ON A REGULARIZED FULL DISPERSION DAVEY-STEWARTSON SYSTEM.

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ABSTRACT. This paper which continues [24] is concerned with the Cauchy problem for the Full-Dispersion Davey-Stewartson system introduced in [18]. More precisely, in order to invert the equation for the mean mode ϕ we introduce an approximate system by a suitable truncation of the high frequencies of ϕ and we prove that the solutions to the truncated systems converge to the solutions of the classical Davey-Stewartson system.

1. INTRODUCTION

The Davey-Stewartson system was introduced in [9] (see also [2, 10]) to model the evolution of wave packets of surface water waves. It is actually derived for a more general system, the Benney-Roskes system, introduced in [3] in the context of water waves and also known as the Zakharov-Rubenchik system, introduced in [34] as a universal Hamiltonian system to describe the interaction of short and (acoustic type) long waves.

Our interest in the present paper is a full dispersion version of the classical Davey-Stewartson system introduced by David Lannes in [18] and that we describe now.

The framework is that of the motion of a irrotational inviscid fluid in an infinite layer of finite depth of an Euler fluid bounded by a free surface and a flat bottom.

The specific regime here is the so-called modulation ("Schrödinger") one. In order to define the key small parameter we introduce h a typical depth of the fluid layer, a a typical amplitude of the wave and λ a typical wavelength in the horizontal directions (assumed to be isotropic). Then we set $\epsilon = a/h$ and $\mu = h^2/\lambda^2$.

In the modulation regime, the relevant small parameter is the *wave steepness*, $\varepsilon = a/\lambda = \epsilon\sqrt{\mu}$. The starting point of the analysis is the water waves system coupling the wave elevation ζ and the trace ψ of the velocity potential at the free surface, written in dimensionless variables (see [18])

$$\begin{cases} \partial_t \zeta - \dfrac{1}{\sqrt{\mu}}\mathcal{G}\psi = 0, \\ \partial_t \psi + \zeta + \dfrac{\varepsilon}{2}|\nabla\psi|^2 - \varepsilon \dfrac{\left(\frac{1}{\sqrt{\mu}}\mathcal{G}\psi + \varepsilon\nabla\zeta\cdot\nabla\psi\right)^2}{2(1+\varepsilon^2|\nabla\zeta|^2)} = 0, \end{cases} \tag{1.1}$$

where $\mathcal{G}$ is the Dirichlet to Neumann operator which links the trace of the velocity potential at the free surface to its normal derivative at the same surface. We denote $X = (x, y)$ or x the spatial variable.

One introduces a fixed wave vector $\mathbf{k} \in \mathbb{R}^d, d = 1, 2$. Setting $\omega = \omega(\mathbf{k}) = \big(|\mathbf{k}| \tanh(\sqrt{\mu}|\mathbf{k}|)\big)^{1/2}$ the dispersion relation of surface gravity waves, one looks for an approximate solution of the water wave system of the form

$$U_{\text{app}}(X,t) = U_0(X,t) + \varepsilon U_1(X,t) + \varepsilon U^2(X,t), \tag{1.2}$$

where $X = (x, y)$ or x and U_0 is a sum of a modulated wave packet and of an induced mean mode ϕ

$$U_0(X,t) = \begin{pmatrix} i\omega\psi(\varepsilon X, \varepsilon t) \\ \psi(\varepsilon X, \varepsilon t) \end{pmatrix} e^{i(\mathbf{k}\cdot X - \omega t)} + \text{c.c.} + \begin{pmatrix} 0 \\ \phi(\varepsilon X, \varepsilon t) \end{pmatrix}. \tag{1.3}$$

In [18], D. Lannes derived the following Full Dispersion Benney-Roskes (FDBR) system coupling ψ, ϕ and the leading term ζ of the surface elevation (see [23] for the case of capillary-gravity waves)

$$\begin{cases} \partial_t'\psi + i\dfrac{\omega(\mathbf{k} + \varepsilon D') - \omega(\mathbf{k})}{\varepsilon}\psi \\ \quad + \varepsilon i[\mathbf{k}\cdot\nabla'\phi + \dfrac{|\mathbf{k}|^2}{2\omega}(1-\sigma^2)\zeta + 2\dfrac{|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi|^2]\psi = 0, \\ \partial_t'\zeta - |D'|\dfrac{\tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon}\phi = -2\omega\mathbf{k}\cdot\nabla'(|\psi|^2), \\ \partial_t'\phi + \zeta = -|\mathbf{k}|^2(1-\sigma^2)|\psi|^2, \end{cases} \tag{1.4}$$

where $\mathbf{k}$ is the given wave vector, $\sigma = \tanh(\sqrt{\mu}|\mathbf{k}|)$ and $\alpha = -\frac{9}{8\sigma^2}(1-\sigma^2)^2$.

Here the time and space derivatives (indicated by a $'$) are taken with respect to the slow time and space variables $t' = \varepsilon t$ and $X' = \varepsilon X$ and D' denotes the operator $\frac{1}{i}\nabla'$.

Without loss of generality one may assume that $\mathbf{k}$ is oriented in the x direction, that is $\mathbf{k} = |\mathbf{k}|e_x$. We will thus write (1.4) as

$$\begin{cases} \partial_t'\psi + i\dfrac{\omega(\mathbf{k} + \varepsilon D') - \omega(\mathbf{k})}{\varepsilon}\psi \\ \quad + \varepsilon i[|\mathbf{k}|\partial_x'\phi + \dfrac{|\mathbf{k}|^2}{2\omega}(1-\sigma^2)\zeta + 2\dfrac{|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi|^2]\psi = 0, \\ \partial_t'\zeta - |D'|\dfrac{\tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon}\phi = -2\omega|\mathbf{k}|\partial_x'(|\psi|^2), \\ \partial_t'\phi + \zeta = -|\mathbf{k}|^2(1-\sigma^2)|\psi|^2, \end{cases} \tag{1.5}$$

Remark 1.1. *It is proven in* [18] *that* (1.4) *(or* (1.5)*) is consistent with the full water wave system. We recall that the notion of consistency is a "static" (not dynamic) one. It amounts to saying that an assumed solution of the water waves system on some time interval* $[0, T]$ *solves the BRFD or FDBR system on the same time interval up to some error term. A full justification of the BRFD system requires in particular that solutions of* (1.4) *exist on a time scale of order* $O(1/\varepsilon)$ *which is an open problem.*

It also necessitates to prove the well-posedness of the full water -wave system in the modulation regime on the correct time scales, which has been proven in one and two horizontal spatial dimensions in [33, 32] *in the infinite depth case. The finite depth case is up to our knowledge open.*

The obvious advantage of (1.4) (or (1.5)) on the classical [1] *Benney-Roskes system (see [3, 18]) is its a priori validity over an extended range of frequencies.*

The full dispersion Benney-Roskes system with surface tension (capillary-gravity waves) is given by (see [23])

$$
\begin{cases}
\partial_t'\psi + i\dfrac{\omega_S(\mathbf{k}+\varepsilon D') - \omega_S(\mathbf{k})}{\varepsilon}\psi \\
\quad + i\varepsilon[|\mathbf{k}|\partial_x'\phi + \dfrac{|\mathbf{k}|^2\kappa}{2\omega_S}(1-\sigma^2)\zeta - \dfrac{2|\mathbf{k}|^4}{\omega_S}\gamma|\psi|^2]\psi = 0, \\
\partial_t'\zeta - |D'|\dfrac{\tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon}\phi = -2\dfrac{\omega_S}{\kappa}|\mathbf{k}|\partial_x'(|\psi|^2), \\
\partial_t'\phi + \zeta = -|\mathbf{k}|^2(1-\sigma^2)|\psi|^2.
\end{cases}
\tag{1.6}
$$

Here, the dispersion relation for the water waves with surface tension is given by

$$\omega_S(\xi) = [|\xi|\tanh(\sqrt{\mu}|\xi|)(1+B|\xi|^2)]^{1/2},$$

where $B > 0$ measures the surface tension effects (Bond number).

As for purely gravity waves, $\mathbf{k} = |\mathbf{k}|e_x$ is a given wave vector. We will use the notations $\omega_S = \omega_S(\mathbf{k})$, $\kappa = \kappa(\mathbf{k}) = 1 + B|\mathbf{k}|^2$ and $\sigma = \tanh(\sqrt{\mu}|\mathbf{k}|)$. γ is a constant depending on $\mathbf{k}$ and B.

The classical Benney-Roskes system [3] is obtained from (1.4) or (1.6) by approximating the nonlocal operators *eg* $i\omega(\mathbf{k}+\varepsilon D')$ and $\frac{\tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon}$ as (denoting $\mathcal{H}_\omega(\mathbf{k})$ the Hessian of ω at $\mathbf{k}$)

$$i\omega(\mathbf{k}+\varepsilon D') \sim \varepsilon\nabla\omega(\mathbf{k})\cdot\nabla' - \frac{i\varepsilon^2}{2}\nabla'\mathcal{H}_\omega(\mathbf{k})\nabla',$$

and

$$|D'|\frac{\tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon} \sim \sqrt{\mu}|D'|^2.$$

The Cauchy problem for the Benney-Roskes system[2] is proven in [28] to be locally well-posed, but on time scales which do not reach the order $O(1/\varepsilon)$ that would be required (together with uniform bounds in ε) for the full justification of the system, see [18]. We refer to [22] for some results on the Cauchy problem for the full dispersion Benney-Roskes system.

As explained in [18] in the "classical" case, the Davey-Stewartson model is obtained from the Benney-Roskes system by observing that at leading order, ψ travels at the group speed velocity $c_g = \nabla\omega(\mathbf{k})$. One thus looks for solutions of (1.4) under the form

$$
\begin{aligned}
\psi(X',t') &= \tilde{\psi}(X' - c_g t, \varepsilon t'),\\
\phi(X',t') &= \tilde{\phi}(X' - c_g t, \varepsilon t') + \phi^*(X',t',\varepsilon t'),\\
\zeta(X',t') &= \tilde{\zeta}(X' - c_g t', \varepsilon t) + \zeta^*(X',t',\varepsilon t').
\end{aligned}
$$

Neglecting $O(\varepsilon)$ correctors, one can replace ∂_t' by $-c_g\cdot\nabla'$, and one is finally led (see details in [18]) to the *Full Dispersion Davey-Stewartson System* (FDDS) for

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[1] We will use the term "classical" to refer to those systems where the nonlocal dispersion has been Taylor-expanded yielding *local* differential operators instead of *nonlocal* ones.

[2] a similar system is derived in [34] in the general context of interactions between low and high-frequency waves. See also [30], Chap. 15, for a similar system describing the interaction of a laser beam with plasma low-frequency fluctuations driven by ponderomotive forces.

gravity waves, where $\tau = \varepsilon t' = \varepsilon^2 t$ (from now on, we will drop the tildas in the FDDS systems)

$$\begin{cases} \partial_\tau \psi + \dfrac{i}{\varepsilon^2}[\omega(\mathbf{k} + \varepsilon D') - \omega - \varepsilon \omega' D'_x]\psi \\ + i(\beta \partial'_x \phi + 2\dfrac{|\mathbf{k}|^4}{\omega}(1-\tilde{\alpha})|\psi|^2)\psi = 0, \\ \left(\dfrac{|D'|\tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon} + \omega'^2 \partial'^2_x\right)\phi = 2\omega\beta\partial'_x(|\psi|^2). \end{cases} \tag{1.7}$$

We have used here the following notations (see [18])

$$\begin{cases} \mathbf{k} = |\mathbf{k}|\mathbf{e}_x, \quad \omega(\mathbf{k}) = \tilde{\omega}(|\mathbf{k}|), \quad \text{with} \quad \tilde{\omega}(r) = (r\tanh(\sqrt{\mu}r))^{1/2}, \\ \omega = \tilde{\omega}(|\mathbf{k}|), \quad \omega' = \tilde{\omega}'(|\mathbf{k}|), \quad \omega'' = \tilde{\omega}''(|\mathbf{k}|), \\ \sigma = \tanh(\sqrt{\mu}|\mathbf{k}|), \quad \alpha = -\dfrac{9}{8\sigma^2}(1-\sigma^2)^2, \quad \tilde{\alpha} = \alpha + \dfrac{1}{4}(1-\sigma^2)^2, \\ \beta = |\mathbf{k}|\left(1 + (1-\sigma^2)\dfrac{\omega'|\mathbf{k}|}{2\omega}\right). \end{cases} \tag{1.8}$$

We also recall that the Fourier multiplier operator $\omega(\mathbf{k} + \varepsilon D')$ is given by

$$\omega(\mathbf{k} + \varepsilon D') = (|\mathbf{k} + \varepsilon D'|\tanh(\sqrt{\mu}|\mathbf{k} + \varepsilon D'|))^{1/2}.$$

Note (see [18]) that the complete rigorous justification of the full dispersion Davey-Stewartson system requires just the well-posedness of the Cauchy problem for (1.16) on time scales $\tau = O(1)$.

For capillary-gravity waves, the full dispersion Davey-Stewartson system writes

$$\begin{cases} \psi_\tau + \dfrac{i}{\varepsilon^2}[\omega_S(\mathbf{k} + \varepsilon D') - \omega_S - \varepsilon \omega'_S D'_x]\psi \\ + i(\beta_S \partial'_x \phi - 2\dfrac{|\mathbf{k}|^4}{\omega_S}\tilde{\gamma}|\psi|^2)\psi = 0, \\ \left(|D'|\dfrac{\tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon} + (\omega'_S)^2 {\partial'_x}^2\right)\phi = 2\dfrac{\omega_S}{\kappa}\beta_S\partial'_x(|\psi|^2), \end{cases} \tag{1.9}$$

with

$$\beta_S = |\mathbf{k}|(1 + (1-\sigma^2)\frac{\omega'_S|\mathbf{k}|}{2\omega_S}), \qquad \tilde{\gamma} = \gamma + \frac{\kappa}{4}(1-\sigma^2)^2.$$

The classical Davey-Stewartson system which was derived formally in [9] for gravity waves and in [2, 10] for capillary-gravity waves has been justified rigorously in the sense of consistency in [7, 6, 8].[3] It is obtained by approximating the non-local operators in (1.16) by the relevant local ones as indicated above for the Full dispersion Benney-Roskes system.

We recall (see [11]) that the Davey-Stewartson systems have the general form

$$\begin{cases} i\partial_t \psi + a\partial_x^2 \psi + b\partial_y^2 \psi = (\nu_1|\psi|^2 + \nu_2 \partial_x \phi)\psi, \\ c\partial_x^2 \phi + \partial_y^2 \phi = -\delta \partial_x(|\psi|^2). \end{cases} \tag{1.10}$$

We refer to [2] for an explicitation of $a, b, c, \nu_1, \nu_2, \delta$ in terms of the physical parameters. It is also convenient to use the normalized form (see [11])

[3]Its "universal" character as an asymptotic system of quadratic hyperbolic systems in the modulation regime has been proved in [5].

$$
\begin{aligned}
i\partial_t\psi + \partial_x^2\psi + \delta\partial_y^2\psi &= (\chi|\psi|^2 + b\partial_x\phi)\psi, \\
\partial_x^2\phi + m\partial_y^2\phi &= \partial_x|\psi|^2,
\end{aligned}
\tag{1.11}
$$

where the parameters δ, χ, b and m re real and can assume both signs, and δ, χ have been normalized in such a way that $|\delta| = |\chi| = 1$.

The Davey-Stewartson systems can be classified as "elliptic-elliptic", hyperbolic-elliptic", "elliptic-hyperbolic" and "hyperbolic-hyperbolic" depending on the sign of δ, m.

The "hyperbolic-hyperbolic" case does not occur in the context of water waves. The "elliptic-elliptic" case may lead to a blow-up phenomenon reminiscent of that of the cubic focusing NLS (see [11] for a proof of blow-up and [26] for numerical simulations analyzing the dynamics of the singularity formation).

In the case of purely gravity waves, one has $b > 0$ and $\delta > 0$. Moreover, $a = \frac{1}{2}\omega'' < 0$ since $\omega'' = \tilde{\omega}''(|\mathbf{k}|) < 0$ by the concavity of $\tilde{\omega}$.

On the other hand, $c = \sqrt{\mu} - \omega'^2 > 0$ (see Section 2). This is a *subsonic* condition. Actually this amounts to saying that $M < 1$ where the Mach number M is defined as $M = C_g/\sqrt{gh}$, where $c_g = \frac{d}{dr}\sqrt{gr\ \tanh(hr)}_{|r=|\mathbf{k}|}$.

One can thus invert in this case the equation for ϕ, writing (one can assume $c = 1$ without loss of generality) $\phi_x = \delta R_1^2(|\psi|^2)$, where R_1 is the Riesz transform $R_1 = \partial_x(-\Delta)^{-1/2}$, and reduce (1.20) to a single semilinear (cubic) Schrödinger type equation

$$
i\partial_t\psi + a\partial_x^2\psi + b\partial_y^2\psi = (\nu_1|\psi|^2 + \nu_2\delta R_1^2(|\psi|^2))\psi,
\tag{1.12}
$$

which is studied in [11]. In particular, whatever the signs of a and b, the Cauchy problem is locally well-posed for initial data in $L^2(\mathbb{R}^2)$ or $H^1(\mathbb{R}^2)$ and globally for small data, as the corresponding cubic NLS. The global Cauchy problem is much less understood. For instance, a global well-posedness result for the *hyperbolic-elliptic* case ($b > 0,\ a < 0$) in (1.12) is still missing for arbitrary large initial data.

However, for a very specific choice of coefficients[4], (1.12) is integrable by the inverse scattering transform method leading to qualitative informations on the flow. After scaling, this integrable version of the DS system writes

$$
\begin{cases}
i\partial_t\psi + \psi_{xx} - \psi_{yy} = \alpha|\psi|^2\psi + \beta\psi\phi_x, \\
\Delta\phi = \partial_x(|\psi|^2),
\end{cases}
\tag{1.13}
$$

with

$$
\alpha + \frac{\beta}{2} = 0.
$$

It is then known as the DS II system, with two subcases, the focussing ($\beta > 0$) and the defocussing one ($\beta < 0$). For the defocussing DS II, one has global solutions that behaves asymptotically in time as the solution of the linear case (see [31, 27, 21]). In particular global well-posedness (and scattering) is proven in [21] with arbitrary L^2 initial data.

On the other hand, in the focussing integrable DS II system, the presence of an explicit localized *lump* solitary wave leads to blow-up in finite time by a pseudo-conformal law [25]. This later behavior does not seem to be generic, that is does not seem to persist in the non integrable case (see the numerical simulations in

[4] with a limited physical relevance

[16]). In particular, one can prove rigorously that no localized solitary waves exist "not too close" to the integrable case (see [12]). However, the "scattering" behavior of arbitrary large solutions to the defocussing integrable DS II system proven in [31, 27] might be generic. In both situations, the numerical simulations in [17, 16] are illuminating.

When surface tension is strong enough however, c is no more positive (see [2]) and ϕ is inverted (with suitable conditions at infinity) via a wave equation and $\partial_x\phi$ is expressed in terms of $|\psi|^2$ via an order one nonlocal operator. The integrable version is known as the DS I system. We refer to [1] and the references therein for a description of the results obtained on DS I system by inverse scattering techniques, in particular the existence of special localized solutions called *dromions*. The resulting Schrödinger type equation is no more semilinear (it involves an order one nonlinear term) which makes the study by PDE techniques delicate. The local associated Cauchy problem is studied in particular in [19, 13, 14]. Scattering of small solutions is proven in [15].

For the FDDS system, in contrast to the classical system (1.20) for gravity waves ("hyperbolic-elliptic"), one cannot invert in a straightforward fashion the equation for ϕ which makes the associated Cauchy problem delicate.

As suggested in [18] one could consider as well the full dispersion version of only the first or the second of the two equations of the FDDS system (1.16), namely

$$\begin{cases}\partial_\tau\psi + \dfrac{i}{\varepsilon^2}[\omega(\mathbf{k}+\varepsilon D') - \omega - \varepsilon\omega' D'_x]\psi \\ \quad + i(\beta\partial'_x\phi + 2\dfrac{|\mathbf{k}|^4}{\omega}(1-\tilde{\alpha})|\psi|^2)\psi = 0, \\ [(\sqrt{\mu} - {\omega'}^2)\partial_x'^2 + \sqrt{\mu}\partial_y'^2]\phi = -2\omega\beta\partial'_x(|\psi|^2),\end{cases} \tag{1.14}$$

or

$$\begin{cases}\partial_\tau\psi - \dfrac{i}{2}(\omega''\partial_x'^2 + \dfrac{\omega'}{|\mathbf{k}|}\partial_y'^2)\psi + i(\beta\partial'_x\phi + 2\dfrac{|\mathbf{k}|^4}{\omega}(1-\tilde{\alpha})|\psi|^2)\psi = 0, \\ \left(\dfrac{|D'|\tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon} + {\omega'}^2\partial_x'^2\right)\phi = 2\omega\beta\partial'_x(|\psi|^2).\end{cases} \tag{1.15}$$

This has the advantage of "splitting the difficulties" : in the first case, one can invert the equation for ϕ as for the hyperbolic-elliptic classical DS case, but one keeps the "bad" dispersive properties of the Schrödinger part. In the later case one has the "good" properties of the hyperbolic NLS and the "bad" invertibility properties of the equation for ϕ.

We consider here the full dispersion Davey-Stewartson system written as

$$\begin{cases}\partial_t\psi - \dfrac{i}{2}(\omega''\partial_x^2 + \dfrac{\omega'}{|\mathbf{k}|}\partial_y^2)\psi + i(\beta\partial_x\phi + \dfrac{2|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi|^2)\psi = 0, \\ (\dfrac{|D|\tanh(\varepsilon\sqrt{\mu}|D|)}{\varepsilon} + \omega'^2\partial_x^2)\phi = 2\omega\beta\partial_x(|\psi|^2).\end{cases} \tag{1.16}$$

Which is a 2-d model in space for the unknown $\psi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$ and $\phi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$.

As noticed in [24] the "subsonic" condition $\sqrt{\mu} - \omega'^2 > 0$ does not ensures the solvability of the equation for ϕ since, denoting ξ the dual variable of X', the frequencies ξ such that

$$\frac{\tanh(\varepsilon\sqrt{\mu}|\xi|)}{\varepsilon|\xi} = \omega'^2 \frac{\xi_1^2}{|\xi|^2}$$

will cancel the symbol.

The parameters $\omega', \omega'', |\mathbf{k}|, \alpha, \beta, \mu$ are defined as previously.
We will only consider the case $\sqrt{\mu} > \omega'^2$, so that we can find r_0 such that:

$$\tanh(\sqrt{\mu}r_0) = \omega'^2 r_0.$$

In order to invert easily the equation for ϕ we introduce an approximate system by truncating the high frequencies.

More precisely, let δ be fixed such that $0 < \delta < r_0$, for each $\varepsilon > 0$, we define $R_\varepsilon = \frac{r_0 - \delta}{\varepsilon}$. We then replace the mean mode ϕ by $\phi_\varepsilon := \mathcal{F}^{-1}(\chi(|\xi| \leq R_\varepsilon)\widehat{\phi})$ so that [5]

$$\textit{supp}\ \widehat{\phi_\varepsilon} \subset \{\xi \in \mathbb{R}^2;\ |\xi| \leq R_\varepsilon\} =: \Omega_\varepsilon, \tag{1.17}$$

to get a family of systems depending on the parameter ε as follows

$$\begin{cases} \partial_t \psi_\varepsilon - \dfrac{i}{2}(\omega'' \partial_x^2 + \dfrac{\omega'}{|\mathbf{k}|}\partial_y^2)\psi_\varepsilon + i(\beta \partial_x \phi_\varepsilon + \dfrac{2|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi_\varepsilon|^2)\psi_\varepsilon = 0, \\ (\dfrac{|D| \tanh(\varepsilon\sqrt{\mu}|D|)}{\varepsilon} + \omega'^2 \partial_x^2)\phi_\varepsilon = 2\omega\beta\partial_x(|\psi_\varepsilon|^2). \end{cases} \tag{1.18}$$

Since the supports of $\widehat{\phi_\varepsilon}$ and $\widehat{|\psi_\varepsilon|^2}$ are in $\{\xi \in \mathbb{R}^2; |\xi| < R_\varepsilon\}$, we need to construct a suitable initial data for the Cauchy problem for the $\varepsilon - FDDS$ which converge to the initial data of the Cauchy problem for the corresponding classical DS system. We proceed as follows.

First, let $\psi_0 \in H^1$ be the intial data of the classical DS system, then we define

$$\psi_{0\varepsilon} = f_\varepsilon \star \psi_0 \text{ where } f_\varepsilon = \mathcal{F}^{-1}(\chi(|\xi| < \frac{R_\varepsilon}{2})). \tag{1.19}$$

Then we have that supp $\widehat{\psi_{0\varepsilon}} \subset \{|\xi| < \frac{R_\varepsilon}{2}\}$. It follows

$$\text{supp } \widehat{|\psi_{0\varepsilon}|^2} \subset \text{supp } \widehat{\psi_{0\varepsilon}} + \text{supp } \overline{\widehat{\psi_{0\varepsilon}}(-.)} \subset \{|\xi| < R_\varepsilon\}.$$

Our purpose is to prove that when ε tends to 0, the solution of (1.18) with initial data $\psi_{0\varepsilon}$ tends to the solution of the following classical Davey-Stewartson system with initial data ψ_0:

$$\begin{cases} \partial_t \psi - \dfrac{i}{2}(\omega'' \partial_x^2 + \dfrac{\omega'}{|\mathbf{k}|}\partial_y^2)\psi + i(\beta \partial_x \phi + \dfrac{2|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi|^2)\psi = 0, \\ (\sqrt{\mu}|D|^2 + \omega'^2 \partial_x^2)\phi = 2\omega\beta\partial_x(|\psi|^2). \end{cases} \tag{1.20}$$

The paper is organized as follows. As a preliminary step we solve the equation for $\partial_x \phi_\varepsilon$ reducing the approximate FDDS system to a NLS type equation. The corresponding Cauchy problem is classically solved in the next section, on a time interval independent of ε. We then pass to the limit as $\varepsilon \to 0$ by compactness arguments.

1.1. **Notations**. The norm in L^2 based Sobolev spaces H^s will be denoted $||u||_s$.. The norm in the Lebesgue spaces L^p will be denoted $|u|_p$. The Fourier transform in spatial space will be denoted by $\widehat{(.)}$ or $\mathcal{F}$ and by $\mathcal{F}^{-1}$ for the inverse Fourier

[5] $\chi(A)$ denote the characteristic function of the set A.

transform. $|D|$ will denote the Fourier multiplier defined by $\widehat{|D|f}(\xi) = |\xi|\hat{f}(\xi)$. Throughout this paper, C denotes a general constant independent of ε if there is no further specification.

2. Solving the equation for $\partial_x \phi_\varepsilon$

We first rewrite the second equation of (1.18) in Fourier space

$$(\frac{|\xi|\tanh(\varepsilon\sqrt{\mu}|\xi|)}{\varepsilon} - \omega'^2\xi_1^2)\widehat{\phi_\varepsilon}(\xi) = 2i\omega\beta\xi_1\widehat{|\psi_\varepsilon|^2}(\xi),$$

then, since the support of $\widehat{\phi_\varepsilon}$ does not contain any zero of the equation $\frac{|\xi|\tanh(\varepsilon\sqrt{\mu}|\xi|)}{\varepsilon} - \omega'^2\xi_1^2 = 0$, one can write

$$\widehat{\partial_x\phi_\varepsilon}(\xi) = \frac{2\omega\beta\xi_1^2}{\omega'^2\xi_1^2 - \frac{|\xi|\tanh(\varepsilon\sqrt{\mu}|\xi|)}{\varepsilon}}\widehat{|\psi_\varepsilon|^2}(\xi). \tag{2.1}$$

We now consider the function

$$\begin{aligned} f(\xi) &= \left|\omega'^2 - \frac{|\xi|\tanh(\varepsilon\sqrt{\mu}|\xi|)}{\varepsilon\xi_1^2}\right| \\ &= \left|\omega'^2 - \frac{|\xi|^2}{\xi_1^2}\frac{\tanh(\varepsilon\sqrt{\mu}|\xi|)}{\varepsilon|\xi|}\right|, \end{aligned}$$

when $|\xi| \le R_\varepsilon$.
Since the function: $x \mapsto \frac{\tanh x}{x}$ is strictly decreasing in $\mathbb{R}_+$ and $R_\varepsilon < \frac{r_0}{\varepsilon}$ we have

$$\sqrt{\mu} \ge \frac{\tanh(\varepsilon\sqrt{\mu}|\xi|)}{\varepsilon|\xi|} \ge \frac{\tanh(\varepsilon\sqrt{\mu}R_\varepsilon)}{\varepsilon R_\varepsilon} > \frac{\tanh(\sqrt{\mu}r_0)}{r_0} = \omega'^2.$$

Then

$$f(\xi) \ge \frac{\tanh(\varepsilon\sqrt{\mu}R_\varepsilon)}{\varepsilon R_\varepsilon} - \omega'^2 = \frac{\tanh(\sqrt{\mu}(r_0-\delta))}{r_0-\delta} - \omega'^2 > 0.$$

Therefore,

$$|\widehat{\partial_x\phi_\varepsilon}| \le C_0|\widehat{|\psi_\varepsilon|^2}|, \tag{2.2}$$

where C_0 is independent of ε, namely

$$C_0 = \frac{|2\omega\beta|}{\frac{\tanh(\sqrt{\mu}(r_0-\delta))}{r_0-\delta} - \omega'^2}.$$

The above observation leads to the following lemma.

Lemma 2.1. *The second equation in* (1.18) *is uniquely solvable and*

$$\|\partial_x\phi_\varepsilon\|_{L^2(\mathbb{R}^2)} \le C_0\,\|\psi_\varepsilon\|^2_{L^4(\mathbb{R}^2)}\,. \tag{2.3}$$

Note that besides the L^2 norm, by a similiar calculation as in the case of the classical Davey-Stewartson system (see [11]), the energy $E(\psi)$ is formally conserved by the flow of (1.16) where

$$\begin{aligned} E(\psi) = \int_{\mathbb{R}^2} \Big(&\omega''|\psi_x|^2 + \frac{\omega'}{|\mathbf{k}|}|\psi_y|^2 + 2\frac{|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi|^4 + \frac{\omega'^2}{4\omega}(\phi_x)^2 \\ &- \frac{1}{4\omega}(\frac{\tanh(\epsilon\sqrt{\mu}|D|)}{\epsilon}\phi)\,(|D|\phi)\Big)\,dx\,dy. \end{aligned}$$

Setting $\partial_x \phi_\varepsilon = F_\varepsilon(|\psi_\varepsilon|^2)$, one can rewrite (1.18) as a single NLS type equation,

$$\partial_t \psi_\varepsilon - \frac{i}{2}(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|}\partial_y^2)\psi_\varepsilon + i(\beta F_\varepsilon(|\psi_\varepsilon|^2) + \frac{2|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi_\varepsilon|^2)\psi_\varepsilon = 0. \tag{2.4}$$

3. Cauchy problem for the $\varepsilon-$FDDS

In this section, we will consider the Cauchy problem for (1.18) (or (2.4)) with initial data $\psi_\varepsilon(t=0) = \psi_{0\varepsilon} \in H^1(\mathbb{R}^2)$ defined as in (1.19). In particular we obtain the following theorem.

Theorem 3.1. *Let $\psi_\varepsilon(t=0) = \psi_{0\varepsilon} \in H^1(\mathbb{R}^2)$, then there exists a unique maximal solution $(\psi_\varepsilon, \phi_\varepsilon)$ solution of (1.18) on $[0, T_\varepsilon^*[$ where $T_\varepsilon^* = O(1/\|\psi_{0\varepsilon}\|_{H^1}^2)$, such that*

$$\psi_\varepsilon \in L_t^\infty(0, T_\varepsilon^*; H^1(\mathbb{R}^2)),\ \nabla\psi_\varepsilon \in L_t^4(0, T_\varepsilon^*; L^4(\mathbb{R}^2)),$$
$$\partial_x \phi_\varepsilon \in L_t^\infty(0, T_\varepsilon^*; L^2(\mathbb{R}^2)) \cap L_t^2(0, T_\varepsilon^*; H^1(\mathbb{R}^2))$$

One has the conservation of the L^2 norm and of the energy :

$$|\psi_\varepsilon(\cdot, t)|_2 = |\psi_{0\varepsilon}|_2, \quad E(\psi_\varepsilon(\cdot, t) = E(\psi_{0\varepsilon}), 0 \le t < T_\varepsilon^*.$$

Moreover one has the persistency property : if $\psi_0 \in H^s(\mathbb{R}^2), s > 1$, then

$$\psi_\varepsilon \in L_t^\infty(0, T_\varepsilon^*; H^s(\mathbb{R}^2)),\ \partial_x \phi_\varepsilon \in L_t^\infty(0, T_\varepsilon^*; H^s(\mathbb{R}^2)).$$

Proof. Let us denote

$$U(t) := e^{it(\omega''\partial_x^2 + \frac{\omega'}{|\mathbf{k}|}\partial_y^2)},\ \text{(unitary in all Sobolev spaces)}$$

then we can rewrite (2.4) using the Duhamel formulation as an integral equation

$$\psi_\varepsilon = U(t)\psi_{0\varepsilon} - i\int_0^t U(t-s)(\beta F_\varepsilon(|\psi_\varepsilon|^2) + \frac{2|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi_\varepsilon|^2)\psi_\varepsilon(s)ds. \tag{3.1}$$

We denote the operator on the right hand side by $\mathcal{T}(\psi_\varepsilon)$ and recall the two-dimensional Strichartz estimate for U(t)

$$\left\| \int_0^t U(t-s)f(s)ds \right\|_{L_t^\infty(0,T;L^2)\cap L_t^4(0,T;L^4)} \le C\, \|f\|_{L_t^{4/3}(0,T;L^{4/3})}. \tag{3.2}$$

Set

$$X = \{u \in L_t^\infty(0, T; H^1), \nabla u \in L_t^4(0, T; L^4)\},$$

and

$$B = \{\|u\|_X \le 3\,\|\psi_{0\varepsilon}\|_{H^1}\},$$

T can be chosen later.
The proof is standard and relies classically on the Banach fixed point theorem. We only derive here suitable bounds in order to emphasize that the existence time and the norm of solution are independent of ε.

We now prove that B is stable under $\mathcal{T}$. Using estimate (2.3), Strichartz's estimate (3.2), Sobolev's embedding and Hölder's inequality, we get the following estimates:

$$
\begin{aligned}
&\|\mathcal{T}(\psi_\varepsilon)\|_X \\
=&\|\mathcal{T}(\psi_\varepsilon)\|_{L_t^\infty(0,T;L^2)} + \|\nabla\mathcal{T}(\psi_\varepsilon)\|_{L_t^\infty(0,T;L^2)} + \|\nabla\mathcal{T}(\psi_\varepsilon)\|_{L_t^4(0,T;L^4)} \\
\le& 2\|\psi_{0\varepsilon}\|_{H^1} + C\Bigg(\left\|F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon\right\|_{L_t^{4/3}(0,T;L^{4/3})} + \left\|\nabla(F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon)\right\|_{L_t^{4/3}(0,T;L^{4/3})} \\
&\quad + \left\||\psi_\varepsilon|^2\psi_\varepsilon\right\|_{L_t^{4/3}(0,T;L^{4/3})} + \left\|\nabla(|\psi_\varepsilon|^2\psi_\varepsilon)\right\|_{L_t^{4/3}(0,T;L^{4/3})}\Bigg).
\end{aligned} \tag{3.3}
$$

And,

$$
\begin{aligned}
\left\|F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon\right\|_{L_t^{4/3}(0,T;L^{4/3})} \le& \left\|F_\varepsilon(|\psi_\varepsilon|^2)\right\|_{L_t^2(0,T;L^2)} \|\psi_\varepsilon\|_{L_t^4(0,T;L^4)} \\
\le& C_0T^{3/4}\|\psi_\varepsilon\|^3_{L_t^\infty(0,T;L^4)} \\
\le& CT^{3/4}\|\psi_\varepsilon\|^3_{L_t^\infty(0,T;H^1)},
\end{aligned} \tag{3.4}
$$

$$
\left\||\psi_\varepsilon|^2\psi_\varepsilon\right\|_{L_t^{4/3}(0,T;L^{4/3})} \le CT^{3/4}\|\psi_\varepsilon\|_{L_t^\infty(0,T;H^1)}, \tag{3.5}
$$

$$
\begin{aligned}
\left\|\nabla(|\psi_\varepsilon|^2\psi_\varepsilon)\right\|_{L_t^{4/3}(0,T;L^{4/3})} =& \left\|\psi_\varepsilon^2\nabla\bar{\psi}_\varepsilon + 2|\psi_\varepsilon|^2\nabla\psi_\varepsilon\right\|_{L_t^{4/3}(0,T;L^{4/3})} \\
\le& CT^{1/2}\|\psi_\varepsilon\|^2_{L_t^\infty(0,T;H^1)}\|\nabla\psi_\varepsilon\|_{L_t^4(0,T;L^4)}.
\end{aligned} \tag{3.6}
$$

And, for $h \in \mathbb{R}^2$

$$
\begin{aligned}
&\left\|F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon(X+h) - F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon(X)\right\|_{L_t^{4/3}(0,T;L^{4/3})} \\
=&\Bigg\|(\psi_\varepsilon(X+h)-\psi_\varepsilon(X))F_\varepsilon(|\psi_\varepsilon(X+h)|^2) \\
&\quad + (F_\varepsilon(|\psi_\varepsilon(X+h)|^2) - F_\varepsilon(|\psi_\varepsilon(X)|^2))\psi_\varepsilon(X)\Bigg\|_{L_t^{4/3}(0,T;L^{4/3})} \\
\le& 3C_0T^{1/2}\|\psi_\varepsilon\|^2_{L_t^\infty(0,T;L^4)}\|\psi_\varepsilon(X+h)-\psi_\varepsilon(X)\|_{L_t^4(0,T;L^4)},
\end{aligned}
$$

Dividing by $|h|$ and letting $|h|$ tends to 0, we deduce that $\nabla(F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon) \in L_t^{4/3}(0,T;L^{4/3})$ and

$$
\left\|\nabla(F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon)\right\|_{L_t^{4/3}(0,T;L^{4/3})} \le 3C_0T^{1/2}\|\psi_\varepsilon\|^2_{L_t^\infty(0,T;L^4)}\|\nabla\psi_\varepsilon\|_{L_t^4(0,T;L^4)} \tag{3.7}
$$

Combining (3.3)-(3.7), we see that there exists $T = O(1/\|\psi_{0\varepsilon}\|^2_{H^1})$ such that B is invariant under $\mathcal{T}$.

Proving that $\mathcal{T}$ is a contraction on B proceeds as in the previous step by using the following expression.

$$
\psi_1F_\varepsilon(|\psi_1|^2) - F_\varepsilon(|\psi_2|^2)\psi_2 = (\psi_1-\psi_2)F_\varepsilon(|\psi_1|^2) + \psi_2F_\varepsilon((|\psi_1|-|\psi_2|)(|\psi_1|+|\psi_2|))
$$

and

$$
\psi_1|\psi_1|^2 - \psi_2|\psi_2|^2 = (\psi_1-\psi_2)|\psi_1|^2 + \psi_2(|\psi_1|-|\psi_2|)(|\psi_1|+|\psi_2|)
$$

where ψ_1 and ψ_2 are in X.
Therefore from these two steps, we can choose the existence time $T^*_\varepsilon = O(1/\|\psi_{0\varepsilon}\|^2_{H^1})$.
□

Remark 3.1. *From* (1.19) *we have*

$$\|\psi_{0\varepsilon}\|_{H^1} = \left\| \sqrt{1+|\xi|^2}\,\chi(|\xi| < \frac{R_\varepsilon}{2})\,\widehat{\psi_0} \right\|_{L^2} \leq \|\psi_0\|_{H^1}.$$

Then, we can choose an existence time T^* of the solution of (1.18) *which is independent of ε and the solutions are also bounded by $3\|\psi_0\|_{H^1}$ in that time interval. Therefore, in the next section, we will prove that the solutions tend to a solution of the classical Davey-Stewartson system when ε tends to zero.*

4. From ε-FDDS to classical DS

We prove here that the solutions of the ε-FDDS system converge to that of the classical Davey-Stewartson system as ε tends to 0. We first recall a very classical result in [20], and some useful estimates.

Lemma 4.1. *(Aubin-Lions's Lemma) Let B_0, B, B_1 be three reflexive Banach spaces. Assume that $B_0$is compactly embedded in B and B is continuously embedded in B_1 . Let*

$$W = \{V \in L^{p_0}(0,T,B_0),\ \frac{\partial V}{\partial t} \in L^{p_1}(0,T,B_1)\}, \quad T<\infty,\ 1<p_0,p_1<\infty.$$

W is a Banach space with norm

$$\|V\|_W = \|V\|_{L^{p_0}(0,T,B_0)} + \|V_t\|_{L^{p_1}(0,T,B_1)}.$$

Then the embedding $W \hookrightarrow L^{p_0}(0,T,B)$ is compact.

When $p_0=\infty$, $p_1>1$, the above statement is also true, see [29].

Lemma 4.2. *Let Ω be an open set of $\mathbb{R}^n$ and let $g, g_\varepsilon \in L^p(\mathbb{R}^n)$, $1<p<\infty$, such that*

$$g_\varepsilon \to g \text{ almost everywhere in } \Omega \text{ and } \|g_\varepsilon\|_{L^p(\Omega)} \leq C.$$

Then $g_\varepsilon \to g$ weakly in $L^p(\Omega)$.

Lemma 4.3. *If $t_0>1$ and $s \geq 0$, one has*

(4.1)
$$\|fg\|_{H^s} \leq C\left(\|f\|_{H^{t_0}}\|g\|_{H^s} + \langle \|f\|_{H^s}\|g\|_{H^{t_0}}\rangle_{s>t_0}\right), \quad \forall f, g \in H^s \cap H^{t_0}(\mathbb{R}^2).$$

Note that, the condensed notation $A_s = B_s + \langle C_s\rangle_{s>s^*}$ means that $A_s = B_s$ if $s \leq s^*$ and $A_s = B_s + C_s$ if $s > s^*$.

Theorem 4.1. *Let $\psi_0 \in H^1(\mathbb{R}^2)$, $\psi_{0\varepsilon}$ is defined as in* (1.19) *and $(\psi_\varepsilon, \phi_\varepsilon)$ be the solution of* (1.18) *given by theorem 3.1. Then $(\psi_\varepsilon, \partial_x\phi_\varepsilon)$ tends to $(\psi, \partial_x\phi)$ in $L^\infty_t(0,T^*;H^1(\mathbb{R}^2)) \times L^\infty_t(0,T^*;L^2(\mathbb{R}^2))$ weak-star, where (ψ,ϕ) is the solution of* (1.20) *with initial data ψ_0.*

Furthermore, If $\psi_0 \in H^s(\mathbb{R}^2)$, $s>3$ then $(\psi_\varepsilon, \partial_x\phi_\varepsilon)$ tends to $(\psi, \partial_x\phi)$ strongly in $L^\infty_t(0,T^*;H^{s-\delta}_{loc}(\mathbb{R}^2))$ for $0<\delta<2$.
(T^* is the existence time interval for all ε and is defined as in Remark 3.1.)

Proof. We first denote

$$F(u) = \mathcal{F}_X^{-1}\left(\frac{2\omega\beta\xi_1^2}{\omega'^2\xi_1^2 - \sqrt{\mu}|\xi|^2}\widehat{u}(\xi)\right).$$

It is not difficult to see that $\psi_{0\varepsilon} \to \psi_0$ in H^1 when ε tends to zero.
From theorem 3.1 and Remark 3.1, we know that $\|\psi_\varepsilon\|_{L^\infty_t(0,T^*;H^1)}$, $\|\partial_x\phi_\varepsilon\|_{L^\infty_t(0,T^*;L^2)}$

are uniformly bounded in ε, then by using Hölder inequality and Sobolev embedding theorem, we also have that $\big\||\psi_\varepsilon|^2\psi_\varepsilon\big\|_{L_t^\infty(0,T^*;L^2)}$, $\|(\partial_x\phi_\varepsilon)\psi_\varepsilon\|_{L_t^\infty(0,T^*;H^{-1})}$ are uniformly bounded in ε. Using the first equation of (1.18), it is also true for $\|\partial_t\psi_\varepsilon\|_{L_t^\infty(0,T^*;H^{-1})}$. Therefore, we have (up to an extraction of subsequences)

$$\psi_\varepsilon \to \psi \text{ in } L_t^\infty(0,T^*;H^1) \quad \text{weak-star.} \tag{4.2}$$

$$\partial_x\phi_\varepsilon \text{ or } F_\varepsilon(|\psi_\varepsilon|^2) \to l \text{ in } L_t^\infty(0,T^*;L^2) \quad \text{weak-star.} \tag{4.3}$$

$$\partial_t\psi_\varepsilon \to \psi \text{ in } L_t^\infty(0,T^*;H^{-1}) \quad \text{weak-star.} \tag{4.4}$$

$$(\omega''\partial_x^2 + \frac{\omega'}{|\mathbf{k}|}\partial_y^2)\psi_\varepsilon \to (\omega''\partial_x^2 + \frac{\omega'}{|\mathbf{k}|}\partial_y^2)\psi \text{ in } L_t^\infty(0,T^*;H^{-1}) \quad \text{weak-star.} \tag{4.5}$$

$$|\psi_\varepsilon|^2\psi_\varepsilon \to h \text{ in } L_t^\infty(0,T^*;L^2) \quad \text{weak-star.} \tag{4.6}$$

$$(\partial_x\phi_\varepsilon)\psi_\varepsilon \text{ or } F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon \to g \text{ in } L_t^\infty(0,T^*;H^{-1}) \quad \text{weak-star.} \tag{4.7}$$

Thus, we need to prove

$$h = |\psi|^2\psi, \tag{4.8}$$

$$g = F(|\psi|^2)\psi, \tag{4.9}$$

and

$$l = F(|\psi|^2) \text{ or } \partial_x\phi. \tag{4.10}$$

We will use, for any bounded open subset Ω of $\mathbb{R}^2$, the compact embedding

$$H^1(\Omega) \hookrightarrow L^4(\Omega),$$

and the continuous injection

$$L_t^\infty(0,T^*;X) \hookrightarrow L_t^2(0,T^*;X), \text{ for any Banach space } X.$$

Let denote

$$W = \{v \in L_t^2(0,T^*;H^1(\Omega)); \frac{\partial v}{\partial t} \in L_t^2(0,T^*;H^{-1}(\Omega))\}.$$

By Aubin-Lions theorem, we have

$$W \hookrightarrow L_t^2(0,T^*;L^4(\Omega)) \text{ compactly }.$$

Hence, $\{\psi_\varepsilon\}$ is relatively compact in $L_t^2(0,T^*;L^4(\Omega))$ for any bounded subdomain Ω in $\mathbb{R}^2$. Therefore, $\psi_\varepsilon \to \psi$ strongly in $L_t^2(0,T^*;L^4_{loc}(\mathbb{R}^2))$ up to a subsequence. Thus,

$$\psi_\varepsilon \to \psi \text{ almost everywhere in } (0,T^*)\times\mathbb{R}^2,$$

or

$$|\psi_\varepsilon|^2\psi_\varepsilon \to |\psi|^2\psi \text{ almost everywhere in } (0,T^*)\times\mathbb{R}^2.$$

Combining with lemma 4.2 we get (4.8).

We now prove (4.9) by proving that $F_\varepsilon(|\psi_\varepsilon|^2\psi_\varepsilon) \to F(|\psi|^2)\psi$ in $L^2_t(0,T^*;H^{-1})$ weak star. First, let v be some test function in $L^2_t(0,T^*;H^1)$ vanishing outside of a compact set $\Omega \in \mathbb{R}^2$. Then

$$\begin{aligned}
&\int_0^{T^*}\int_{\mathbb{R}^2}(F_\varepsilon(|\psi_\varepsilon|^2)\psi_\varepsilon - F(|\psi|^2)\psi)v\,dX\,dt\\
&=\int_0^{T^*}\int_{\Omega} F_\varepsilon(|\psi_\varepsilon|^2)(\psi_\varepsilon-\psi)v\,dX\,dt + \int_0^{T^*}\int_{\mathbb{R}^2}(F_\varepsilon(|\psi_\varepsilon|^2)-F(|\psi|^2))\psi v\,dX\,dt\\
&= I^1_\varepsilon + I^2_\varepsilon
\end{aligned}$$

Estimate for I^1_ε:

$$|I^1_\varepsilon| \le \left\|F_\varepsilon(|\psi_\varepsilon|^2)\right\|_{L^\infty_t(0,T^*;L^2(\mathbb{R}^2))}\|\psi_\varepsilon-\psi\|_{L^2_t(0,T^*;L^4(\Omega))}\|v\|_{L^2_t(0,T^*;L^4)}.$$

Since $\psi_\varepsilon \to \psi$ strongly in $L^2_t(0,T^*;L^4)$, $I^1_\varepsilon \to 0$ when $\varepsilon \to 0$.
Estimate for I^2_ε: Note that we can rewrite

$$\begin{aligned}
&F_\varepsilon(|\psi_\varepsilon|^2) - F(|\psi|^2)\\
=&C\mathcal{F}^{-1}\left(\frac{\widehat{|\psi_\varepsilon|^2}(\xi)}{\omega'^2 - \frac{|\xi|\tanh(\varepsilon\sqrt{\mu}|\xi|)}{\varepsilon\xi_1^2}} - \frac{\widehat{|\psi|^2}(\xi)}{\omega'^2-\sqrt{\mu}\frac{|\xi|^2}{\xi_1^2}}\right)\\
:=&C\mathcal{F}^{-1}\left(q_\varepsilon(\xi)\widehat{|\psi_\varepsilon|^2}(\xi) - q_0(\xi)\widehat{|\psi|^2}(\xi)\right).\\
=&C\mathcal{F}^{-1}\left((q_\varepsilon-q_0)(\xi)\,\widehat{|\psi_\varepsilon|^2}(\xi)\right) + C\mathcal{F}^{-1}\left(q_0(\xi)\,(\widehat{|\psi_\varepsilon|^2}-\widehat{|\psi|^2})(\xi)\right)
\end{aligned}$$

Then by Plancherel theorem, we have

$$\begin{aligned}
&|I^2_\varepsilon|\\
\le& C\left|\int_0^{T^*}\int_{\mathbb{R}^2}(q_\varepsilon-q_0)(\xi)\widehat{|\psi_\varepsilon|^2}(\xi)\overline{\widehat{\psi v}}(\xi)\,d\xi\,dt\right| + C\left|\int_0^{T^*}\int_{\mathbb{R}^2}q_0(\xi)(\widehat{|\psi_\varepsilon|^2}-\widehat{|\psi|^2})(\xi)\overline{\widehat{\psi v}}(\xi)\,d\xi\,dt\right|\\
:=&I^3_\varepsilon + I^4_\varepsilon
\end{aligned}$$

Estimate for I^3_ε: By Hölder inequality and

$$I^3_\varepsilon \le \left\|\widehat{|\psi_\varepsilon|^2}\right\|_{L^2_t(0,T^*;L^2)}\left\|\chi(\Omega_\varepsilon)(q_\varepsilon-q_0)\widehat{\psi v}\right\|_{L^2_t(0,T^*;L^2)}.$$

Since $\chi(\Omega_\varepsilon)(q_\varepsilon-q_0)\widehat{\psi v}(t,\xi)\to 0$ pointwise in $(0,T^*)\times\mathbb{R}^2$ and $|\chi(\Omega_\varepsilon)(q_\varepsilon-q_0)\widehat{\psi v}| < C|\widehat{\psi v}|$ and $\widehat{\psi v}\in L^2_t(0,T^*;L^2)$, dominated convergence theorem follows

$$\left\|\chi(\Omega_\varepsilon)(q_\varepsilon-q_0)\widehat{\psi v}\right\|_{L^2_t(0,T^*;L^2)} \to 0$$

when ε tends to 0. Hence, $I^3_\varepsilon \to 0$ when ε tends to 0.

Estimate for I^4_ε: By Plancherel identity

$$I^4_\varepsilon = \left|\int_0^{T^*}\int_{\mathbb{R}^2}(|\psi_\varepsilon|^2-|\psi|^2)\,\overline{\mathcal{F}^{-1}(q_0(\xi)\widehat{\psi v}(\xi))}\,dX\,dt\right|.$$

Since $\left\||\psi_\varepsilon|^2\right\|_{L^\infty_t(0,T^*;L^2)}$ is uniformly bounded then by similar argument of getting (4.8), we get that $|\psi_\varepsilon|^2 \to |\psi|^2$ in $L^\infty_t(0,T^*;L^2)$ weak-star. Therefore we need to show that $\mathcal{F}^{-1}(q_0(\xi)\widehat{\overline{\psi}v}(\xi)) \in L^1_t(0,T^*;L^2)$. Indeed, by Plancherel's identity

$$\begin{aligned}\left\|\mathcal{F}^{-1}(q_0(\xi)\widehat{\overline{\psi}v}(\xi))\right\|_{L^1_t(0,T^*;L^2)} &= \left\|q_0(\xi)\widehat{\overline{\psi}v}(\xi)\right\|_{L^1_t(0,T^*;L^2)} \\ &\leq C\left\|\widehat{\overline{\psi}v}(\xi)\right\|_{L^1_t(0,T^*;L^2)} \\ &= C\left\|\psi v\right\|_{L^1_t(0,T^*;L^2)} \\ &\leq C\left\|\psi\right\|_{L^2_t(0,T^*;L^4)}\left\|v\right\|_{L^2_t(0,T^*;L^4)} < \infty.\end{aligned}$$

The proof of (4.10) is similar to that of (4.9), and this concludes the proof of weak convergence.

In order to prove the strong convergence, we note that by a similar argument as above, $(\psi_\varepsilon, \partial_x\phi_\varepsilon)$ tends to $(\psi, \partial_x\phi)$ in $L^\infty_t(0,T^*;H^s(\mathbb{R}^2))$ weak-star. Furthermore, by (4.1), Plancherel identity and using the fact that $s > 3$, we have

$$\begin{aligned}\left\|\partial_t\partial_x\phi_\epsilon\right\|_{L^\infty_t(0,T^*;H^{s-2})} =& \left\|F_\epsilon(2Re(\partial_t\psi_\epsilon\overline{\psi_\epsilon}))\right\|_{L^\infty_t(0,T^*;H^{s-2})} \\ \leq& C\left\|\langle\xi\rangle^{s-2}\mathcal{F}(Re(\partial_t\psi_\epsilon\overline{\psi_\epsilon}))\right\|_{L^\infty_t(0,T^*;L^2)} \\ =& C\left\|J^{s-2}Re(\partial_t\psi_\epsilon\overline{\psi_\epsilon})\right\|_{L^\infty_t(0,T^*;L^2)} \\ \leq& C(\left\|J^{s-2}(\partial_t\psi_\epsilon\overline{\psi_\epsilon})\right\|_{L^\infty_t(0,T^*;L^2)} + \left\|J^{s-2}(\overline{\partial_t\psi_\epsilon}\psi_\epsilon)\right\|_{L^\infty_t(0,T^*;L^2)}) \\ \leq& C\left\|\psi_\epsilon\right\|_{L^\infty_t(0,T^*;H^s)}\left\|\partial_t\psi_\epsilon\right\|_{L^\infty_t(0,T^*;H^{s-2})}.\end{aligned}$$

Therefore, Simon's lemma 4.2 and the identity of the weak and strong limit follow that $(\psi_\varepsilon, \partial_x\phi_\varepsilon)$ tends to $(\psi, \partial_x\phi)$ in $L^\infty_t(0,T^*;H^{s-\delta}_{loc}(\mathbb{R}^2))$ strongly, where $0 < \delta < 2$. □

5. Remarks on the complete full dispersion system

One can of course apply the same truncation process on the ϕ equation for the complete full dispersion Davey-Stewartson systems (1.16), (1.9) yielding approximations of them. Using Lemma 2.1, the fact that the linear dispersive part is skew-adjoint and standard arguments, one obtains straightforwardely the those two systems are locally well posed for initial data in $H^s(\mathbb{R}^2), s > 1$ on a time interval $[0, T^*)$ where T^* does not depend on ε. Note that this result does not depends on the dispersion.

In presence of surface tension, the resulting equation in ψ_ε is reminiscent of a fractional nonlinear Schrödinger equation.

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