AN INTEGRAL THEOREM FOR PLURISUBHARMONIC FUNCTIONS

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Abstract. In this paper, we prove an integral theorem for Cegrell class $\mathcal{F}(f)$ and use this result to study the $\mathcal{F}$-equivalence relation.

Introduction

Let $\Omega \subset \mathbb{C}^n$ $(n \geq 2)$ be a bounded hyperconvex domain. Following [Ceg98, Ceg04, ACCP09], we denote

$$\mathcal{E}_0(\Omega) = \{u \in PSH^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} u(z) = 0, \int_\Omega (dd^c u)^n < \infty\},$$
$$\mathcal{F}(\Omega) = \{u \in PSH^-(\Omega) : \exists \{u_j\} \subset \mathcal{E}_0(\Omega),\ u_j \searrow u,\ \sup_j \int_\Omega (dd^c u_j)^n < \infty\},$$
$$\mathcal{E}(\Omega) = \{u \in PSH^-(\Omega) : \forall K \Subset \Omega, \exists u_K \in \mathcal{F}(\Omega) \text{ such that } u_K = u \text{ on } K\},$$

and for every $f \in PSH^-(\Omega)$,

$$\mathcal{F}(\Omega, f) = \{u \in PSH^-(\Omega) : \exists v \in \mathcal{F} \text{ such that } v + f \leq u \leq f\}.$$

The class $\mathcal{E}$ is the largest subclass of $PSH^-(\Omega)$ on which the complex Monge-Ampère operator is well-defined [Ceg04, Blo06]. The class $\mathcal{F}$ is the subclass of $\mathcal{E}$ containing those functions with smallest maximal plurisubharmonic majorant identically zero and with finite total Monge-Ampère mass. If $f \in \mathcal{E}$ then $\mathcal{F}(f) \subset \mathcal{E}$.

Our main result is the following:

Theorem 1. *Suppose (X, d, μ) is a totally bounded metric probability space and $u, f : \Omega \times X \to [-\infty, 0]$ are measurable functions such that*

(i) *For every $a \in X$, $f(\cdot, a) \in \mathcal{E}$.*

(ii) *For every $a \in X$, $u(\cdot, a) \in \mathcal{F}(f(., a))$ and*
$$\int_{U_a} (dd^c u(z, a))^n \leq (M(a))^n,$$
where $M \in L^1(X)$ is given and $U_a = \Omega \cap \overline{\{z \in \Omega : u(z, a) < f(z, a)\}}$.

(iii) *The function $a \mapsto u(z, a)$ is upper semicontinuous in X for every $z \in \Omega$.*

(iv) *The function $a \mapsto e^{f(z,a)}$ is lower semicontinuous in X for every $z \in \Omega$.*

(v) *The function $\tilde{f}(z) := \int_X f(z, a) d\mu(a)$ is not identically $-\infty$.*

Then $\tilde{u}(z) := \int_X u(z, a) d\mu(a) \in \mathcal{F}(\tilde{f})$. In particular, if $\tilde{f} \in \mathcal{E}$ then $\tilde{u} \in \mathcal{E}$ and $\int_\Omega (dd^c \tilde{u})^n < \infty$.

This result follows the plurisubharmonic version of [Kli91, Theorem 2.6.5] in the direction of focusing on the conservation of the existence of Monge-Ampère measures. We are not sure that the conditions (iii) and (iv) are necessary but we need these

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conditions in our proof. Our method is as follows: we solve the problem for the case $f \equiv 0$, then we use plurisubharmonic envelopes to reduce the problem to the case $f \equiv 0$. In the first step, we consider a decreasing sequence of functions $u_j \in \mathcal{F}$ and prove that $\lim_{j\to\infty} u_j \in \mathcal{F}$. Then we use the condition (iii) to show that $u = \lim_{j\to\infty} u_j$. In the last step, we need the conditions (iii) and (iv) to reduce the problem to the case $f \equiv 0$.

For $u_1, u_2 \in \mathcal{E}(\Omega)$, we say that u_1 is $\mathcal{F}$-equivalent to u_2 if there exist $v_1, v_2 \in \mathcal{F}$ such that $u_1 + v_1 \leq u_2$ and $u_2 + v_2 \leq u_1$. Observe that u_1 is $\mathcal{F}$-equivalent to u_2 iff $u_1, u_2 \in \mathcal{F}(\max\{u_1, u_2\})$. The following result is an immediate corollary of Theorem 1:

Corollary 2. *Suppose (X, d, μ) is a totally bounded metric probability space and $u, v : \Omega \times X \to [-\infty, 0]$ are measurable functions such that*

(i) *For every $a \in X$, $u(\cdot, a), v(\cdot, a) \in \mathcal{E}(\Omega)$ and*
$$\int_{U_a} (dd^c u(z,a))^n + \int_{V_a} (dd^c v(z,a))^n \leq (M(a))^n,$$
where $M \in L^1(X)$ is given, $U_a = \Omega \cap \overline{\{z \in \Omega : u(z,a) < v(z,a)\}}$ and $V_a = \Omega \cap \overline{\{z \in \Omega : v(z,a) < u(z,a)\}}$.

(ii) *For every $a \in X$, $u(\cdot, a)$ is $\mathcal{F}$-equivalent to $v(\cdot, a)$.*

(iii) *The functions $a \mapsto e^{u(z,a)}$ and $a \mapsto e^{v(z,a)}$ are continuous in X for every $z \in \Omega$.*

Then $\tilde{u}(z) := \int_X u(z,a) d\mu(a) \in \mathcal{E}$ iff $\tilde{v}(z) := \int_X v(z,a) d\mu(a) \in \mathcal{E}$. Moreover, if $\tilde{u}, \tilde{v} \in \mathcal{E}$ then $\tilde{u}$ is $\mathcal{F}$-equivalent to $\tilde{v}$.

In the next section, we recall briefly some properties of the class $\mathcal{F}$ and plurisubharmonic envelopes that will be used to prove the main theorem.

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1. Preliminaries

1.1. **The class $\mathcal{F}$.**

We recall some properties of the class $\mathcal{F}$. The reader can find more details in [Ceg04, NP09].

The following proposition is a corollary of [Ceg04, Proposition 5.1]:

Proposition 3. *Suppose $u \in \mathcal{F}(\Omega)$. If $u_j \in \mathcal{E}_0(\Omega)$ decreases to u as $j \to \infty$ then*
$$\lim_{j\to\infty} \int_\Omega (dd^c u_j)^n = \int_\Omega (dd^c u)^n.$$
In particular, $\int_\Omega (dd^c u)^n < \infty$.

Proposition 4. [Ceg04, Corollary 5.6] *Suppose $u_1, ..., u_n \in \mathcal{F}(\Omega)$. Then*
$$\int_\Omega dd^c u_1 \wedge ... \wedge dd^c u_n \leq \left(\int_\Omega (dd^c u_1)^n\right)^{1/n} ... \left(\int_\Omega (dd^c u_n)^n\right)^{1/n}.$$

Proposition 5. *a) If $u, v \in \mathcal{F}(\Omega)$ then $u + v \in \mathcal{F}(\Omega)$.*
b) If $u \in \mathcal{F}(\Omega)$ and $v \in PSH^-(\Omega)$ then $\max\{u, v\} \in \mathcal{F}(\Omega)$.

The part a) of Proposition 5 can be obtained by using [Ceg04, Lemma 5.4], Proposition 3 and the definition of the class $\mathcal{F}$. The part b) can be obtained by using the definition of the class $\mathcal{F}$ and the Bedford-Taylor Comparison Principle [BT82].

By [NP09, Theorem 3.7], we have:

Proposition 6. *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and $u \in PSH^-(\Omega)$. Assume that there are $u_j \in \mathcal{F}(\Omega)$, $j \in \mathbb{N}$, such that u_j converges almost everywhere to u as $j \to \infty$. If $\sup_{j>0} \int_\Omega (dd^c u_j)^n < \infty$ then $u \in \mathcal{F}(\Omega)$.*

By [NP09, Proposition 3.1], we have:

Proposition 7. *Let $u, v \in \mathcal{F}$ such that $u \leq v$ in Ω. Then*

$$\int\limits_\Omega (dd^c v)^n \leq \int\limits_\Omega (dd^c u)^n.$$

1.2. **Plurisubharmonic envelopes.** Let $D \Subset \mathbb{C}^n$ be a bounded domain. If $u : D \to \mathbb{R}$ is a bounded function then the plurisubharmonic envelope $P_D(u)$ of u in D is defined by

$$P_D(u) = (\sup\{v \in PSH(\Omega) : v \leq u\})^*,$$

where $(\sup\limits_{v \in S} v(z))^*$ is the upper envelope of $\sup\limits_{v \in S} v(z)$.

Lemma 8. *a) Let $u : D \to \mathbb{R}$ be a bounded function. Then $P_D(u) \leq u$ quasi everywhere, i.e., the set $\{z \in D : P_D(u)(z) > u(z)\}$ is pluripolar. Moreover,*

$$P_D(u) = \sup\{v \in PSH(D) : v \leq u \text{ quasi everywhere on } D\}.$$

b) Let $u_j, u : D \to \mathbb{R}$ be bounded functions such that $u_j \searrow u$ as $j \to \infty$. Then $P_D(u_j)$ decreases to $P_D(u)$.

The proof of Lemma 8 is the same as the proof of the parts 1), 2) of [GLZ19, Proposition 2.2]. For every domain $W \Subset D$, we also consider

$$P_{\overline{W}}(u) := (\sup\{v \in \mathrm{PSH}(W) : \hat{v} \leq u \text{ on } \overline{W}\})^*,$$

where $\hat{v}$ is the upper semicontinuous extension of v to $\overline{W}$ defined by

$$\hat{v}(\xi) := \lim_{r \to 0^+} \sup_{B(\xi, r) \cap W} v, \ \forall \xi \in \partial W.$$

The following results are also proved in [GLZ19]:

Lemma 9. [GLZ19, Lemma 3.11] *Let (D_j) be an increasing sequence of relatively compact domains in D such that $\cup D_j = D$. Assume that u is a bounded lower semi-continuous function in D. Then $P_{\overline{D_j}}(u)$ decreases to $P_D(u)$.*

Lemma 10. [GLZ19, Lemma 3.10] *Let (u_j) be an increasing sequence of continuous functions on D which converges pointwise to a bounded function u. Let W be a relatively compact domain in D. Then $P_W(u_j)$ increases almost everywhere to $P_{\overline{W}}(u)$.*

Proposition 11. [GLZ19, Theorem 3.9] *Let $D \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Assume that a bounded lower semi-continuous function u is a viscosity supersolution (see* [EGZ11] *for the definition) of the equation*

$$(dd^c u)^n = f dV, \tag{1}$$

in D. Then $P_D(u)$ is a pluripotential supersolution of (1) *in D.*

2. Proof of the main result

We first prove Theorem 1 for the case $f=0$.

Proposition 12. *Let $\Omega\subset\mathbb{C}^n$ be a bounded hyperconvex domain and (X,d,μ) be a totally bounded metric probability space. Let $u:\Omega\times X\to[-\infty,0]$ such that*

(i) *For every $a\in X$, $u(\cdot,a)\in\mathcal{F}(\Omega)$ and*
$$\int_\Omega (dd^c u(z,a))^n\leq (M(a))^n,$$
where $M\in L^1(X)$ is given.

(ii) *For every $z\in\Omega$, the function $u(z,\cdot)$ is upper semicontinuous in X.*

Then $\tilde{u}(z):=\int_X u(z,a)d\mu(a)\in\mathcal{F}(\Omega)$. Moreover
$$\int_\Omega (dd^c\tilde{u})^n\leq (\int_X M(a)d\mu(a))^n.$$

Proof. We will show that there exists a sequence of functions $\tilde{u}_j\in\mathcal{F}(\Omega)$ such that $\tilde{u}_j\searrow\tilde{u}$ as $j\to\infty$
$$\sup_{j\in\mathbb{Z}^+}\int_\Omega (dd^c\tilde{u}_j)^n\leq M(a),$$
for every $a\in X$.

Since X is totally bounded, there exists a finite cover $\{X_k\}_{k=1}^{m_1}$ of X such that the diameter of each X_k is at most $1/2$. Denote
$$U_{1,1}=X_1, U_{1,2}=X_2\setminus X_1, ..., U_{1,m_1}=X_{m_1}\setminus(\cup_{l=1}^{m_1-1}X_l).$$
Then $\{U_{1,k}\}_{k=1}^{m_1}$ is a finite cover of X such that

- $U_{1,k}\cap U_{1,l}=\emptyset$ if $k\neq l$;
- $diam(U_{1,k})\leq 1/2$ for all $k=1,...,m_1$;
- $U_{1,k}$ is totally bounded for all $k=1,...,m_1$.

By using induction, for every $j\in\mathbb{Z}^+$, we can find a finite cover $\{U_{j,k}\}_{k=1}^{m_j}$ of X such that

- For every $1\leq k\leq m_{j+1}$, there exists $1\leq l\leq m_j$ such that $U_{j+1,k}\subset U_{j,l}$;
- $U_{j,k}\cap U_{j,l}=\emptyset$ if $k\neq l$;
- $diam(U_{j,k})\leq 2^{-j}$ for all $k=1,...,m_1$.

For every $j\in\mathbb{Z}^+$, we define
$$u_j(z)=\sum_{k=1}^{m_j}\mu(U_{j,k})\sup_{a\in U_{j,k}}u(z,a)\qquad\text{and}\qquad\tilde{u}_j=(u_j)^*.$$

Then $\tilde{u}_j\in\mathcal{F}(\Omega)$. Let $a_{j,k}$ be an arbitrary element of $U_{j,k}$ for $j\in\mathbb{Z}^+$ and $k=1,...,m_j$. By using Proposition 7 for $\tilde{u}_j$ and $\sum_{k=1}^{m_j}\mu(U_{j,k})u(z,a_{j,k})$ and by applying Proposition 4, we have
$$\int_\Omega (dd^c\tilde{u}_j)^n\leq\int_\Omega (dd^c(\sum_{k=1}^{m_j}\mu(U_{j,k})u(z,a_{j,k})))^n$$
$$=\sum_{k_1+...+k_{m_j}=n}\frac{n!}{k_1!...k_{m_j}!}\left(\prod_{l=1}^{m_j}\mu(U_{j,l})^{k_l}\right)\int_\Omega (dd^c u(z,a_{j,1}))^{k_1}\wedge...\wedge(dd^c u(z,a_{j,m_j}))^{k_{m_j}}$$

$$\leq \sum_{k_1+\ldots+k_{m_j}=n} \frac{n!}{k_1!\ldots k_{m_j}!} \left(\prod_{l=1}^{m_j} \mu(U_{j,l})^{k_l}\right) \prod_{l=1}^{m_j} (\int_\Omega (dd^c u(z,a_{j,l}))^n)^{k_l/n}$$
$$\leq \sum_{k_1+\ldots+k_{m_j}=n} \frac{n!}{k_1!\ldots k_{m_j}!} \left(\prod_{l=1}^{m_j} \mu(U_{j,l})^{k_l}\right) \prod_{l=1}^{m_j} M(a_{j,l})^{k_l}$$
$$\leq \sum_{k_1+\ldots+k_{m_j}=n} \frac{n!}{k_1!\ldots k_{m_j}!} \left(\prod_{l=1}^{m_j} \mu(U_{j,l})^{k_l}\right) \prod_{l=1}^{m_j} (\int_\Omega (dd^c u(z,a_{j,l}))^n)^{k_l/n}$$
$$= \sum_{k_1+\ldots+k_{m_j}=n} \frac{n!}{k_1!\ldots k_{m_j}!} \prod_{l=1}^{m_j} \mu(U_{j,l}) M(a_{j,l}))^{k_l}$$
$$= (\mu(U_{j,1})M(a_{j,1}) + \ldots + \mu(U_{j,k_{m_j}} M(a_{j,k_{m_j}})))^n,$$

for all $j \in \mathbb{Z}^+$. Since $a_{j,k}$ is arbitrary for every j, k, we have

$$\int_\Omega (dd^c \tilde{u}_j)^n \leq \left(\sum_{k=1}^{m_j} \mu(U_{j,k}) \inf_{U_{j,k}} M(a) \right)^n \leq \left(\int_X M(a) d\mu(a) \right)^n, \tag{2}$$

for all $j \in \mathbb{Z}^+$.

We will show that $\tilde{u}_j$ is decreasing to $\tilde{u}$ and use Proposition 6 to prove that $\tilde{u} \in \mathcal{F}(\Omega)$.

For every $z \in \Omega, a \in X$ and $j \in \mathbb{Z}^+$, we define

$$\phi_j(z,a) = \sum_{k=1}^{m_j} \chi_{U_{j,k}}(a) \sup_{a \in U_{j,k}} u(z,a) = \sup_{\xi \in U_{j,k(j,a)}} u(z,\xi),$$

where $\chi_{U_{j,k}}$ is the characteristic function of $U_{j,k}$ and $k(j,a)$ is given by $a \in U_{j,k(j,a)}$. Then, we have

$$u_j(z) = \int_X \phi_j(z,a) d\mu(a) \geq \int_X u(z,a) d\mu(a) = \tilde{u}(z), \tag{3}$$

for every $z \in \Omega$ and $j \in \mathbb{Z}^+$.

Note that $a \in U_{j+1,k(j+1,a)} \cap U_{j,k(j,a)} \neq \emptyset$. Then, by the construction of the sets $U_{j,k}$, we have $U_{j+1,k(j+1,a)} \subset U_{j,k(j,a)}$. Hence

$$u_j(z) = \int_X \phi_j(z,a) d\mu(a) \geq \int_X \phi_{j+1}(z,a) d\mu(a) = u_{j+1}(z), \tag{4}$$

for every $z \in$ and $j \in \mathbb{Z}^+$.

By the semicontinuity of $u(z,\cdot)$, we have,

$$u(z,a) \geq \lim_{j\to\infty} (\sup\{u(z,\xi) : |\xi - a| \leq 2^{-j}\}) \geq \lim_{j\to\infty} \phi_j(z,a), \tag{5}$$

for every $z \in \Omega$ and $a \in X$. By integrating the sides of (5) with respect to a and using Fatou's lemma, we get

$$\tilde{u}(z) \geq \lim_{j\to\infty} u_j(z). \tag{6}$$

Combining (3), (4) and (6), we get that u_j is decreasing to $\tilde{u}$ as $j \to \infty$. Note that $u_j = \tilde{u}_j$ almost everywhere [Kli91, Proposition 2.6.2], and then $\lim\limits_{j\to\infty} \tilde{u}_j = \tilde{u}$ almost everywhere. Since $\lim\limits_{j\to\infty} \tilde{u}_j$ is either plurisubharmonic or identically $-\infty$, we have $\lim\limits_{j\to\infty} \tilde{u}_j = \tilde{u}$ everywhere. Therefore, $\tilde{u}_j$ is decreasing to $\tilde{u}$ as $j \to \infty$.

By Proposition 6, $\max\{\tilde{u}, -k\} \in \mathcal{F}(\Omega)$ for $k > 0$ and it implies that $\tilde{u}$ is not identically $-\infty$. Then, by using Proposition 6 for $\tilde{u}$, we get that $\tilde{u} \in \mathcal{F}(\Omega)$. Moreover, since the sequence $\tilde{u}_j$ is decreasing, we have

$$\int\limits_{\Omega}(dd^c\tilde{u})^n \leq \liminf_{j\to\infty} \int\limits_{\Omega}(dd^c\tilde{u}_j)^n \leq (\int\limits_{X} M(a)d\mu(a))^n.$$

□

In order to prove Theorem 1, we need the following proposition:

Proposition 13. *Let $\varphi \in \mathcal{E}(\Omega)$ and $u \in \mathcal{F}(\varphi)$. Define*

$$\phi(u) := (\sup\{v \in PSH^-(\Omega) : v + \varphi \leq u\})^*.$$

Then $\phi(u) \in \mathcal{F}$, $\phi(u) + \varphi \leq u$ and $(dd^c\phi(u))^n \leq \chi_U(dd^c u)^n$, where $U = \Omega \cap \overline{\{u < \varphi\}}$.

We proceed through some lemmas.

Lemma 14. *Let $u \in C(\overline{\Omega}) \cap PSH(\Omega)$ and $v \in L^\infty(\Omega) \cap PSH(\Omega)$. Then, for every relatively compact pseudoconvex domain W in Ω, $P_{\overline{W}}(u-v) \in L^\infty(W) \cap PSH(W)$ and $(dd^c P_{\overline{W}}(u - v))^n \leq (dd^c u)^n$ on W.*

Proof. Since $u|_W - \sup_W v \leq P_{\overline{W}}(u-v) \leq u|_W - \inf_W v$, we have $P_{\overline{W}}(u-v) \in L^\infty(W)$. It remains to show that $(dd^c P_{\overline{W}}(u - v))^n \leq (dd^c u)^n$ on W.

Let u_j, v_j be sequences of smooth plurisubharmonic functions on a neighborhood of $\overline{W}$ such that $u_j \searrow u$ and $v_j \searrow v$ as $j \to \infty$. Then, for every $j, k \geq 1$, the function $u_j - v_k$ is a viscosity supersolution to the equation

$$(dd^c w)^n = (dd^c u_j)^n, \tag{7}$$

on W. It follows from Proposition 11 that the function $P_W(u_j - v_k) \in L^\infty(W) \cap PSH(W)$ satisfies

$$(dd^c P_W(u_j - v_k))^n \leq (dd^c u_j)^n, \tag{8}$$

on W in the pluripotential sense. Moreover, by Lemma 8, we have

$$P_W(u_j - v_k) \searrow P_W(u - v_k), \tag{9}$$

as $j \to \infty$. Combining (8) and (9), we have

$$(dd^c P_W(u - v_k))^n \leq (dd^c u)^n. \tag{10}$$

By Lemma 10, we also have $P_W(u - v_k) \nearrow P_{\overline{W}}(u - v)$ almost everywhere as $k \to \infty$. Therefore, by (10), we have

$$(dd^c P_{\overline{W}}(u - v))^n \leq (dd^c u)^n.$$

□

Lemma 15. *Let $u \in C(\overline{\Omega}) \cap PSH(\Omega)$ and $v \in L^\infty(\Omega) \cap PSH(\Omega)$. Then $P_\Omega(u - v) \in L^\infty(\Omega) \cap PSH(\Omega)$ and $(dd^c P_\Omega(u - v))^n \leq (dd^c u)^n$.*

Proof. Since $u - \sup_\Omega v \leq P_\Omega(u-v) \leq u - \inf_\Omega v$, we have $P_\Omega(u-v) \in L^\infty(\Omega) \cap PSH(\Omega)$. Let (Ω_j) be an increasing sequence of relatively compact pseudoconvex domains in Ω such that $\cup_{j\in\mathbb{Z}^+}\Omega_j = \Omega$. It follows from Lemma 14 that

$$(dd^c P_{\overline{\Omega_j}}(u-v))^n \leq (dd^c u)^n,$$

on Ω_j for every $j \in \mathbb{Z}^+$. Moreover, by Lemma 9, we have $P_{\overline{\Omega_j}}(u-v)$ decreases to $P_\Omega(u-v)$. Hence, we have

$$(dd^c P_\Omega(u-v))^n \leq (dd^c u)^n, \text{ on } \Omega.$$

□

Proof of Proposition 13. By the assumption, there exists $v \in \mathcal{F}$ such that $v + \varphi \leq u \leq \varphi$. Then $v \leq \phi(u) \leq 0$. It follows from Proposition 5 that $\phi(u) \in \mathcal{F}$. By the definition of $\phi(u)$, we have $\phi(u) + \varphi \leq u$ almost everywhere. Therefore, by the subharmonicity of $\phi(u) + \varphi$ and u, we have $\phi(u) + \varphi \leq u$. It remains to show that $(dd^c\phi(u))^n \leq (dd^c u)^n$.

Since $u \in PSH^-(\Omega)$, it follows from [Ceg04, Theorem 2.1] that there exists a sequence of functions $u_j \in \mathcal{E}_0(\Omega) \cap C(\overline{\Omega})$ such that $u_j \searrow u$ as $j \to \infty$. For every $j \in \mathbb{Z}^+$, we denote

$$w_j = u_j - \max\{\varphi, u_j\}.$$

We have

$$w_j = u_j - \frac{\varphi + u_j + |\varphi - u_j|}{2} = \frac{u_j - \varphi - |\varphi - u_j|}{2} = \min\{-\varphi + u_j, 0\}.$$

Then

$$\phi(u) \leq w_{j+1} \leq w_j \leq 0, \tag{11}$$

for every $j \in \mathbb{Z}^+$. Hence

$$\phi(u) \leq P_\Omega(w_{j+1}) \leq P_\Omega(w_j) \leq 0, \tag{12}$$

for every $j \in \mathbb{Z}^+$. In particular, $P_\Omega(w_j) \in \mathcal{F}(\Omega)$ for every j.

Since $P_\Omega(w_j) + \max\{\varphi, u_j\}$ and u_j are plurisubharmonic and $P_\Omega(w_j) + \max\{\varphi, u_j\} \leq w_j$ almost everywhere, we have $P_\Omega(w_j) + \max\{\varphi, u_j\} \leq u_j$ for all $z \in \Omega$. Letting $j \to \infty$, we get

$$\lim_{j\to\infty} P_\Omega(w_j) + \varphi \leq u. \tag{13}$$

Combining (12) and (13), we get $P_\Omega(w_j) \searrow \phi(u)$ as $j \to \infty$. Moreover, by Lemma 15, we have $(dd^c P_\Omega(w_j))^n \leq (dd^c u_j)^n$. Therefore, by letting $j \to \infty$, we obtain $(dd^c\phi(u))^n \leq (dd^c u)^n$. Observe that $\phi(u)$ is maximal plurisubharmonic (see [Sad81], [Kli91] for the definition) on $\Omega \setminus \overline{\{u < \phi\}} = Int\{u = \phi\}$. Then, we have $(dd^c\phi(u))^n = 0$ on $\Omega \setminus \overline{\{u < \phi\}}$. Thus $(dd^c\phi(u))^n \leq \chi_U (dd^c u)^n$. □

Proof of Theorem 1. As in the proposition 13, for all $a \in X$, we define

$$\begin{aligned} \phi(u)(\cdot, a) &:= (\sup\{v \in PSH^-(\Omega) : v + f \leq u(\cdot, a)\})^* \\ &= \sup\{v \in \mathcal{F}(\Omega) : v + \varphi \leq u(\cdot, a)\}. \end{aligned}$$

For every $a \in X$, we have

$$\begin{aligned} u(z,a) \geq \limsup_{\xi\to a} u(z,\xi) &\geq \limsup_{\xi\to a}(\phi(u)(z,\xi)+f(z,\xi)) \\ &\geq \limsup_{\xi\to a}\phi(u)(z,\xi)+\liminf_{\xi\to a} f(z,\xi) \\ &\geq \limsup_{\xi\to a}\phi(u)(z,\xi)+f(z,a). \end{aligned}$$

Hence

$$\phi(u)(z,a) \geq \limsup_{\xi\to a}\phi(u)(z,\xi).$$

Moreover, by Proposition 13, we have

$$\int_\Omega (dd^c\phi(u)(z,a))^n \leq \int_{U_a}(dd^c u(z,a))^n \leq (M(a))^n.$$

Hence, the function $\phi(u)$ satisfies the assumptions of Proposition 12. Then $\widetilde{\phi(u)} := \int_X \phi(u)d\mu(a) \in \mathcal{F}(\Omega)$. In the other hand, we have

$$\widetilde{\phi(u)} + \tilde{f} \leq \tilde{u} \leq \tilde{f}.$$

Thus $\tilde{u} \in \mathcal{F}(\tilde{f})$. □

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