# Brill-Noether conjecture on cactus graphs

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**Abstract**

The divisor theory for graph was introduced by Baker and Norine in a study of the interaction between algebraic curves theory and graph theory. Baker then formulated the Brill-Noether conjecture for graph on the existence of a divisor whose degree and rank satisfies a certain condition. Since then, this conjecture has attracted many researchers and it has been proved for some special classes of graphs. We prove the validity of this conjecture for cactus graphs. Our proof, based on the Chip Firing Game theory, explicitly construct the divisor mentioned in the conjecture.

**Keywords:** Brill – Noether conjecture, cactus graph, chip firing game, cycle, rank of divisor on graph.

## 1 Introduction

In 2007, Baker and Norine developed research on the interplay between Riemann surfaces and graphs by introducing the concept of divisor on graph and proving the discrete version of the Riemann-Roch theorem on the rank of divisor [3]. Then Baker formulated the Brill-Noether theorem on algebraic

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curves into a conjecture for graphs [2]. This conjecture is still open, and attracts many researchers [1, 6, 7, 8, 13, 16]. Before presenting these theorem and conjecture, we will give an explicit definition of the rank of divisor on graph.

In this whole paper, all considered graphs are (multiple) undirected connected graphs without loops. Let $G = (V, E)$ be a graph. We always denote by $n$ the number of vertices and by $m$ the number of edges of $G$. The *genus* of $G$ is the quantity $g = m - n + 1$. For a subset $U$ of $V$, we denote by $G(U)$ the subgraph of $G$ induced by $U$. The number of edges between $u$ and $v$ is denoted by $e(u, v)$.

A *divisors* $D$ on $G$ is a function $D : V \to \mathbb{Z}$ (or can be considered as a vector $D \in \mathbb{Z}^V$). The degree of $D$ is $deg(D) = \sum_{v \in V} D(v)$. A divisor $D$ is called *effective*, and written $D \geq 0$, if $D(v) \geq 0$ for all $v \in V$ (by convention, the vector zero is written 0). The *group of divisors* of $G$ - denoted by $Div(G)$ - is the set of all divisors on $G$, and it is a free abelian group on $V$ with respect to the pointwise addition. The *basic vector* $\epsilon_v$ is defined by: $\epsilon_v(v) = 1$ and $\epsilon_v(u) = 0$ for all $u \neq v$.

The *Laplacian matrix* $\Delta_G$ of graph $G$, where the coordinates are indexed by $V \times V$, is defined by:

$$\Delta_G(u, v) = \begin{cases} deg(u) & \text{if } u = v, \\ -e(u, v) & \text{if } u \neq v. \end{cases}$$

We write $\Delta_G(v)$ the row vector indexed by vertex $v$ of the graph. Note that $\sum_{v \in V} \Delta_G(v) = 0$.

The *linear equivalence* is a relation on $Div(G)$ defined by: $D \sim E$ if there exists $x \in \mathbb{Z}^V$ such that $E = D - x\Delta_G$.

If $D$ is linearly equivalent with an effective divisor $E$, we say that $D$ is *L-effective*. It is clear that the $L$-effectiveness is an invariance of linear equivalence classes. For a divisor $D$, the *linear system associated to* $D$ is the set $|D|$ of all effective divisors linearly equivalent to $D$:

$$|D| = \{E \in Div(G) : E \geq 0, E \sim D\}.$$

Formally, the definition of the rank of divisor can be written as follows.

**Definition 1.** [3] For a divisor $D \in Div(G)$, the *rank* of $D$, denoted by $r(D)$, is equal to

- $-1$ if $D$ is not $L$-effective,

- the largest integer $r$ such that for any effective divisor $\lambda$ of degree $r$ the divisor $D - \lambda$ is $L$-effective.

In some cases when there is different graphs, to precise the graph $G$, one write $D_G$ instead of $D$, $r_G$ instead of $r$ and $L_G$-effective instead of $L$-effective.

The problem of divisor on graph can also be considered in the context of the Chip Firing Game (CFG) theory [4, 5, 11]. For instance, (in a general sense) in the CFG on a given graph $G$, a configuration is a distribution of chips on vertices, and the firing of a vertex $v$ consists of moving one chip from $v$ along each of its edges. Each divisor $D$ can be considered as a configuration of $CFG$ in which $D(v)$ is the number (possibly negative) of chips at $v$. And the substraction of $\Delta_v$ from $D$ corresponds to the firing of the vertex $v$ on $D$. Similarly, the linear equivalence between two divisors $D$ and $E$ corresponds to the existence of a firing sequence from $D$ to $E$.

Baker and Norine proved the following Riemann Roch theorem for graph (see [3] and also [9] for a proof of this theorem).

**Theorem 1.** *[3] Let $G$ be a graph. Let $\kappa$ be the divisor such that $\kappa(v) = deg(v) - 2$ for all $v \in V$. Then any divisor $D$ satisfies:*

$$r(D) - r(\kappa - D) = deg(D) - g + 1,$$

*where $g$ being the genus of $G$.*

And the the Brill-Noether conjecture on graphs can be stated as follows.

**The Brill–Noether conjecture for graphs** [2] Fix integers $g, r, d \geq 0$, and set $\rho(g, r, d) = g - (r + 1)(g - d + r)$. Then

- If $\rho(g, r, d) \geq 0$ then every graph of genus $g$ has a divisor $D$ with the rank $r(D) = r$ and $deg(D) \leq d$.
- If $\rho(g, r, d) < 0$ then there exists a graph of genus $g$ for which there is no divisor with $r(D) = r$ and $deg(D) \leq d$.

The second part of this conjecture has been proved to be correct in [2] while the first part is still open so far. Even in a special case when $r = 1$, the problem is also unresolved, and considered separately under the name "Gonality conjecture".

**The Gonality conjecture for graphs** [2] For any graph $G$ of genus $g$, there exists a divisor $D$ of degree $\lfloor (g + 3)/2 \rfloor$ such that the rank of $D$ is at least 1.

In recent years, these two conjectures were considered for certain classes of graphs. The Brill-Noether conjecture has been proved for graphs with genus at most five [1]. On the other direction, this conjecture holds for metric graphs and tropical graphs [2, 7]. Some ideas to decompose graph for computing the rank of divisors on a graph from that of its subgraphs was showed in [13, 16].

In this paper, we investigate this problem on the class of cactus graphs - a connected graph in which any two simple cycles have at most one vertex in common. Cactus graph was introduced in the 1950's [12] and can be used to represent models on different research domains [17, 14, 15]. Especially, chain of loops (a special case of cactus graph) was used to study a tropical proof of the Brill Noether theorem on curves [8]. With its treelike structure, cactus graph is a special case of sparse graph on which several NP-hardness problem on general graphs can be solved in polynomial time [10, 18].

We will prove that these two conjectures are correct for the class of cactus graphs. Our idea is to construct a divisor satisfying the condition of each conjecture. The construction is a recursive procedure on the genus of the cactus graph in consideration.

# 2 Transferring chips and Gonality conjecture

We first examine the structure of cactus graphs and introduce the notion of transferring chips. These results will helps us to prove the Gonality conjecture for cactus graphs.

## 2.1 Cactus graph and tree of cycles

Let us first give the formal definition of cactus graph.

**Definition 2.** A cactus graph is a connected graph in which any two simple cycles have at most one vertex in common.

It is evident that the number of cycles of a cactus graph is equal to its genus.

We say that a graph $G'$ is equivalent to a graph $G$ if the problem of rank of divisor on $G'$ is equivalent to that on $G$, which means that the linearly equivalent classes of divisors on $G'$ are in a bijection with that on $G$ and this bijection reserves the firing operation.

First, we will show that we only need to consider cactus graphs in which there is no common vertices of cycles.

To do that, for a cactus graph $G$ with a vertex $v$ belonging to $k$ cycles $C_1, C_2, \dots, C_k$, we will construct a graph $G'$ from $G$ as follows (see Figure 1). Creat a new vertex $u$, separate the cycles $C_i$ and replace the vertex $v$ in each cycle $C_i$ by a new vertex $u_i$, then connect $u_i$ with $u$. By this way, each edge $u_i u$ is a cut of $G'$ into $X_i$ containing $C_i$ and $Y_i = V(G') \backslash X_i$. Consider a map $\phi$ from $Div(G')$ to $Div(G)$ which maps a divisor $D'$ on $G'$ to a divisor $D$ on $G$ as follows: $D(x) = D'(x)$ for all $x \neq v$ and $d(v) = \sum_{x \in \{u_1, \dots, u_k, u\}} D'(x)$. It is easy to check that $D' \sim E'$ in $Div(G')$ if and only if $\phi(D') \sim \phi(E')$ in $Div(G)$. On the other hand, firing $v$ in $G$ is equivalent to firing $\{u_1, \dots, u_k, u\}$ in $G$; firing $u_i$ in $G'$ is equivalent to firing $Y_i \cup \{u_i\}$; finally firing $u$ in $G'$ does not change the corresponding divisor in $G$.

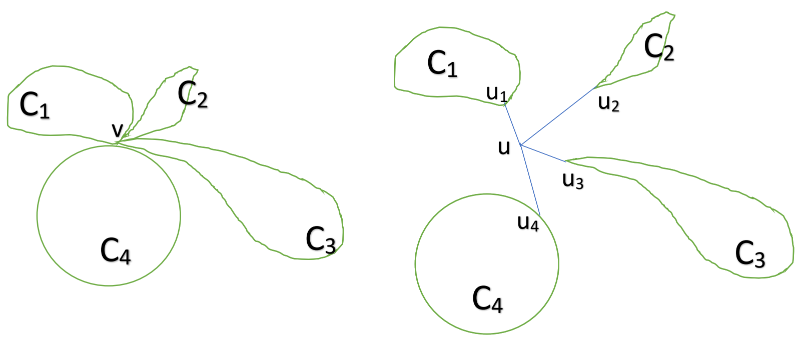


Figure 1: Two equivalent graphs

After applying all the above operations if needed, from a cactus graph, we obtain an equivalent cactus graph where there is no common vertices of cycles. Hence from now on, we consider only cactus graph with no common vertices of cycles.

Now we define the *cycle contraction* of cactus graph $G$ the graph obtained from $G$ by contracting each cycle to one of its vertices (this vertex is called *the representing vertex of the cycle*). It is clear that this contraction is a tree $T$, called *the representing tree of the graph.* If a cycle (resp. vertex) of $G$ is represented by a leaf in $T$, we call it a cycle leaf (resp. vertex leaf).

By the way, we call a *path of cycles* (resp. *star of cycles*) a cactus graph such that its representing tree is a path (resp. star).

## 2.2 Rank on trees and cycles

We apply the Riemann-Roch theorem for computing the rank of divisor on trees and cycles.

In a tree, $m = n-1, g = 0$, and $deg(\kappa) = 2(m-n) = -2$. So for every divisor $D$ of degree non negative, we have $deg(\kappa - D) < 0$, and $r(\kappa - D) = -1$, which implies that $r(D) = deg(D)$.

In a cycle of $n$ vertices $C_n = \{v_1, \ldots, v_n\}$, $m = n, g = 1$ then $\deg(\kappa) = 0$. So for every divisor $D$ of positive degree, we have $deg(\kappa - D) < 0$, and $r(\kappa - D) = -1$, which implies that $r(D) = deg(D) - 1$. In the case $deg(D) = 0$, we have $r(D) = 0$ if $D \sim 0$ (that means $D$ is $L$-effective), otherwise $r(D) = -1$. We analyze this case below.

On the cycle $C_n$, we can write a divisor $D$ as a vector $D = (D_1, D_2, \ldots, D_n)$. We have $D \sim 0$ if and only if there exists $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ such that $D - x\Delta_{C_n} = 0$. Because $\sum_{i=1}^{i=n} \Delta_{C_n}(v_i) = 0$ then

$$D \sim 0 \Leftrightarrow \exists x = (x_1, x_2, \ldots, x_{n-1}, 0) \in \mathbb{Z}^{n-1} \times \{0\} : D - x\Delta_{C_n} = 0.$$

$$\Leftrightarrow \exists x : (D_1, D_2, \ldots, D_n) = (x_1, \ldots x_{n-1}, 0) \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 & -1 \\ -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ \ldots & & & & & & \\ 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ -1 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix}$$

$$\Leftrightarrow D_1 + 2D_2 + \ldots + (n-2)D_{n-2} + (n-1)D_{n-1} \equiv 0 \mod n.$$

So we have the following result.

**Proposition 2.** *Let $D = (D_1, D_2, \ldots, D_n)$ be a divisor on the cycle $C_n$, then the rank of $D$ is computed as follows.*

$$r(D) = \begin{cases} -1 & \text{if } deg(D) \leq -1, \\ deg(D) - 1 & \text{if } deg(D) \geq 1, \\ 0 & \text{if } deg(D) = 0 \text{ and } D_1 + 2D_2 + \ldots + (n-1)D_{n-1} \equiv 0 \mod n, \\ -1 & \text{if } deg(D) = 0 \text{ and otherwise }. \end{cases}$$

The following result is straightforward.

**Corollary 3.** *If the divisor $D$ on the cycle $C_n$ (with $n \geq 3$) has degree 0 and rank 0, then for all $v \neq w$ in $C_n$, $r(D - \epsilon_v + \epsilon_w) = -1$.*

## 2.3 Transferring chips

For computing the rank of divisors, it is important to check if a divisor is $L$-effective or not. To do so, we need to consider whether we can transfer chips in that divisor from vertices with multiple chips to vertices with negative numbers of chips. So we will come up with the notion of transferring chips.

**Definition 3.** We say that in an effective divisor $D$, one can transfert one chip from a vertex $v$ to a vertex $u$ if there exists a firing sequence from $D$ to an effective divisor $D'$ (in other words, if there exists an effective divisor $D' \sim D$) with $D'(v) = D(v) - 1$ and $D'(u) = D(u) + 1$.

More generally, we say that in $D$ one can transfert $k$ chips from $v$ to get $l$ chips on $u$ if there exists an effective divisor $D' \sim D$ with $D'(v) = D(v) - k$ and $D'(u) = D(u) + l$.

Independently of divisors, we say that one can transfert $k$ chips from $v$ to get $l$ chips on $u$ on a graph $G$ if the divisor $D$ defined in $G$ by $D(v) = k, D(u) = -l$ and $D(w) = 0$ for all $w \neq v, u$, is $L$-effective ($D$ is nothing but $k\epsilon_v - l\epsilon_u$).

This notion is very important for the Gonality conjecture (and the Brill-Noerther conjecture also) because an effective divisor $D$ is good for the gonality conjecture if and only if in $D$ one can transfert a chip to every vertex $u$ with $D(u) = 0$.

Before considering the transferring of chips on graphs with simple structure, we note that firing a set $A$ of vertices corresponds to moving one chip along each outgoing edge of $A$.

- If $v$ and $u$ are the two extremities of a cut edge of a graph $G = (S, T)$ ($v \in S, u \in T$), then firing $S$ correponds to transferring one chip from $v$ to $u$. Hence for any positive integer $k$, one can trasfert $k$ chips from $v$ to get $k$ chips on $u$.

  More generally, if there is a path from $v$ to $u$ such that every edge of this path is a cut edge, then one can trasfert $k$ chips from $v$ to $u$. (We call such a path *a simple path.*)

- If $v$ and $u$ are the two vertices of a cycle graph $C$. The divisor $D = k\epsilon_v - k\epsilon_u$ can have rank 0 or $-1$ depending on the position of $v$ and $u$. That means the transferring of $k$ chips from $v$ to get $k$ chips in $u$ is not guaranteed.

However the divisor $D = k\epsilon_v - (k-1)\epsilon_u$ has degree 1, then has rank 0. Hence the transferring of $k$ chips from $v$ to get $k-1$ chips in $u$ is guaranteed.

So we say that the transferring chips from $v$ to $u$ "loses" one chip on the cycle.

- Now, we consider a cycle $C$ of $G$ in which two vertices $v$ and $u$ may have degree greater than two and all other vertices of $C$ have degree two. The vertex set of $G$ consists of three parts $S$, $C$ and $T$ (eventually empty) where $v$ connects $C$ to $S$ and $u$ connects $C$ to $T$. The firing of $S \cup \{v\}$ (resp. $T \cup \{u\}$) on $G$ corresponds to the firing of $v$ on $C$ (resp. $u$ on $C$). Then a transferring chips from $v$ and $u$ in $G$ can be considered as an internal firing sequence on $C$, and then it loses one chip on $C$.

- If $v$ and $u$ be two extremities of a path of $k$ cycles, then to transfer chips in the path from $v$ to $u$, one loses one chip to go through each cycle.

  We can state the following lemma.

  **Lemma 4.** *If there is a path of $p$ cycles from $v$ to $u$ then one can transfert $l + p$ chips from $v$ to get $l$ chips on $u$.*

Furthermore, a first simple result on cactus graph is the following.

**Lemma 5.** *Every divisor of degree $g$ on a cactus graph $G$ (of genus $g$) are L-effective.*

*By consequence, every divisor of degree $g + r$ on $G$ has a rank at least $r$.*

*Proof.* Indeed, if on a cycle of $G$ there is at least 2 chips then this cycle can transfer chips to other cycles by keeping only 1 chip for itself. Then by transferring chips from cycle having more than one chips to cycles of less than one chips, we can distribute one chip on each cycle and zero chip every elsewhere, which give us an effective divisor. □

## 2.4 Gonality conjecture on cactus graph

**Theorem 6.** *The Gonality conjecture is true for cactus graph.*

*Proof.* Let $G$ be a cactus graph of genus $g$. We will construct a divisor of degree $\lfloor \frac{g+3}{2} \rfloor$ and of rank at least 1 on $G$.

Because a cactus is a tree of cycles and because any tree has one center or two centers (a center of a graph is a vertex $O$ where the greatest distance $d(O, v)$ to other vertices $v$ is minimal), then $G$ has a cycle $O$ such that the path of cycles from $O$ to any cycle of $G$ contains at most $\lfloor \frac{g+1}{2} \rfloor$ cycles. Let consider a vertex $\omega \in O$. We defince the divisor $D$ as follows: $D$ has $\lfloor \frac{g+3}{2} \rfloor$ chips at $\omega$ and zero chips elsewhere. By Lemma 4, one can transfert $\lfloor \frac{g+3}{2} \rfloor$ from $\omega$ to get at least $\lfloor \frac{g+3}{2} \rfloor - \lfloor \frac{g+1}{2} \rfloor = 1$ chip on $v$ for every vertex $v$ of $G$. This implies that the rank of $D$ is at least 1. Hence the conjecture is correct. □

Moreover, from the above proof, we can compute the gonality of a cactus graph. Recall that the gonality of a graph $G$, denoted by $gon(G)$, is the minimum degree of a divisor of rank 1. For that, we use the notion of radius of a graph - the minimum of eccentricities of all vertices of $G$ - where eccentricity of a vertex $v$ is the maximum of distance from $v$ to other vertices.

**Corollary 7.** *Let $G$ be a cactus graph. Let $T$ be the representing tree of $G$. Then the gonality of $G$ is equal to the radius of $T$ plus* 2.

# 3 Brill-Noether conjecture on cactus graph

**Theorem 8.** *The Brill-Noether conjecture is true for cactus graph.*

The Brill-Noether number $\rho(g, r, d) = g - (r+1)(g-d+r)$ is non negative if and only if $d \geq g + r - \frac{g}{r+1}$. We now fix $d = g + r - \lfloor \frac{g}{r+1} \rfloor$. Then to prove the Brill-Noether conjecture, it suffices to find a divisor of degree $d$ and of rank at least $r$.

We will first analyze these numbers.

Put $p = \lfloor \frac{g}{r+1} \rfloor$, we write $g = p(r+1) + t$ with $t = g \mod (r+1)$.

Then $d = g + r - \lfloor \frac{g}{r+1} \rfloor = p(r+1) + t + r - p = (p+1)r + t$.

**Definition 4.** In this Section we fix the cactus graph $G$ with genus $g$. Fix a positive number $r$. We define $p = \lfloor \frac{g}{r+1} \rfloor$, $d = g + r - p$, and $t = g - p(r+1)$.

Let $H$ be a connected subgraph of $G$ (and then $H$ is a cactus graph). We denote $g_H$ the genus of $H$. We define $t_H = g_H \mod p$ and $r_H = \lfloor \frac{g_H}{p} \rfloor - 1$ then $g_H = (r_H + 1)p + t_H$.

We define $d_H = g_H + r_H - p$.

A divisor $D_H$ on $H$ is called a *good divisor* on $H$ if $deg(D_H) = d_H$ and $r(D_H) \geq r_H$.

Our purpose is to construct a good divisor for cactus graphs.

However, note that for the original graph $G$, evan $g_G = g$ but $r_G$ may not be equal to $r$ and $d_G$ may not be equal to $d$, because by definition $t = g \mod (r+1)$ but $t_G = g_G \mod p$. Then the good divisor $D_G$ on $G$ (of degree $d_G$ and of rank at least $r_G$) may not be a divisor $D$ satisfying the Brill-Noether conjecture (of degree $d$ and of rank at least $r$). We can adjust this difference as follows.

If $t < p$ then $t_G = t$ and $r_G = r$, $d_G = d$, hence $D_G$ is also the desired divisor $D$ on $G$.

If $t \geq p$ then $t_G < t$, and $r_G > r, d_G > d$. We know that $d = g + r - p$ and $d_G = g + r_G - p$, then $d_G - d = r_G - r$. Now $D_G$ is a divisor of rank $r_G$, then if we define a divisor $D$ from $D_G$ by subtracting $r_G - r$ chips (in some vertices), the rank of $D$ decreases by at most $r_G - r$, that implies $r(D) \geq r_G - (r_G - r) = r$. Moreover the degree of $D$ is equal to $d_G - (r_G - r) = d$.

So to find a divisor $D$ satisfying the Brill-Noether conjecture on $G$, it is sufficient to find a good divisor $D_G$ on $G$.

We know that the cactus graph $G$ is a tree of cycles, then to find a good divisor for $G$, we begin by finding good divisor for simpler structure like path of cycles or star of cycles.

Our purpose is to prove the following result which is useful for the recursive construction of good divisor.

**Let $T$ be a tree of cycles rooted at $v$ and with subtrees $T_1, T_2, \ldots T_l$. If for each subtree $T_i$, there is a good divisor, then there is a good divisor for $T$.**

For that, from a good divisor $D_{T_i}$ on $T_i$, we will construct a symmetric divisor $S(T_i, v)$ and an asymmetric divisor $A(T_i, v)$ on $T_i$ oriented to $v$. Then by using these divisors on all $T_i$, we will construct a good divisor $D_T$ for $T$.

First of all, we consider a path of cycle.

**Lemma 9.** *Let $P$ be a path of cycles which is an induced subgraph of $G$. Then there exists a good divisor on $P$.*

*Proof.* Let $P$ be a path of $g_P$ cycles: $C_1, C_2, \ldots C_{g_P}$. On each $C_i$ there is two vertices $x_i, y_i$ - possibly the same - of degree greater than 2 such that for $1 \leq i \leq p-1$, $C_i$ and $C_{i+1}$ are connected by a simple path from $y_i$ to $x_{i+1}$.

First, we write $g_P = (r_P + 1)p + t_P$ with $0 \leq t_P \leq p - 1$. We will find a divisor $D_P$ of degree $d_P = g_P + r_P - p$ and of rank at least $r_P$ (defined as in Definition 4).

We call *node* the $r_P + 1$ vertices : $y_p, y_{2p}, \ldots, y_{r_P p}, y_{(r_P+1)p}$, and specially *keynode* the vertex $y_{(r_P+1)p}$.

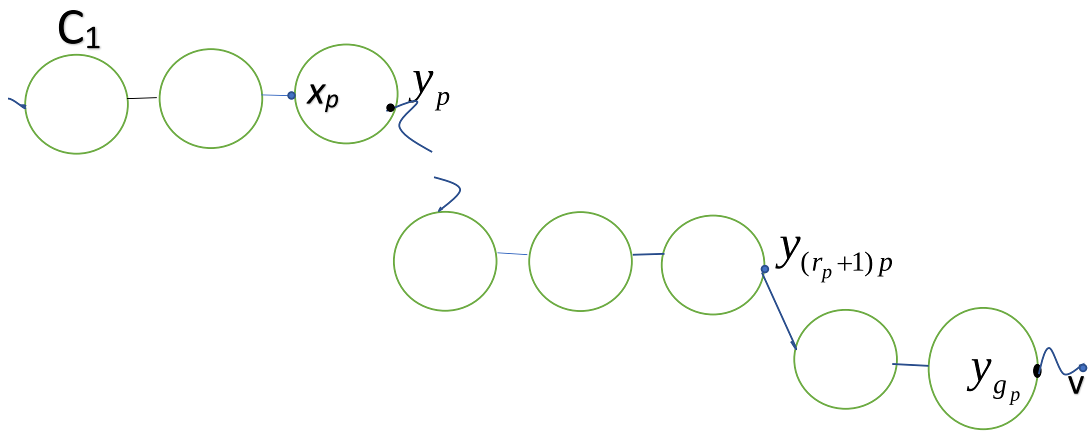


Figure 2: Path of cycles and its nodes

The divisor $D_P$ is defined as follows: put $t_P$ chips at the keynode and put $p+1$ chips at each other node. It is clear that $deg(D_P) = d_P$. We will prove that $D_P - \lambda$ is $L$-effective for any effective divisor of degree $r_P$ $\lambda$. Because we have $r_P + 1$ nodes, then there exists an index $0 \leq j \leq r_P$ such that $\lambda$ has exactly $j$ chips on the interval $I_1 = [x_1, y_{jp}]$, $r_P - j$ chips on the interval $I_3 = [x_{(j+1)p+1}, y_{(r_P+1)p+t}]$ and zero chips on the interval $I_2 = [x_{jp+1}, y_{(j+1)p}]$.

The divisor $D_P$ is equivalent to $D'_P$ obtained from $D_P$ by transfering $p+1$ chips from $y_{(j+1)p}$ to $x_{(j+1)p+1}$. Now, the divisor $D'_P$ has $g_1 + j$ chip on $I_1$, zero chips in $I_2$ and $g_3 + r_P - j$ chips in $I_3$ (where $g_1$ and $g_3$ are the genus of $I_1$ and $I_3$ respectively). Then the divisor $D_P - \lambda$ has $g_1$ chips on $I_1$, $g_3$ chips on $I_3$ and 0 chips on each vertex of $I_2$, then it is $L$-effective on $I_1$, on $I_2$ and on $I_3$, hence $D_P - \lambda$ is $L$-effective.

This implies that $D_P$ is of rank at least $r_P$, hence a good divisor on $P$. □

Keeping the notations of the above lemma. Let $v$ be the extreme vertex of $P$ closed to the last cycle $C_{g_P}$. We call $(P, v)$ the *path of cycles oriented* *to* $v$. From the good divisor $D_P$ on $P$, we define the asymmetric divisor and the symmetric divisor on $(P, v)$.

**Definition 5.**
- The *asymmetric divisor oriented to* $v$ on $P$ $A(P, v)$ is nothing but $D_P$ (defined in the proof of Lemma 4).
- The *symmetric divisor oriented to* $v$ on $P$ $S(P, v)$ is obtained from $A(P, v)$ by putting $p+1$ (instead of $t_P$) chips at the keynode.

The following properties of these new divisors can be deduced directly from the proof of Lemma 4.

**Corollary 10.**
- *deg*$(A(P, v)) = d_P$ *and* $r(A(P, v)) \geq r_P$. *Consequently from* $A(P, v)$, *one can send* $r_P - s$ *chips to* $v$ *and* $s$ *chips to anywhere in* $P$, *for* $0 \leq s \leq r_P$.
- *deg*$(S(P, v)) = d_P + p + 1 - t_P$ *and* $r(S(P, v)) \geq r_P + 1$. *Moreover from* $S(P, v)$, *one can send* $p + 1 - t_P + r_P - s$ *chips to* $v$ *and* $s$ *chips to anywhere in* $P$, *for* $0 \leq s \leq r_P$.

We now consider a *star of cycles*.

**Lemma 11.** *Let* $S$ *be a star of cycles which is an induced subgraph of* $G$. *Then there exists a good divisor on* $S$.

*Proof.* The root of a star of cycles can be a vertex or a cycle.

**Let consider the first case: the root of $S$ is a vertex.**

Now let vertex $v$ be the root and $B_1, B_2, \ldots, B_l$ be branches of $S$. Each branch $B_i$ is a path of $g_i$ cycles. We define for each $B_i$ the nodes and keynode (as the proof of Lemma 4).

The genus of $S$ is $g_S = \sum_{i=1}^{l} g_i$. And we write $g_S = (r_S + 1)p + t_S$ with $t_S \leq p-1$. We will find a divisor $D_S$ of degree $d_S = g_S + r_S - p = r_S(p+1) + t_S$ and of rank at least $r_S$.

We first write $g_i = p(r_i+1) + t_i$ with $0 \leq t_i \leq p-1$. Put $r' = (r_S - \sum_{i=1}^{l} r_i)$ and $t' = (t_S - \sum_{i=1}^{l} t_i) = p(l - 1 - r')$.

The good divisor $D_S$ is constructed as follows. We define the restriction of $D_S$ on each branch $B_i$ to be the symmetric divisor on $B_i$ if $t' < 0$, and the asymmetric divisor on $B_i$ if $t' \geq 0$. At the end, we put the remain chips at $v$ (see Figure 3).

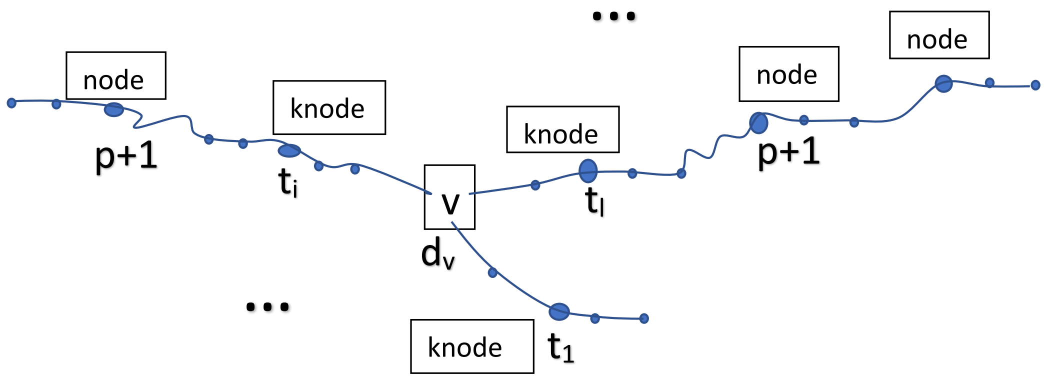


Figure 3: Star of cycles $S$ and the good divisor defined on it in the case $t' \geq 0$

To ensure that $D_S$ is a good divisor, we must prove that $D_S - \lambda$ is $L$-effecitve for any divisor $\lambda$ of degree $r_S$. Let $\lambda_i$ be the number of chips of $\lambda$ in branch $B_i$, and put $r'_i = \lambda_i - r_i$.

We consider two cases: $t' \geq 0$ or $t' < 0$.

- If $t' \geq 0$, then $r' \leq l - 1$. The remain chips to put at $v$ is

$$d(v) = (p+1)r_S + t_S - (\sum_{i=1}^{l} r_i(p+1) + t_i) = (l-1)p + r'.$$

  Because there are $l$ branches and $\sum_{i=1}^{l} r'_i = r' \leq l - 1$, then there is some index $j$ such that $r'_j \leq 0$. Applying Corollary 10, we can send $-r'_j$ chips from $B_j$ to $v$ and ensure that $D_S - \lambda$ is $L$-effective on $B_j$ (and the number of chips of $D_S - \lambda$ in $B_j$ is equal to $g_j - p$).

  Now, restrited on the graph $S \backslash B_j$, the number of chips of $D_S - \lambda$ is equal to the genus, then it is $L$-effective (see Lemma 5). This implies that $D_S - \lambda$ is $L$-effecitve.

- If $t' = p(l - 1 - r') < 0$. Because $t_i < p$ for each $i$, then $t' \geq -p(l-1)$, and $l \leq r' \leq 2l - 2$.

  The remain chips to put at $v$ is

$$d(v) = (p+1)r_S + t_S - \sum_{i=1}^{l}(r_i + 1)(p+1) = (p+1)(r' - l) + t_S.$$

Because there are $l$ branches and $\sum_{i=1}^{l} r'_i = r' \leq 2l-2$, then there is some index $j$ such that $r'_j \leq 1$.

If there is some $r'_j \leq 0$. Similar to the previous case, we can send $-r'_j + p + 1 - t_j$ chips from $B_j$ to $v$ and ensure that $D_S - \lambda$ is $L$-effective on $B_j$ (and the number of chips of $D_S - \lambda$ in $B_j$ is equal to $g_j - p$). And the result is the same as in the previous case.

If all the $r'_i \geq 1$ then there are some branches $j$ such that $r'_j = 1$; and we denote by $J$ the set of there indices $j$. We denote also $H$ the graph $\cup_{j\in J} B_j$. To show that $D_S$ is a good divisor, it is sufficient to prove that $D_S - \lambda$ on $H$ is $L$-effective and that $deg_H(D_S - \lambda) \leq g(H) - p$. In fact, for $B_j$ with $j \in J$, by Corrolary 10, the restriction of $D_S$ on $B_j$ has rank at least $r_j + 1 = \lambda_j$, so $D_S - \lambda$ is $L$-effective on $B_j$. Hence $D_S - \lambda$ is $L$-effective on $H$.

Now, let us compute:

$$deg_H(D_S - \lambda) - g_H = \sum_{j\in J}(p+1)(r_j+1) - \sum_{j\in J}(r_j+1) - \sum_{j\in J}(p(r_j+1)+t_j) = \sum_{j\in J} -t_j.$$

We must prove that $\sum_{j\in J} t_j \geq p$. For that, we analyze $g_S$ by two ways. First,

$$g_S = (r_S+1)p + t_S \geq \sum_i r_i p + \sum_{j\in J} r'_i p + \sum_{i\notin J} r'_i p + p$$

$$\geq \sum_i r_i p + |J|p + 2(l - |J|)p + p = \sum_i r_i p + p(2l - |J| + 1).$$

On the other side,

$$g_S = \sum_i g_i = \sum_i (r_i+1)p + \sum_{i\notin J} t_i + \sum_{j\in J} t_j$$

$$\leq \sum_i r_i p + lp + (l - |J|)p + \sum_{j\in J} t_j = \sum_i r_i p + (2l - |J|)p + \sum_{j\in J} t_j.$$

So, we have $\sum_{j\in J} t_j \geq p$, which implies that $deg_H(D_S - \lambda) \leq g(H) - p$. This completes our proof.

**Let us consider now the second case: the root of $S$ is a cycle $C$.**

Recall that $g_S = (r_S + 1)p + t_S$ and $d_S = r_S(p+1) + t_S$. We consider the star $S'$ obtained from $S$ by replace the cycle $C$ by a vertex $v$.

We analyze two subcases.

If $t_S \geq 1$, then $g_{S'} = g_S - 1$ and $d_{S'} = d_S - 1$. We construct a good divisor $D_{S'}$ on $S'$ as in the first case for a star with a vertex root. Then, we define $D_S$ the divisor obtained from $D_{S'}$ by putting one chip at the cycle $C$.

If $t_S = 0$, the divisor $D_S$ is defined as the good divisor on $S'$ in the case $t' < 0$, that means we put $p+1$ chips at each node of every branch (and the remain chips at $C$).

We let the reader check that $D_S$ is a good divisor on $S$.

□

We now define the symmetric divisor and the asymmetric divisor of a star oriented to a vertex.

**Definition 6.** Let $S$ be the star defined as in Lemma 11. Let $w$ be the extreme vertex of $B_l$ (on the opposite side of the root of $S$, which is not included in $B_l$). We call $(S, w)$ the star oriented to $w$.

We write $t_l = \tau_l + \omega_l$ such that $\omega_l = g \mod p$, or equivalently $\tau_l + \sum_{i=1}^{l-1} t_l$ is a multiple of $p$.

The branch $B_l$ is reorganized as follows. Write $B_l$ as a path from $v$ to $w$ of $g_l$ cycles: $C_1, C_2, \ldots C_{g_l}$.

We call the vertices $y_{\tau_l + ip}$, with $0 \leq i \leq r_l + 1$, nodes, and specially $y_{\tau_l}$ the "keynode", and $y_{\tau_l + (r_l+1)p}$ the *uppernode* of $B_l$.

Let $D_S$ be the good divisor defined in the Lemma 11.

- The asymmetric divisor (oriented to $w$) on $S$, denoted by $A(S, w)$, is defined by taking the good divisor $D_S$ with the modification on $B_l$ as follows:

  - put $\omega_l$ chips at the uppernode of $B_l$,

  - put $p + 1$ chips at each other node of $B_l$ (including the keynode),

  - at the end, put the remain chips at $v$.

- The symmetric divisor (oriented to $w$) on $S$, denoted by $S(S, w)$, is obtained from $A(S, w)$ by putting $p + 1$ (instead of $\omega_l$) chips at the uppernode of $B_l$.

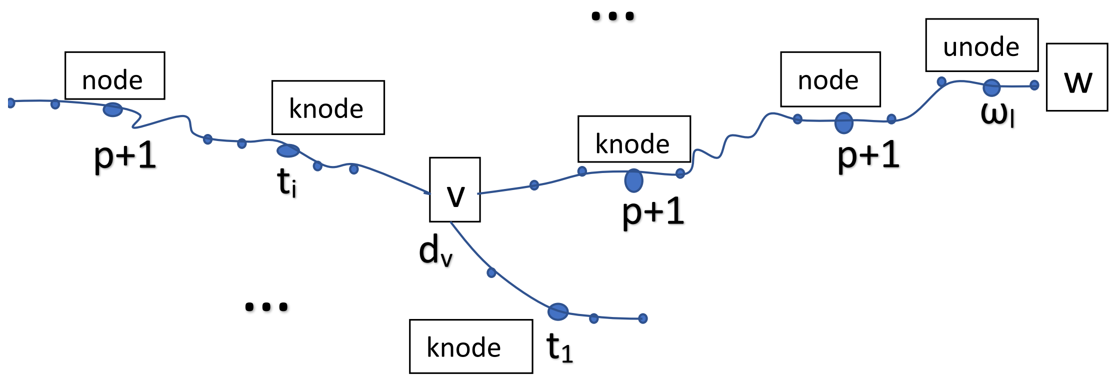


Figure 4: Asymmetric divisor of a star of cycles, in the case $t' \geq 0$

From the proof of the above lemma, we can state the following properties of the asymmetric and symmetric divisors on star.

**Corollary 12.**
- *For the asymmetric divisor $A(S, w)$: $deg(A(S, w)) = d_S$ and $r(A(S, w)) \geq r_S$. Consequently, for $0 \leq s \leq r_S$, one can send $r_S - s$ chips to $w$ and $s$ chips in anywhere in $S$.*
- *For the symmetric divisor $S(S, w)$: $deg(S(S, w)) = d_S + p + 1 - t_S$ and $r(S(S, w)) \geq r_S + 1$. Moreover, for $0 \leq s \leq r_S$, one can send $p + 1 - t + r_S - s$ chips to $w$ and $s$ chips in anywhere in $S$.*

Now, we have all ingredients to build a recursive process for determining the good divisor for a tree of cycles.

**Proposition 13.** *Let $T$ be a tree of cycles which is an induced subgraph of $G$. $T$ is rooted at $v$ and with subtrees $T_1, T_2, \ldots T_l$. If for each subtree $T_i$, we has a good divisor, then there exists a good divisor for $T$.*

*Proof.* We consider the case where the root of $T$ is a vertex, the other case can be proved similarly.

Similar to the above lemmas, we write $g_T$ the genus of $T$ and $g_i$ the genus of $T_i$. We use always the notations of $r, p, r_i, t_i, r', t'$.

For each subtree $T_i$, from the good divisor of $T_i$, we can construct the asymmetric divisor $A(T_i, v)$ and the symmetric divisor $S(T_i, v)$ on $T_i$ oriented to $v$.

We define the good divisor $D_T$ of $T$ as follows. If $t' < 0$, then $D_T$ restricted on each subtree $T_i$ is the symmetric divisor on $D_i$. Otherwise $t' \geq 0$, $D_T$ restricted on each branch $T_i$ is the asymmetric divisor on $T_i$. At the end, we put the remain chips at $v$.

The proof of the goodness of $D_T$ on the tree of cycles is similar to that on a star of cycles. □

Now, the above proposition give us a recursive method to construct the good divisor for the tree of cycles $G$ from the good divisors of its sub-trees. And then, because we have a good divisor for $G$, the Brill-Noerther conjecture holds.

**Discussion** We can prove the Brill-Noether conjecture for cactus graphs because these graphs have a structure of tree of cycles. We hope to apply some of our techniques for more general classes of graphs, for example graph with special ear decompositions, and in particular for series- parallel graphs.

On the other side, we think that by using an explicit analysis of the structure of the representing tree of a cactus graph, one can find not only its gonality, but furthermore a lower bound for the degree of a divisor of rank $r$.

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