On the exponential stability for a class of stochastic differential delay equations with fractional Brownian noises

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Abstract

We investigate the asymptotic stability of the trivial solution of a stochastic differential delay equation which is driven by a fractional Brownian noise. Under the strong dissipative condition, we prove the exponential stability of the trivial solution under a small intensity of noise.

Keywords: fractional Brownian motion, stochastic differential equations (SDE), Young integral, exponential stability, random attractor.

1 Introduction

In this paper, we study the asymptotic stability of the following stochastic differential delay equation

$$dy(t) = [Ay(t) + f(y(t-r))]dt + Cy(t)dB^H(t), \tag{1.1}$$

given initial function $\eta \in \mathcal{C}([-r,0],\mathbb{R}^d)$, where $A, C \in \mathbb{R}^{d\times d}$ are matrices, r is a constant delay; $f(0) = 0$ and f is a Lipschitz function, i.e.

$$\|f(y_1) - f(y_2)\| \leq C_f\|y_1 - y_2\|. \tag{1.2}$$

B^H is an one-dimensional fractional Brownian motion [12] on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the Hurst index $H > 1/2$. Equation (1.1) is understood and solved in the pathwise sense as a Young differential equation with delay

$$dy(t) = [Ay(t) + f(y(t-r))]dt + Cy(t)d\omega(t). \tag{1.3}$$

When $r = 0$, system (1.1) has the non-delay form, and the exponential stability of the trivial solution has been studied in recent papers [4], [7], [6] for Young equations. The stability problem for the Young differential delay equation (1.1) is however still in a fancy stage and is only studied in a few references, e.g. [11], although solutions of the Young differential equations have been well studied, [3], [5], or [6]. Since it is not sure how to apply the semigroup technique developed in [6] for the delay situation, our aim in this paper is to apply the techniques developed in [7] for nondelay equations to prove the exponential stability for the trivial solution of (1.1).
We present here a brief introduction to Young differential equations for Hölder continuous functions

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to match the delay situation (a detailed explaination can be found in [3] or [5]). For each $0 < \alpha < 1$, denote by $C^\alpha([a,b],\mathbb{R}^d)$ the space of Hölder continuous functions on $[a,b]$ equipped with the norm

$$\|u\|_{\infty,\alpha,[a,b]} := \|u\|_{\infty,[a,b]} + |\!|\!|u|\!|\!|_{\alpha,[a,b]}$$

where $\|\cdot\|_{\infty,[a,b]}$ is the sup norm of continuous functions on $[a,b]$ and

$$|\!|\!|u|\!|\!|_{\alpha,[a,b]} := \sup_{a\le s<t\le b} \frac{\|u(t)-u(s)\|}{(t-s)^\alpha}.$$

We also introduce the space

$$C^{0,\alpha}([a,b],\mathbb{R}^d) := \Big\{u \in C^\alpha([a,b],\mathbb{R}^d)\Big|\lim_{h\to 0} \sup_{\substack{a\le s<t\le b,\\ |t-s|<h}} \frac{\|u(t)-u(s)\|}{(t-s)^\alpha} = 0\Big\}.$$

Given $y \in C^\beta([a,b],\mathbb{R}^d)$ and $\omega \in C^\nu([a,b],\mathbb{R})$, $\beta+\nu>1$, it is well known that the *Young integral* $\int_a^b y(t)d\omega(t)$ exists (see [13, p. 264–265], [5, p. 316]). This integral satisfies additive property by the construction, and the so-called Young-Loeve estimate [8, Theorem 6.8, p. 116]

$$\Big\|\int_s^t y(u)d\omega(u) - y(s)[\omega(t)-\omega(s)]\Big\| \le K|t-s|^{\beta+\nu}\, |\!|\!|y|\!|\!|_{\beta,[s,t]}\, |\!|\!|\omega|\!|\!|_{\nu,[s,t]}\,, \tag{1.4}$$

where

$$K := (1-2^{1-\theta})^{-1}, \qquad \theta := \beta+\nu > 1. \tag{1.5}$$

A result given in [1] shows that, under some regularity conditions of f, i.e. f is Lipchitz continuous, then there exists a unique solution $y(\cdot,\omega,\eta)$ of (1.1) on $\mathcal{C}^\beta([-r,T],\mathbb{R}^d)$ for $H>\beta>1-H$ almost surely. A similar result can be found in [5]. In fact, (1.1) can be solved by induction in each interval $[kr,(k+1)r]$ as a Young differential equation. Namely, given $y_0 = \eta \in \mathcal{C}^\beta([-r,0],\mathbb{R}^d)$, then (1.3) derives

$$dy(kr+t) = [Ay(kr+t) + f(y((k-1)r+t))]dt + Cy(kr+t)d\theta_{kr}\omega(t), \quad \forall t \in [0,r].$$

The solution of (1.3) can be written by induction in the explicit form as $y_0(\cdot) = \eta(\cdot) \in \mathcal{C}^\beta$ and

$$y(kr+t) = \Phi(t,\theta_{kr}\omega)y(kr) + \int_0^t \Phi(t-s,\theta_{kr+s}\omega)f(y((k-1)r+s))ds,\ t\in[0,r] \tag{1.6}$$

for all $k\ge 1$, where $\Phi(t,x)$ is the matrix solution of the linear Young differential equation

$$dz(t) = Az(t)dt + Cz(t)d\omega(t) \tag{1.7}$$

and satisfies $\Phi(0,\omega) = Id$.
Due to [2, Remark 2.3] $\Phi(\cdot,\theta_{kr}\omega)y(kr) \in C^\beta([0,r],\mathbb{R}^d)$. Hence (1.6) implies that $y(kr+\cdot) \in C^\beta([0,r],\mathbb{R}^d)$. By induction, one can prove that $y(\cdot) \in C^\beta([-r,T],\mathbb{R}^d)$ for all $T \ge -r$. As a result, $y_t(\cdot)$ belongs to $C^\beta([-r,0],\mathbb{R}^d)$ for all $t\ge 0$.

2 Exponential stability

In this section we would like to study the exponential stability of the zero solution of (1.1). We will work with the canonical space $(\Omega,\mathcal{F},\mathbb{P},\theta)$ of fBm as introduced in [3]. Namely, for fixed $1/2<\nu'<H$, denote by $C_0^{\nu'}(\mathbb{R},\mathbb{R})$ the space of all paths ω which belong to $C^{\nu'}(I,\mathbb{R})$ for all closed interval $I\subset\mathbb{R}$ and receive value 0 at time 0. Then $(\Omega,\mathcal{F},\mathbb{P},\theta)$ is the induced space on $C_0^{\nu'}(\mathbb{R},\mathbb{R})$ of

$(C_0(\mathbb{R},\mathbb{R}),\mathcal{B},\mathbb{P},\theta)$ where $(C_0(\mathbb{R},\mathbb{R})$ is the space of all continuous paths ω on $\mathbb{R}$ such that $\omega(0)=0$, $\mathcal{B}$ is the Borel $-\sigma$ algebra generated by the compact open topology, $\mathbb{P}$ is the Wiener measure on $\mathcal{B}$ generated by a fractional Brownian motion and θ is the Wiener shift define by $\theta_t\omega(\cdot)=\omega(t+\cdot)-\omega(\cdot)$. Note that the Wiener shift is ergodic due to [9]. It is proved in [3] that equation (1.1) then generates a so-called *random dynamical system* $\varphi:\mathbb{R}_+\times(\Omega,\mathcal{F},\mathbb{P},\theta)\times C^{0,\beta}([-r,0],\mathbb{R}^d)\to C^{0,\beta}([-r,0],\mathbb{R}^d)$ given by

$$\varphi(t,\omega,\eta)(s)=y(t+s,\omega,\eta),\quad \forall t\in\mathbb{R}_+,s\in[-r,0],\omega\in\Omega,\eta\in C^{0,\beta}([-r,0],\mathbb{R}^d).$$

From now on, we fix $\nu\in(1/2,\nu')$ and $\beta\in(1-\nu,\nu)$. It is known that $B^H(\omega)|_{[a,b]}\in C^{0,\nu}([a,b],\mathbb{R}),\ \forall\omega\in\Omega$ and $E\left\|B^H\right\|^m_{\nu,[a,b]}$ is bounded for all $[a,b]\subset\mathbb{R}$ and $m>0$ (see for instance [11]).
To obtain the main results we need the following lemmas.

Lemma 2.1 *Assume that for fixed T, y satisfies*

$$\|y\|_{\beta,[a,b]}\le M\Big[(b-a)^{1-\beta}+(b-a)^{\nu-\beta}\|\omega\|_{\nu,[a,b]}\Big]\Big[L+(b-a)^{\beta}\|y\|_{\beta,[a,b]}\Big] \tag{2.1}$$

for all $0\le a<b\le T$, in which M,L are constants. Then

$$\|y\|_{\beta,[0,T]}\le L[2\vee(4M)]^{1+\frac{1}{\nu}}T^{-\beta}\left(1+T^{1+\frac{1}{\nu}}+T^{1+\nu}\|\omega\|^{1+\frac{1}{\nu}}_{\nu,[0,T]}\right), \tag{2.2}$$

in which $a\vee b:=\max\{a,b\}$.

Proof: Fix $\mu=\frac{1}{2M}$, we construct a sequence $\{t_i\}$ such that $t_0=0$ and

$$t_{i+1}=\inf\{t\ge t_i\mid\ (t-t_i)+(t-t_i)^{\nu}\|\omega\|_{\nu,[t_i,t]}\ge\mu\}.$$

Since $|\,\|\omega\|_{\nu,[t_i,t']}-\|\omega\|_{\nu,[t_i,t]}\,|\le\|\omega\|_{\nu,[t,t']}$ for all $t_i\le t\le t'$ (see [5, p. 316]) and $\omega\in\mathcal{C}^{0,\nu}(I,\mathbb{R})$ for all closed interval I, $|\,\|\omega\|_{\nu,[t_i,t']}-\|\omega\|_{\nu,[t_i,t]}\,|\le\|\omega\|_{\nu,[t,t']}\to0$ as $t'\to t$ (see [8, Corollary 5.31, p. 96]). It follows that for given t_i the function $f(t)=(t-t_i)+(t-t_i)^{\nu}\|\omega\|_{\nu,[t_i,t]}$ is continuous on $[t_i,+\infty)$. Moreover, f is strictly increasing, $f(t_i)=0$ and $\lim_{t\to+\infty}f(t)=+\infty$. Therefore, there exists t_{i+1} such that

$$(t_{i+1}-t_i)+(t_{i+1}-t_i)^{\nu}\|\omega\|_{\nu,[t_i,t_{i+1}]}=\mu. \tag{2.3}$$

Assign $N:=\sup\{n\mid t_n\le T\}<\infty$.

For $i=0,...,N-1$, (2.3) leads to

$$\mu=(t_{i+1}-t_i)^{\nu}\Big[(t_{i+1}-t_i)^{1-\nu}+\|\omega\|_{\nu,[t_i,t_{i+1}]}\Big]\le(t_{i+1}-t_i)^{\nu}\left(T^{1-\nu}+\|\omega\|_{\nu,[0,T]}\right).$$

This implies

$$t_{i+1}-t_i\ge\left(\frac{\mu}{T^{1-\nu}+\|\omega\|_{\nu,[0,T]}}\right)^{\frac{1}{\nu}},$$

and

$$T\ge\sum_{i=0}^{N-1}(t_{i+1}-t_i)\ge N\left(\frac{\mu}{T^{1-\nu}+\|\omega\|_{\nu,[0,T]}}\right)^{\frac{1}{\nu}}.$$

It follows that

$$N\le T\left(\frac{T^{1-\nu}+\|\omega\|_{\nu,[0,T]}}{\mu}\right)^{\frac{1}{\nu}}. \tag{2.4}$$

By the asumption, for $0 \le i \le N-1$ we have

$$\|y\|_{\beta,[t_i,t_{i+1}]} \le M\Big[(t_{i+1}-t_i)^{1-\beta} + (t_{i+1}-t_i)^{\nu-\beta}\,\|\omega\|_{\nu,[t_i,t_{i+1}]}\Big]\Big[L + (t_{i+1}-t_i)^{\beta}\,\|y\|_{\beta,[t_i,t_{i+1}]}\Big]$$

or

$$\|y\|_{\beta,[t_i,t_{i+1}]} \le \frac{ML}{1-M\mu}\Big[T^{1-\beta} + T^{\nu-\beta}\,\|\omega\|_{\nu,[0,T]}\Big].$$

We also have

$$\|y\|_{\beta,[t_N,T]} \le \frac{ML}{1-M\mu}\Big[T^{1-\beta} + T^{\nu-\beta}\,\|\omega\|_{\nu,[0,T]}\Big].$$

Because $[0,T] \subset \cup_{i=0}^{N-1}[t_i,t_{i+1}] \cup [t_N,T]$ and Hölder norm is of the superadditivity, we obtain

$$\begin{aligned}
\|y\|_{\beta,[0,T]} &\le \sum_{i=0}^{N-1}\|y\|_{\beta,[t_i,t_{i+1}]} + \|y\|_{\beta,[t_N,T]} \\
&\le (N+1)\frac{ML}{1-M\mu}\Big[T^{1-\beta} + T^{\nu-\beta}\,\|\omega\|_{\nu,[a,b]}\Big]. \qquad (2.5)
\end{aligned}$$

Combining this with (2.4) we get

$$\begin{aligned}
\|y\|_{\beta,[0,T]} &\le 2ML\Big[1 + T\Big(\frac{T^{1-\nu} + \|\omega\|_{\nu,[0,T]}}{\mu}\Big)^{\frac{1}{\nu}}\Big]\Big[T^{1-\beta} + T^{\nu-\beta}\,\|\omega\|_{\nu,[0,T]}\Big] \\
&\le L(2M)^{1+\frac{1}{\nu}}T^{-\beta}\Big[\frac{1}{(2M)^{\frac{1}{\nu}}} + \Big(T + T^{\nu}\,\|\omega\|_{\nu,[0,T]}\Big)^{\frac{1}{\nu}}\Big]\Big[T + T^{\nu}\,\|\omega\|_{\nu,[0,T]}\Big] \\
&\le L[1 \vee (2M)]^{1+\frac{1}{\nu}}T^{-\beta}\Big[1 + \Big(T + T^{\nu}\,\|\omega\|_{\nu,[0,T]}\Big)^{\frac{1}{\nu}}\Big]\Big[T + T^{\nu}\,\|\omega\|_{\nu,[0,T]}\Big] \\
&\le L[1 \vee (2M)]^{1+\frac{1}{\nu}}T^{-\beta}\Big[T + T^{\nu}\,\|\omega\|_{\nu,[0,T]} + (T + T^{\nu}\,\|\omega\|_{\nu,[0,T]})^{1+\frac{1}{\nu}}\Big]. \qquad (2.6)
\end{aligned}$$

Using the following inequalities

$$x^p + p - 1 \ge px,\ \forall x \ge 0, p \ge 1, \qquad (2.7)$$

and

$$(a+b)^p \le (a^p + b^p)\max\{1, 2^{p-1}\},\ \forall a,b,p > 0, \qquad (2.8)$$

we have

$$\begin{aligned}
\|y\|_{\beta,[0,T]} &\le L[1 \vee (2M)]^{1+\frac{1}{\nu}}T^{-\beta}\Big[\frac{(T + T^{\nu}\,\|\omega\|_{\nu,[0,T]})^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} + \frac{\frac{1}{\nu}}{1+\frac{1}{\nu}} + (T + T^{\nu}\,\|\omega\|_{\nu,[0,T]})^{1+\frac{1}{\nu}}\Big] \\
&\le L[1 \vee (2M)]^{1+\frac{1}{\nu}}T^{-\beta}\Big[2(T + T^{\nu}\,\|\omega\|_{\nu,[0,T]})^{1+\frac{1}{\nu}} + 1\Big] \\
&\le L[2 \vee (4M)]^{1+\frac{1}{\nu}}T^{-\beta}\left(1 + T^{1+\frac{1}{\nu}} + T^{1+\nu}\,\|\omega\|_{\nu,[0,T]}^{1+\frac{1}{\nu}}\right). \qquad (2.9)
\end{aligned}$$

The proof is completed. □

The following lemma is a Hölder-version of [7, Corrolary 3.5], which facilitates the proof of the stability of the delay system.

Lemma 2.2 *Assuming that in (1.7) A is negative definite, i.e. there exists constant $h_0 > 0$ such that*

$$\langle y, Ay\rangle \le -h_0\|y\|^2, \quad \forall y \in \mathbb{R}^d. \qquad (2.10)$$

Then for all $0 \le s < t \le r$*, the following inequalities hold*

$$(i) \qquad \|\Phi(t,\omega)\| \le \exp\{-h_0 t + \|C\| t Q(r,\omega) + 2KG\|C\|\}, \tag{2.11}$$

$$(ii) \qquad \|\Phi(t-s,\theta_s\omega)\| \le \exp\{-h_0(t-s) + \|C\|(t-s)Q(r,\omega) + 2KG\|C\|\}, \tag{2.12}$$

where

$$G \quad := \quad \Big(\max\{2, 8\|A\|, 16K\|C\|\}\Big)^{1+\frac{1}{\nu}} \tag{2.13}$$

and

$$Q(t,\omega) := 2KG\Big[\frac{\nu(1+2KG)}{2KG} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]}^{\frac{1}{\nu}} + t^{\nu+\frac{1}{\nu}} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]} + t^{2\nu} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]}^{2+\frac{1}{\nu}}\Big]. \tag{2.14}$$

Proof: (i) In the proof of [7, Theorem 3.4], we obtain

$$\begin{aligned}
\log\|\Phi(t,\omega)\| &\le -h_0 t + \Big|\int_0^t \langle y(s), Cy(s)\rangle d\omega(s)\Big| \\
&\le -h_0 t + \|C\| t^\nu \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]} \Big(1 + 2Kt^\beta \left|\!\left|\!\left|y\right|\!\right|\!\right|_{\beta,[0,t]}\Big).
\end{aligned}$$

in which $y(t) = \frac{\Phi(t,\omega)}{\|\Phi(t,\omega)\|}$ satisfies

$$\left|\!\left|\!\left|y\right|\!\right|\!\right|_{\beta,[s,t]} \le (2\|A\| \vee 4K\|C\|)\Big[(t-s)^{1-\beta} + (t-s)^{\nu-\beta} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[s,t]}\Big]\Big[1 + (t-s)^\beta \left|\!\left|\!\left|y\right|\!\right|\!\right|_{\beta,[s,t]}\Big], \ \forall s < t.$$

Apply Lemma 2.1 for $M = 2\|A\| \vee 4K\|C\|$ and $L = 1$, we get

$$\left|\!\left|\!\left|y\right|\!\right|\!\right|_{\beta,[0,t]} \le Gt^{-\beta}\Big(1 + t^{1+\frac{1}{\nu}} + t^{1+\nu} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]}^{1+\frac{1}{\nu}}\Big).$$

Therefore, for $0 \le t \le r$

$$\begin{aligned}
\log\|\Phi(t,\omega)\| &\le -h_0 t + \|C\| t^\nu \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]} \Big[1 + 2KG\Big(1 + t^{1+\frac{1}{\nu}} + t^{1+\nu} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]}^{1+\frac{1}{\nu}}\Big)\Big] \\
&\le -h_0 t + \|C\| t^\nu \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]} \Big[1 + 2KG + 2KGt^{1+\frac{1}{\nu}} + 2KGt^{1+\nu} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]}^{1+\frac{1}{\nu}}\Big] \\
&\le -h_0 t + (1+2KG)\|C\| t^\nu \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]} \\
&\quad + 2KG\|C\| t^\nu \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]} \Big(t^{1+\frac{1}{\nu}} + t^{1+\nu} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]}^{1+\frac{1}{\nu}}\Big) \\
&\le -h_0 t + (1+2KG)\|C\| \frac{t \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]}^{\frac{1}{\nu}} + \frac{1}{\nu} - 1}{\frac{1}{\nu}} \\
&\quad + 2KG\|C\| t^\nu \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]} \Big(t^{1+\frac{1}{\nu}} + t^{1+\nu} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]}^{1+\frac{1}{\nu}}\Big) \\
&\le -h_0 t + 2KG\|C\| \\
&\quad + 2KG\|C\| t\Big[\frac{\nu(1+2KG)}{2KG} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]}^{\frac{1}{\nu}} + t^{\nu+\frac{1}{\nu}} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]} + t^{2\nu} \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,t]}^{2+\frac{1}{\nu}}\Big].
\end{aligned}$$

This implies (2.11).
(ii) For $0 \le s < t \le r$,

$$\|\Phi(t-s,\theta_s\omega)\| \le \exp\Big\{-h_0(t-s) + \|C\|(t-s)Q(t-s,\theta_s\omega) + 2KG\|C\|\Big\}.$$

On the other hand

$$\begin{aligned}
\left|\!\left|\!\left|\theta_s\omega\right|\!\right|\!\right|_{\nu,[0,t-s]} &= \sup_{0\le v<u\le t-s} \frac{\|\omega(u+s) - \omega(v+s)\|}{|(u+s)-(v+s)|^\nu} \\
&= \sup_{s\le v<u\le t} \frac{\|\omega(u) - \omega(v)\|}{|u-v|^\nu} \le \left|\!\left|\!\left|\omega\right|\!\right|\!\right|_{\nu,[0,r]}.
\end{aligned}$$

As a result $Q(t-s,\theta_s\omega) \le Q(r,\omega)$, which proves (2.12). □

To state our main result, let us introduce a Gronwall-Bellman type estimation, see the proof in [10].

Proposition 2.3 *Let $t_0 \in \mathbb{R}, 0 < t_0 \leq \infty, c \geq 0$ and $a : [t_0, T] \to \mathbb{R}_+$ be locally integrable. Assume $r \geq 0$, and $\tau : [t_0, T) \to \mathbb{R}_+$ is a measurable function such that $t_0 - r \leq t - \tau(t), t_0 \leq t < T$. If $x : [t_0 - r, T) \to \mathbb{R}_+$ is Borel measurable and locally bounded such that*

$$x(t) \leq c + \int_{t_0}^{t} a(u)x(u-\tau)du,\ t_0 \leq t < T, \tag{2.15}$$

then

$$x(t) \leq K \exp\Big\{ \int_{t_0}^{t} \gamma(s)ds \Big\},\ t_0 \leq t < T, \tag{2.16}$$

where the function $\gamma : [t_0 - r, T) \to \mathbb{R}_+$ is locally integrable, and satisfies the characteristic inequality

$$a(t) \exp\Big\{ - \int_{t-\tau(t)}^{t} \gamma(s)ds \Big\} \leq \gamma(t),\ t_0 \leq t < T, \tag{2.17}$$

and

$$K := \max\Big\{ c \exp\Big\{ \int_{t_0 - r}^{t_0} \gamma(u)du \Big\}, \sup_{t_0 - r \leq s \leq t_0} \exp\Big\{ \int_{s}^{t_0} \gamma(u)du \Big\} \Big\}. \tag{2.18}$$

When a and τ are constant functions, we get the following corollary

Corollary 2.4 *Let $t_0 \in \mathbb{R}, t_0 < T \leq \infty$, and $c, a, \tau \geq 0$. If $y : [t_0 - \tau, T) \to \mathbb{R}_+$ is Borel measurable and locally bounded such that*

$$y(t) \leq c + \int_{t_0}^{t} ay(u-\tau)du,\ t_0 \leq t < T, \tag{2.19}$$

then

$$y(t) \leq Ke^{\gamma(t-t_0)},\ t_0 \leq t < T, \tag{2.20}$$

where the nonnegative number γ satisfies the inequality

$$a \leq \gamma e^{\gamma\tau}, \tag{2.21}$$

and

$$K := \max\Big\{ ce^{\gamma\tau}, \sup_{t_0 - r \leq s \leq t_0} y(s)e^{\gamma(t_0 - s)} \Big\}. \tag{2.22}$$

Our first main result can be formulated as follows.

Theorem 2.5 *Assume A is negative definite, i.e. there exists a $h_0 > 0$ such that*

$$\langle y, Ay \rangle \leq -h_0 \|y\|^2. \tag{2.23}$$

and $C_f < \frac{h_0}{2} e^{-\frac{h_0 r}{2}}$. Then there exists $\varepsilon > 0$ such that if $\|C\| < \varepsilon$ the trivial solution of system (1.1) *is forward exponentially stable almost surely, i.e. for any solution y of* (1.1) *we have*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|y_t(\omega, y_0)\|_{\infty,\beta,[-r,0]} < 0, \tag{2.24}$$

for almost surely all the realizations ω.

Proof: First, we fix ω, η and denote by $y(t),\ t \in [-r, \infty)$ the solution of (1.3). Since the assumption of the function f, it follows that

$$\|f(y)\| \leq C_f \|y\|. \tag{2.25}$$

Due to (1.6), (2.11), (2.12) and (2.25), for all $t \in [0, r]$

$$\begin{aligned} \|y(kr+t)\| &\leq \exp\{-h_0 t + \|C\| t Q(r, \theta_{kr}\omega) + 2KG\|C\|\}\|y(kr)\| \\ &\quad + C_f \int_0^t \exp\{-h_0(t-s) + \|C\|(t-s)Q(r, \theta_{kr}\omega) + 2KG\|C\|\}\|y(kr+s-r)\| ds. \end{aligned} \tag{2.26}$$

For each $k \in \mathbb{N}$ is fixed. Put

$$z^k(t) = \|y(kr+t)\| \exp\{h_0 t - \|C\| t Q(r, \theta_{kr}\omega)\}, t \in [-r, r]. \tag{2.27}$$

We have

$$\begin{aligned} z^k(t) &\leq e^{2KG\|C\|}\|y(kr)\| + e^{2KG\|C\|} C_f \int_0^t \exp\{h_0 s - \|C\| s Q(r, \theta_{kr}\omega)\}\|y(kr+s-r)\| ds \\ &\leq e^{2KG\|C\|}\|y(kr)\| + C_f \exp\{h_0 r - \|C\| r Q(r, \theta_{kr}\omega) + 2KG\|C\|\} \int_0^t z^k(s-r) ds. \end{aligned} \tag{2.28}$$

By the assumption $C_f < \frac{h_0}{2} e^{-\frac{h_0 r}{2}}$, there exists $\varepsilon_1 > 0$ such that $C_f < \frac{h_0}{2} \exp\{-\frac{h_0 r}{2} - 2KG\varepsilon_1\}$. Then for $\|C\| < \varepsilon_1$

$$C_f \exp\{h_0 r + 2KG\|C\|\} < \frac{h_0}{2} \exp\left\{\frac{h_0 r}{2} + 2KG(\|C\| - \varepsilon_1)\right\} < \frac{h_0}{2} e^{\frac{h_0 r}{2}}.$$

One can choose $\gamma \in (0, \frac{h_0}{2})$ not depending on k and ω such that

$$C_f \exp\{h_0 r - \|C\| r Q(r, \theta_{kr}\omega) + 2KG\|C\|\} \leq C_f \exp\{h_0 r + 2KG\|C\|\} \leq \gamma e^{\gamma r}.$$

Combining this with Corollary 2.4 we have for each $t \in [0, r]$

$$z^k(t) \leq M_k e^{\gamma t}, \ \text{ or } \ \|y(kr+t)\| \leq M_k \exp\{-h_0 t + \gamma t + \|C\| t Q(r, \theta_{kr}\omega)\}, \tag{2.29}$$

where

$$M_k = \max\left\{e^{\gamma r + 2KG\|C\|}\|y(kr)\|, \sup_{s \in [-r, 0]} \|y(kr+s)\| \exp\{h_0 s - \gamma s - \|C\| s Q(r, \theta_{kr}\omega)\}\right\}. \tag{2.30}$$

Next, we introduce the function

$$\rho^k(t) = M_k \exp\{-h_0 t + \gamma t + \|C\| t Q(r, \theta_{kr}\omega)\}$$

then $\|y(kr+t)\| \leq \rho^k(t)$ for all $t \in [0, r]$ and

$$\rho^k(r) = M_k \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \theta_{kr}\omega)\}.$$

Next we will prove that

$$M_k \leq \rho^{k-1}(r) \exp\{\gamma r + \|C\| r Q(r, \theta_{kr}\omega) + 2KG\|C\|\}. \tag{2.31}$$

Firstly, by the definition of M_k and (2.29), for $s \in [-r, 0]$,

$$\begin{aligned}
& \|y(kr+s)\| \exp\{h_0 s - \gamma s - \|C\| s Q(r, \theta_{kr}\omega)\} \\
\leq\ & \|y((k-1)r + s + r)\| \exp\{h_0(s+r) - \gamma(s+r)\} \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \theta_{kr}\omega)\} \\
\leq\ & \|y((k-1)r + s')\| \exp\{h_0 s' - \gamma s'\} \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \theta_{kr}\omega)\} \\
\leq\ & M_{k-1} \exp\{\|C\| r Q(r, \theta_{(k-1)r}\omega)\} \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \theta_{kr}\omega)\}, \qquad (2.32)
\end{aligned}$$

in which $0 \leq s + r = s' \leq r$.

On the other hand,

$$M_{k-1} \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \theta_{(k-1)r}\omega)\} = \rho^{k-1}(r)$$

then

$$\sup_{[-r,0]} \|y(ks+r)\| \exp\{h_0 s - \gamma s - \|C\| s Q(r, \theta_{kr}\omega) s\} \leq \rho^{k-1}(r) e^{\|C\| r Q(r, \theta_{kr}\omega)}. \qquad (2.33)$$

Secondly,

$$e^{\gamma r + 2KG\|C\|} \|y(kr)\| = e^{\gamma r + 2KG\|C\|} \|y((k-1)r + r)\| \leq \rho^{k-1}(r) e^{\gamma r + 2KG\|C\|}.$$

Hence

$$M_k \leq \rho^{k-1}(r) \exp\{\gamma r + \|C\| r Q(r, \theta_{kr}\omega) + 2KG\|C\|\}. \qquad (2.34)$$

This implies

$$\begin{aligned}
\rho^k(r) &= M_k \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \theta_{kr}\omega)\} \\
&\leq \rho^{k-1}(r) \exp\{-h_0 r + 2\gamma r + 2\|C\| r Q(r, \theta_{kr}\omega) + 2KG\|C\|\} \\
&\quad \cdots \\
&\leq \rho^0(r) \prod_{j=1}^{k} \exp\{-h_0 r + 2\gamma r + 2\|C\| r Q(r, \theta_{jr}\omega) + 2KG\|C\|\} \\
&= \rho^0(r) \exp\Big\{\Big[-(h_0 - 2\gamma) r + 2\|C\| r \frac{1}{k} \sum_{j=1}^{k} Q(r, \theta_{jr}\omega) + 2KG\|C\|\Big] k\Big\}. \qquad (2.35)
\end{aligned}$$

Applying Birkhoff ergodic theorem

$$\begin{aligned}
& \lim_{k \to \infty} \Big\{-(h_0 - 2\gamma) r + 2KG\|C\| + \frac{2\|C\| r}{k} \sum_{j=1}^{k} Q(r, \theta_{jr}\omega)\Big\} \\
=\ & -(h_0 - 2\gamma) r + 2KG\|C\| + 2\|C\| r E[Q(r, \omega)] =: -\lambda \qquad (2.36)
\end{aligned}$$

holds for almost all ω. Choose

$$\epsilon = \min\Big\{\epsilon_1, \frac{(h_0 - 2\gamma) r}{2KG + 2rE[Q(r, \omega)]}\Big\}.$$

Then $\lambda > 0$ provided that $\|C\| < \varepsilon$. Moreover, $\rho^0(r) = M_0 \exp\{-h_0 r + \gamma r + \|C\| r Q(r, \omega)\}$, in which

$$\begin{aligned}
M_0 &= \max\left\{e^{\gamma r + 2KG\|C\|} \|\eta(0)\|, \sup_{s \in [-r,0]} \|\eta(s)\| \exp\{h_0 s - \gamma s - \|C\| s Q(r, \omega)\}\right\} \\
&\leq \|\eta\|_{\infty, [-r,0]} \exp\Big\{\gamma r + \|C\| Q(r, \omega) + 2KG\|C\|\Big\}.
\end{aligned}$$

Hence

$$\rho^0(r) \leq \|\eta\|_{\infty, [-r,0]} \exp\Big\{2\gamma r + 2\|C\| r Q(r, \omega) + 2KG\|C\|\Big\}.$$

Hence when k is large enough, (2.36) deduce

$$\Big[-(h_0 - 2\gamma)r + 2KG\|C\| + 2\|C\| r \frac{1}{k}\sum_{j=1}^{k} Q(r,\theta_{jr}\omega)\Big] k < \frac{-\lambda k}{2}. \tag{2.37}$$

As a result

$$\rho^k(r) \le \rho^0(r) e^{\frac{-\lambda k}{2}}.$$

On the other hand, it is obvious that, for any $0 \le t \le r$

$$\|y(kr+t)\| \le \rho^k(t) \le e^{h_0 r}\rho^k(r) \le e^{-\frac{\lambda k}{2} + h_0 r}\rho^0(r).$$

Therefore

$$\limsup_{t\to\infty} \frac{1}{t}\log \|y_t\|_{\infty,[-r,0]} < -\lambda/2. \tag{2.38}$$

Moreover, for any $0 \le t_1 < t_2 \le r$,

$$\begin{aligned}
& \|y(kr+t_2,\omega,\eta) - y(kr+t_1,\omega,\eta)\| \\
\le\ & \Big\| \Big(\Phi(t_2,\theta_{kr}\omega) - \Phi(t_1,\theta_{kr}\omega)\Big) y(kr,\omega,\eta)\Big\| \\
& + \Big\| \int_{t_1}^{t_2} \Phi(t_2-s,\theta_{s+kr}\omega) f(y(kr+s-r,\omega,\eta)) ds \Big\| \\
& + \Big\| \int_0^{t_1} \Big(\Phi(t_2-s,\theta_{s+kr}\omega) - \Phi(t_1-s,\theta_{s+kr}\omega)\Big) f(y(kr+s-r,\omega,\eta)) ds \Big\| \\
\le\ & \|\Phi(t_2,\theta_{kr}\omega) - \Phi(t_1,\theta_{kr}\omega)\| \|y(kr,\omega,\eta)\| \\
& + C_f \int_{t_1}^{t_2} \|\Phi(t_2-s,\theta_{s+kr}\omega)\| \|y(kr+s-r,\omega,\eta)\| ds \\
& + C_f \int_0^{t_1} \|\Phi(t_2-s,\theta_{s+kr}\omega) - \Phi(t_1-s,\theta_{s+kr}\omega)\| \|y(kr+s-r,\omega,\eta)\| ds \\
\le\ & (t_2-t_1)^\beta \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \Phi(\cdot,\theta_{kr}\omega) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[0,r]} \\
& + (t_2-t_1) C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \|\Phi(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \\
& + C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \int_0^{t_1} \Big\| \Big(\Phi(t_2,\theta_{kr}\omega)) - \Phi(t_1,\theta_{kr}\omega))\Big) \Phi^{-1}(s,\theta_{kr}\omega) \Big\| ds \\
\le\ & (t_2-t_1)^\beta \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \Phi(\cdot,\theta_{kr}\omega) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[0,r]} \\
& + (t_2-t_1) C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \|\Phi(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \\
& + (t_2-t_1)^\beta r C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \Phi(\cdot,\theta_{kr}\omega) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[0,r]}.
\end{aligned} \tag{2.39}$$

We then obtain for k large enough

$$\begin{aligned}
\left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert y_{(k+1)r} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[-r,0]} \le\ & \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \Phi(\cdot,\theta_{kr}\omega) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[0,r]} \\
& + r^{1-\beta} C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \|\Phi(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \\
& + r C_f \|y(kr+\cdot,\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \Phi(\cdot,\theta_{kr}\omega) \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert_{\beta,[0,r]} \\
\le\ & e^{-\frac{\lambda(k-1)}{2} + h_0 r} \rho^0(r) \Big[1 + C_f (r^{1-\beta} + r) \|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]}\Big] \|\Phi(\cdot,\theta_{kr}\omega)\|_{\infty,\beta,[0,r]}.
\end{aligned} \tag{2.40}$$

On the other hand, due to [2, Proposition 2.4], Φ and $\Psi := \Phi^{-1}$ satisfy

$$\begin{aligned}
d\Phi(t,\omega) &= A\Phi(t,\omega)dt + C\Phi(t,\omega)d\omega(t) \\
d\Psi(t,\omega) &= -A^T\Psi(t,\omega)dt - C^T\Psi(t,\omega)d\omega(t).
\end{aligned}$$

Now, apply [6, Theorem 2.1] we get

$$\begin{aligned}
\|\Phi(\cdot,\omega)\|_{\infty,[0,r]} &\leq e^{2\|A\|r}\left(\frac{K+2}{K+1}\right)^{1+r[2(K+1)\|C\|\|\omega\|_{\nu,[0,r]}]^{1/\nu}} \\
\|\Phi^{-1}(\cdot,\omega)\|_{\infty,[0,r]} &\leq e^{2\|A\|r}\left(\frac{K+2}{K+1}\right)^{1+r[2(K+1)\|C\|\|\omega\|_{\nu,[0,r]}]^{1/\nu}}.
\end{aligned}$$

Moreover for all $s,t\in[0,r]$

$$\begin{aligned}
\|\Phi(\cdot,\omega)\|_{\beta,[s,t]} &\leq (\|A\|\vee K\|C\|)\Big[(t-s)^{1-\beta}+(t-s)^{\nu-\beta}\,\|\omega\|_{\nu,[s,t]}\Big]\times \\
&\quad \times\Big[\|\Phi(\cdot,\omega)\|_{\infty,[0,r]}+(t-s)^{\beta}\,\|\Phi(\cdot,\omega)\|_{\beta,[s,t]}\Big] \\
&\leq (\|A\|\vee K\|C\|)\Big[(t-s)^{1-\beta}+(t-s)^{\nu-\beta}\,\|\omega\|_{\nu,[s,t]}\Big]\Big[L+(t-s)^{\beta}\,\|y\|_{\beta,[s,t]}\Big]
\end{aligned}$$

with $L=e^{2\|A\|r}\left(\frac{K+2}{K+1}\right)^{1+r[2(K+1)\|C\|\|\omega\|_{\nu,[0,r]}]^{1/\nu}}$.
Due to Lemma 2.1

$$\|\Phi(\cdot,\omega)\|_{\beta,[0,r]}\leq L[2\vee 4\|A\|\vee 4K\|C\|]^{1+\frac{1}{\nu}}r^{-\beta}\left(1+r^{1+\frac{1}{\nu}}+r^{1+\nu}\,\|\omega\|^{1+\frac{1}{\nu}}_{\nu,[0,r]}\right).$$

Since $\limsup_{k\to\infty}\frac{\|\theta_{kr}\omega\|_{\nu,[0,r]}}{k}=0$, it is evident that

$$\limsup_{k\to\infty}\log\left(\Big[1+C_f(r^{1-\beta}+r)\|\Phi^{-1}(\cdot,\theta_{kr}\omega)\|_{\infty,[0,r]}\Big]\|\Phi(\cdot,\theta_{kr}\omega)\|_{\infty,\beta,[0,r]}\right)=0$$

for all ω in a set Ω' with probability 1. Combining this with (2.40), $\left\|y_{(k+1)r}\right\|_{\beta,[-r,0]}\leq e^{-\lambda k/4}$ for k large enough. Now for $t\in[kr,(k+1)r]$

$$\|y_t\|_{\beta,[-r,0]}\leq\|y_{kr}\|_{\beta,[-r,0]}+\left\|y_{(k+1)r}\right\|_{\beta,[-r,0]}\leq 2e^{-\lambda(k-1)/4}.$$

Therefore

$$\limsup_{t\to\infty}\frac{\log\|y_t\|_{\beta,[-r,0]}}{t}<-\lambda/4 \tag{2.41}$$

From (2.38) and (2.41) we obtain (2.24). The proof is completed. □

In the second main result, we prove that the trivial solution of (1.1) is also exponentially stable almost surely but in the pullback sense.

Theorem 2.6 *Under the assumptions of Theorem 2.5, there exists $\varepsilon>0$ such that if $\|C\|<\varepsilon$ the trivial solution of system* (1.1) *is pullback exponentially stable almost surely with respect to the $\|\cdot\|_{\infty,\beta,[-r,0]}$ norm, i.e.*

$$\limsup_{t\to\infty}\frac{1}{t}\log\|y_t(\theta_{-t}\omega,y_0)\|_{\infty,\beta,[-r,0]}<0\,,$$

for almost surely all the realizations ω.

Proof: Fristly, we write $y(t,\omega,\eta)$ and $\rho^k(t,\omega,\eta)$ to stress the dependence of y, ρ^k in Theorem 2.5 on ω and η. The proof of this theorem follows the arguments of Theorem 2.5 with some modifications. Namely, fix $u\in[kr,(k+1)r]$ and replace ω in (2.35) by $\theta_{-u}\omega$, we obtain

$$\begin{aligned}
\rho^k(r,\theta_{-u}\omega,\eta) &\leq \rho^0(r,\theta_{-u}\omega,\eta)\exp\Big\{\Big[-(h_0-2\gamma)r+\frac{2\|C\|r}{k}\sum_{j=1}^{k}Q(r,\theta_{jr-u}\omega)+2KG\|C\|\Big]k\Big\} \\
&\leq \rho^0(r,\theta_{-u}\omega,\eta)\exp\Big\{\Big[-(h_0-2\gamma)r+\frac{2\|C\|r}{k}\sum_{j=0}^{k-1}P(r,\theta_{-jr}\omega)+2KG\|C\|\Big]k\Big\},
\end{aligned}$$

where

$$P(r,\omega) := 2KG\Big[\frac{\nu(1+2KG)}{2KG}\,|||\omega|||_{\nu,[-r,r]}^{\frac{1}{\nu}} + r^{\nu+\frac{1}{\nu}}\,|||\omega|||_{\nu,[-r,r]} + r^{2\nu}\,|||\omega|||_{\nu,[-r,r]}^{2+\frac{1}{\nu}}\Big].$$

Put

$$\epsilon' = \min\Big\{\epsilon_1, \frac{(h_0-2\gamma)r}{2KG+2rE[P(r,\omega)]}\Big\}$$

and consider $\|C\| < \epsilon'$ then for almost all ω

$$\begin{aligned}
&\lim_{k\to\infty}\Big\{-(h_0-2\gamma)r + 2KG\|C\| + \frac{2\|C\|r}{k}\sum_{j=0}^{k-1}P(r,\theta_{-jr}\omega)\Big\} \\
= \quad & -(h_0-2\gamma)r + 2KG\|C\| + 2\|C\|rE[P(r,\omega)] =: -\lambda' < 0. \qquad (2.42)
\end{aligned}$$

Proof similarly to Theorem 2.5, it is easy to see

$$\rho^0(r,\theta_{-u}\omega,\eta) \le \|\eta\|_{\infty,[-r,0]}\exp\Big\{2\gamma r + 2\|C\|rP(r,\theta_{-kr}\omega) + 2KG\|C\|\Big\}.$$

It follows that when k is large enough,

$$\Big[-(h_0-2\gamma)r + 2KG\|C\| + 2\|C\|r\frac{1}{k}\sum_{j=0}^{k-1}P(r,\theta_{-jr}\omega)\Big]k < \frac{-\lambda' k}{2}, \qquad (2.43)$$

and

$$\rho^0(r,\theta_{-u}\omega,\eta) \le e^{\frac{\lambda' k}{4}}.$$

As a result

$$\rho^k(r,\theta_{-u}\omega,\eta) \le \rho^0(r,\theta_{-u}\omega,\eta)e^{\frac{-\lambda' k}{2}} \le e^{\frac{-\lambda' k}{4}}.$$

On the other hand, it is obvious that, for any $0 \le t \le r$

$$\begin{aligned}
\|y(kr+t,\theta_{-u}\omega,\eta)\| &\le \rho^k(t,\theta_{-u}\omega,\eta) \\
&\le e^{h_0 r}\rho^k(r,\theta_{-u}\omega,\eta) \\
&\le e^{-\frac{\lambda' k}{2}+h_0 r}.
\end{aligned}$$

Also, one can choose k is greater than $n(\omega)$ large enough so that

$$\|y(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[(k-2)r,(k+1)r]} \le e^{-\frac{\lambda'(k-2)}{2}+h_0 r}. \qquad (2.44)$$

Next, in the analog of (1.6), for $u \in [kr,(k+1)r]$ and $u' = u - r$

$$y(u'+t,\omega,\eta) \;=\; \Phi(t,\theta_{u'}\omega)y(u',\omega,\eta) + \int_0^t \Phi(t-s,\theta_{s+u'}\omega)f(y(u'+s-r,\omega,\eta))ds.$$

Therefore, for $k \ge n(\omega)$ and $u \in [kr,(k+1)r]$, we have

$$y(u'+t,\theta_{-u}\omega,\eta) \;=\; \Phi(t,\theta_{-r}\omega)y(u',\theta_{-u}\omega,\eta) + \int_0^t \Phi(t-s,\theta_{s-r}\omega)f(y(u'+s-r,\theta_{-u}\omega,\eta))ds.$$

For any $0 \leq t_1 < t_2 \leq r$, by similar computations to (2.39), we obtain

$$
\begin{aligned}
& \|y(u'+t_2,\theta_{-u}\omega,\eta) - y(u'+t_1,\theta_{-u}\omega,\eta)\| \\
\leq\ & \Big\|\Big(\Phi(t_2,\theta_{-r}\omega) - \Phi(t_1,\theta_{-r}\omega)\Big) y(u',\theta_{-u}\omega,\eta)\Big\| \\
& + \Big\| \int_{t_1}^{t_2} \Phi(t_2-s,\theta_{s-r}\omega) f(y(u'+s-r,\theta_{-u}\omega,\eta)) ds \Big\| \\
& + \Big\| \int_0^{t_1} \Big(\Phi(t_2-s,\theta_{s-r}\omega) - \Phi(t_1-s,\theta_{s-r}\omega)\Big) f(y(u'+s-r,\theta_{-u}\omega,\eta)) ds \Big\| \\
\leq\ & \|\Phi(t_2,\theta_{-r}\omega) - \Phi(t_1,\theta_{-r}\omega)\| \|y(u',\theta_{-u}\omega,\eta)\| \\
& + C_f \int_{t_1}^{t_2} \|\Phi(t_2-s,\theta_{s-r}\omega)\| \|y(u'+s-r,\theta_{-u}\omega,\eta)\| ds \\
& + C_f \int_0^{t_1} \|\Phi(t_2-s,\theta_{s-r}\omega) - \Phi(t_1-s,\theta_{s-r}\omega)\| \|y(u'+s-r,\theta_{-u}\omega,\eta)\| ds \\
\leq\ & (t_2-t_1)^\beta \,|\!|\!| \Phi(\cdot,\theta_{-r}\omega) |\!|\!|_{\beta,[0,r]} \|y_u(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]} \\
& + (t_2-t_1) C_f \|y_{u-r}(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \|\Phi(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \\
& + C_f \|y_{u-r}(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]} \int_0^{t_1} \Big\| \Big(\Phi(t_2,\theta_{-r}\omega)) - \Phi(t_1,\theta_{-r}\omega))\Big) \Phi^{-1}(s,\theta_{-r}\omega) \Big\| ds \\
\leq\ & (t_2-t_1)^\beta \,|\!|\!| \Phi(\cdot,\theta_{-r}\omega) |\!|\!|_{\beta,[0,r]} \|y_u(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]} \\
& + (t_2-t_1) C_f \|y_{u-r}(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \|\Phi(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \\
& + (t_2-t_1)^\beta r C_f \|y_{u-r}(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]} \|\Phi^{-1}(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \,|\!|\!| \Phi(\cdot,\theta_{-r}\omega) |\!|\!|_{\beta,[0,r]}.
\end{aligned}
$$

Due to (2.44) it follows that

$$
\begin{aligned}
\|y_u(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,\beta,[-r,0]} &= \|y_u(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,[-r,0]} + |\!|\!| y_u(\cdot,\theta_{-u}\omega,\eta) |\!|\!|_{\beta,[-r,0]} \\
&\leq e^{-\frac{\lambda'(k-2)}{2} + h_0 r} \times \Big[1 + |\!|\!| \Phi(\cdot,\theta_{-r}\omega) |\!|\!|_{\beta,[0,r]} \\
&\quad + r^{1-\beta} C_f \|\Phi^{-1}(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \|\Phi(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \\
&\quad + r C_f \|\Phi^{-1}(\cdot,\theta_{-r}\omega)\|_{\infty,[0,r]} \,|\!|\!| \Phi(\cdot,\theta_{-r}\omega) |\!|\!|_{\beta,[0,r]} \Big].
\end{aligned}
$$

Consequently,

$$
\lim_{u\to+\infty} \frac{1}{u} \log \|y_u(\cdot,\theta_{-u}\omega,\eta)\|_{\infty,\beta,[-r,0]} = -\frac{\lambda'}{2} < 0.
$$

That means the trivial solution of system (1.1) is pullback exponentially stable almost surely. □

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