18) imply that $\bar{s}(t) = 0$ for almost every $t \in [t^*, T]$. Invoking Lemma 3.4 for $t_1 = t^*$, $t_2 = T$, and $\tau = t^*$, we obtain $\bar{x}_1(t) = \bar{x}_1(t^*)e^{-\sigma(t-t^*)}$ for all $t \in [t^*, T]$. If $t_0 = t^*$, then we get $\bar{s}(t) = 0$ for a.e. $t \in [t_0, T]$ and $\bar{x}_1(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$. If $t_0 < t^*$, then either $\psi(t_0) < 0$ or $\psi(t_0) > 0$.

• If $\psi(t_0) < 0$, then $\psi(t) < 0$ for all $t \in [t_0, t^*)$ by the continuity of $\psi(\cdot)$. So, by (3.18) one has $\bar{s}(t) = 0$ for a.e. $t \in [t_0, t^*]$; therefore, $\bar{s}(t) = 0$ for a.e. $t \in [t_0, T]$. Applying Lemma 3.4 for $t_1 = t_0$, $t_2 = T$, and $\tau = t_0$, we get $\bar{x}_1(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0, T]$.

• If $\psi(t_0) > 0$, then $\psi(t) > 0$ for all $t \in [t_0, t^*)$ also by the continuity of $\psi(\cdot)$. Thus, from (3.18) we have $\bar{s}(t) = 1$ for a.e. $t \in [t_0, t^*]$. Applying Lemma 3.4 for $t_1 = t_0$, $t_2 = t^*$, and $\tau = t_0$, we get $\bar{x}_1(t) = k_0 e^{(A-\sigma)(t-t_0)}$ for all $t \in [t_0, t^*]$. Combining this with the obtained formulas for $\bar{x}_1(\cdot)$ and $\bar{s}(\cdot)$ on $[t^*, T]$, we get

$$\bar{s}(t) = \begin{cases} 1, & \text{a.e.} \quad t \in [t_0, t^*] \\ 0, & \text{a.e.} \quad t \in (t^*, T] \end{cases} \quad \text{and} \quad \bar{x}_1(t) = \begin{cases} k_0 e^{(A-\sigma)(t-t_0)}, & t \in [t_0, t^*] \\ \bar{x}_1(t^*)e^{-\sigma(t-t^*)}, & t \in (t^*, T]. \end{cases}$$

Now, our task is to find a formula for computing the value of t^*. To that aim, we fix a value $\tau \in [t_0, T]$ and consider the control function $s^\tau(\cdot)$ defined by

$$s^\tau(t) = \begin{cases} 1, & t \in [t_0, \tau] \\ 0, & t \in (\tau, T]. \end{cases} \tag{3.24}$$

Denote the unique trajectory of (GP_{1a}) corresponding to $s^\tau(\cdot)$ (see Lemma 3.1) by $x^\tau(\cdot)$. Then one has

$$x_1^\tau(t) = \begin{cases} k_0 e^{(A-\sigma)(t-t_0)}, & t \in [t_0, \tau] \\ x_1^\tau(\tau)e^{-\sigma(t-\tau)}, & t \in (\tau, T]. \end{cases} \tag{3.25}$$

From (3.12) and (3.24), we have $x_2^\tau(T) = -\int_\tau^T x_1^\tau(t)e^{-\lambda t}dt$ Thus, by using (3.25), we get

$$x_2^\tau(T) = \frac{k_0}{\sigma + \lambda} e^{(\sigma-A)t_0} e^{A\tau} \left[e^{-(\sigma+\lambda)T} - e^{-(\sigma+\lambda)\tau} \right]. \tag{3.26}$$

Clearly, $(x^{t^*}, s^{t^*}) = (\bar{x}, \bar{s})$. Thus, applying (3.26) with $\tau = t^*$ one has

$$\bar{x}_2(T) = \frac{k_0}{\sigma+\lambda} e^{(\sigma-A)t_0} e^{At^*} \left[e^{-(\sigma+\lambda)T} - e^{-(\sigma+\lambda)t^*} \right]. \tag{3.27}$$

It follows from (3.26) and (3.27) that

$$\bar{x}_2(T) - x_2^\tau(T) = \frac{k_0}{\sigma+\lambda} e^{(\sigma-A)t_0} \times \left\{ \left[e^{At^*} e^{-(\sigma+\lambda)T} - e^{(A-\sigma-\lambda)t^*} \right] - \left[e^{A\tau} e^{-(\sigma+\lambda)T} - e^{(A-\sigma-\lambda)\tau} \right] \right\}.$$

So, setting

$$\varphi(t) := \left[e^{At^*} e^{-(\sigma+\lambda)T} - e^{(A-\sigma-\lambda)t^*} \right] - \left[e^{At} e^{-(\sigma+\lambda)T} - e^{(A-\sigma-\lambda)t} \right], \ t \in \mathbb{R}, \tag{3.28}$$

we have

$$\bar{x}_2(T) - x_2^\tau(T) = \frac{k_0}{\sigma+\lambda} e^{(\sigma-A)t_0} \times \varphi(\tau). \tag{3.29}$$

Observe that $\varphi(t^*) = 0$ and $\varphi(\cdot)$ is differentiable on $\mathbb{R}$. In addition, we have

$$\dot{\varphi}(t) = -Ae^{At} e^{-(\sigma+\lambda)T} + (A-\sigma-\lambda) e^{(A-\sigma-\lambda)t}, \quad \forall t \in \mathbb{R}. \tag{3.30}$$

As $\sigma + \lambda > 0$ and $A - \sigma - \lambda > 0$ by (2.5), the number ρ in (2.6) is well-defined and positive. Since $\bar{t} = T - \rho$, one has $\bar{t} < T$. By (3.30), $\dot{\varphi}(t) = 0$ if and only if $t = \bar{t}$, $\dot{\varphi}(t) < 0$ if and only if $t > \bar{t}$, and $\dot{\varphi}(t) > 0$ if and only if $t < \bar{t}$. One must have $t^* = \bar{t}$. Indeed, if this is false, then either $t^* < \bar{t}$ or $t^* > \bar{t}$.

If $t^* < \bar{t}$, then we have $\dot{\varphi}(t) > 0$ for all $t \in [t^*, \bar{t})$, which means that the function $\varphi(\cdot)$ is strictly increasing on $[t^*, \bar{t})$. So, for any $\tau \in (t^*, \bar{t})$, one has $\varphi(\tau) > \varphi(t^*) = 0$. From the latter and (3.29), one gets $\bar{x}_2(T) > x_2^\tau(T)$. This contradicts the assumption that $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer of (GP_{1a}), because τ can be taken as close to t^* as one wishes.

If $t^* > \bar{t}$, then we have $\dot{\varphi}(t) < 0$ for all $t \in (\bar{t}, t^*]$, meaning that $\varphi(\cdot)$ is strictly decreasing on $(\bar{t}, t^*]$. So, for any $\tau \in (\bar{t}, t^*)$, one has $\varphi(\tau) > \varphi(t^*) = 0$. Thus, from (3.29), one obtains $\bar{x}_2(T) > x_2^\tau(T)$, a contradiction to the fact that $(\bar{x}, \bar{u})$ is a $W^{1,1}$ local minimizer of (GP_{1a}) as $\tau \in (\bar{t}, t^*)$ can be chosen arbitrarily.

We have proved that $\bar{t} = t^*$. Thus, since $t^* \in (t_0, T)$, one has $\bar{t} \in (t_0, T)$. The latter can be rewritten equivalently as $T - t_0 > \rho$.

From the above analysis we can infer that (GP_{1a}) may have at most two candidates for $W^{1,1}$ local minimizers. Using the above definition of the process (x^τ, s^τ) corresponding to s^τ given in (3.24) for $\tau \in [t_0, T]$, we see that the first candidate for the $W^{1,1}$ local minimizers of (GP_{1a}) is (x^{t_0}, s^{t_0}) and the second one is $(x^{\bar{t}}, s^{\bar{t}})$, where $\bar{t} = T - \rho$. Since the second one may exist only if $T - t_0 > \rho$ we can assert that if $T - t_0 \le \rho$, then the unique candidate for $W^{1,1}$ local minimizers of (GP_{1a}) is (x^{t_0}, s^{t_0}). Now, in the case where $T - t_0 > \rho$, we will prove that the unique candidate for $W^{1,1}$ local minimizers of (GP_{1a}) is $(x^{\bar{t}}, s^{\bar{t}})$. To do so, we claim that (x^{t_0}, s^{t_0}) is not a $W^{1,1}$ local minimizer of (GP_{1a}). Indeed, for any $\delta > 0$, one can find some $t_1 \in (t_0, \bar{t})$ such that the feasible process (x^{t_1}, s^{t_1}) has the property $\|x^{t_0} - x^{t_1}\|_{W^{1,1}} \leq \delta$. Since $T - t_0 > \rho$, one has $t_0 < \bar{t}$. Thus, the function $\varphi(\cdot)$ defined by (3.28) has $\dot{\varphi}(t) > 0$ for all $t \in (-\infty, \bar{t})$. So, $\varphi(t)$ is strictly increasing on $(-\infty, \bar{t})$; in particular, $\varphi(t_0) < \varphi(t_1)$. Applying (3.29) for $\tau := t_0$ and $\tau = t_1$ and recalling that $(x^{\bar{t}}, s^{\bar{t}}) = (\bar{x}, \bar{s})$ in that formula, we have

$$x_2^{\bar{t}}(T) - x_2^{t_0}(T) = \frac{k_0}{\sigma + \lambda} e^{(\sigma - A)t_0} \times \varphi(t_0) \quad \text{and} \quad x_2^{\bar{t}}(T) - x_2^{t_1}(T) = \frac{k_0}{\sigma + \lambda} e^{(\sigma - A)t_0} \times \varphi(t_1).$$

Thus, $x_2^{t_0}(T) - x_2^{t_1}(T) = \dfrac{k_0}{\sigma + \lambda} e^{(\sigma - A)t_0} \times [\varphi(t_1) - \varphi(t_0)]$. This and the inequality $\varphi(t_0) < \varphi(t_1)$ yield $x_2^{t_0}(T) > x_2^{t_1}(T)$. As (x^{t_1}, s^{t_1}) is a feasible process of (GP_{1a}) with $\|x^{t_0} - x^{t_1}\|_{W^{1,1}} \leq \delta$, the last inequality yields that is not a $W^{1,1}$ local minimizer of (GP_{1a}). The claim is proved.

We have just show that, the set of $W^{1,1}$ local minimizers of (GP_{1a}) contains at most one element $(\bar{x}, \bar{s})$. Moreover, if $T - t_0 \leq \rho$, then $(\bar{x}, \bar{s}) = (x^{t_0}, s^{t_0})$; and if $T - t_0 > \rho$, then $(\bar{x}, \bar{s}) = (x^{\bar{t}}, s^{\bar{t}})$. Thus, the set of global minimizers of (GP_{1a}) also contains at most such the element $(\bar{x}, \bar{s})$. By the equivalence between (GP_{1a}) and (GP_1) and by the fact that (GP_1) has a global solution (see Theorem 4.1 in [2] or Theorem B in the Appendix), we get the claims (a) and (b) in Theorem 2.2.

Acknowledgements This work was supported by National Foundation for Science & Technology Development (Vietnam) under grant number 101.01-2018.308. The author would like to thank Professor Nguyen Dong Yen and the anonymous referee for their insightful comments and valuable suggestions.

Appendix: Solution Existence of (GP_1)

In [2, Theorem 4.1], exploiting the concavity of both the per capital production function and the utility function, the fact that (GP_1) has a solution was derived from the solution existence theorem for (GP) as a corollary. Since the present paper focuses on the characterization of the solution, rather than on the solution existence, we will now provide a direct proof for the solution existence of (GP_1) to make the paper self-contained.

To recall a solution existence theorem for optimal control problems of the Lagrange type, which is a special version of the one for problems of the Bolza type in [19], suppose that $A \subset \mathbb{R} \times \mathbb{R}^n$ is a subset and $U : A \rightrightarrows \mathbb{R}^m$ is a set-valued map defined on A. Let

$$M := \{(t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \ : \ (t, x) \in A, \ u \in U(t, x)\},$$

$f_0(t, x, u)$ and $f(t, x, u) = (f_1, f_2, \ldots, f_n)$ be functions defined on M. Let there be given an interval $[t_0, T] \subset \mathbb{R}$ and a point $x_0 \in \mathbb{R}^n$. By $\mathcal{L}$ we denote the following optimal control problem:

$$\text{Minimize } \; J(x,u) := \int_{t_0}^{T} f_0(t,x(t),u(t))dt$$

over pairs (x,u) such that $x(\cdot) : [t_0,T] \to \mathbb{R}^n$ is absolutely continuous, $u(\cdot) : [t_0,T] \to \mathbb{R}^m$ is measurable, $f_0(.,x(\cdot),u(\cdot)) : [t_0,T] \to \mathbb{R}$ is Lebesgue integrable, and

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t)), & \text{a.e. } t \in [t_0,T] \\ u(t) \in U(t,x(t)), & \text{a.e. } t \in [t_0,T] \\ x(t_0 = x_0 & \\ (t,x(t)) \in A, & \forall t \in [t_0,T]. \end{cases}$$

Such a pair (x,u) is called a *feasible process*. The set of all the feasible processes for $\mathcal{L}$ is denoted by Ω. A feasible process $(\bar{x},\bar{u})$ is said to be a *global solution* for $\mathcal{L}$ if one has $J(\bar{x},\bar{u}) \leq J(x,u)$ for any feasible process (x,u).

Let $A_0 := \{t : \exists x \in \mathbb{R}^n \text{ s.t. } (t,x) \in A\}$. Set $A(t) = \{x \in \mathbb{R}^n : (t,x) \in A\}$ for $t \in A_0$, and $\widetilde{Q}(t,x) = \{(z^0,z) \in \mathbb{R}^{n+1} : z^0 \geq f_0(t,x,u),\ z = f(t,x,u) \text{ for some } u \in U(t,x)\}$ for $(t,x) \in A$.

Theorem A (Filippov's Theorem; see [19, Theorem 9.3.i, p. 317, and Section 9.5]) *Suppose that Ω is nonempty, f_0 and f is continuous on M and, for almost every $t \in [t_0,T]$, the sets $\widetilde{Q}(t,x)$, $x \in A(t)$, are convex. Moreover, assume that A is closed and contained in $[t_0,T] \times \mathbb{R}^n$ and the following conditions are fulfilled:*

(a) *For any $\varepsilon \geq 0$, the set $M_\varepsilon := \{(t,x,u) \in M : \|x\| \leq \varepsilon\}$ is compact;*

(b) *There exists $c \geq 0$ such that $\|f(t,x,u)\| \leq c(\|x\|+1)$ for all $(t,x,u) \in M$.*

Then, $\mathcal{L}$ has a global solution.

Theorem B (GP_1) *has a global solution.*

Proof. To apply Theorem A, we have to interpret (GP_1) in the form of $\mathcal{L}$. For doing so, we let the variable k (resp., the variable s) play the role of the phase variable x in $\mathcal{L}$ (resp., the control variable u in $\mathcal{L}$). Then, (GP_1) has the form of $\mathcal{L}$ with $n = m = 1$, $A = [t_0,T] \times \mathbb{R}_+$, $U(t,k) = [0,1]$ for all $(t,k) \in A$, $M = [t_0,T] \times \mathbb{R}_+ \times [0,1]$, $f_0(t,k,s) = -[1-s(t)]^\beta k^{\alpha\beta}(t)e^{-\lambda t}$, and $f(t,k,s) = Ak^\alpha s - \sigma k$ for all $(t,k,s) \in M$.

It is not hard to see that the pair (k,s) with $s(t) = 0$ and $k(t) = k_0 e^{-\sigma(t-t_0)}$ for all $t \in [t_0,T]$ is a feasible process for (GP). Thus, the set Ω is nonempty. The continuity of f_0 and f are continuous on M are obvious. Besides, by the formula for A, one has $A_0 = [t_0,T]$ and $A(t) = \mathbb{R}_+$ for all $t \in A_0$. In addition, it follows from the formulas for f_0, f and U that $\widetilde{Q}(t,k) = \{(z^0,z) \in \mathbb{R}^2 : \exists s \in [0,1] \text{ s.t. } z^0 \geq -[1-s]^\beta k^{\alpha\beta}e^{-\lambda t},\ z = Ak^\alpha s - \sigma k\}$, for any $(t,k) \in A$. Let us show that, for any $t \in [t_0, T]$ and $k \in A(t)$, the set $\widetilde{Q}(t,k)$ is convex. Indeed, given any (z_1^0, z_1), $(z_2^0, z_2) \in \widetilde{Q}(t,k)$ and $\mu \in [0,1]$, one can find $s_1, s_2 \in [0,1]$ such that $z_1^0 \geq -[1-s_1]^\beta k^{\alpha\beta} e^{-\lambda t}$, $z_1 = Ak^\alpha s_2 - \sigma k$, $z_2^0 \geq -[1-s_2]^\beta k^{\alpha\beta} e^{-\lambda t}$, $z_2 = Ak^\alpha s_2 - \sigma k$. Therefore, it holds that

$$\mu z_1^0 + (1-\mu) z_2^0 \geq -\mu[1-s_1]^\beta k^{\alpha\beta} e^{-\lambda t} - (1-\mu)[1-s_2]^\beta k^{\alpha\beta} e^{-\lambda t} \quad (1)$$

and

$$\mu z_1 + (1-\mu) z_2 = \mu[Ak^\alpha s_2 - \sigma k] + (1-\mu)[Ak^\alpha s_2 - \sigma k]. \quad (2)$$

Setting $s_\mu = \mu s_1 + (1-\mu) s_2$, one has $s_\mu \in [0,1]$ and it follows from (2) that

$$\mu z_1 + (1-\mu) z_2 = Ak^\alpha s_\mu - \sigma k. \quad (3)$$

As $\beta \in (0,1]$, the function $s \mapsto -(1-s)^\beta$ is convex on $[0,1]$. So,

$$-\mu(1-s_1)^\beta - (1-\mu)(1-s_2)^\beta \geq -(1-s_\mu)^\beta.$$

Hence, by (1) we obtain $\mu z_1^0 + (1-\mu) z_2^0 \geq -(1-s_\mu)^\beta k^{\alpha\beta} e^{-\lambda t}$, which together with (3) implies that $\mu(z_1^0, z_1) + (1-\mu)(z_2^0, z_2) \in \widetilde{Q}(t,k)$.

Now, since $A = [t_0, T] \times \mathbb{R}_+$ is closed, it remains to check the conditions (a) and (b) in Theorem B. For any $\varepsilon \geq 0$, the set M_ε is compact because

$$M_\varepsilon = \{(t,k,s) \in [t_0,T] \times \mathbb{R}_+ \times [0,1] \ : \ |k| \leq \varepsilon\} = [t_0,T] \times [0,\varepsilon] \times [0,1].$$

So, condition (a) is satisfied. By the formula for M and f, condition (b) is satisfied if there exists a constant $c > 0$ such that

$$|Ask^\alpha - \sigma k| \leq c(k+1), \quad \forall k \geq 0, s \in [0,1]. \quad (4)$$

Fix any $k \geq 0$ and $s \in [0,1]$. We have

$$|Ask^\alpha - \sigma k| \leq Ask^\alpha + \sigma k < Ak^\alpha + \sigma k. \quad (5)$$

Besides, as $\alpha \in (0,1]$, the function $k \mapsto y(k) := Ak^\alpha + \sigma k$ is concave on $\mathbb{R}_+$ and differential on $\mathbb{R}_+ \setminus \{0\}$. So, $y(k) \leq \dot{y}(1)(k-1) + y(1)$. Equivalently, $Ak^\alpha + \sigma k \leq (A+\sigma)(k-1) + A + \sigma$. Thus, with $c := A + \sigma$, the last inequality and (5) imply (4). Condition (b) is verified. □

References

[1] Huong VT, Yao JC, Yen ND. Optimal processes in a parametric optimal economic growth model. Taiwanese J. Math. 2020; https://doi.org/10.11650/tjm/200203.

[2] Huong VT. Solution existence theorems for finite horizon optimal economic growth problems. Preprint arXiv:2001.03298v2. (Submitted)

[3] Vinter R. Optimal Control. Boston: Birkhäuser; 2000.

[4] Ramsey FP. A mathematical theory of saving. Econ. J. 1928; l38: 543–559.

[5] Harrod RF. An essay in dynamic theory. The Economic Journal 1939; 49: 14–33.

[6] Domar ED. Capital expansion, rate of growth, and employment. Econometrica 1946; 14: 137–147.

[7] Solow RM. A contribution to the theory of economic growth. Quart. J. Econom. 1956; 70: 65–94.

[8] Swan TW. Economic growth and capital accumulation. Economic Record 1956; 32: 334–361.

[9] Barro RJ, Sala-i-Martin X. Economic Growth. MIT Press; 2004.

[10] Acemoglu D. Introduction to Modern Economic Growth. Princeton University Press; 2009.

[11] Cass D. Optimum growth in an aggregative model of capital accumulation. Review of Economic Studies 1965; 32: 233–240.

[12] Koopmans T.C. On the concept of optimal economic growth. In The Econometric Approach to Development Planning. Amsterdam: North-Holland; 1965; pp. 225–295.

[13] Takayama A. Mathematical Economics. Illinois: The Dryden Press, Hinsdale; 1974.

[14] Huong VT, Yao JC, Yen ND. Analyzing a maximum principle for finite horizon state constrained problems via parametric examples. Part 1: Unilateral state constraints. J. Nonlinear Convex Anal. 2020; 21: 157–182.

[15] Huong VT, Yao JC, Yen ND. Analyzing a maximum principle for finite horizon state constrained problems via parametric examples. Part 2: Bilateral state constraints. Preprint 2019. (Submitted)

[16] Ioffe AD, Tihomirov VM. Theory of Extremal Problems. Amsterdam: North-Holland Publishing Company; 1979.

[17] Pierre NVT. Introductory Optimization Dynamics. Optimal Control with Economics and Management Science Applications. Berlin: Springer-Verlag; 1984.

[18] Chiang AC, Wainwright K. Fundamental Methods of Mathematical Economics. McGraw-Hill; 2005.

[19] Cesari L. Optimization Theory and Applications. New York: Springer-Verlag; 1983.

[20] Mordukhovich BS. Variational Analysis and Generalized Differentiation. Vol. I: Basic Theory. New York: Springer; 2006.

[21] Mordukhovich BS. Variational Analysis and Generalized Differentiation. Vol. II: Applications. New York: Springer; 2006.

[22] Mordukhovich BS. Variational Analysis and Applications. Berlin, Switzerland: Springer; 2018.

[23] Kolmogorov AN, Fomin SV. Introductory Real Analysis. New York: Dovers Publications, Inc.; 1970.

[24] Luenberger DG. Optimization by Vector Space Methods. New York: John Wiley & Sons; 1969.

[25] Ferreira MMA., Vinter RB. When is the maximum principle for state constrained problems nondegenerate?. J. Math. Anal. Appl. 1994; 187: 438–467.

[26] Frankowska H. Normality of the maximum principle for absolutely continuous solutions to Bolza problems under state constraints. Control Cybernet. 2009; 38: 1327–1340.

[27] Fontes FACC, Frankowska H. Normality and nondegeneracy for optimal control problems with state constraints. J. Optim. Theory Appl. 2015; 166: 115–136.