ON REGULARITY OF WEAK SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS IN GENERAL DOMAINS

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ABSTRACT. Let u be a weak solution of the instationary Navier-Stokes equations in a completely general domain $\Omega \subseteq \mathbb{R}^3$ which additionally satisfies the strong energy inequality. Firstly, we prove that u is regular if the kinetic energy $\frac{1}{2}\|u(t)\|_2^2$ is left-side Hölder continuous with Hölder exponent $\frac{1}{2}$ and with a sufficiently small Hölder seminorm. This result extends the previous ones by several authors [6, 7, 8] in which the domain Ω is additionally supposed to be bounded. Secondly, we show that if $u(t) \in \mathbb{D}(A^{\frac{1}{4}})$ and $\underline{\lim}_{\delta \to 0^+} \|A^{\frac{1}{4}}\big(u(t-\delta) - u(t)\big)\|_2 < C$ for all $t \in [0, T)$ with a sufficiently small positive constant C then u is regular in $[0, T)$. Our proofs use the theory about the existence of local strongs solutions and uniqueness arguments in the general domain.

1. INTRODUCTION AND MAIN RESULTS

We consider the instationary problem of the Navier-Stokes system

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \\ \text{div } u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0, x) = u_0 \end{cases} \tag{1.1}$$

in a general domain $\Omega \subseteq \mathbb{R}^3$, i.e a nonempty connected open subset of $\mathbb{R}^3$, not necessarily bounded, with boundary $\partial\Omega$ and a time interval $[0, T), 0 < T \leq \infty$ and with the initial value u_0, where $u = (u_1, u_2, u_3)$; $u \cdot \nabla u = \text{div}(uu), uu = (u_i u_j)_{i,j=1}$, if div $u = 0$.

We recall some well-known function spaces, the definitions of weak and strong solutions to (1.1) and introduce some notations before describing the main results. Throughout the paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant C. The expression $\langle \cdot, \cdot \rangle_\Omega$ denotes the pairing of functions, vector fields, etc. on Ω and $\langle \cdot, \cdot \rangle_{\Omega,T}$ means the corresponding pairing on $[0, T) \times \Omega$. For $1 \leq q \leq \infty$ we use the well-known Lebesgue and Sobolev spaces $L^q(\Omega)$, $W^{k,p}(\Omega)$, with norms $\| \cdot \|_{L^q(\Omega)} = \| \cdot \|_q$ and $\| \cdot \|_{W^{k,p}(\Omega)} = \| \cdot \|_{k,p}$. Further, we use the Bochner spaces $L^s\big(0, T; L^p(\Omega)\big)$, $1 \leq s, p \leq \infty$ with the norm

$$\| \cdot \|_{L^s\left(0,T;L^p(\Omega)\right)} := \left(\int_0^T \|\cdot\|_p^s \, \mathrm{d}\tau \right)^{1/s} = \| \cdot \|_{p,s,T}.$$

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To deal with solenoidal vector fields we introduce the spaces of divergence - free smooth compactly supported functions $C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega), \mathrm{div}(u) = 0\}$, and the spaces $L^2_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_2}$, $W^{1,2}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{1,2}}}$, and $W^{1,2}_{0,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}(\Omega)}}$.

Let $\mathbb{P} : L^2(\Omega) \longrightarrow L^2_\sigma(\Omega)$ be the Helmholtz projection. Let the Stokes operator

$$A :\ \mathbb{D}(A) \longrightarrow L^2_\sigma(\Omega)$$

with the domain of definition

$$\mathbb{D}(A) = \{u \in W^{1,2}_{0,\sigma}(\Omega), \exists f \in L^2_\sigma(\Omega) : \langle \nabla u, \nabla \varphi\rangle_\Omega = \langle f, \varphi\rangle_\Omega, \quad \forall\, \varphi \in W^{1,2}_{0,\sigma}(\Omega)\}$$

be defined as

$$Au = f, u \in \mathbb{D}(A).$$

As in [19], we define the fractional powers

$$A^\alpha : \mathbb{D}(A^\alpha) \longrightarrow L^2_\sigma(\Omega),\ -1 \le \alpha \le 1.$$

We have $\mathbb{D}(A) \subset \mathbb{D}(A^\alpha) \subset L^2_\sigma(\Omega)$ for $\alpha \in (0, 1]$. It is known that for any domain $\Omega \subseteq \mathbb{R}^3$ the operator A is self-adjoint and generates a bounded analytic semigroup $e^{-tA}, t \ge 0$ on $L^2_\sigma(\Omega)$.

The following embedding properties play a basic role in the theory of the Navier-Stokes system

$$\|u\|_q \le C\|A^\alpha u\|_2,\ u \in \mathbb{D}(A^\alpha),\ 0 \le \alpha \le \frac{1}{2},\ \frac{3}{2} = \frac{3}{q} + 2\alpha, \tag{1.2}$$

and

$$\|A^{-\alpha}\mathbb{P}u\|_2 \le C\|u\|_q,\ u \in L^q_\sigma(\Omega),\ 0 \le \alpha \le \frac{1}{2},\ \frac{3}{q} = \frac{3}{2} + 2\alpha. \tag{1.3}$$

We also have

$$\|A^{\frac{1}{2}}u\|_2 = \|\nabla u\|_2,\ u \in W^{1,2}_{0,\sigma}(\Omega) = \mathbb{D}(A^{\frac{1}{2}}). \tag{1.4}$$

Furthermore, we mention the Stokes semigroup estimates

$$\|A^\alpha e^{-tA}u\|_2 \le t^{-\alpha}\ \|u\|_2,\ u \in L^2_\sigma(\Omega),\ 0 \le \alpha \le 1 \tag{1.5}$$

and

$$\|A^{\frac{1}{s}}e^{-tA}u\|_{2,s;T} \le\ \|u\|_2,\ u \in L^2_\sigma(\Omega),\ 2 \le s < \infty. \tag{1.6}$$

We use the following Lorentz spaces and some important inequalities in these spaces.

Definition 1.1. (Lorentz spaces). (See [1].)
Let $U \subseteq \mathbb{R}^d,\ d \ge 1,\ 1 \le p, r \le \infty$. The Lorentz spaces $L^{p,r}(U)$ is defined as follows: A measurable function $f \in L^{p,r}(U)$ if and only if

$\big\|f\big\|_{L^{p,r}}(U) := \big(\frac{r}{p}\int_0^\infty (t^{\frac{1}{p}}f^*(t))^r \frac{\mathrm{d}t}{t}\big)^{\frac{1}{r}} < \infty$ when $1 \le r < \infty$,

$\big\|f\big\|_{L^{p,\infty}}(U) := \sup\limits_{t>0} t^{\frac{1}{p}}f^*(t) < \infty$ when $r = \infty$,

where $f^*(t) = \inf\big\{\tau : \mathcal{M}^d(\{x \in U : |f(x)| > \tau\}) \le t\big\}$, with $\mathcal{M}^d$ being the Lebesgue measure in $\mathbb{R}^d$.

Lemma 1.1. (*Hölder's inequality in Lorentz spaces*).
Let $1 < r, p, q < \infty$ and $1 \le \bar{r}, \bar{p}, \bar{q} \le \infty$ satisfy the relations

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}\ \textit{and}\ \frac{1}{\bar{r}} = \frac{1}{\bar{p}} + \frac{1}{\bar{q}}.$$

Suppose that $f \in L^{p,\bar{p}}(U)$ *and* $g \in L^{q,\bar{q}}(U)$. *Then* $fg \in L^{r,\bar{r}}(U)$ *and we have the inequality*

$$\|fg\|_{L^{r,\bar{r}}} \lesssim \|f\|_{L^{p,\bar{p}}}\|g\|_{L^{q,\bar{q}}}. \tag{1.7}$$

Proof. See Proposition 2.3 in ([18], p. 19). □

Lemma 1.2. (*Young's inequality for convolution in Lorentz spaces*).
Let $1 < r, p, q < \infty$ *and* $1 \leq \bar{r}, \bar{p}, \bar{q} \leq \infty$ *satisfy the relations*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \text{ and } \frac{1}{\bar{r}} = \frac{1}{\bar{p}} + \frac{1}{\bar{q}}.$$

Suppose that $f \in L^{p,\bar{p}}(\mathbb{R}^d)$, $d \geq 1$ *and* $g \in L^{q,\bar{q}}(\mathbb{R}^d)$. *Then* $f * g \in L^{r,\bar{r}}(\mathbb{R}^d)$ *and the following inequality holds*

$$\|f * g\|_{L^{r,\bar{r}}} \lesssim \|f\|_{L^{p,\bar{p}}}\|g\|_{L^{q,\bar{q}}}. \tag{1.8}$$

Proof. See Proposition 2.4 in ([18], p. 20). □

Now we recall the definitions of weak and strong solutions to (1.1).

Definition 1.2. (See [19].) Let $u_0 \in L^2_\sigma(\Omega)$.
1. A vector field

$$u \in L^\infty\big(0,T; L^2_\sigma(\Omega)\big) \cap L^2_{\text{loc}}([0,T); W^{1,2}_{0,\sigma}(\Omega)\big) \tag{1.9}$$

is called a *weak solution* (in the sense of Leray-Hopf) of the Navier-Stokes system (1.1) with the initial value $u(0,x) = u_0$ if the relation

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w\rangle_\Omega \tag{1.10}$$

is satisfied for all test functions $w \in C_0^\infty\big([0,T); C^\infty_{0,\sigma}(\Omega)\big)$, and additionally the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 \mathrm{d}\tau \leq \frac{1}{2}\|u_0\|_2^2 \tag{1.11}$$

is satisfied for all $t \in [0,T)$.
A weak solution u is called a *strong solution* of the Navier-Stokes equation (1.1) if additionally local *Serrin's condition*

$$u \in L^s_{\text{loc}}\big([0,T); L^q(\Omega)\big) \tag{1.12}$$

is satisfied with $2 < s < \infty$, $3 < q < \infty$ where $\frac{2}{s} + \frac{3}{q} \leq 1$.

As is well known, in the case the domain Ω is bounded, it is not difficult to prove the existence of a weak solution u as in Definition 1.2 which additionally satisfies the *strong energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t'}^t \|\nabla u(\tau)\|_2^2 \mathrm{d}\tau \leq \frac{1}{2}\|u(t')\|_2^2 \tag{1.13}$$

for almost all $t' \in [0,T)$ and all $t \in [t',T)$, see [19], p. 340. For further results in this context for unbounded domains we refer to [5].

Definition 1.3. (See [8].) A weak solution u is called *regular* in some interval $(a,b) \subseteq (0,T)$ if Serrin's condition

$$u \in L^s_{\text{loc}}\big(a,b\,;L^q(\Omega)\big) \tag{1.14}$$

is satisfied with $2 < s < \infty$, $3 < q < \infty$, $\dfrac{2}{s} + \dfrac{3}{q} = 1$.

A time $t \in (0,T)$ is called a *regular point* of u if there exists an interval $(a,b) \subseteq (0,T)$ such that u is regular in (a,b) with $a < t < b$.

Now we can state our main results

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain. Then there exists a positive constant C such that if u is a weak solution of the Navier-Stokes system* (1.1) *on* $(0,T)$ *verifying the strong energy inequality* (1.13) *and at $t_0 \in (0,T)$ the kinetic energy satisfying*

$$\varliminf_{\delta\to 0^+} \frac{\left|\frac{1}{2}\left\|u(t_0-\delta)\right\|_2^2 - \frac{1}{2}\left\|u(t_0)\right\|_2^2\right|}{\delta^{\frac{1}{2}}} < C, \tag{1.15}$$

then u is regular at t_0.

Remark 1.1. In 2008 and 2009, see [6, 7], Farwig, Kozono, and Sohr obtained the same results as in Theorem 1.1 but the domain Ω is additionally supposed to be bounded. They proved the regularity of u under a condition

$$\varliminf_{\delta\to 0^+} \frac{\left|\frac{1}{2}\left\|u(t_0-\delta)\right\|_2^2 - \frac{1}{2}\left\|u(t_0)\right\|_2^2\right|}{\delta^{\alpha}} < \infty, \tag{1.16}$$

where $\frac{1}{2} < \alpha < 1$. In 2010, see [8], they improved the their results, in which the condition (1.16) is replaced by the weaker condition (1.15) and the domain Ω is bounded. Finally, In 2016, see [9], Farwig and Riechwald proved Theorem 1.1 for Ω being not necessarily an unbounded domain with uniform C^2-boundary $\partial\Omega$. Our result improves the previous ones. Here we obtain the same result but under a much weaker condition on the domain Ω. Our approach is to establish the existence of a local strong solution with the initial value in $\mathbb{D}(A^{\frac{1}{2}})$.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain. Then there exists a positive constant C such that if u is a weak solution of the Navier-Stokes system* (1.1) *on* $(0,T)$ *satisfying $u(t) \in \mathbb{D}(A^{\frac{1}{4}})$ for all $t \in [0,T)$ and*

$$\varliminf_{\delta\to 0^+} \left\|A^{\frac{1}{4}}\big(u(t-\delta) - u(t)\big)\right\|_2 < C \textit{ for all } t \in (0,T) \tag{1.17}$$

then $u \in L^4_{\text{loc}}([0,T);L^6(\Omega))$.

Remark 1.2. In Theorem 1.2, if the function u is left-continuous from $[0,T)$ to $\mathbb{D}(A^{\frac{1}{4}})$ then $\varliminf_{\delta\to 0^+} \left\|A^{\frac{1}{4}}\big(u(t-\delta) - u(t)\big)\right\|_2 = 0$ for all $t \in [0,T)$. Therefore, the condition (1.17) holds.

Remark 1.3. In 2012, Farwig, Sohr, and Varnhorn showed that if u is a weak solution of the Navier-Stokes system (1.1) satisfying $u \in L^\infty_{\text{loc}}\big([0,T),L^3(\Omega)\big)$ with a bounded domain Ω or $u \in L^\infty_{\text{loc}}\big([0,T),\mathbb{D}(A^{\frac{1}{4}})\big)$ with a general domain Ω then u satisfies the local right-hand side Serrin condition in $[0,T)$, see [3]. If the domain Ω is a general domain then the statement in Theorem 1.2 is stronger than that of [3] but under the condition (1.17) which is not stronger than the condition in

[3]. Indeed, let $u(t,x) = f(t)a(x)$, where $a \in \mathbb{D}(A^{\frac{1}{4}})$, $\|a\|_{\mathbb{D}(A^{\frac{1}{4}})} > 0$ and $f(t) = 0$ for $0 \le t \le 1$, $f(t) = 1/(t-1)$ for $1 < t < \infty$. Then u is left-continuous from $[0,\infty)$ to $\mathbb{D}(A^{\frac{1}{4}})$, and so u satisfies the condition (1.17) but u does not belong to $L^{\infty}_{\mathrm{loc}}\big([0,\infty), \mathbb{D}(A^{\frac{1}{4}})\big)$.

2. Proof of Theorems

Let us construct a solution of the following integral equation

$$u(t) = e^{-tA}u_0 - A^{\frac{1}{2}} \int_0^t e^{-(t-\tau)A} A^{-\frac{1}{2}} \mathbb{P}(u \cdot \nabla u) \mathrm{d}\tau. \tag{2.1}$$

We know that

$$u \in L^{\infty}\big(0,T; L^2_{\sigma}(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T); W^{1,2}_{0,\sigma}(\Omega)\big)$$

is a weak solution of the system (1.1) with the initial value u_0 iff u is a solution of the integral equation (2.1), see [19].

First, we define an auxiliary space $\mathcal{K}^{\overline{s}}_{\tilde{s},T}$ which is made up of the functions u such that

$$t^{\frac{\alpha}{2}} u \in BC\big([0,T); \mathbb{D}(A^{\frac{\overline{s}}{2}})\big)$$

and

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \big\|A^{\frac{\overline{s}}{2}} u(t)\big\|_2 = 0 \tag{2.2}$$

with $-1 < \tilde{s} \le \overline{s} < \infty$, $\alpha = \overline{s} - \tilde{s}$. The auxiliary space $\mathcal{K}^{\overline{s}}_{\tilde{s},T}$ is equipped with the norm

$$\|u\|_{\mathcal{K}^{\overline{s}}_{\tilde{s},T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}} \big\|A^{\frac{\overline{s}}{2}} u(t)\big\|_2. \tag{2.3}$$

In the case $\overline{s} = \tilde{s}$, it is also convenient to define the space $\mathcal{K}^{\overline{s}}_{\tilde{s},T}$ as the natural subspace $BC\big([0,T); \mathbb{D}(A^{\frac{\overline{s}}{2}})\big)$ with the additional condition that its elements $u(t,x)$ satisfy

$$\lim_{t\to 0} \big\|A^{\frac{\overline{s}}{2}} u(t)\big\|_2 = 0.$$

We define

$$\mathcal{G}^{\overline{s}}_{\tilde{s},T} := \mathcal{K}^{\overline{s}}_{\tilde{s},T} \cap L^{\infty}\big(0,T; L^2_{\sigma}(\Omega)\big) \cap L^4\big([0,T); W^{1,2}_{0,\sigma}(\Omega)\big).$$

To prove the main theorems, we need the following lemmas

Lemma 2.1. *Let $\theta < 1, \gamma < 1$, and $t > 0$ then*

$$\int_0^t (t-\tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau = K_{\gamma,\theta} t^{1-\gamma-\theta}, \text{ where } K_{\gamma,\theta} = \int_0^1 (1-\tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau < \infty.$$

The proof of this lemma is elementary and may be omitted. □

We prove the following result on solutions of a quadratic equation in Banach spaces which is a generalization of Theorem 22.4 in ([18], p. 227).

Lemma 2.2. *Let E and F be two normed spaces such that $E \cap F$ is a Banach space with the norm $\|x\|_{E\cap F} := \|x\|_E + \|x\|_F$. Assume that B is a bilinear operator from $(E\cap F) \times (E \cap F)$ to $E \cap F$ such that there exists a positive constant $\eta > 0$ satisfying*

$$\|B(x,y)\|_E \le \eta \|x\|_E \|y\|_E, \text{ for all } x, y \in E \cap F,$$

$$\|B(x,y)\|_F \le \eta \|x\|_E \|y\|_F, \text{ for all } x, y \in E \cap F,$$

$$\|B(x,y)\|_F \le \eta \|x\|_F \|y\|_E, \text{ for all } x, y \in E \cap F.$$

Then for any fixed $y \in E \cap F$ such that $\|y\|_E < \frac{1}{4\eta}$, the equation $x = y - B(x,x)$ has a unique solution $\overline{x} \in E \cap F$ satisfying $\|\overline{x}\|_E < \frac{1}{2\eta}$.

Proof. The uniqueness of $\bar{x}$ in $E \cap F$ is obvious it is even unique in E. Thus, we need to prove the existence of $\bar{x}$ in $E \cap F$. Let x_n be defined by

$$x_0 = y \text{ and } x_{n+1} = y - B(x_n, x_n).$$

By induction, we can easily prove that

$$\|x_n\|_E < 2\|y\|_E$$

for any n. It follows that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence in E. We will show that $\{x_n\}_{n=0}^\infty$ is also a Cauchy sequence in F. Indeed, we have

$$\begin{aligned}
\|x_1 - x_0\|_F &= \|B(y,y)\|_F \le \eta\|y\|_E\|y\|_F,\\
\|x_{n+1} - x_n\|_F &= \|B(x_n, x_n - x_{n-1}) + B(x_n - x_{n-1}, x_{n-1})\|_F\\
&\le \eta\|x_n\|_E\|x_n - x_{n-1}\|_F + \eta\|x_{n-1}\|_E\|x_n - x_{n-1}\|_F\\
&< 4\eta\|y\|_E\|x_n - x_{n-1}\|_F, \text{ with } 4\eta\|y\|_E < 1.
\end{aligned}$$

An elementary computation leads to

$$\lim_{m,n\to\infty}\|x_n - x_m\|_F = 0.$$

This proves that $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence in F. Therefore $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence in $E\cap F$, so $\{x_n\}_{n=0}^\infty$ converges in $E\cap F$ to an element $\overline{x} \in E\cap F$. We thus obtain, from $\|x_n\|_E < 2\|y\|_E$, that $\|\overline{x}\|_E \le 2\|y\|_E < \frac{1}{2\eta}$. The proof of Lemma 2.2 is complete. $\square$

In the following four lemmas a particular attention will be devoted to the study of the bilinear operator $B(u,v)$ defined by

$$B(u,v) = A^{\frac{1}{2}}\int_0^t e^{-(t-\tau)A}A^{-\frac{1}{2}}\mathbb{P}(u\cdot\nabla v)\mathrm{d}\tau.$$

Lemma 2.3. *Let $s_1, s_2, s_3, \overline{s}$ and $T \in \mathbb{R}$ be such that*

$$-1 < s_1, s_2 \le 1,\ s_1 + s_2 > 0,\ -1 < s_3 \le s_1 + s_2 - \frac{1}{2},$$
$$\max\Big\{s_3, -\frac{1}{2}\Big\} \le \overline{s} < \frac{3}{2},\quad T > 0.$$

Then the operator B is a bilinear operator from $\mathcal{G}^1_{s_1,T} \times \mathcal{G}^1_{s_2,T}$ to $\mathcal{K}^{\overline{s}}_{s_3,T}$ and the following inequality holds

$$\Big\|B(u,v)\Big\|_{\mathcal{K}^{\overline{s}}_{s_3,T}} \lesssim T^{\frac{s_1+s_2-s_3-1/2}{2}}\|u\|_{\mathcal{K}^1_{s_1,T}}\|v\|_{\mathcal{K}^1_{s_2,T}}. \tag{2.4}$$

Proof. Let $u \in \mathcal{K}^1_{s_1,T},\ v \in \mathcal{K}^1_{s_2,T}$. Applying the inequalities (1.5), (1.3), Hölder inequality, (1.2), (1.4), and Lemma 2.1 in order to obtain

$$\Big\|A^{\frac{\overline{s}}{2}}B(u,v)\Big\|_2 = \Big\|\int_0^t e^{-(t-\tau)A}A^{-\frac{1}{2}}\mathbb{P}(u\cdot\nabla v)\mathrm{d}\tau\Big\|_{\mathbb{D}(A^{\frac{\overline{s}+1}{2}})}$$
$$\le \int_0^t \Big\|e^{-(t-\tau)A}A^{-\frac{1}{2}}\mathbb{P}(u\cdot\nabla v)\Big\|_{\mathbb{D}(A^{\frac{\overline{s}+1}{2}})}\mathrm{d}\tau = \int_0^t \Big\|A^{\frac{\overline{s}+1}{2}}e^{-(t-\tau)A}A^{-\frac{1}{2}}\mathbb{P}(u\cdot\nabla v)\Big\|_2\mathrm{d}\tau$$
$$= \int_0^t \Big\|A^{\frac{\overline{s}+1/2}{2}}e^{-(t-\tau)A}A^{-\frac{1}{4}}\mathbb{P}(u\cdot\nabla v)\Big\|_2\mathrm{d}\tau \le \int_0^t (t-\tau)^{-\frac{\overline{s}+1/2}{2}}\Big\|A^{-\frac{1}{4}}\mathbb{P}(u\cdot\nabla v)\Big\|_2\mathrm{d}\tau$$

$$\lesssim \int_0^t (t-\tau)^{-\frac{\overline{s}+1/2}{2}} \Big\|u\cdot\nabla v\Big\|_{\frac{3}{2}} \mathrm{d}\tau \quad \leq \quad \int_0^t (t-\tau)^{-\frac{\overline{s}+1/2}{2}} \|u\|_6 \|\nabla v\|_2 \mathrm{d}\tau$$
$$\lesssim \int_0^t (t-\tau)^{-\frac{\overline{s}+1/2}{2}} \|A^{\frac{1}{2}}u\|_2 \|A^{\frac{1}{2}}v\|_2 \mathrm{d}\tau \tag{2.5}$$
$$\leq \sup_{0<\xi<t} \xi^{\frac{1-s_1}{2}} \|A^{\frac{1}{2}}u(\xi)\|_2 \sup_{0<\xi<t} \xi^{\frac{1-s_2}{2}} \|A^{\frac{1}{2}}v(\xi)\|_2 \int_0^t (t-\tau)^{-\frac{\overline{s}+1/2}{2}} \tau^{\frac{s_1+s_2}{2}-1} \mathrm{d}\tau$$
$$\lesssim t^{\frac{s_1+s_2-\overline{s}-1/2}{2}} \sup_{0<\xi<t} \xi^{\frac{1-s_1}{2}} \|A^{\frac{1}{2}}u(\xi)\|_2 \sup_{0<\xi<t} \xi^{\frac{1-s_2}{2}} \|A^{\frac{1}{2}}v(\xi)\|_2.$$

Thus

$$t^{\frac{\overline{s}-s_3}{2}} \Big\|A^{\frac{\overline{s}}{2}}B(u,v)\Big\|_2 \lesssim t^{\frac{s_1+s_2-s_3-1/2}{2}} \sup_{0<\xi<t} \xi^{\frac{1-s_1}{2}} \|A^{\frac{1}{2}}u(\xi)\|_2 \sup_{0<\xi<t} \xi^{\frac{1-s_2}{2}} \|A^{\frac{1}{2}}v(\xi)\|_2.$$

The estimate (2.4) is deduced from the above inequality. Let us now check the validity of the condition (2.2) for the bilinear term B. Indeed, we have

$$\lim_{t\to 0} t^{\frac{\overline{s}-s_3}{2}} \big\|A^{\frac{\overline{s}}{2}}B(u,v)\big\|_2 = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{1-s_1}{2}} \big\|A^{\frac{1}{2}}u(t)\big\|_2 = 0 \;\text{ or }\; \lim_{t\to 0} t^{\frac{1-s_2}{2}} \big\|A^{\frac{1}{2}}u(t)\big\|_2 = 0.$$

This proves the continuity of $t^{\frac{\overline{s}-s_3}{2}} A^{\frac{\overline{s}}{2}} B(u,v)$ at $t=0$. The continuity elsewhere follows from carefully rewriting the expression $\int_0^{t+\epsilon} - \int_0^t$ and applying the same argument. □

Lemma 2.4. *Let p and $s \in \mathbb{R}$ be such that*

$$\frac{1}{2} \leq s < 1, \; \frac{2}{1+s} < p < \infty.$$

Then the operator B is a bilinear operator from

$$\Big(\mathcal{G}^1_{s,T} \cap L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)\Big) \times \Big(\mathcal{G}^1_{s,T} \cap L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)\Big) \text{ to } L^p([0,T);\mathbb{D}(A^{\frac{1}{2}}))$$

and the following inequalities holds

$$\Big\|B(u,v)\Big\|_{L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)} \lesssim T^{\frac{s-1/2}{2}} \|u\|_{\mathcal{K}^1_{s,T}} \|v\|_{L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)}$$

and

$$\Big\|B(u,v)\Big\|_{L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)} \lesssim T^{\frac{s-1/2}{2}} \|v\|_{\mathcal{K}^1_{s,T}} \|u\|_{L^p\big([0,T);\mathbb{D}(A^{\frac{1}{2}})\big)}.$$

Proof. Applying the inequality (2.5) with $\overline{s}=1$, we have

$$\begin{aligned} \Big\|A^{\frac{1}{2}}B(u,v)\Big\|_2 &\lesssim \int_0^t (t-\tau)^{-\frac{3}{4}} \|A^{\frac{1}{2}}u(\tau)\|_2 \|A^{\frac{1}{2}}v(\tau)\|_2 \mathrm{d}\tau \\ &\leq \|u\|_{\mathcal{K}^1_{s,T}} \int_0^t (t-\tau)^{-\frac{3}{4}} \tau^{-\frac{1-s}{2}} \Big\|A^{\frac{1}{2}}v(\tau)\Big\|_2 \mathrm{d}\tau. \end{aligned} \tag{2.6}$$

From the inequality (2.6), applying the inequalities (1.8) and (1.7) in order to obtain

$$\Big\|\|A^{\frac{1}{2}}B(u,v)\|_2\Big\|_{L^p([0,T])} = \Big\|\|A^{\frac{1}{2}}B(u,v)\|_2\Big\|_{L^{p,p}([0,T])}$$

$$\begin{aligned}
&\lesssim \|u\|_{\mathcal{K}^1_{s,T}} \left\| 1_{[0,T]} \, |\cdot|^{-\frac{3}{4}} \right\|_{L^{\frac{2}{1+s},\infty}} \left\| |\cdot|^{-\frac{1-s}{2}} \left\| A^{\frac{1}{2}} v(\cdot) \right\|_2 \right\|_{L^{\frac{1}{\frac{1-s}{2}+\frac{1}{p}},p}([0,T])} \\
&\lesssim T^{\frac{s-1/2}{2}} \|u\|_{\mathcal{K}^1_{s,T}} \left\| |\cdot|^{-\frac{1-s}{2}} \right\|_{L^{\frac{2}{1-s},\infty}} \left\| \left\| A^{\frac{1}{2}} v(\cdot) \right\|_2 \right\|_{L^{p,p}([0,T])} \\
&\lesssim T^{\frac{s-1/2}{2}} \|u\|_{\mathcal{K}^1_{s,T}} \|v\|_{L^p([0,T);\mathbb{D}(A^{\frac{1}{2}}))},
\end{aligned}$$

where $1_{[0,T]}$ is the indicator function of the set $[0,T]$ on $\mathbb{R}$ and note that

$$\left\| 1_{[0,T]} |\cdot|^{-\alpha} \right\|_{L^{p,\infty}} = C T^{\frac{1}{p}-\alpha}$$

with $0 < \alpha \leq 1,\ 1 \leq p \leq \frac{1}{\alpha}$, and C is a positive constant independent of T. By a similar argument, we get the following estimate

$$\left\| \left\| A^{\frac{1}{2}} B(u,v) \right\|_2 \right\|_{L^p([0,T))} \lesssim T^{\frac{s-1/2}{2}} \|v\|_{\mathcal{K}^1_{s,T}} \|u\|_{L^p\left([0,T);\mathbb{D}(A^{\frac{1}{2}})\right)}.$$

The proof is complete. $\square$

Lemma 2.5. *Let*

$$E = \mathcal{K}^1_{\frac{1}{2},T}, \ \ F = \mathcal{G}^1_{\frac{1}{2},T} \cap \mathcal{G}^1_{0,T},$$

where $0 < T \leq \infty$. The space F is equipped with the norm

$$\|u\|_F := \|u\|_{\mathcal{K}^1_{\frac{1}{2},T}} + \|u\|_{\mathcal{K}^1_{0,T}} + \|u\|_{L^4\left([0,T);\mathbb{D}(A^{\frac{1}{2}})\right)} + \|u\|_{L^\infty\left([0,T);L^2_\sigma(\Omega)\right)}.$$

Then the operator B is a bilinear operator from $F \times F$ to F satisfying

$$\|B(u,v)\|_E \ \leq \ \eta \|u\|_E \, \|v\|_E, \quad \forall u,v \in F, \tag{2.7}$$

$$\|B(u,v)\|_F \ \leq \ \eta \|u\|_E \, \|v\|_F, \quad \forall u,v \in F, \tag{2.8}$$

$$\|B(u,v)\|_F \ \leq \ \eta \|u\|_F \, \|v\|_E, \quad \forall u,v \in F, \tag{2.9}$$

where η is a positive constant independent of T.

Proof. The estimate (2.7) is directly deduced from applying Lemma 2.3 with $s_1 = s_2 = s_3 = \frac{1}{2},\ \overline{s} = 1$.
Applying Lemma 2.3 with $s_1 = \frac{1}{2}, s_2 = s_3 = \overline{s} = 0$, we have

$$\left\| B(u,v) \right\|_{L^\infty([0,T);L^2_\sigma(\Omega))} = \left\| B(u,v) \right\|_{\mathcal{K}^0_{0,T}} \lesssim \ \left\| u \right\|_E \left\| v \right\|_{\mathcal{K}^1_{0,T}}. \tag{2.10}$$

In view of Lemma 2.3 with $s_1 = \frac{1}{2},\ s_2 = s_3 = 0$, and $\overline{s} = 1$, we have

$$\left\| B(u,v) \right\|_{\mathcal{K}^1_{0,T}} \ \lesssim \ \|u\|_{\mathcal{K}^1_{\frac{1}{2},T}} \|v\|_{\mathcal{K}^1_{0,T}} = \|u\|_E \left\| v \right\|_{\mathcal{K}^1_{0,T}}. \tag{2.11}$$

Finally, applying Lemma 2.4 with $p = 4, s = \frac{1}{2}$, we have

$$\|B(u,v)\|_{L^4\left([0,T);\mathbb{D}(A^{\frac{1}{2}})\right)} \ \lesssim \ \|u\|_E \|v\|_{L^4\left([0,T);\mathbb{D}(A^{\frac{1}{2}})\right)}. \tag{2.12}$$

The estimate (2.8) is deduced from the inequalities (2.7), (2.10), (2.11), and (2.12). By an argument analogous similar to the previous one, we get (2.9). $\square$

Lemma 2.6.
(a) *If $u_0 \in \mathbb{D}(A^{\frac{s}{2}}),\ 0 \leq s < 1$ then $e^{-tA}u_0 \in \mathcal{K}^1_{s,\infty}$.*
(b) *If $u_0 \in \mathbb{D}(A^{\frac{1}{4}})$ then $e^{-tA}u_0 \in F$.*

Proof. (a) Applying the inequality (1.5), we have

$$t^{\frac{1-s}{2}}\left\|A^{\frac{1}{2}}e^{-tA}u_0\right\|_2 = t^{\frac{1-s}{2}}\left\|A^{\frac{1-s}{2}}e^{-tA}A^{\frac{s}{2}}u_0\right\|_2 \le t^{\frac{1-s}{2}}t^{\frac{s-1}{2}}\left\|A^{\frac{s}{2}}u_0\right\|_2 = \left\|A^{\frac{s}{2}}u_0\right\|_2 < \infty.$$

We now claim the validity of the condition (2.2). Since $C^\infty_{0,\sigma}(\Omega)$ is dense in $\mathbb{D}(A^{\frac{s}{2}})$, then for all $\varepsilon > 0$, there exists an $u_\varepsilon \in C^\infty_{0,\sigma}(\Omega) \subseteq \mathbb{D}(A^{\frac{1}{2}})$ such that

$$\left\|A^{\frac{s}{2}}(u_\varepsilon - u_0)\right\|_2 < \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned} t^{\frac{1-s}{2}}\left\|A^{\frac{1}{2}}e^{-tA}u_0\right\|_2 &\le t^{\frac{1-s}{2}}\left\|A^{\frac{1}{2}}e^{-tA}(u_\varepsilon - u_0)\right\|_2 + t^{\frac{1-s}{2}}\left\|A^{\frac{1}{2}}e^{-tA}u_\varepsilon\right\|_2 \\ &\le \left\|A^{\frac{s}{2}}(u_\varepsilon - u_0)\right\|_2 + t^{\frac{1-s}{2}}\left\|e^{-tA}A^{\frac{1}{2}}u_\varepsilon\right\|_2 \\ &< \frac{\varepsilon}{2} + t^{\frac{1-s}{2}}\left\|A^{\frac{1}{2}}u_\varepsilon\right\|_2. \end{aligned}$$

We can choose $t_0(\varepsilon) = t_0(u_\varepsilon)$ small enough that

$$t^{\frac{1-s}{2}}\left\|A^{\frac{1}{2}}u_\varepsilon\right\|_2 < \frac{\varepsilon}{2} \quad \text{for all } t < t_0(\varepsilon).$$

Hence

$$t^{\frac{1-s}{2}}\left\|A^{\frac{1}{2}}e^{-tA}u_0\right\|_2 < \varepsilon \quad \text{for all } t < t_0(\varepsilon).$$

The proof of the part (a) is complete.
(b) From the inequality (1.5), it follows that

$$\|e^{-tA}u_0\|_2 \le \|u_0\|_2.$$

Therefore $e^{-tA}u_0 \in L^\infty([0,T); L^2(\Omega))$. In view of the inequality (1.6) , we have

$$\left\|A^{\frac{1}{2}}e^{-tA}u_0\right\|_{2,4,\infty} = \left\|A^{\frac{1}{4}}e^{-tA}A^{\frac{1}{4}}u_0\right\|_{2,4,\infty} \le \|A^{\frac{1}{4}}u_0\|_2 < \infty.$$

Hence $e^{-tA}u_0 \in L^4\big([0,T); \mathbb{D}(A^{\frac{1}{2}})\big)$. Applying the part (a) of this lemma for $u_0 \in \mathbb{D}(A^{\frac{1}{2}}) \subset L^2_\sigma(\Omega)$, we get $e^{-tA}u_0 \in \mathcal{K}^1_{\frac{1}{2},T} \cap \mathcal{K}^1_{0,T}$, so $e^{-tA}u_0 \in F$. The proof of part (b) is complete. $\square$

The proof of Theorem 1.1 is based on the existence of local strong solutions and global strong solutions in the following theorem. There are many papers treating global strong solutions and local strong solutions of the Navier-Stokes equations in unbounded domains. For example, for Ω being the whole space $\mathbb{R}^n, n \ge 2$ see [2, 12, 13, 14, 15, 16] ; for Ω being the n-dimensional half space $\mathbb{R}^n_+, n \ge 2$ see [10, 11, 21] ; for Ω being not necessarily an unbounded domain with uniform C^3-boundary $\partial\Omega$ see [17]; and for Ω being a completely general domain $\Omega \subseteq \mathbb{R}^3$ see [4].

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain. Then*
(a) *There exists a positive constant D such that for all $u_0 \in \mathbb{D}(A^{\frac{1}{4}})$ and $0 < T \le \infty$ satisfying*

$$\sup_{0\le t\le T} t^{\frac{1}{4}}\left\|A^{\frac{1}{2}}e^{-tA}u_0\right\|_2 < D \tag{2.13}$$

the Navier-Stokes system (1.1) *has a strong solution u in time interval $[0,T)$ with the following properties:*

$$u \in L^4\big([0,T); \mathbb{D}(A^{\frac{1}{2}})\big) \tag{2.14}$$

and

$$(1+t)^{\frac{1}{4}} u \in BC\big([0,T); \mathbb{D}(A^{\frac{1}{4}})\big). \tag{2.15}$$

In particular, for arbitrary $u_0 \in \mathbb{D}(A^{\frac{1}{4}})$, *there exists* $T = T(u_0)$ *small enough such that the inequality* (2.13) *holds.*

(b) *Suppose* $u_0 \in \mathbb{D}(A^{\frac{s}{2}})$, $\frac{1}{2} \le s \le 1$. *Then the inequality* (2.13) *holds if*

$$T^{\frac{1}{2}(s-\frac{1}{2})} \big\| A^{\frac{s}{2}} u_0 \big\|_2 < D. \tag{2.16}$$

Proof. (a) From Lemmas 2.2, 2.5, and 2.6(b), we deduce that there exists a constant $D > 0$ such that if

$$\big\| e^{-tA} u_0 \big\|_E = \sup_{0<t<T} t^{\frac{1}{4}} \big\| A^{\frac{1}{2}} e^{-tA} u_0 \big\|_2 < D,$$

then the integral equation (2.1) has a unique solution u on the interval $(0,T)$ so that

$$\begin{aligned} u \in F &\subseteq L^\infty\big([0,T); L^2(\Omega)\big) \cap L^4([0,T); \mathbb{D}(A^{\frac{1}{2}})) \\ &\subseteq L^\infty\big(0,T; L^2(\Omega)\big) \cap L^2_{\text{loc}}\big([0,T); W^{1,2}_{0,\sigma}(\Omega)\big) \end{aligned}$$

and u satisfies the integral equation (2.1) i. e. u is a weak solution of (1.1). Since

$$u \in L^4\big([0,T); \mathbb{D}(A^{\frac{1}{2}})\big) \subseteq L^4\big([0,T); L^6(\Omega)\big),$$

it follows that the solution u satisfies Serrin's condition, hence the solution u is the strong solution of the Navier-Stokes equations (1.1). Applying Lemma 2.3 with $s_1 = s_2 = s_3 = \overline{s} = \frac{1}{2}$, we have

$$\sup_{0<t<T} \big\| A^{\frac{1}{4}} B(u,u) \big\|_2 = \big\| B(u,u) \big\|_{\mathcal{K}^{\frac{1}{2}}_{\frac{1}{2},T}} \lesssim \big\| u \big\|^2_{\mathcal{K}^{1}_{\frac{1}{2},T}} < \infty,$$

and

$$\sup_{0<t<\infty} \big\| A^{\frac{1}{4}} e^{-tA} u_0 \big\|_2 = \sup_{0\le t<\infty} \big\| e^{-tA} A^{\frac{1}{4}} u_0 \big\|_2 \le \big\| A^{\frac{1}{4}} u_0 \big\|_2 < \infty.$$

From the above estimates, we have

$$\begin{aligned} \sup_{0<t<T} \big\| A^{\frac{1}{4}} u(t) \big\|_2 &= \sup_{0<t<T} \big\| A^{\frac{1}{4}} \big(e^{-tA} u_0 - B(u,u) \big) \big\|_2 \\ &\le \sup_{0<t<T} \big\| A^{\frac{1}{4}} e^{-tA} u_0 \big\|_2 + \sup_{0<t<T} \big\| A^{\frac{1}{4}} B(u,u) \big\|_2 < \infty. \end{aligned} \tag{2.17}$$

Applying Lemma 2.3 with $s_1 = \frac{1}{2}, s_2 = s_3 = 0$, and $\overline{s} = \frac{1}{2}$, we have

$$\sup_{0<t<T} t^{\frac{1}{4}} \big\| A^{\frac{1}{4}} B(u,u) \big\|_2 = \big\| B(u,u) \big\|_{\mathcal{K}^{\frac{1}{2}}_{0,T}} \lesssim \big\| u \big\|_{\mathcal{K}^{1}_{\frac{1}{2},T}} \big\| u \big\|_{\mathcal{K}^{1}_{0,T}} < \infty.$$

From the estimate (1.5), it follows that

$$\sup_{0\le t\le T} t^{\frac{1}{4}} \| A^{\frac{1}{4}} e^{-tA} u_0 \|_2 \le \| u_0 \|_2 < \infty.$$

From the above estimates, we obtain

$$\begin{aligned}
&\sup_{0<t<T} t^{\frac{1}{4}}\left\|A^{\frac{1}{4}}u(t)\right\|_2 = \sup_{0<t<T} t^{\frac{1}{4}}\left\|A^{\frac{1}{4}}\left(e^{-tA}u_0 - B(u,u)\right)\right\|_2\\
&\le \sup_{0<t<T} t^{\frac{1}{4}}\left\|A^{\frac{1}{4}}e^{-tA}u_0\right\|_2 + \sup_{0<t<T} t^{\frac{1}{4}}\left\|A^{\frac{1}{4}}B(u,u)\right\|_2 < \infty. \qquad (2.18)
\end{aligned}$$

The property (2.15) is now deduced from the inequalities (2.17) and (2.18). Now we show that the condition (2.13) is valid when T is small enough. Indeed, from the definition of $\mathcal{K}^{\overline{s}}_{\tilde{s},T}$ and Lemma 2.6 (a) with $s=\frac{1}{2}$, we deduce that the left-hand side of the inequality (2.13) converges to 0 when T tends to 0. Therefore the condition (2.13) holds for arbitrary $u_0 \in \mathbb{D}(A^{\frac{1}{4}})$ when $T = T(u_0)$ is small enough.
(b) We shall estimate the left-hand side of the inequality (2.13). From the estimate (1.5), for $u_0 \in \mathbb{D}(A^{\frac{s}{2}})$ we have

$$\begin{aligned}
&\sup_{0\le t\le T} t^{\frac{1}{4}}\left\|A^{\frac{1}{2}}e^{-tA}u_0\right\|_2 = \sup_{0\le t\le T} t^{\frac{1}{4}}\left\|A^{\frac{1-s}{2}}e^{-tA}A^{\frac{s}{2}}u_0\right\|_2\\
&\le \sup_{0\le t\le T} t^{\frac{1}{2}(s-\frac{1}{2})}\left\|A^{\frac{s}{2}}u_0\right\|_2 = T^{\frac{1}{2}(s-\frac{1}{2})}\left\|A^{\frac{s}{2}}u_0\right\|_2.
\end{aligned}$$

This proves (b). □

Remark 2.1. Sohr [19] showed that the existence of a strong solution in $L^8\big(0,T;L^4(\Omega)\big)$ under the following condition

$$\left\|(I-e^{-2TA})A^{\frac{1}{4}}u_0\right\|_2^{\frac{1}{8}}\left\|A^{\frac{1}{4}}u_0\right\|_2^{\frac{7}{8}} < K \qquad (2.19)$$

where K is a positive constant independent of Ω, T, and u_0. On the other hand, it is easy to see that

$$\left\|(I-e^{-2TA})A^{\frac{1}{4}}u_0\right\|_2 \le 2^{\frac{9}{4}}T^{\frac{1}{4}}\left\|A^{\frac{1}{2}}u_0\right\|_2. \qquad (2.20)$$

It follows from inequality (2.20) that if the condition

$$T^{\frac{1}{32}}\left\|A^{\frac{1}{2}}u_0\right\|_2^{\frac{1}{8}}\left\|A^{\frac{1}{4}}u_0\right\|_2^{\frac{7}{8}} < 2^{-\frac{9}{32}}K$$

is satisfied, then condition (2.19) holds. This condition is different from condition (2.16). In the proof Theorem 1.1, we use Theorem 2.1 (b) with $s=1$.

Proof of Theorem 1.1

Proof. We can take $C = D^2$, where D is the constant in Theorem 2.1. Since u satisfies the strong energy inequality (1.13) and the inequality (1.15), it follows that there exist $\delta_0 > 0$ small enough such that

$$\delta_0^{-\frac{1}{2}}\int_{t_0-\delta_0}^{t_0}\|\nabla u(\tau)\|_2^2\,\mathrm{d}\tau \le \frac{\left|\frac{1}{2}\|u(t_0-\delta_0)\|_2^2 - \frac{1}{2}\|u(t_0)\|_2^2\right|}{\delta_0^{\frac{1}{2}}} < D^2$$

and there exists a null set $N \subset (0,T)$ such that for each $t' \in (0,T)\backslash N$ the following inequality holds

$$\frac{1}{2}\left\|u(t)\right\|_2^2 + \int_{t'}^{t}\left\|\nabla u(\tau)\right\|_2^2\mathrm{d}\tau \le \frac{1}{2}\left\|u(t')\right\|_2^2, \text{ for all } t \ge t'. \qquad (2.21)$$

On the other hand, there exists $t' \in (t_0 - \delta_0, t_0) \backslash N$ such that

$$\delta_0^{\frac{1}{2}} \left\|\nabla u(t')\right\|_2^2 \leq \delta_0^{-\frac{1}{2}} \int_{t_0-\delta_0}^{t_0} \left\|\nabla u(\tau)\right\|_2^2 \mathrm{d}\tau < D^2.$$

Thus

$$\delta_0^{\frac{1}{4}} \left\|A^{\frac{1}{2}} u(t')\right\|_2 = \delta_0^{\frac{1}{4}} \left\|\nabla u(t')\right\|_2 < D. \tag{2.22}$$

From the inequality (2.22), applying Theorem 2.1(b) with $s = 1$ we obtain the strong solution v of the Navier-Stokes system (1.1) with the initial $u(t')$ on the interval $[0, \delta_0]$ satisfying Serrin's condition $v \in L^4\big([0, \delta_0); L^6(\Omega)\big)$. Since inequality (2.21) holds, it follows that $u(t+t')$ is a weak solution of the Navier-Stokes equation (1.1) with the initial $u(t')$ on the interval $[0, \delta_0]$. Using Serrin's uniqueness criterion, see [19, 20], we obtain that $u(t) = v(t - t')$ on $[t', t' + \delta_0)$, therefore we conclude that u belongs to Serrin's class $L^4\big([t', t'+\delta_0); L^6(\Omega)\big)$, hence u is regular at $t_0 \in (t', t' + \delta_0)$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2

Proof. We can take $C = D$, where D is the constant in Theorem 2.1. Applying Theorem 2.1, there exists a strong solution v of the Navier-Stokes system (1.1) with the initial u_0 on some interval $(0, T')$, where $0 < T' < T$, so that $v \in L^4\big([0, T'); L^6(\Omega)\big)$. Using Serrin's uniqueness criterion, we obtain that $u = v$ on $[0, T']$, and so u belongs to Serrin's class $L^4\big([0, T'); L^6(\Omega)\big)$. Let

$$T^* = \sup\{T' > 0 : u \in L^4\big([0, T'); L^6(\Omega)\big)\}.$$

Then $0 < T^* \leq T$ and $u \in L^4_{loc}\big([0, T^*); L^6(\Omega)\big)$, we only need prove that $T^* = T$. Suppose that $T^* < T$, since $u \in L^4_{loc}\big([0, T^*); L^6(\Omega)\big)$, it follows that the energy equality

$$\frac{1}{2} \left\|u(t)\right\|_2^2 + \int_{t_0}^{t} \left\|\nabla u(\tau)\right\|_2^2 \mathrm{d}\tau = \frac{1}{2} \left\|u(t_0)\right\|_2^2$$

holds for $0 \leq t_0 < T^*,\ t_0 \leq t < T^*$. We have by the above equality and inequality (1.11) that the following inequality

$$\frac{1}{2} \left\|u(t)\right\|_2^2 + \int_{t_0}^{t} \left\|\nabla u(\tau)\right\|_2^2 \mathrm{d}\tau \leq \frac{1}{2} \left\|u(t_0)\right\|_2^2 \tag{2.23}$$

is satisfied for $0 \leq t_0 < T^*,\ t_0 \leq t \leq T$. From Lemma 2.6(a), it follows that $e^{-tA}u(T^*) \in \mathcal{K}^1_{1/2,\infty}$ and there exists $\delta_1 > 0$ small enough such that

$$\sup_{0<t<\delta_1} t^{\frac{1}{4}} \left\|A^{\frac{1}{2}} e^{-tA} u(T^*)\right\|_2 < \frac{1}{2}\big(D - \varliminf_{\delta \to 0^+} \left\|A^{\frac{1}{4}}\big(u(T^* - \delta) - u(T^*)\big)\right\|_2\big).$$

Applying inequality (1.17) with $C = D$, there exists a positive number $\delta_2 \leq \delta_1/2$ small enough such that

$$\left\|A^{\frac{1}{4}}\big(u(T^* - \delta_2) - u(T^*)\big)\right\|_2 < \frac{1}{2}\big(\varliminf_{\delta \to 0^+} \left\|A^{\frac{1}{4}}\big(u(T^* - \delta) - u(T^*)\big)\right\|_2 + D\big).$$

From the above two inequalities, we deduce that

$$\sup_{0<t<2\delta_2} t^{\frac{1}{4}} \left\|A^{\frac{1}{2}} e^{-tA} u(T^* - \delta_2)\right\|_2$$

$$\le \sup_{0<t<2\delta_2} t^{\frac{1}{4}} \left\| A^{\frac{1}{2}} e^{-tA} \big(u(T^* - \delta_2) - u(T^*) \big) \right\|_2 + \sup_{0<t<2\delta_2} t^{\frac{1}{4}} \left\| A^{\frac{1}{2}} e^{-tA} u(T^*) \right\|_2$$
$$\le \left\| A^{\frac{1}{4}} \big(u(T^* - \delta_2) - u(T^*) \big) \right\|_2 + \sup_{0<t<\delta_1} t^{\frac{1}{4}} \left\| A^{\frac{1}{2}} e^{-tA} u(T^*) \right\|_2 < D.$$

Applying Theorem 2.1, there exists a strong solution v of the Navier-Stokes system (1.1) with the initial value $u(T^* - \delta_2)$ on the interval $[0, 2\delta_2]$ so that

$$v \in L^4\big([0, 2\delta_2]; \mathbb{D}(A^{\frac{1}{2}})\big) \subset L^4\big([0, 2\delta_2]; L^6(\Omega)\big).$$

In view of the inequality (2.23), it follows that $u(t + T^* - \delta_2)$ is a weak solution of the Navier-Stokes system (1.1) on $[0, T - T^* + \delta_2)$ with the initial $u(T^* - \delta_2)$. Using Serrin's uniqueness, we obtain that $u(t) = v(t - T^* + \delta_2)$ for $t \in [T^* - \delta_2, T^* + \delta_2]$, and so $u \in L^4\big([0, T^* + \delta_2); L^6(\Omega)\big)$, which constitutes a contradiction. Thus, u belongs to the space $L^4_{\text{loc}}\big([0, T); L^6(\Omega)\big)$. □

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