Bohr inequality and Paley-Wiener type theorem for functions with value in Banach spaces

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Abstract

In this paper, we prove Bohr inequality and Paley-Wiener type theorem for functions with value in Banach spaces.

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1. Introduction.

The relations between properties of functions and their spectrum (the support of their Fourier transform) are interested for many mathematicians. The classical results such as inequalities of Bernstein, Bohr, Nikol'skii, Paley-Wiener theorem, etc. ([1], [2], [3], [4]) belong to this direction. In this paper we provide Bohr inequality and Paley-Wiener type theorem for functions with value in Banach spaces. Note that the mentioned classical results were stated for usual L^p- functions.
We recall Bohr inequality ([2]):
Let $1 \le p \le \infty, f \in C^m(\mathbb{R}), \varrho > 0$, *supp*$\hat{f} \cap (-\varrho, \varrho) = \emptyset$ *and* $D^m f \in L^p(\mathbb{R})$. *Then* $f \in L^p(\mathbb{R})$ *and*

$$\|f\|_p \le \varrho^{-m} K_m \|D^m f\|_p,$$

where the Favard constants K_m *are best possible when* $p = 1$ *and defined as follows*

$$K_m = 8(2\pi)^{-(m+1)} \sum_{j=0}^{\infty} ((-1)^j(2j+1))^{-(m+1)}.$$

The Favard constants have the following properties

$$1 = K_0 \le K_2 < ... < \frac{4}{\pi} < ... < K_3 \le K_1 = \frac{\pi}{2}.$$

The initial Bohr inequality was proved for $p = \infty$ and generalized in [5], [6].
The Paley-Wiener theorem was proved first for L^2-functions and has many generalizations (see, for example, [7]- [18]).

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Notations. Let $f \in L^1(\mathbb{R})$ and $\hat{f} = \mathcal{F}f$ be its Fourier transform

$$\hat{f}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\zeta} f(x)dx,$$

and $\check{f} = \mathcal{F}^{-1}f$ be its inverse Fourier transform

$$\check{f}(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ix\zeta} f(x)dx.$$

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ denote a complex Banach space and $L(\mathbb{X}) = BC(\mathbb{R} \to \mathbb{X})$ be the set of all $\mathbb{X}$-value bounded continuous functions $f : \mathbb{R} \to \mathbb{X}$. For a given function $f \in BC(\mathbb{R} \to \mathbb{X})$, we define $\|f\|_{L(\mathbb{X})} = \sup\{\|f(t)\|_{\mathbb{X}} : t \in \mathbb{R}\}$. Then $(L(\mathbb{X}), \|.\|_{L(\mathbb{X})})$ itself is a Banach space. We define the derivative Df of $f \in L(\mathbb{X})$, as usual,

$$Df(s) = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h},$$

and $D^m f$ is the derivative of order m of f. The convolution $\varphi * f$ of f with a Schwartz function is defined as follows

$$\varphi * f(s) = \int_{-\infty}^{+\infty} \varphi(s-t)f(t)dt.$$

The Beurling spectrum Spec(f) of a function $f \in L(\mathbb{X})$ is defined by

$$\text{Spec}(f) = \{\zeta \in \mathbb{R} : \forall \epsilon > 0, \exists \varphi \in \mathcal{S}(\mathbb{R}) : \text{supp}\hat{\varphi} \subset (\zeta - \epsilon, \zeta + \epsilon), \varphi * f \neq 0\}.$$

Note that, Spec(f) is always a closed subset of $\mathbb{R}$. Let $K \subset \mathbb{R}$ and $\epsilon > 0$. We put $K_{(\epsilon)} := \{\zeta \in \mathbb{C} : \quad \exists x \in K : |x - \zeta| < \epsilon\}$, which is the $\epsilon-$ neighborhood in $\mathbb{C}$ of K and $K_\epsilon := \{\zeta \in \mathbb{R} : \quad \exists x \in K : |x - \zeta| < \epsilon\}$, $\mathbb{Z}_+ = \{0, 1, 2, ...\}$.

Let $P(x)$ be a polynomial. The differential operator $P(D)$ is obtained from $P(x)$ by substituting $x \to D = -i\partial/\partial x$,

2. Bohr inequality for functions with value in Banach spaces

Now, we state the Bohr inequality for functions with value in Banach spaces.

Theorem 2.1. *Let $\Delta > 1, \varrho > 0$, $(D^m f)_{m=0}^{\infty} \subset L(\mathbb{X})$, Spec$(f) \subset [-\Delta\varrho, \Delta\varrho]$ and Spec$(f) \cap (-\varrho, \varrho) = \emptyset$. Then there exists a constant $C > 0$ independent of f, m, ϱ such that*

$$\|D^m f\|_{L(\mathbb{X})} \geq C\varrho^m \|f\|_{L(\mathbb{X})}. \tag{1}$$

To obtain the theorem, we need the following results.

Lemma 2.2 (Young inequality for Banach spaces). *Let $f \in L(\mathbb{X})$, and $\varphi \in \mathcal{S}(\mathbb{R})$. Then $\varphi * f \in L(\mathbb{X})$ and*

$$\|\varphi * f\|_{L(\mathbb{X})} \leq \|f\|_{L(\mathbb{X})} \|\varphi\|_{L^1}.$$

Proof. We see that

$$\|\varphi * f\|_{L(\mathbb{X})} = \sup_{t\in\mathbb{R}} \| \int_{-\infty}^{+\infty} \varphi(s-t)f(t)dt\|_{\mathbb{X}} \le \sup_{t\in\mathbb{R}} \int_{-\infty}^{+\infty} \|\varphi(s-t)f(t)\|_{\mathbb{X}} dt$$
$$\le \|f\|_{L(\mathbb{X})} \sup_{t\in\mathbb{R}} \int_{-\infty}^{+\infty} |\varphi(s-t)|dt = \|f\|_{L(\mathbb{X})}\|\varphi\|_{L^1},$$

which completes the proof.

Lemma 2.3. *Let $\Delta > 2$, $(D^m f)_{m=0}^{\infty} \subset L(\mathbb{X})$, Spec$(f) \subset [-\Delta, \Delta]$ and Spec$(f) \cap (-1,1) = \emptyset$. Then there exists a constant $C > 0$ independent of f, m such that*

$$\|D^m f\|_{L(\mathbb{X})} \ge C\|f\|_{L(\mathbb{X})}. \tag{2}$$

Proof. Put $K := [-\Delta, -1] \cup [1, \Delta]$ and

$$\top(\zeta) = \begin{cases} C_1 e^{\frac{1}{\zeta^2 - 1}} & \text{if } |\zeta| < 1, \\ 0 & \text{if } |\zeta| \ge 1, \end{cases}$$

where C_1 is chosen such that $\int_{\mathbb{R}} \top(\zeta)d\zeta = 1$. We define the sequence of functions $(\phi_m(\zeta))_{m\in\mathbb{N}}$ via the formula

$$\phi_m(\zeta) = (1_{K_{3/(4m)}} * \top_{1/(4m)})(\zeta),$$

where

$$\top_{1/(4m)}(\zeta) = 4m\top(4m\zeta).$$

Then $\top_{1/(4m)}(\zeta) = 0$ for all $\zeta \notin [-1/(4m), 1/(4m)]$, $\int_{\mathbb{R}} \top_{1/(4m)}(\zeta)d\zeta = 1$. Hence, for all $m \in \mathbb{N}$ we have $\phi_m(\zeta) \in C^{\infty}(\mathbb{R})$, and

$$\phi_m(\zeta) = 1 \quad \forall\zeta \in K_{1/(2m)}, \phi_m(\zeta) = 0 \quad \forall\zeta \notin K_{1/m}. \tag{3}$$

So, it follows from Spec$(f) \subset K$ that

$$\begin{aligned} f &= (2\pi)^{-1/2}(D^m f) * \mathcal{F}^{-1}(\phi_m(-\zeta)(-i\zeta)^m) \\ &= (2\pi)^{-1/2}(D^m f) * \mathcal{F}(\phi_m(\zeta)/(i\zeta)^m). \end{aligned} \tag{4}$$

Therefore, since (4) and Lemma 2.2, we have $f \in L(\mathbb{X})$ and the following estimate

$$\|f\|_{L(\mathbb{X})} \le (2\pi)^{-1/2}\|D^m f\|_{L(\mathbb{X})}\|\mathcal{F}(\phi_m(\zeta)/\zeta^m)\|_{L^1}. \tag{5}$$

Therefore, since $0 < \|\mathcal{F}(\phi_m(\zeta)/\zeta^m)\|_{L^1} < \infty$ for all $m = 1, 2, 3$, we conclude $\|D^m f\|_{L(\mathbb{X})} \ge C\|f\|_{L(\mathbb{X})}$ for all $m = 1, 2, 3$. So, to complete the proof, it is sufficient to prove (2) only for $m \ge 4$. To do that, we define

$$k_m := 1 + \frac{1}{m}, g_m(\zeta) = \phi_m(k_m\zeta), \Phi_m(\zeta) = \phi_m(\zeta) - g_m(\zeta).$$

Hence,

$$(\mathcal{F}(g_m(\zeta)/\zeta^m))(x) = (k_m)^m(\mathcal{F}(\phi_m(k_m\zeta)/(k_m\zeta)^m)(x) = (k_m)^{m-1}(\mathcal{F}(\phi_m(\zeta)/\zeta^m))(x/k_m).$$

So,

$$\Big\|\mathcal{F}(g_m(\zeta)/\zeta^m)\Big\|_{L^1} = k_m^m\Big\|\mathcal{F}(\phi_m(\zeta)/\zeta^m)\Big\|_{L^1}.$$

Then it follows from $(k_m)^m = (1+\frac{1}{m})^m \geq 2$ that

$$\Big\|\mathcal{F}(g_m(\zeta)/\zeta^m)\Big\|_{L^1} \geq 2\Big\|\mathcal{F}(\phi_m(\zeta)/\zeta^m)\Big\|_{L^1}.$$

Therefore, since $\Phi_m(\zeta) = \phi_m(\zeta) - g_m(\zeta)$ we get

$$\begin{aligned}\Big\|\mathcal{F}(\Phi_m(\zeta)/\zeta^m)\Big\|_{L^1} &\geq \Big\|\mathcal{F}(g_m(\zeta)/\zeta^m)\Big\|_{L^1} - \Big\|\mathcal{F}(\phi_m(\zeta)/\zeta^m)\Big\|_{L^1} \\ &\geq \Big\|\mathcal{F}(\phi_m(\zeta)/\zeta^m)\Big\|_{L^1}.\end{aligned} \tag{6}$$

From (5)-(6) we obtain

$$\|f\|_{L(\mathbb{X})} \leq (2\pi)^{-1/2}\|D^m f\|_{L(\mathbb{X})}\|\mathcal{F}(\Phi_m(\zeta)/\zeta^m)\|_{L^1}. \tag{7}$$

Now, we will estimate$\|\mathcal{F}(\Phi_m(\zeta)/\zeta^m)\|_{L^1}$. To do that, we put $C_2 = \max\{\|\top^{(j)}\|_{L^1}, \quad j \leq 3\}$. Since $\top_{1/(4m)}(x) = 4m\top(4mx)$, we obtain $\top^{(j)}_{1/(4m)}(x) = (4m)^{j+1}\top^{(j)}(4mx)$ and then

$$\|\top^{(j)}_{1/(4m)}\|_{L^1} = (4m)^j\|\top^{(j)}\|_{L^1} \leq C_2(4m)^j, \quad \forall j \leq 3.$$

Therefore,

$$\|\phi_m^{(j)}\|_{L^\infty} = \|(1_{K_{3/(4m)}} * \top^{(j)}_{1/(4m)})\|_{L^\infty} \leq \Big\|\top^{(j)}_{1/(4m)}\Big\|_{L^1} \leq (4m)^j C_2, \quad \forall j \leq 3. \tag{8}$$

If $|\zeta| < 1-(3/m)$ then $|\zeta| < |k_m\zeta| < 1-(1/m)$, combining this and (3) we have $\phi_m(\zeta) = \phi_m(k_m\zeta) = 0$, which implies $\Phi_m(\zeta) = \phi_m(\zeta) - \phi_m(k_m\zeta) = 0$.
If $|\zeta| > \Delta + 1/m$ then $|k_m\zeta| > |\zeta| > \Delta + 1/m$, combining this and (3) we have $\phi_m(\zeta) = \phi_m(k_m\zeta) = 0$, which implies $\Phi_m(\zeta) = 0$.
If $1 < |\zeta| < \Delta - (\Delta - \frac{1}{2})/m$ then $1 < |\zeta| < |k_m\zeta| < \Delta + 1/(2m)$ and then $\phi_m(\zeta) = \phi_m(k_m\zeta) = 1$, which implies $\Phi_m(\zeta) = 0$.
From these we have

$$\text{supp}\Phi_m \subset \{\zeta : |\zeta| \in [1-(3/m), 1] \cup [\Delta - (\Delta - \frac{1}{2})/m, \Delta + 1/m]. \tag{9}$$

So, for $\zeta \in \text{supp}\Phi_m$ we get $|\zeta| \leq 2\Delta$ and then

$$\Big|\zeta - k_m\zeta\Big| = \Big|\frac{\zeta}{m}\Big| \leq \frac{2\Delta}{m}. \tag{10}$$

From (8) and (10) we have the following estimate for $\zeta \in \text{supp}\Phi_m$

$$\begin{aligned}\big|\Phi_m(\zeta)\big| &= \big|\phi_m(\zeta) - g_m(\zeta)\big| = \big|\phi_m(\zeta) - \phi(k_m\zeta)\big| \\ &\leq |\zeta - k_m\zeta|.\|\phi_m'\|_{L^\infty} \leq \frac{2\Delta}{m}4mC_2 = 8\Delta C_2,\end{aligned} \tag{11}$$

$$\big|\Phi_m'(\zeta)\big| = \big|\phi_m'(\zeta) - g_m'(\zeta)\big| = \big|\phi_m'(\zeta) - k_m\phi_m'(k_m\zeta)\big| \tag{12}$$

$$\leq \big|\phi_m^{'}(\zeta) - \phi_m^{'}(k_m\zeta)\big| + \big|(1-k_m)\phi_m^{'}(k_m\zeta)\big|$$
$$\leq \big|\zeta - k_m\zeta\big|.\|\phi_m^{''}\|_{L^\infty} + \big|1-k_m\big|.\|\phi_m^{'}\|_{L^\infty}$$
$$\leq \frac{2\Delta}{m}(4m)^2C_2 + \big|1-k_m\big|4mC_2 \leq 36\Delta m C_2,$$

and

$$\begin{aligned}\big|\Phi_m^{''}(\zeta)\big| &= \big|\phi_m^{''}(\zeta) - g_m^{''}(\zeta)\big| = \big|\phi_m^{''}(\zeta) - k_m^2\phi_m^{''}(k_m\zeta)\big| \\ &\leq \big|\phi_m^{''}(\zeta) - \phi_m^{''}(k_m\zeta)\big| + \big|(1-k_m^2)\phi_m^{''}(k_m\zeta)\big| \\ &\leq \big|\zeta - k_m\zeta\big|.\|\phi_m^{'''}\|_{L^\infty} + \big|1-k_m^2\big|.\|\phi_m^{''}\|_{L^\infty} \\ &\leq \frac{2\Delta}{m}(4m)^3C_2 + \big|1-k_m^2\big|(4m)^2C_2 \leq 560\Delta m^2 C_2.\end{aligned} \tag{13}$$

Put $\Lambda_m(x) = (\mathcal{F}(\Phi_m(\zeta)/\zeta^m))(x)$. Then

$$\Lambda_m(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\zeta}\Phi_m(\zeta)/\zeta^m d\zeta.$$

Therefore, since (9), we have

$$\sup_{x\in\mathbb{R}}\Big|\Lambda_m(x)\Big| \leq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Big|\Phi_m(\zeta)/\zeta^m\Big|d\zeta = \frac{1}{\sqrt{2\pi}}\int_{\zeta\in\text{supp}\Phi_m}\Big|\Phi_m(\zeta)/\zeta^m\Big|d\zeta$$

and it follows from (11) that

$$\sup_{x\in\mathbb{R}}\Big|\Lambda_m(x)\Big| \leq \frac{8\Delta}{m\sqrt{2\pi}}\sup_{\zeta\in\mathbb{R}}\Big|\Phi_m(\zeta)\Big|(1-\frac{3}{m})^{-m} \leq \frac{(8\Delta)^2e^4C_2}{m\sqrt{2\pi}} := \frac{C_3}{m}. \tag{14}$$

On the other hand, we have

$$\begin{aligned}\big|x^2\Lambda_m(x)\big| &= \frac{1}{\sqrt{2\pi}}\Big|\int_{\mathbb{R}} e^{-ix\zeta}(\Phi_m(\zeta)m(m+1)/\zeta^{m+2} + \Phi_m^{'}(\zeta)2m/\zeta^{m+1} + \Phi_m^{''}(\zeta)/\zeta^m)d\zeta\Big| \\ &\leq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Big|e^{-ix\zeta}(\Phi_m(\zeta)m(m+1)/\zeta^{m+2} + \Phi_m^{'}(\zeta)2m/\zeta^{m+1} + \Phi_m^{''}(\zeta)/\zeta^m)d\zeta\Big|d\zeta\end{aligned}$$

and then

$$\begin{aligned}\sup_{x\in\mathbb{R}}\big|x^2\Lambda_m(x)\big| &\leq \frac{1}{\sqrt{2\pi}}\Big[\int_{\zeta\in\text{supp}\Phi_m}|\Phi_m(\zeta)m(m+1)/\zeta^{m+2} + \Phi^{'}(\zeta)2m/\zeta^{m+1} + \Phi^{''}(\zeta)/\zeta^m|d\zeta \\ &\leq \frac{8\Delta}{m\sqrt{2\pi}}\Big[\sup_{\zeta\in\mathbb{R}}|\Phi(\zeta)|m(m+1)(1-\frac{3}{m})^{-m} + \\ &\qquad \sup_{\zeta\in\mathbb{R}}|\Phi^{'}(\zeta)|2m(1-\frac{3}{m})^{-(m+1)} + \sup_{\zeta\in\mathbb{R}}|\Phi^{''}(\zeta)|(1-\frac{3}{m})^{-m}\Big].\end{aligned} \tag{15}$$

So, since (11)-(15), we get

$$\sup_{x\in\mathbb{R}}\big|x^2\Lambda_m(x)\big| \leq \frac{8\Delta}{m\sqrt{2\pi}}\Big[8\Delta C_2 m(m+1)e^4 + 36\Delta m C_2 2me^4 + 560\Delta m^2 C_2 e^4\Big] \tag{16}$$

$$\leq C_4 m.$$

Using (14) and (16) and

$$\|\Lambda_m\|_{L^1} = \int\limits_{|x|\leq m} |\Lambda_m(x)|dx + \int\limits_{|x|\geq m} |\Lambda_m(x)|dx \leq 2m \sup_{x\in\mathbb{R}} |\Lambda_m(x)| + \frac{4}{m} \sup_{x\in\mathbb{R}} |x^2\Lambda_m(x)|,$$

we obtain that

$$\|\mathcal{F}(\Phi_m(x)x^m)\|_{L^1} = \|\Lambda_m\|_{L^1} \leq 2m\frac{C_3}{m} + \frac{4}{m}C_4 m := C_5. \tag{17}$$

From (7) and (17) we have

$$\|f\|_{L(\mathbb{X})} \leq \|D^m f\|_{L(\mathbb{X})}/C.$$

The proof is complete.

Proof (Proof of Theorem 2.1). Put

$$g(x) = f(\frac{x}{\varrho}).$$

Then it follows from $\mathrm{Spec}(f) \cap (-\varrho, \varrho) = \emptyset, \mathrm{Spec}(f) \subset (-\Delta\varrho, \Delta\varrho)$ that $\mathrm{Spec}(g) \cap (-1,1) = \emptyset, \mathrm{Spec}(f) \subset (-\Delta, \Delta) \subset (-2\Delta, 2\Delta)$. Applying Lemma 2.3, we have

$$\|D^m g\|_{L(\mathbb{X})} \geq C\|g\|_{L(\mathbb{X})}. \tag{18}$$

Since $g(x) = f(\frac{x}{\varrho})$, we have

$$\|g\|_{L(\mathbb{X})} = \|f\|_{L(\mathbb{X})}, \|D^m g\|_{L(\mathbb{X})} = \varrho^{-m}\|D^m f\|_{L(\mathbb{X})}.$$

Hence, it follows from (18) that

$$\varrho^{-m}\|D^m f\|_{L(\mathbb{X})} \geq C\|f\|_{L(\mathbb{X})}.$$

Hence,

$$\|D^m f\|_{L(\mathbb{X})} \geq C\varrho^m\|f\|_{L(\mathbb{X})}.$$

The proof is complete.

3. Paley-Wiener theorem for functions with value in Banach spaces

3.1. Paley-Wiener theorem for arbitrary compact

Theorem 3.1. *Let $f \in L(\mathbb{X})$ and K be an arbitrary compact set in $\mathbb{R}$. Then $\mathrm{Spec}(f) \subset K$ if and only if for any $\delta > 0$ there exists a constant $C_\delta < \infty$ such that*

$$\|P(D)f\|_{L(\mathbb{X})} \leq C_\delta\|f\|_{L(\mathbb{X})} \sup_{x\in K_{(\delta)}} |P(x)| \tag{19}$$

for all polynomials with complex coefficients $P(x)$.

To obtain the theorem, we need the following results.

Lemma 3.2. *[19] Let $f \in L(\mathbb{X})$, and $\varphi, \psi \in \mathcal{S}(\mathbb{R})$. Assume that $\hat{\varphi} = 0$ on $\mathrm{Spec}(f)$, and $\hat{\psi} = 1$ on $\mathrm{Spec}(f)$. Then $\varphi * f = 0$ and $\psi * f = f$.*

In [20], author derives a radial spectral formula.

Lemma 3.3. *Let $f \in L(\mathbb{X})$. Asume that Spec(f) is the compact set. Then there always exists the following limit*

$$\lim_{m\to\infty} \|P^m(D)f\|_{L(\mathbb{X})}^{1/m}$$

and

$$\lim_{m\to\infty} \|P^m(D)f\|_{L(\mathbb{X})}^{1/m} = \sup\{|P(\zeta)| : \zeta \in \mathit{Spec}(f)\}.$$

Proof (Proof of Theorem 3.1). *Necessity.* We choose a function $\vartheta(\zeta) \in C_0^\infty(\mathbb{R})$ such that $\vartheta(\zeta) = 1$ if $\zeta \in K_{\delta/4}$ and $\vartheta(\zeta) = 0$ if $\zeta \notin K_{\delta/2}$. Then it follows from $\mathrm{Spec}(f) \subset K$ and Lemma 3.2 that

$$P(D)f = (2\pi)^{-1/2} f * \mathcal{F}^{-1}(\vartheta(\zeta)P(\zeta)).$$

Therefore, by Lemma 2.2 we have

$$\begin{aligned}\big\|P(D)f\big\|_{L(\mathbb{X})} &\le (2\pi)^{-1/2}\big\|f\big\|_{L(\mathbb{X})}\big\|\mathcal{F}^{-1}\big(\vartheta(\zeta)P(\zeta)\big)\big\|_{L^1} \\ &= (2\pi)^{-1/2}\big\|f\big\|_{L(\mathbb{X})}\big\|\mathcal{F}\big(\vartheta(\zeta)P(\zeta)\big)\big\|_{L^1} = (2\pi)^{-1/2}\big\|f\big\|_{L(\mathbb{X})}\big\|\Psi\big\|_{L^1},\end{aligned}$$

where

$$\Psi(x) := \big(\mathcal{F}\big(\vartheta(\zeta)P(\zeta)\big)\big)(x).$$

Then it follows from

$$\int_{\mathbb{R}} \big|\Psi(x)\big| dx \le \pi \sup_{x\in\mathbb{R}} |(1+x^2)\Psi(x)|,$$

that

$$\big\|P(D)f\big\|_{L(\mathbb{X})} \le (2\pi)^{1-1/2}\big\|f\big\|_{L(\mathbb{X})} \sup_{x\in\mathbb{R}} |(1+x^2)\Psi(x)|. \tag{20}$$

For $\beta \in \{0,1,2\}$ we get the following estimate

$$\begin{aligned}\sup_{x\in\mathbb{R}} |x^\beta \Psi(x)| &= (2\pi)^{-1/2} \sup_{x\in\mathbb{R}} \Big| \int_{\mathbb{R}} e^{-ix\zeta} D^\beta\big(\vartheta(\zeta)P(\zeta)\big) d\zeta \Big| \\ &= (2\pi)^{-1/2} \sup_{x\in\mathbb{R}} \Big| \int_{\zeta\in K_{\delta/2}} e^{-ix\zeta} D^\beta\big(\vartheta(\zeta)P(\zeta)\big) d\zeta \Big| \\ &\le (2\pi)^{-1/2} \int_{\zeta\in K_{\delta/2}} \Big| D^\beta\big(\vartheta(\zeta)P(\zeta)\big)\Big| d\zeta.\end{aligned}$$

Then it follows from the Leibniz rule that

$$\begin{aligned}\sup_{x\in\mathbb{R}} |x^\beta \Psi(x)| &\le (2\pi)^{-1/2} \int_{\zeta\in K_{\delta/2}} \Big| \sum_{\gamma\le\beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} D^\gamma\vartheta(\zeta) D^{\beta-\gamma}P(\zeta)\Big| d\zeta \qquad (21) \\ &\le (2\pi)^{-1/2} \sum_{\gamma\le\beta} \Big(\frac{\beta!}{\gamma!(\beta-\gamma)!} \sup_{x\in K_{\delta/2}} |D^{\beta-\gamma}P(x)| \int_{\zeta\in K_{\delta/2}} |D^\gamma\vartheta(\zeta)| d\zeta \Big)\end{aligned}$$

$$\leq (2\pi)^{-1/2} \max_{\theta \leq 2} \sup_{x \in K_{\delta/2}} |D^\theta P(x)| \sum_{\gamma \leq \beta} \Big(\frac{\beta!}{\gamma!(\beta-\gamma)!} \int\limits_{\zeta \in K_{\delta/2}} |D^\gamma \vartheta(\zeta)| d\zeta \Big).$$

Because the derivatives of the analytic function $P(x)$ can be estimated in $K_{\delta/2}$ by the maximum of the modulus in $K_{(\delta)}$, there exists a constant A_δ independent of $f, P(x)$ such that

$$\sup_{x \in K_{\delta/2}} |D^\theta P(x)| \leq A_\delta \sup_{x \in K_{(\delta)}} |P(x)|, \quad \forall \theta \in \mathbb{Z}_+, \theta \leq 2. \tag{22}$$

From (21) - (22), we have

$$\sup_{x \in \mathbb{R}} |x^\beta \Psi(x)| \leq (2\pi)^{-1/2} \sum_{\gamma \leq \beta} \Big(\frac{\beta!}{\gamma!(\beta-\gamma)!} A_\delta \sup_{x \in K_{(\delta)}} |P(x)| \int\limits_{\zeta \in K_{\delta/2}} |D^\gamma \vartheta(\zeta)| d\zeta \Big) \tag{23}$$

$$\leq (2\pi)^{-1/2} 2^{2n} A_\delta C \sup_{x \in K_{(\delta)}} |P(x)|,$$

where $C := \max_{\gamma \leq 2} \int\limits_{\zeta \in K_{\delta/2}} |D^\gamma \vartheta(\zeta)| d\zeta$. Then it follows from (23) that

$$\int_{\mathbb{R}} \big|\Psi(x)\big| dx \leq C_\delta \sup_{x \in K_{(\delta)}} |P(x)|, \tag{24}$$

where C_δ independent of $f, P(x)$. From (20) and (24) we obtain (19) .

Sufficiency. Assume (19) is true, we need to prove $\mathrm{Spec}(f) \subset K$. Indeed, assume the contrary that there exists $\varrho \in \mathrm{Spec}(f)$ and $\varrho \notin K$. We construct a polynomial $G(x) = t - (x-\varrho)^2$, where $t = \sup_{x \in K} (x-\varrho)^2$. Then applying (19) for $P(x) = G^m(x)$, we get for all $m \in \mathbb{Z}_+$

$$\|G^m(D) f\|_{L(\mathbb{X})} \leq C_\delta \|f\|_{L(\mathbb{X})} \sup_{x \in K_{(\delta)}} |G^m(x)|,$$

which gives

$$\overline{\lim_{m \to \infty}} (\|G^m(D) f\|_{L(\mathbb{X})})^{1/m} \leq \sup_{x \in K_{(\delta)}} |G(x)|.$$

Letting $\delta \to 0$, we obtain

$$\overline{\lim_{m \to \infty}} (\|G^m(D) f\|_{L(\mathbb{X})})^{1/m} \leq \sup_{x \in K} |G(x)|. \tag{25}$$

Then it follows from Lemma 3.3 that

$$\big|G(\varrho)\big| \leq \sup_{x \in K} |G(x)|$$

and then

$$t = |G(\varrho)| \leq \sup_{x \in K} (t - (x-\varrho)^2).$$

This is a contradiction. So, $\mathrm{Spec}(f) \subset K$. The proof is complete.

It follows from Lemma 3.3 that, if $f \in L(\mathbb{X})$ and $\mathrm{Spec}(f) \subset K$ then for any $\delta > 0$ there exists a constant $C_{P,\delta,f} < \infty$ ($C_{P,\delta,f}$ depends on P, δ and f) such that

$$\|P^m(D) f\|_{L(\mathbb{X})} \leq C_{P,\delta,f} \|f\|_{L(\mathbb{X})} \sup_{x \in K_{(\delta)}} |P^m(x)| \quad \forall m \in \mathbb{N}$$

while by Theorem 3.1 we have the stronger result that for any $\delta > 0$ there exists a constant $C_\delta < \infty$ (independent of P, m, f) such that

$$\|P^m(D) f\|_{L(\mathbb{X})} \leq C_\delta \|f\|_{L(\mathbb{X})} \sup_{x \in K_{(\delta)}} |P^m(x)|.$$

3.2. Paley-Wiener theorem for the sets generated by polynomials

Let $P(x)$ be a polynomial with real coefficients. We put

$$Q(P)_r := \{x \in \mathbb{R} : |P(x)| \le r\} \text{ for } r > 0$$

and $Q(P)_r$ is called the set generated by $P(x)$ repect to r. Clearly, for $a, b \in \mathbb{R}, a \le b; \alpha > 0$ then $(a, a+\alpha) \cup (b, b+\alpha)$ is the set generated by polynomial.

Theorem 3.4. *Let $f \in L(\mathbb{X})$ and $P(x)$ be a polynomial. Then* $\operatorname{Spec}(f) \subset Q(P)_r := K$ *if and only if for any $\delta > 0$ there exists a constant $C_\delta < \infty$ independent of f, m such that*

$$\|P^m(D)f\|_{L(\mathbb{X})} \le C_\delta \|f\|_{L(\mathbb{X})}(r+\delta)^m. \tag{26}$$

for all $m \in \mathbb{Z}_+$.

PROOF. *Necessity.* The necessity is follows from Theorem 3.1.
Sufficiency. Assume the contrary that there exists $\sigma \in \operatorname{Spec}(f)$ and $\sigma \notin K$. Combining $\sigma \notin K$ and $K = \{x \in \mathbb{R} : \quad |P(x)| \le r\}$, we have

$$|P(\sigma)| > r.$$

Using (26), we obtain

$$\overline{\lim_{m\to\infty}} \left(\|P^m(D)f\|_{L(\mathbb{X})}\right)^{1/m} \le r + \delta. \tag{27}$$

Applying Lemma 3.3, we have

$$\underline{\lim_{m\to\infty}} \left(\|P^m(D)f\|_{L(\mathbb{X})}\right)^{1/m} \ge |P(\sigma)|. \tag{28}$$

From (27) and (28), we get $|P(\sigma)| \le r + \delta$. Letting $\delta \to 0$, we obtain $|P(\sigma)| \le r$. This is a contradiction. So, $\operatorname{Spec}(f) \subset K$.
The proof is complete.

Since Theorem 3.4 we get the following corollary:

Corollary 1. Let $r > 0$ and $f \in L(\mathbb{X})$. Then $\operatorname{Spec}(f) \subset [-r, r]$ if and only if for any $\delta > 0$ there exists a constant $C_\delta < \infty$ such that

$$\|D^m f\|_{L(\mathbb{X})} \le C_\delta (r+\delta)^m \|f\|_{L(\mathbb{X})}$$

for all $m \in \mathbb{Z}_+$.

In general, for $a, b \in \mathbb{R}, a < b$ then (a, b) is the set generated by polynomial $P(x) = x - \frac{a+b}{2}$ respect to $\frac{b-a}{2}$. Then $\operatorname{Spec}(f) \subset [a, b]$ if and only if for any $\delta > 0$ there exists a constant $C_\delta < \infty$ such that

$$\|(x - \frac{a+b}{2})^m(D)f\|_{L(\mathbb{X})} \le C_\delta (\frac{b-a}{2} + \delta)^m \|f\|_{L(\mathbb{X})}$$

for all $m \in \mathbb{Z}_+$.
Moreover, for $a, b \in \mathbb{R}, a < b; \alpha > 0$ then $(a, a+\alpha) \cup (b, b+\alpha)$ is the set generated by polynomial

$P(x) = x^2 - (a+b+\alpha)x + ab + \frac{(a+b)\alpha}{2}$ respect to $r = \frac{(b-a)\alpha}{2}$. Then $\mathrm{Spec}(f) \subset (a, a+\alpha) \cup (b, b+\alpha)$ if and only if for any $\delta > 0$ there exists a constant $C_\delta < \infty$ such that

$$\|P^m(D)f\|_{L(\mathbb{X})} \leq C_\delta(\frac{(b-a)\alpha}{2} + \delta)^m \|f\|_{L(\mathbb{X})}$$

for all $m \in \mathbb{Z}_+$. Consequently, for $0 < a < b$ and $\mathrm{Spec}(f) \subset (a, b) \cup (-b, -a)$ if and only if for any $\delta > 0$ there exists a constant $C_\delta < \infty$ such that

$$\|(x^2 - \frac{a^2+b^2}{2})^m(D)f\|_{L(\mathbb{X})} \leq C_\delta(\frac{b^2-a^2}{2} + \delta)^m \|f\|_{L(\mathbb{X})}$$

for all $m \in \mathbb{Z}_+$.

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