

# PROBABILISTIC MAPPINGS AND BAYESIAN NONPARAMETRICS

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**ABSTRACT.** In this paper we develop a functorial language of probabilistic mappings and apply it to basic problems in Bayesian nonparametrics. First we extend and unify the Kleisli category of probabilistic mappings proposed by Lawvere and Girshick with the category of statistical models proposed by Chentsov and Morse-Sacksteder. Then we introduce the notion of a Bayesian statistical model that formalizes the notion of a parameter space with a given prior distribution in Bayesian statistics. We give a formula for posterior distributions, assuming that the underlying parameter space of a Bayesian statistical model is a Souslin space and the sample space of the Bayesian statistical model is a subset in a complete connected finite dimensional Riemannian manifold. Then we give a new proof of the existence of Dirichlet measures over any measurable space using a functorial property of the Dirichlet map constructed by Sethuraman.

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## 1. INTRODUCTION

In Bayesian statistics we start with a formulation of a model that we hope is adequate to describe the situation of interest. We then formulate a prior distribution over the unknown parameters of the model, which is meant to capture our beliefs about the situation before seeing the data. The parameter in a Bayesian problem is (or parameterizes) the unknown (regular conditional) probability measure that governs the distribution of the observable. After observing some data we apply Bayes' formula to obtain the posterior distribution for these unknowns. From this posterior distribution we compute predictive distributions for future observations. In particular, this posterior can then be used as a new prior before seeing the next data. To apply the available Bayes' formula, see e.g. [Schervish1997, Theorem 1.31, p. 16] for the most general form, one needs to impose the assumption of a dominated measure on Bayesian parameter models [Schervish1997], [OT2011]. The prior is the theoretically most subtle part of Bayesian statistics. Formally, the prior is a probability distribution on a family of probability distributions, and to handle that formally, functional analytic concepts and methods are needed.

In the present paper we introduce the notion of a Bayesian statistical model, using the functorial language of probabilistic mappings. We derive a new Bayes' formula, relaxing the condition of a dominated measure model. It is known that many infinite dimensional statistical models are not dominated, e.g., the family of posterior distributions of a Dirichlet process on  $\mathbb{R}$  is not dominated, see e.g., [Orbanz2008, Remark 60], in particular the family  $\mathcal{P}(\mathbb{R})$  of all probability measures on  $\mathbb{R}$  is not dominated, see also [HS1949] for a necessary condition for a family of distributions to be a dominated measure family. The only assumption we pose in our new formula is that the underlying parameter space of a Bayesian statistical model is a Souslin space and the underlying sample space  $\mathcal{X}$  is a subset of a finite dimensional complete Riemannian manifold  $(M^n, g)$  with the induced distance generated by  $g$  on  $M^n$ . Recall that a Souslin space is a Hausdorff space admitting a surjective continuous mapping from a complete metrizable space. In particular, every Polish space (a complete separable metrizable space) is a Souslin space, and more general, every standard Borel space (a measurable space admitting a bijective, bimeasurable correspondence with a Borel subset of a Polish space) is a Souslin space [Bogachev2007, Corolary 6.6.7, p. 22, vol.2], see [Bogachev2007, §6.6, vol. 2] for more detailed discussions on Souslin spaces. Finally we revisit the Dirichlet measures over separable metrizable spaces using the functorial language of probabilistic mappings.

To give an overview of the remainder of our paper let us fix some notations. For a measurable space  $\mathcal{X}$ , which sometimes we shall denote by  $(\mathcal{X}, \Sigma_{\mathcal{X}})$  if we wish to specify the underlying  $\sigma$ -algebra  $\Sigma_{\mathcal{X}}$ , let  $\mathcal{S}(\mathcal{X})$  (resp.  $\mathcal{M}(\mathcal{X})$  and  $\mathcal{P}(\mathcal{X})$ ) denote the space of finite signed measures on  $\mathcal{X}$  (resp. the space of non-negative measures and the space of probability measures on  $\mathcal{X}$ ). We also set  $\mathcal{M}^*(\mathcal{X}) := \mathcal{M}(\mathcal{X}) \setminus \{0\}$  and denote by  $L(\mathcal{X})$ <sup>1</sup> the space of measurable bounded functions on  $\mathcal{X}$ . Furthermore, we denote by  $1_A$  the indicator (characteristic) function of a measurable set  $A$ . For a  $\sigma$ -additive measure  $\mu$  we denote by  $\mu^*$  its outer measure.

In the second section we extend the Kleisli category of probabilistic mappings proposed by Lawvere [Lawvere1962] and Giry [Giry1982] and unify it with the category of statistical models proposed by Chentsov [Chentsov1965], [Chentsov1972] and by Morse-Sacksteder [MS1966]. In the third section, using the results in the second section, we introduce the notion of a Bayesian statistical model that formalizes the notion of a parameter space with a prior distribution in Bayesian statistics. Then we give a formula for the posterior distribution without the dominated measure model condition. In the fourth section we give a new proof of the existence of Dirichlet measures over any measurable space using a functorial property of the Dirichlet map constructed by Sethuraman. This paper also contains an Appendix where we give a proof of a theorem used in the third section.

## 2. PROBABILISTIC MAPPINGS AND CATEGORY OF STATISTICAL MODELS

In this section, first, we extend Lawvere's natural  $\sigma$ -algebra on  $\mathcal{P}(\mathcal{X})$  to the spaces  $\mathcal{S}(\mathcal{X})$ ,  $\mathcal{M}(\mathcal{X})$ ,  $\mathcal{M}^*(\mathcal{X})$ ,  $L(\mathcal{X})$ . Then we prove their important properties (Propositions 2.3, 2.9, 2.11) that shall be needed in later sections. We extend and survey related results due to Lawvere, Chentsov, Giry and Morse-Sacksteder (Theorem 2.13, Remark 2.18) concerning the Kleisli category of probabilistic mappings. At the end of this section we embed the Kleisli category of probabilistic mappings into the category of statistical models introduced by Chentsov and Morse-Sacksteder (Definition 2.19, Remark 2.20). We recall the notion of sufficient morphisms, sufficient probabilistic mappings and their relevance to the notion of equivalence between statistical models (Definition 2.21, Examples 2.23, 2.24, Proposition 2.25, Remark 2.26). Finally we prove a structure theorem on the subcategory of equivalent statistical models whose morphisms are defined by sufficient probabilistic mappings (Theorem 2.27, Remark 2.28).

**2.1. Weak topology and  $\sigma$ -algebra on  $\mathcal{M}(\mathcal{X})$ .** Given a measurable space  $\mathcal{X}$ , let  $\mathcal{F}_s(\mathcal{X})$  denote the linear space of simple (step) functions on  $\mathcal{X}$ . There is a natural homomorphism  $I : \mathcal{F}_s(\mathcal{X}) \rightarrow \mathcal{S}^*(\mathcal{X}) := \text{Hom}(S(\mathcal{X}), \mathbb{R})$ ,  $f \mapsto I_f$ , defined by integration:  $I_f(\mu) := \int_{\mathcal{X}} f d\mu$  for  $f \in \mathcal{F}_s(\mathcal{X})$  and  $\mu \in \mathcal{S}(\mathcal{X})$ .

<sup>1</sup>In [Chentsov1972, p. 74] Chentsov used the notation  $L(\mathcal{X}, \Sigma_{\mathcal{X}})$  which is equivalent to our notation, and in [Bogachev2007, p. 371, vol. 2] Bogachev used the notation  $\mathcal{L}_{\Sigma_{\mathcal{X}}}^{\infty}$  instead of our notation  $L(\mathcal{X})$ .

Following Lawvere [Lawvere1962], we shall denote by  $\Sigma_w$  the smallest  $\sigma$ -algebra on  $\mathcal{S}(\mathcal{X})$  such that  $I_f$  is measurable for all  $f \in \mathcal{F}_s(\mathcal{X})$ . We also denote by  $\Sigma_w$  the restriction of  $\Sigma_w$  to  $\mathcal{M}(\mathcal{X})$ ,  $\mathcal{M}^*(\mathcal{X})$  and  $\mathcal{P}(\mathcal{X})$ . Since  $I_f : \mathcal{P}(\mathcal{X}) \rightarrow \mathbb{R}$  is bounded, if  $f$  is bounded, by the Lebesgue dominated convergence theorem, the  $\sigma$ -algebra  $\Sigma_w$  on  $\mathcal{P}(\mathcal{X})$  is the smallest  $\sigma$ -algebra on  $\mathcal{P}(\mathcal{X})$  such that  $I_f$  is measurable for all  $f \in L(\mathcal{X})$ .

**Remark 2.1.** (1) Lawvere [Lawvere1962] and later authors [GH1989], [Kallenberg2017]<sup>2</sup> defined the  $\sigma$ -algebra on  $\mathcal{P}(\mathcal{X})$  as the smallest  $\sigma$ -algebra for which the evaluation map  $ev_A : \mathcal{P} \rightarrow [0, 1]$ ,  $\mu \mapsto \mu(A)$ , is measurable for all  $A \in \Sigma_{\mathcal{X}}$ . It is not hard to see that their definition is equivalent to ours, since  $\{I_{1_A} \mid A \in \Sigma_{\mathcal{X}}\}$  generate the vector space  $\mathcal{F}_s$ . The later space leads directly to the space  $L(\mathcal{X})$ , which is a good object in the Kleisli category of probabilistic mappings as we shall see later.

(2) Any element in  $\mathcal{S}(\mathcal{X})$  (resp. in  $\mathcal{M}(\mathcal{X})$  and in  $\mathcal{P}(\mathcal{X})$ ) can be considered as an element in the space  $\mathbb{R}^{\Sigma_{\mathcal{X}}}$  (resp. in  $(\mathbb{R}_{\geq 0})^{\Sigma_{\mathcal{X}}}$  and in  $[0, 1]^{\Sigma_{\mathcal{X}}}$ ), which carries the canonical product  $\sigma$ -algebra  $\Sigma_{\Pi}$ . Clearly the restriction of  $\Sigma_{\Pi}$  to  $\mathcal{S}(\mathcal{X})$  (resp.  $\mathcal{M}(\mathcal{X})$  and  $\mathcal{P}(\mathcal{X})$ ) is  $\Sigma_w$ . The subsets  $\mathcal{S}(\mathcal{X})$ ,  $\mathcal{M}(\mathcal{X})$  and  $\mathcal{P}(\mathcal{X})$  of  $\mathbb{R}^{\Sigma_{\mathcal{X}}}$  satisfy the  $\sigma$ -additivity constraint and it is known that  $\mathcal{P}(\mathbb{R})$  is not a measurable subset of  $[0, 1]^{\Sigma_{\mathbb{R}}}$  [GR2003, 2.3.2, p. 64].

For a topological space  $\mathcal{X}$  we shall consider the natural Borel  $\sigma$ -algebra  $\mathcal{B}(\mathcal{X})$ , unless otherwise specified. Let  $C_b(\mathcal{X}) \subset L(\mathcal{X})$  be the space of bounded continuous functions on  $\mathcal{X}$ . We denote by  $\tau_v$  the smallest topology on  $\mathcal{S}(\mathcal{X})$  such that for any  $f \in C_b(\mathcal{X})$  the map  $I_f : (\mathcal{S}(\mathcal{X}), \tau_v) \rightarrow \mathbb{R}$  is continuous. We also denote by  $\tau_v$  the restriction of  $\tau_v$  to  $\mathcal{M}(\mathcal{X})$  and  $\mathcal{P}(\mathcal{X})$ , which is also called the weak topology. It is known that  $(\mathcal{P}(\mathcal{X}), \tau_v)$  is separable, metrizable if and only if  $\mathcal{X}$  is [Bogachev2018, Theorem 3.1.4, p. 104], [Parthasarathy1967, Theorem 6.2, p.43]<sup>3</sup>. If  $\mathcal{X}$  is separable and metrizable then the Borel  $\sigma$ -algebra on  $\mathcal{P}(\mathcal{X})$  generated by  $\tau_v$  coincides with  $\Sigma_w$  [GH1989, Theorem 2.3].

In this paper we shall consider only three types of measurable spaces:

- $\mathcal{X}$  is a measurable space with a  $\sigma$ -algebra  $\Sigma_{\mathcal{X}}$ ,
- $\mathcal{X}$  is a separable metrizable space with the associated Borel  $\sigma$ -algebra  $\mathcal{B}(\mathcal{X})$ ,
- Souslin spaces, denoted by  $\Theta$  with Borel  $\sigma$ -algebra  $\mathcal{B}(\Theta)$ , which we shall consider in Subsection 3.2.

**Example 2.2.** Let  $\Omega_k$  be a discrete topological space with  $k$  elements  $\omega_1, \dots, \omega_k$ . We regard  $\Omega_k$  as a measurable space with the Borel  $\sigma$ -algebra  $\mathcal{B}(\Omega_k) = 2^{\Omega_k}$ . The space  $(\mathcal{S}(\Omega_k), \tau_v)$  is homeomorphic to  $\mathbb{R}^k$  with the standard topology. The space  $(\mathcal{M}(\Omega_k), \tau_v)$  is homeomorphic to the quadrant

<sup>2</sup>Kallenberg in [Kallenberg2017, p.1] defined  $\Sigma_w$  on the space of all locally finite measures on  $\mathcal{X}$  as in [Lawvere1962].

<sup>3</sup>If  $\mathcal{X}$  is infinite, then  $(\mathcal{S}(\mathcal{X}), \tau_v)$  is non-metrizable [Bogachev2018, p. 102] (warning: Bogachev's  $\mathcal{M}(\mathcal{X})$  is our  $\mathcal{S}(\mathcal{X})$ ).

$\mathbb{R}_{\geq 0}^n$  and the space  $(\mathcal{P}(\Omega_k), \tau_v)$  is homeomorphic to the simplex  $\Delta_k := \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_i \geq 0 \text{ and } \sum_{i=1}^k x_i = 1\}$ . Furthermore  $\mathcal{F}_s(\Omega_k) = L(\Omega_k) = C_b(\Omega_k) = \mathbb{R}^k$ .

**Proposition 2.3.** (1) Assume that  $\Sigma_{\mathcal{X}}$  has a countable generating algebra  $\mathcal{A}_{\mathcal{X}}$ . Then  $\mathcal{M}(\mathcal{X})$  is a measurable subset of  $\mathcal{S}(\mathcal{X})$ , and  $\mathcal{P}(\mathcal{X})$  and  $\mathcal{M}^*(\mathcal{X})$  are measurable subsets of  $\mathcal{M}(\mathcal{X})$ .

(2) The addition  $\mathbf{a} : (\mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X}), \Sigma_w \otimes \Sigma_w) \rightarrow (\mathcal{M}(\mathcal{X}), \Sigma_w)$ ,  $(\mu, \nu) \mapsto \mu + \nu$ , is a measurable map. If  $\mathcal{X}$  is a topological space, then the map  $\mathbf{a}$  is  $\tau_v$ -continuous, i.e., continuous in the  $\tau_v$ -topology.

*Proof.* 1. The  $\sigma$ -algebra  $\Sigma_w$  on  $\mathcal{S}(\mathcal{X})$  is generated by subsets  $\langle A, B^* \rangle := I_{1_A}^{-1}(B^*)$  where  $A \in \Sigma_{\mathcal{X}}$  and  $B^* \in \mathcal{B}(\mathbb{R})$ . We have

$$\mathcal{M}(\mathcal{X}) = \bigcap_{A \in \mathcal{A}_{\mathcal{X}}} \langle A, \mathbb{R}_{\geq 0} \rangle,$$

because an element of  $\mathcal{M}(\mathcal{X})$  has to be nonnegative on every  $A \in \mathcal{A}_{\mathcal{X}}$ . When  $\Sigma_{\mathcal{X}}$  has a countable generating algebra  $\mathcal{A}_{\mathcal{X}}$ , this is a countable intersection, implying the measurability of  $\mathcal{M}(\mathcal{X})$ . And since

$$\mathcal{M}^*(\mathcal{X}) = \mathcal{M}(\mathcal{X}) \cap \langle \mathcal{X}, \mathbb{R}_{>0} \rangle$$

and

$$\mathcal{P}(\mathcal{X}) = \mathcal{M}(\mathcal{X}) \cap \langle \mathcal{X}, 1 \rangle$$

we then also obtain the measurability of  $\mathcal{M}^*(\mathcal{X})$  and  $\mathcal{P}(\mathcal{X})$ . This proves (1).

2. To prove the measurability of the map  $\mathbf{a}$  it suffices to show that for any  $f \in \mathcal{F}_s(\mathcal{X})$  the composition  $I_f \circ \mathbf{a} : \mathcal{M}(\mathcal{X}) \times \mathcal{M}(\mathcal{X}) \rightarrow \mathbb{R}_{\geq 0}$  is measurable. Using the formula

$$I_f \circ \mathbf{a}(\mu, \nu) = I_{1_{\mathcal{X}}}(f \cdot \mu) + I_{1_{\mathcal{X}}}(f \cdot \nu),$$

we reduce the measurability of  $I_f \circ \mathbf{a}$  to the measurability of the map  $\mathbf{a} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ ,  $(x, y) \mapsto (x + y)$ , which is well-known.

Similarly we prove the continuity of the map  $\mathbf{a}$ , if  $\mathcal{X}$  is a topological space. This proves Proposition 2.3(2).  $\square$

By definition, any measure on  $\mathcal{M}(\mathcal{X})$  (resp. on  $\mathcal{M}^*(\mathcal{X})$  and on  $\mathcal{P}(\mathcal{X})$ ) is obtained by restricting a measure on  $\mathcal{S}(\mathcal{X})$  to  $\mathcal{M}(\mathcal{X})$  (resp. to  $\mathcal{M}^*(\mathcal{X})$  and  $\mathcal{P}(\mathcal{X})$ ). By Proposition 2.3 the restriction of any measure on  $\mathcal{S}(\mathcal{X})$  to  $\mathcal{M}(\mathcal{X})$  (resp. to  $\mathcal{M}^*(\mathcal{X})$  and  $\mathcal{P}(\mathcal{X})$ ) is a measure on  $\mathcal{M}(\mathcal{X})$  (resp. on  $\mathcal{M}^*(\mathcal{X})$  and  $\mathcal{P}(\mathcal{X})$ ) if  $\Sigma_{\mathcal{X}}$  is countably generated.

## 2.2. Probabilistic mappings and associated functors.

**Definition 2.4.** A probabilistic mapping (or an arrow) from a measurable space  $\mathcal{X}$  to a measurable space  $\mathcal{Y}$  is a measurable mapping from  $\mathcal{X}$  to  $(\mathcal{P}(\mathcal{Y}), \Sigma_w)$ .

**Remark 2.5.** Definition 2.4 agrees with the definition of morphisms in the Kleisli category of the Giry probability monad, see Remark 2.18 below. At this stage we have not defined the category where a probabilistic mapping

is a morphism, so we choose the word “arrow”. Furthermore we would like to emphasize that a probabilistic mapping from  $\mathcal{X} \rightarrow \mathcal{Y}$  *is not* a mapping from  $\mathcal{X}$  to  $\mathcal{Y}$  but defines an *arrow* from  $\mathcal{X}$  to  $\mathcal{Y}$ . A probabilistic mapping is equivalent to a Markov kernel, see also Remark 2.18.

We shall denote by  $\overline{T} : \mathcal{X} \rightarrow (\mathcal{P}(\mathcal{Y}), \Sigma_w)$  the measurable mapping defining/generating a probabilistic mapping  $T : \mathcal{X} \rightsquigarrow \mathcal{Y}$ . Similarly, for a measurable mapping  $\mathbf{p} : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{Y})$  we shall denote by  $\underline{\mathbf{p}} : \mathcal{X} \rightsquigarrow \mathcal{Y}$  the generated probabilistic mapping. Note that a probabilistic mapping is denoted by a curved arrow and a measurable mapping by a straight arrow.

**Example 2.6.** (1) Assume that  $\mathcal{X}$  is separable and metrizable. Then the identity mapping  $Id_{\mathcal{P}} : (\mathcal{P}(\mathcal{X}), \tau_v) \rightarrow (\mathcal{P}(\mathcal{X}), \tau_v)$  is continuous, and hence measurable w.r.t. the Borel  $\sigma$ -algebra  $\Sigma_w = \mathcal{B}(\tau_v)$ . Consequently  $Id_{\mathcal{P}}$  generates a probabilistic mapping  $ev : (\mathcal{P}(\mathcal{X}), \mathcal{B}(\tau_v)) \rightsquigarrow (\mathcal{X}, \mathcal{B}(\mathcal{X}))$  and we write  $\overline{ev} = Id_{\mathcal{P}}$ . Similarly, for any measurable space  $\mathcal{X}$ , we also have an arrow (a probabilistic mapping)  $ev : (\mathcal{P}(\mathcal{X}), \Sigma_w) \rightsquigarrow \mathcal{X}$  generated by the measurable mapping  $\overline{ev} = Id_{\mathcal{P}}$ .

(2) Let  $\delta_x$  denote the Dirac measure concentrated at  $x$ . It is known that the map  $\delta : \mathcal{X} \rightarrow (\mathcal{P}(\mathcal{X}), \Sigma_w)$ ,  $x \mapsto \delta(x) := \delta_x$ , is measurable [GR2003, Proposition 2.2.4, p. 62]. If  $\mathcal{X}$  is a topological space, then the map  $\delta : \mathcal{X} \rightarrow (\mathcal{P}(\mathcal{X}), \tau_v)$  is continuous, since the composition  $I_f \circ \delta : \mathcal{X} \rightarrow \mathbb{R}$  is continuous for any  $f \in C_b(\mathcal{X})$ . Hence, if  $\kappa : \mathcal{X} \rightarrow \mathcal{Y}$  is a measurable mapping between measurable spaces (resp. a continuous mapping between separable metrizable spaces), then the map  $\overline{\kappa} : \mathcal{X} \xrightarrow{\delta \circ \kappa} \mathcal{P}(\mathcal{Y})$  is a measurable mapping (resp. a continuous mapping). We regard  $\kappa$  as a probabilistic mapping defined by  $\delta \circ \kappa : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{Y})$ . In particular, the identity mapping  $Id : \mathcal{X} \rightarrow \mathcal{X}$  of a measurable space  $\mathcal{X}$  is a probabilistic mapping generated by  $\delta : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{X})$ . Graphically speaking, any straight arrow (a measurable mapping)  $\kappa : \mathcal{X} \rightarrow \mathcal{Y}$  between measurable spaces can be seen as a curved arrow (a probabilistic mapping).

(3) We would like to present here a non-trivial construction of probabilistic mappings, which comes from the theory of random mappings [Kif86], [Kif88]. Consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a *random mapping*  $T$ , which is a measurable mapping  $T : \Omega \times \mathcal{X} \rightarrow \mathcal{Y}$  (in case  $\mathcal{X} = \mathcal{Y}$ , the random mapping  $T$  is the source to generate a random dynamical system [Arnold1998]). Such a random mapping  $T : \Omega \times \mathcal{X} \rightarrow \mathcal{Y}$  then generates a measurable mapping  $\overline{T} : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{Y})$ , defined by

$$(2.1) \quad \overline{T}(x)(B) := \mathbb{P}\{\omega \in \Omega : T(\omega, x) \in B\} = \mathbb{P}\{T^{-1}(\cdot, x)(B)\}, \quad \forall B \in \Sigma_{\mathcal{Y}}.$$

Relation (2.1) shows that once a probabilistic sample space  $(\Omega, \mathcal{F}, \mathbb{P})$  is given, any random mapping would generate a probabilistic mapping. However, since probabilistic mappings need not be constructed this way, the concept of probabilistic mappings can be seen as a generalization of random mappings.

We are also interested in the inverse problem on the representation of probability measures, which can be formulated as follows: Given a probabilistic mapping  $T : \mathcal{X} \rightsquigarrow \mathcal{Y}$ , is there a probabilistic sample space  $(\Omega, \mathcal{F}, \mathbb{P})$  such that  $T$  can be written as a random mapping from  $\Omega \times \mathcal{X}$  to  $\mathcal{Y}$ ? This question appears in several contexts in Ergodic Theory [Kif86], [Kif88]. The answer is affirmative if  $\mathcal{Y}$  is a compact, oriented, connected manifold of class  $C^2$  and  $\mathcal{X}$  is a  $C^0$  manifold with some additional assumptions on the decay of the densities, where one can choose  $\Omega := \mathcal{Y}$  (see details in [JJRMJPCR16, Theorem A]).

Given a probabilistic mapping  $T : \mathcal{X} \rightsquigarrow \mathcal{Y}$ , we define a linear map  $T^* : L(\mathcal{Y}) \rightarrow L(\mathcal{X})$  as follows

$$(2.2) \quad T^*(f)(x) := I_f(\overline{T}(x)) = \int_{\mathcal{Y}} f d\overline{T}(x)$$

which coincides with the classical formula (5.1) in [Chentsov1972, p. 66] for the transformation of a bounded measurable  $f$  under a Markov morphism (i.e., a probabilistic mapping)  $T$ . In particular, if  $\kappa : \mathcal{X} \rightarrow \mathcal{Y}$  is a measurable mapping, then we have  $\kappa^*(f)(x) = f(\kappa(x))$ , since  $\overline{\kappa} = \delta \circ \kappa$ .

**Remark 2.7.** If  $\kappa : \mathcal{X} \rightarrow \mathcal{Y}$  is a measurable mapping then  $\kappa^*(\mathcal{F}_s(\mathcal{Y})) \subset \mathcal{F}_s(\mathcal{X})$ . For a general probabilistic mapping  $T : \mathcal{X} \rightsquigarrow \mathcal{Y}$ , we don't have the inclusion  $T^*(\mathcal{F}_s(\mathcal{Y})) \subset \mathcal{F}_s(\mathcal{X})$ , since the cardinal  $\#(T^*(1_B)(x) | x \in \mathcal{X})$  can be infinite. For example, consider  $T = ev : \mathcal{X} := \mathcal{P}(\mathcal{Z}) \rightsquigarrow \mathcal{Y} := \mathcal{Z}$  for some measurable space  $\mathcal{Z}$  and  $\emptyset \neq B \neq \mathcal{Z}$ . Then  $ev^*(1_B)(\mu) = \mu(B)$  for any  $\mu \in \mathcal{X} = \mathcal{P}(\mathcal{Z})$ . Therefore  $\#(ev^*(1_B)(\mu) | \mu \in \mathcal{P}(\mathcal{Z})) = \#(\mu(B) | \mu \in \mathcal{P}(\mathcal{Z})) = \infty$ , if  $\mathcal{Z} = [0, 1]$  and  $B = [0, 1/2]$ .

**Definition 2.8.** Let  $\Sigma_*$  be the smallest  $\sigma$ -algebra on  $L(\mathcal{X})$  such that for any  $\mu \in \mathcal{P}(\mathcal{X})$  the evaluation function  $ev_\mu : L(\mathcal{X}) \rightarrow \mathbb{R}, h \mapsto I_h(\mu)$ , is measurable. We shall regard  $L(\mathcal{X})$  as a measurable space endowed with the  $\sigma$ -algebra  $\Sigma_*$ .

**Proposition 2.9.** *Assume that  $T : \mathcal{X} \rightsquigarrow \mathcal{Y}$  is a probabilistic mapping.*

(1) *Then  $T$  induces a linear bounded map  $S_*(T) : \mathcal{S}(\mathcal{X}) \rightarrow \mathcal{S}(\mathcal{Y})$  w.r.t. the total variation norm  $\|\cdot\|_{TV}$ . The restriction  $M_*(T)$  of  $S_*(T)$  to  $\mathcal{M}(\mathcal{X})$  (resp.  $P_*(T)$  of  $S_*(T)$  to  $\mathcal{P}(\mathcal{X})$ ) maps  $\mathcal{M}(\mathcal{X})$  to  $\mathcal{M}(\mathcal{Y})$  (resp.  $\mathcal{P}(\mathcal{X})$  to  $\mathcal{P}(\mathcal{Y})$ ).*

(2) *If  $T$  is a measurable mapping, then  $S_*(T) : (\mathcal{S}(\mathcal{X}), \Sigma_w) \rightarrow (\mathcal{S}(\mathcal{Y}), \Sigma_w)$  is a measurable map. It is  $\tau_v$ -continuous if  $T$  is a continuous map between separable metrizable spaces. Hence the maps  $M_*(T)$  and  $P_*(T)$  are measurable and they are  $\tau_v$ -continuous if  $T$  is a continuous map between separable metrizable spaces.*

(3) *If  $T_1 : \mathcal{X} \rightarrow \mathcal{Y}$  and  $T_2 : \mathcal{Y} \rightarrow \mathcal{Z}$  are measurable mappings then we have*

$$(2.3) \quad S_*(T_2 \circ T_1) = S_*(T_2) \circ S_*(T_1), \quad P_*(T_2 \circ T_1) = P_*(T_2) \circ P_*(T_1).$$

(4) *The map  $T$  also induces a linear measurable map  $T^* : L(\mathcal{Y}) \rightarrow L(\mathcal{X})$ .*

*Proof.* 1. Proposition 2.9 (1) is due to Chentsov [Chentsov1972, Lemma 5.9, p.72] by setting

$$(2.4) \quad S_*(T)(\mu)(B) := \int_{\mathcal{X}} \overline{T}(x)(B) d\mu(x)$$

for any  $\mu \in \mathcal{S}(\mathcal{X})$  and  $B \in \Sigma_{\mathcal{Y}}$ .

2. Assume that  $T$  is a measurable mapping. To prove that  $S_*(T)$  is a measurable mapping, it suffices to show that for any  $f \in \mathcal{F}_s(\mathcal{Y})$  the composition  $I_f \circ S_*(T) : \mathcal{S}(\mathcal{X}) \rightarrow \mathbb{R}$  is measurable. The latter assertion follows from the identity  $I_f \circ S_*(T) = I_{T^*f}$ , taking into account that  $T^*(f) \in \mathcal{F}_s(\mathcal{X})$ . In the same way we prove that  $S_*(T)$  is  $\tau_v$ -continuous, if  $T$  is continuous map between separable metrizable spaces. The statement concerning  $M_*(T)$  and  $S_*(T)$  is a consequence of the first two assertions of Proposition 2.9(2). Note that the measurability of  $P_*(T)$  has been first noticed by Lawvere [Lawvere1962] by letting  $P_*(T)(\mu)$  be defined by the same formula (2.4), where  $\mu \in \mathcal{P}(\mathcal{X})$ .

3. The first identity in (2.3) is obvious, the second one is a consequence of the first one.

4. The linearity of  $T^*$  and the inclusion  $T^*(L(\mathcal{Y})) \subset L(\mathcal{X})$  have been proved in [Chentsov1972, Corollary, p. 66]. Here we provide an alternative shorter proof. Let  $T^*$  be defined by (2.2) and  $f \in L(\mathcal{Y})$ . Since

$$\sup_{x \in \mathcal{X}} |T^*(f)(x)| \leq \sup_{y \in \mathcal{Y}} |f(y)|$$

the function  $T^*(f)$  is bounded on  $\mathcal{X}$ , which is what we needed to prove.

Next we shall show that  $T^*$  is a measurable function. It suffices to show that for any  $f \in L(\mathcal{Y})$  the map  $T^*(f) : \mathcal{X} \rightarrow \mathbb{R}$  is measurable, i.e., for any Borel set  $I \subset \mathbb{R}$  we have  $T^*(f)^{-1}(I) \in \Sigma_{\mathcal{X}}$ . Note that

$$T^*(f)^{-1}(I) = \{x \in \mathcal{X} \mid T^*(f)(x) \in I \iff I_f(\overline{T}(x)) \in I\}.$$

Since  $I_f : \mathcal{P}(\mathcal{Y}) \rightarrow \mathbb{R}$  is measurable, the set  $I_f^{-1}(I)$  is measurable. Since  $\overline{T} : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{Y})$  is measurable, the set  $(T^*(f))^{-1}(I) = \overline{T}^{-1}(I_f^{-1}(I))$  is measurable.

This completes the proof of Proposition 2.9.  $\square$

**Example 2.10.** By [GH1989, Theorem 2.3] the  $\sigma$ -algebra  $\Sigma_w$  on  $\mathcal{S}(\Omega_k)$  is the usual Borel  $\sigma$ -algebra on  $\mathbb{R}^k$ . Identifying  $L(\Omega_k) = C_b(\Omega_k)$  with  $\mathcal{S}(\Omega_k)$  via the Euclidean metric, it is not hard to see that  $\Sigma_*$  coincides with the Borel  $\sigma$ -algebra on  $\mathbb{R}^k$  and  $\tau_v^*$  coincides with the usual topology on  $\mathbb{R}^k$ .

**Proposition 2.11.** (1) *The projection  $\pi_{\mathcal{X}} : \mathcal{M}^*(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X}), \mu \mapsto \mu(\mathcal{X})^{-1} \cdot \mu$ , is a measurable retraction. If  $\mathcal{X}$  is a topological space then  $\pi_{\mathcal{X}}$  is  $\tau_v$ -continuous.*

(2) If  $\Sigma_{\mathcal{X}}$  is countably generated, then we have the following commutative diagram

$$\begin{array}{ccccc}
 \mathcal{P}^2(\mathcal{X}) & \xrightarrow{i_{p,m}} & \mathcal{M}(\mathcal{P}(\mathcal{X})) & \xrightarrow{i_{m,s}} & \mathcal{S}(\mathcal{P}(\mathcal{X})) \\
 \downarrow P_*(i_{p,m}) & & \downarrow M_*(i_{p,m}) & & \downarrow S_*(i_{p,m}) \\
 \mathcal{P}(\mathcal{M}(\mathcal{X})) & \xrightarrow{i_{p,m}} & \mathcal{M}^2(\mathcal{X}) & \xrightarrow{i_{m,s}} & \mathcal{S}(\mathcal{M}(\mathcal{X})) \\
 \downarrow P_*(i_{m,s}) & & \downarrow M_*(i_{m,s}) & & \downarrow S_*(i_{m,s}) \\
 \mathcal{P}(\mathcal{S}(\mathcal{X})) & \xrightarrow{i_{p,m}} & \mathcal{M}(\mathcal{S}(\mathcal{X})) & \xrightarrow{i_{m,s}} & \mathcal{S}^2(\mathcal{X})
 \end{array}$$

where all arrows are measurable embeddings. If  $\mathcal{X}$  is a separable metrizable space then the arrows are  $\tau_v$ -continuous embeddings.

*Proof.* 1. Clearly  $\pi_{\mathcal{X}}$  is a retraction. The map  $\pi_{\mathcal{X}}$  is measurable, since the mapping  $I_{1_{\mathcal{X}}} : \mathcal{M}^*(\mathcal{X}) \rightarrow \mathbb{R}_{>0}$ ,  $\mu \mapsto \mu(\mathcal{X})$ , is measurable, and hence the map  $(I_{1_{\mathcal{X}}})_{-1} : \mathcal{M}^*(\mathcal{X}) \rightarrow \mathbb{R}_{>0}$ ,  $\mu \mapsto \mu(\mathcal{X})^{-1}$ , is also measurable. Similarly we prove that  $\pi_{\mathcal{X}}$  is continuous in the  $\tau_v$ -topology, if  $\mathcal{X}, \mathcal{Y}$  are topological spaces.

2. The measurability of the horizontal mappings  $i_{p,m}$  and  $i_{m,s}$  follows from Proposition 2.3(1). If  $\mathcal{X}$  is a separable metrizable space, these maps are inclusion maps, and hence they are continuous by definition.

The measurability (resp. the continuity) of the vertical mappings  $S_*(i_{p,m})$  and  $S_*(i_{m,s})$  follows from the measurability (resp. the continuity) of the maps  $i_{p,m}, i_{m,s}$  and Proposition 2.9(2). The measurability (resp. the continuity) of the vertical mappings  $P_*(i_{p,m})$  and  $S_*(i_{m,s})$  (resp.  $M_*(i_{p,m}), M_*(i_{m,s})$ ) follows from the corresponding assertion concerning the functor  $P_*$  and the measurability (continuity) for the horizontal mappings.

3. The commutativity of the diagram is obvious. This completes the proof of Proposition 2.11.  $\square$

Let us recall the following result due to Giry.

**Lemma 2.12.** ([Giry1982]) *Let  $ev_{\mathcal{P}} : \mathcal{P}^2(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X})$  be defined by*

$$(2.5) \quad ev_{\mathcal{P}}(\nu_{\mathcal{P}})(A) := \int_{\mathcal{P}(\mathcal{X})} I_{1_A}(\mu) d\nu_{\mathcal{P}}(\mu)$$

for all  $\nu_{\mathcal{P}} \in \mathcal{P}^2(\mathcal{X})$  and  $A \in \Sigma_{\mathcal{X}}$ .

(1) *The composition  $ev_{\mathcal{P}} \circ \delta : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X})$  is the identity map.*

(2) *Assume that  $\kappa : \mathcal{X} \rightarrow \mathcal{Y}$  is a measurable mapping. Then we have the following commutative diagrams*

$$\begin{array}{ccc}
 \mathcal{P}^2(\mathcal{X}) & \xrightarrow{P_*^2(\kappa)} & \mathcal{P}^2(\mathcal{Y}) \\
 \downarrow ev_{\mathcal{P}} & & \downarrow ev_{\mathcal{P}} \\
 \mathcal{P}(\mathcal{X}) & \xrightarrow{P_*(\kappa)} & \mathcal{P}(\mathcal{Y}).
 \end{array}$$

(3) The mapping  $ev_P$  is a measurable mapping. It is  $\tau_v$ -continuous, if  $\mathcal{X}$  is a separable metrizable space.

Note that Giry considered the smaller category of Polish spaces but his proof is also valid for the category of separable metrizable spaces.

**Theorem 2.13.** (1) Probabilistic mappings  $T$  are morphisms in the category of measurable spaces  $\mathcal{X}$ . Furthermore  $M_*$  and  $P_*$  are functors from the category of measurable spaces whose morphisms are probabilistic mappings to the category of nonnegative finite measure spaces and the category probability measure spaces resp., whose morphisms are measurable mappings. If  $\bar{T} : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{Y})$  is a continuous mapping between separable metrizable spaces then  $M_*(T) : \mathcal{M}(\mathcal{X}) \rightarrow \mathcal{M}(\mathcal{Y})$  and  $P_*(T) : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{Y})$  are  $\tau_v$ -continuous.

(2) If  $\nu \ll \mu \in \mathcal{M}^*(\mathcal{X})$  then  $M_*(T)(\nu) \ll M_*(T)(\mu)$ .

*Proof.* 1. The statement of Theorem 2.13(1) concerning the functor  $P_*$  is a consequence of Giry's theorem stating that the triple  $(P_*, \delta, ev_P)$  is a monad in the category of measurable spaces whose morphisms are measurable mappings (resp. in the category of Polish spaces whose morphisms are continuous mappings, see also the remark before Theorem 2.13 and Remark 2.18 below) [Giry1982, Theorem 1, p. 70] and his observation that the Kleisli category of the monad  $(P_*, \delta, ev_P)$  is the category of measurable spaces whose morphisms are probabilistic mappings. In other words, Giry's theorem says that Theorem 2.13 is valid in the subcategory of measurable spaces whose morphisms are measurable mappings,  $\delta$  is a natural transformation of the functor  $Id_{\mathcal{X}}$  to the functor  $P_*$  on this subcategory and  $ev_P$  is a natural transformation of the functor  $P_*^2$  to the the functor  $P_*$  on the same subcategory. Note that the last assertion is equivalent to the statement of Lemma 2.12(2) (the associativity of  $ev_P$  follows from the identity (2.11) below).

Taking into account Giry's theorem, the statement of Theorem 2.13 concerning  $P_*$  is a consequence of [MacLane1994, Theorem 1, p. 43] on the structure of the Kleisli category of a monad. Since the proof of [MacLane1994, Theorem 1, p.143] is only sketched, we shall prove Proposition 2.14 below, which proves the functoriality of  $P_*$ . Since  $M_*(T)(c \cdot \mu) = c \cdot M_*(T)(\mu)$  for any  $c \in \mathbb{R}_{\geq 0}$  and  $\mu \in \mathcal{P}(\mathcal{X})$ , the functoriality of  $M_*(T)$  is a consequence of the functoriality of  $P_*(T)$ , and the measurability of  $M_*(T)$  is a consequence of measurability of  $P_*(T)$ . We provide furthermore a new categorical proof for the associativity of the composition of the Markov kernels (Proposition 2.16).

**Proposition 2.14.** *We have*

$$(2.6) \quad P_*(T_3 \circ T_2) = P(T_3) \circ P_*(T_2).$$

*Proof.* The proof consists of two steps. In the first step, using (2.4) we shall prove relations (2.7) and (2.8) below. (If we want to construct the Kleisli category of a monad, we take these formulas as defining relations for the functor  $P_*$  and for the composition rule of morphisms in the Kleisli category.)

$$(2.7) \quad P_*(T) = ev_P \circ P_*(\overline{T}),$$

$$(2.8) \quad \overline{T_2 \circ T_1} = P_*(T_2) \circ \overline{T_1}.$$

Note that (2.7) is a consequence of the following straightforward computation for any  $B \in \Sigma_{\mathcal{Y}}$  and any  $\mu \in \mathcal{P}(\mathcal{X})$ ,

$$ev_P \circ P_*(\overline{T})(\mu)(B) = \int_{\mathcal{P}(\mathcal{Y})} I_{1_B} dP_*(\overline{T})\mu = \int_{\mathcal{X}} \overline{T}(x)(B) d\mu(x) = P_*(T)(\mu)(B).$$

We leave the reader to verify (2.8), which has been verified by Giry [Giry1982] as we mentioned above, (we prefer to consider (2.8) as a defining relation and hence we did not recall the known composition rule for Markov kernels in this paper).

In the second step we examine the following diagram:

$$(2.9) \quad \begin{array}{ccccc} & & & & \mathcal{P}^3(\mathcal{X}_4) \\ & & & & \downarrow P_*(ev_P) \\ & & & & \mathcal{P}^2(\mathcal{X}_4) \\ & & & \nearrow P_*^2(\overline{T}_3) & \\ & & \mathcal{P}^2(\mathcal{X}_3) & \xrightarrow{P_*^2(T_3)} & \mathcal{P}^2(\mathcal{X}_4) \\ & & \downarrow P_*^2(T_3) & & \downarrow ev_P \\ & & \mathcal{P}(\mathcal{X}_3) & \xrightarrow{P_*(T_3)} & \mathcal{P}(\mathcal{X}_4) \\ & \nearrow P_*(\overline{T}_2) & & \nearrow P_*(\overline{T}_3) & \\ \mathcal{P}(\mathcal{X}_2) & \xrightarrow{P_*(T_2)} & \mathcal{P}(\mathcal{X}_3) & \xrightarrow{P_*(T_3)} & \mathcal{P}(\mathcal{X}_4) \\ \uparrow \overline{T}_1 & \downarrow ev & \uparrow \overline{T}_2 & \downarrow ev & \uparrow \overline{T}_3 \\ X_1 & \xrightarrow{T_1} & X_2 & \xrightarrow{T_2} & X_3 & \xrightarrow{T_3} & X_4, \end{array}$$

By (2.7), (2.8), and using (2.3), we have

$$(2.10) \quad \begin{aligned} P_*(T_3 \circ T_2) &= ev_P \circ P_*(\overline{T_3 \circ T_2}) \\ &= ev_P \circ P_*(P_*(T_3) \circ \overline{T_2}) \\ &= ev_P \circ P_*(ev_P) \circ P_*^2(\overline{T_3}) \circ P_*(\overline{T_2}). \end{aligned}$$

**Lemma 2.15.** ([Giry1982, p. 71]) *We have the following identity*

$$(2.11) \quad ev_P \circ P_*(ev_P) = ev_P \circ ev_P.$$

From Lemma 2.12 (3) we obtain immediately the following identity

$$(2.12) \quad ev_P \circ P_*^2(\overline{T}) = P_*(\overline{T}) \circ ev_P.$$

Plugging (2.11) and (2.12) into (2.10), taking into account (2.7), we obtain immediately the first identity of (2.6), which also implies the second one. This completes the proof of Proposition 2.14.  $\square$

**Proposition 2.16.** *We have*

$$(2.13) \quad \overline{T_3 \circ (T_2 \circ T_1)} = \overline{(T_3 \circ T_2) \circ T_1}.$$

*Proof.* Proposition 2.16 follows from known properties of Markov kernels. Here we give another short algebraic (categorical) proof. Straightforward computations using (2.8) and (2.7) yield

$$(2.14) \quad \overline{T_3 \circ (T_2 \circ T_1)} = P_*(T_3) \circ \overline{(T_2 \circ T_1)} = P_*(T_3) \circ P_*(T_2) \circ \overline{T_1},$$

$$(2.15) \quad \overline{(T_3 \circ T_2) \circ T_1} = P_*(T_3 \circ T_2) \circ \overline{T_1}.$$

By Proposition 2.14 the LHS of (2.14) equals the LHS of (2.15). This completes the proof of Proposition 2.16.  $\square$

Combining Proposition 2.14 with Propositions 2.16 and 2.9, we complete the proof of Theorem 2.13(1).

2. Theorem 2.13(2) has been proved by Morse-Sacksteder [MS1966, Proposition 5.1]. We also have an alternative proof that is a bit shorter. By the (2.7) and the validity of Theorem 2.13(2) in the case  $T$  is a measurable mapping, it suffices to verify that  $ev_P(\nu) \ll ev_P(\mu)$  which is obvious. This completes the proof of Theorem 2.13.  $\square$

Finally, using relations (2.7), (2.8) we shall prove the following Lemma, which clarifies formula (2.7).

**Lemma 2.17.** *For any probabilistic mapping  $T : \mathcal{X} \rightsquigarrow \mathcal{Y}$  we have  $T = ev \circ \overline{T}$ .*

*Proof.* To prove Lemma 2.17 it suffices to show the following identity

$$(2.16) \quad \overline{T} = \overline{(ev \circ \overline{T})}.$$

Applying (2.7) and (2.8), and noting that  $P_*(ev) = ev_P$  by (2.7), we conclude that

$$(2.17) \quad \overline{(ev \circ \overline{T})} = P_*(ev) \circ \overline{(\overline{T})} = ev_P \circ \overline{(\overline{T})}.$$

Using the formula  $\overline{\kappa} = \delta \circ \kappa$  from Example 2.6(2), we have

$$(2.18) \quad ev_P \circ \overline{(\overline{T})} = ev_P \circ \delta \circ \overline{T}.$$

By Lemma 2.12(2) the RHS of (2.18) is equal to  $\overline{T}$ . This proves (2.16) and completes the proof of Lemma 2.17.  $\square$

**Remark 2.18.** (1) Lawvere introduced the  $\sigma$ -algebra  $\Sigma_w$  on  $\mathcal{P}(\mathcal{X})$  and defined the probabilistic map  $ev$  in terms of Markov kernels [Lawvere1962]. He coined the term of a “probabilistic mapping” and noted that it is equivalent to the notion of a Markov kernel. He also defined  $P_*(T)$  by a formula equivalent to (2.4). In [Giry1982] Giry considered the category of measurable spaces whose morphisms are measurable mappings and the category of Polish spaces  $\mathcal{X}$  whose morphisms are continuous mappings. He noticed that the space  $(\mathcal{P}(\mathcal{X}), \tau_v)$  is again Polish cf. [GH1989, Theorem 2.3]. He proved Theorem 2.13 for the case that  $P$  is a measurable mapping and for the case that  $\mathcal{X}$  is a Polish space, respectively. The triple  $(P_*, \delta, ev_P)$  is now called *Giry’s probability monad*. Giry noticed that the Kleisli category of the probability

monad  $(P_*, \delta, ev_P)$  is the category of measurable spaces whose morphisms are probabilistic mappings. We refer the interested reader to [FP2017] and references therein for other probability monads, to [Panangaden1999] for an exposition of a part of Giry's work [Giry1982], and to [MacLane1994, p. 143] for the Kleisli category of a monad.

(2) Chentsov called the category of Markov kernels *the statistical category* and their morphisms - *Markov morphisms*. Chentsov also showed that  $S_*(T)$  is a linear operator with  $\|S_*(T)\| = 1$  for any Markov kernel  $T$  and that  $S_*(T)$  sends a probability measure to a probability measure [Chentsov1972, Lemma 5.9, p.72]. Markov morphisms  $S_*(T)$  have been investigated further in [AJLS2015], [AJLS2017], [AJLS2018].

(3) Historically, Blackwell was the first who considered probabilistic mappings, which he called *stochastic mappings*, between parametrized statistical models, which he called statistical experiments [Blackwell1953]. LeCam [LeCam1964] used the equivalent terminology *randomized mapping* and Chentsov [Chentsov1965] used the equivalent notion of *transition measure*. All of them are in use today. We decide to use the term *probabilistic mapping*, since we consider the  $\sigma$ -algebra introduced by Lawvere important for the functorial language of probabilistic mappings and its application in Bayesian statistics.

**2.3. Category of statistical models and sufficient probabilistic mappings.** Given a probabilistic mapping  $T$  (in particular a measurable mapping  $\kappa$ ), we shall also use the short notation  $T_*$  (resp.  $\kappa_*$ ) for  $M_*(T)$  (resp. for  $M_*(\kappa)$ ).

**Definition 2.19.** A *statistical model* is a pair  $(\mathcal{X}, P_{\mathcal{X}})$  where  $\mathcal{X}$  is a measurable space and  $P_{\mathcal{X}} \subset \mathcal{P}(\mathcal{X})$ . The *category of statistical models* consists of statistical models as its objects whose morphisms  $\varphi : (\mathcal{X}, P_{\mathcal{X}}) \rightsquigarrow (\mathcal{Y}, P_{\mathcal{Y}})$  are probabilistic mappings  $T : \mathcal{X} \rightsquigarrow \mathcal{Y}$  such that  $T_*(P_{\mathcal{X}}) \subset P_{\mathcal{Y}}$ . A morphism  $T : (\mathcal{X}, P_{\mathcal{X}}) \rightsquigarrow (\mathcal{X}, P_{\mathcal{X}})$  will be called a *unit*, if  $T_* : P_{\mathcal{X}} \rightarrow P_{\mathcal{X}}$  is the identity. Two statistical models  $(\mathcal{X}, P_{\mathcal{X}})$  and  $(\mathcal{Y}, P_{\mathcal{Y}})$  are called *equivalent*, if there exist morphisms  $T_1 : (\mathcal{X}, P_{\mathcal{X}}) \rightsquigarrow (\mathcal{Y}, P_{\mathcal{Y}})$  and  $T_2 : (\mathcal{Y}, P_{\mathcal{Y}}) \rightsquigarrow (\mathcal{X}, P_{\mathcal{X}})$  such that  $T_1 \circ T_2$  and  $T_2 \circ T_1$  are units. In this case  $T_1$  and  $T_2$  will be called *equivalences*.

**Remark 2.20.** (1) The Kleisli category of probabilistic mappings can be realized as a subcategory of the category of statistical models by assigning each measurable space  $\mathcal{X}$  the pair  $(\mathcal{X}, \mathcal{P}(\mathcal{X}))$ .

(2) The notion of statistical models and their morphisms in Definition 2.19 is almost equivalent to the notion of statistical systems and their morphisms introduced by Morse-Sacksteder [MS1966], except that Morse-Sacksteder allowed morphisms that need not to be induced from probabilistic mappings, and almost coincides with the Markov category of family of probability distributions introduced by Chentsov [Chentsov1972, §6, p. 76], except that Chentsov considered only morphisms between statistical models whose probability sets  $P_{\mathcal{X}}, P_{\mathcal{Y}}$  are parameterized by the same set  $\Theta$ .

Among equivalences between statistical models there is an important class of sufficient morphisms, introduced by Morse-Sacksteder, which we reformulate as follows.

**Definition 2.21.** (cf. [MS1966]) A morphism  $T : (\mathcal{X}, P_{\mathcal{X}}) \rightsquigarrow (\mathcal{Y}, P_{\mathcal{Y}})$  between statistical models will be called *sufficient* if there exists a probabilistic mapping  $\underline{\mathbf{p}} : \mathcal{Y} \rightsquigarrow \mathcal{X}$  such that for all  $\mu \in P_{\mathcal{X}}$  and  $h \in L(\mathcal{X})$  we have

$$(2.19) \quad T_*(h\mu) = \underline{\mathbf{p}}^*(h)T_*(\mu) \iff \underline{\mathbf{p}}^*(h) = \frac{dT_*(h\mu)}{dT_*(\mu)} \in L^1(\mathcal{Y}, T_*(\mu)).$$

In this case we shall call  $T : \mathcal{X} \rightsquigarrow \mathcal{Y}$  a *probabilistic mapping sufficient for  $P_{\mathcal{X}}$*  and we shall call the measurable mapping  $\mathbf{p} : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{X})$  defining the probabilistic mapping  $\underline{\mathbf{p}} : \mathcal{Y} \rightsquigarrow \mathcal{X}$  a *conditional mapping for  $T$* .

**Remark 2.22.** (1) We call  $\mathbf{p} : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{X})$  a conditional mapping because of the interpretation of  $\mathbf{p}$  as a regular conditional probability in the case  $T$  is a measurable mapping, see (2.20) below.

(2) As in Remark 2.1 (1), we note that to prove the sufficiency of a probabilistic mapping  $T$  w.r.t.  $P_{\mathcal{X}} \subset \mathcal{P}(\mathcal{X})$  it suffices to verify (2.19) for all  $\mu \in P_{\mathcal{X}}$  and for all  $h = 1_A$  where  $A \in \Sigma_{\mathcal{X}}$ .

**Example 2.23.** Assume that  $\kappa : (\mathcal{X}, P_{\mathcal{X}}) \rightsquigarrow (\mathcal{Y}, P_{\mathcal{Y}})$  is a sufficient morphism, where  $\kappa : \mathcal{X} \rightarrow \mathcal{Y}$  is a statistic. Let  $\mathbf{p} : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{X})$ ,  $y \mapsto \mathbf{p}_y$ , be a conditional mapping for  $\kappa$ . By (2.2),  $\mathbf{p}^*(1_A)(y) = \mathbf{p}_y(A)$ , and we rewrite (2.19) as follows

$$(2.20) \quad \mathbf{p}_y(A) = \frac{d\kappa_*(1_A\mu)}{d\kappa_*\mu} \in L^1(\mathcal{Y}, \kappa_*(\mu)).$$

The RHS of (2.20) is the conditional measure of  $\mu$  applied to  $A$  w.r.t. the measurable mapping  $\kappa$ . The equality (2.20) implies that this conditional measure is regular and independent of  $\mu$ . Thus the notion of sufficiency of  $\kappa$  for  $P_{\mathcal{X}}$  coincides with the classical notion of sufficiency of  $\kappa$  for  $P_{\mathcal{X}}$ , see e.g., [Chentsov1972, p. 28], [Schervish1997, Definition 2.8, p. 85]. We also note that the equality in (2.20) is understood as equivalence class in  $L^1(\mathcal{Y}, \kappa_*(\mu))$  and hence every statistic  $\kappa'$  that coincides with a sufficient statistic  $\kappa$  except on a zero  $\mu$ -measure set, for all  $\mu \in P_{\mathcal{X}}$ , is also a sufficient statistic for  $P_{\mathcal{X}}$ .

**Example 2.24.** (cf. [Chentsov1972, Lemma 2.8, p. 28]) Assume that  $\mu \in \mathcal{P}(\mathcal{X})$  has a regular conditional distribution w.r.t. to a statistic  $\kappa : \mathcal{X} \rightarrow \mathcal{Y}$ , i.e., there exists a measurable mapping  $\mathbf{p} : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{X})$ ,  $y \mapsto \mathbf{p}_y$ , such that

$$(2.21) \quad \mathbb{E}_{\mu}^{\sigma(\kappa)}(1_A|y) = \mathbf{p}_y(A)$$

for any  $A \in \Sigma_{\mathcal{X}}$  and  $y \in \mathcal{Y}$ . Let  $\Theta \subset \mathcal{P}(\mathcal{X})$  and  $P := \{\nu_{\theta} | \theta \in \Theta\}$  be a family of probability measures dominated by  $\mu$ . If there exist a function  $h : \mathcal{Y} \times \Theta \rightarrow \mathbb{R}$  such that for all  $\theta \in \Theta$  and we have

$$(2.22) \quad \nu_{\theta} = h(\kappa(x))\mu$$

then  $\kappa$  is sufficient for  $P$ , since for any  $\theta \in \Theta$

$$\mathbf{p}^*(1_A) = \frac{d\kappa_*(1_A\nu_\theta)}{d\kappa_*\nu_\theta}$$

does not depend on  $\theta$ . The condition (2.22) is the Fisher-Neymann sufficiency condition for a family of dominated measures [Neyman1935].

Let us assume that  $T : (\mathcal{X}, P_{\mathcal{X}}) \rightsquigarrow (\mathcal{Y}, P_{\mathcal{Y}})$  is a sufficient morphism for  $\mu \in \mathcal{P}(\mathcal{X})$ . Then for any  $A \in \Sigma_{\mathcal{X}}$  we have

$$(2.23) \quad \mu(A) \stackrel{(2.4)}{=} T_*(1_A\mu)(Y) \stackrel{\text{Theorem 2.13(2)}}{=} \int_{\mathcal{Y}} \frac{dT_*(1_A\mu)}{dT_*\mu} dT_*\mu \stackrel{(2.19)}{=} \int_{\mathcal{Y}} \mathbf{p}_y(A) dT_*(\mu).$$

Comparing (2.23) with (2.4), we conclude that  $\mu = \underline{\mathbf{p}}_* \circ T_*(\mu)$ . Hence  $T_* \circ \underline{\mathbf{p}}_*(T_*\mu) = T_*\mu$ . It follows that  $(\mathcal{X}, \mu)$  is equivalent to  $(\mathcal{Y}, T_*(\mu))$ . Hence we get the following

**Proposition 2.25.** ([MS1966, Proposition 5.2]) *Two statistical models  $(\mathcal{X}, P_{\mathcal{X}})$  and  $(\mathcal{Y}, P_{\mathcal{Y}})$  are equivalent if there exists a probabilistic mapping  $T : \mathcal{X} \rightsquigarrow \mathcal{Y}$  such that  $T$  is sufficient for  $P_{\mathcal{X}}$  and  $T_*(P_{\mathcal{X}}) = P_{\mathcal{Y}}$ .*

**Remark 2.26.** (1) The converse of Proposition 2.25 is valid, if  $P_{\mathcal{X}}$  is a dominated family [Sacksteder1967, Theorem 2.1], and it is false without the dominated family condition [Sacksteder1967, §6].

(2) We refer the reader to [Pfanzagl2017, Chapter 2] for the history of the concept of sufficiency in statistics. As Pfanzagl emphasized, the concept of sufficiency of a statistic related to the question of the existence of a regular conditional distribution [Pfanzagl2017, p. 13]. We also would like to refer the reader to Theorem 3.6 below and [Bogachev2007, Chapter 10, vol. 2] for discussions on the existence of regular conditional distributions. We also do not discuss the related notion of Bayesian sufficiency in our paper and refer the interested reader to [Schervish1997, Definition 2.4, p.84, Theorem 2.20, p.89].

(3) There are many characterizations of sufficiency of a statistic, see e.g. [Pfanzagl2017, §2.9, p. 35]. In [AJLS2017, Proposition 5.6, p. 266] Ay-Jost-Lê-Schwachhöfer give a characterization of a sufficient statistic w.r.t. to a regular set  $P_{\mathcal{X}}$  of dominated measures that is parameterized by a smooth manifold via the notion of “information loss”.

We complete this section with the following theorem on the category of sufficient morphisms.

**Theorem 2.27.** (1) *A composition of sufficient morphisms between statistical models is a sufficient morphism.*

(2) *Assume that  $T : (\mathcal{X}, P_{\mathcal{X}}) \rightsquigarrow (\mathcal{Y}, P_{\mathcal{Y}})$  is a sufficient morphism and  $T_*(P_{\mathcal{X}}) = P_{\mathcal{Y}}$ . Let  $\mathbf{p} : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{X})$  be a conditional mapping for  $T$ . Then  $\underline{\mathbf{p}} : \mathcal{Y} \rightsquigarrow \mathcal{X}$  is a sufficient probabilistic mapping for  $P_{\mathcal{Y}}$  and  $\overline{T} : \mathcal{X} \rightarrow \mathcal{P}(\mathcal{Y})$  is a conditional mapping for  $\underline{\mathbf{p}}$ .*

*Proof.* 1. Assume that  $T_1 : (\mathcal{X}, P_{\mathcal{X}}) \rightsquigarrow (\mathcal{Y}, P_{\mathcal{Y}})$  and  $T_2 : (\mathcal{Y}, P_{\mathcal{Y}}) \rightsquigarrow (\mathcal{Z}, P_{\mathcal{Z}})$  are sufficient morphisms with conditional mappings  $\mathbf{p}_1 : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{X})$  and  $\mathbf{p}_2 : \mathcal{Z} \rightarrow \mathcal{P}(\mathcal{Y})$  for  $T_1$  and  $T_2$  respectively. Then we have for any  $h \in L(\mathcal{X})$  and  $\mu \in P_{\mathcal{X}}$

$$\begin{aligned} (T_2 \circ T_1)_*(h\mu) &= (T_2)_* \circ (T_1)_*(h\mu) \\ &= (T_2)_*(\mathbf{p}_1^*(h)(T_1)_*(\mu)) = \mathbf{p}_2^* \circ \mathbf{p}_1^*(h)(T_2 \circ T_1)_*(\mu) \end{aligned}$$

which proves the assertion (1) of Theorem 2.27.

2. Assume that  $T : (\mathcal{X}, P_{\mathcal{X}}) \rightarrow (\mathcal{Y}, P_{\mathcal{Y}})$  is a sufficient morphism and  $\mathcal{P}_{\mathcal{Y}} = T_*(P_{\mathcal{X}})$ . Let  $\mathbf{p} : \mathcal{Y} \rightarrow \mathcal{P}(\mathcal{X})$  be a conditional mapping for  $T$ , i.e., we have for any  $h \in L(\mathcal{X})$  and  $\mu \in P_{\mathcal{X}}$

$$(2.24) \quad \mathbf{p}^*(h) = \frac{dT_*(h\mu)}{dT_*(\mu)}.$$

To prove the assertion (2) of Theorem 2.27 it suffices to establish the following equality

$$(2.25) \quad \mathbf{p}_*(1_B T_* \mu)(A) = T^*(1_B)\mu(A)$$

for any  $\mu \in P_{\mathcal{X}}$ ,  $B \in \Sigma_{\mathcal{Y}}$ ,  $A \in \Sigma_{\mathcal{X}}$  and  $h \in L(\mathcal{X})$ .

Straightforward computations yield

$$\begin{aligned} \mathbf{p}_*(1_B T_* \mu)(A) &\stackrel{(2.4)}{=} \int_{\mathcal{Y}} \mathbf{p}_y(A) d(1_B T_* \mu) \\ &= \int_B \mathbf{p}_y(A) dT_* \mu \stackrel{(2.24)}{=} \int_B \frac{dT_*(1_A \mu)}{dT_* \mu} dT_* \mu \\ &= T_*(1_A \mu)(B) \stackrel{(2.4)}{=} \int_{\mathcal{X}} \bar{T}_x(B) d1_A \mu \\ (2.26) \quad &= \int_A \bar{T}_x(B) d\mu. \end{aligned}$$

On the other hand we have

$$(2.27) \quad T^*(1_B)\mu(A) = \int_A T^*(1_B)(x) d\mu \stackrel{(2.2)}{=} \int_A \bar{T}_x(B) d\mu.$$

Comparing (2.26) with (2.27), we obtain (2.25). This completes the proof of Theorem 2.27.  $\square$

**Remark 2.28.** Theorem 2.27 implies that in the subcategory of equivalent statistical models, whose morphisms are sufficient morphisms, every object is a terminal object, so we don't have the notion of a "minimal sufficient probabilistic mapping" like the notion of a minimal sufficient statistic. Note that a minimal sufficient statistic does not always exist [Pfanzagl2017, §2.6, p. 24].

## 3. BAYESIAN STATISTICAL MODELS AND BAYES' FORMULA

In this section we formalize the notion of a parameter space of a probabilistic model in Bayesian machine learning via the notion of a Bayesian statistical model (Definition 3.1, Example 3.2, Remark 3.3). Then we define the notion of posterior distributions (Definition 3.4) and compare it with the classical notion of a posterior distribution (Remark 3.5). Finally we give a new formula for the posterior distributions (Theorem 3.6). We derive a consequence (Corollary 3.14), compare our new formula with the classical Bayes' formula as well as other related results (Remarks 3.15, 3.17, 3.18) and consider an example (Example 3.19).

## 3.1. Bayesian statistical models.

**Definition 3.1.** A Bayesian statistical model is a quadruple  $(\Theta, \mu_\Theta, \mathbf{p}, \mathcal{X})$ , where  $\mathbf{p} : \Theta \rightarrow \mathcal{P}(\mathcal{X})$  is a measurable mapping and  $\mu_\Theta \in \mathcal{P}(\Theta)$ , called a *prior measure*.

Note that we can also regard a Bayesian statistical model as a morphism  $\underline{\mathbf{p}} : (\Theta, \mu_\Theta) \rightsquigarrow (\mathcal{X}, \mathcal{P}(\mathcal{X}))$  of statistical models.

Recall that if  $\mathcal{X}$  is a topological space then  $\Sigma_{\mathcal{X}}$  is assumed to be the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathcal{X})$  and hence  $\mathcal{P}(\mathcal{X})$  consists of Borel measures, if not otherwise specified.

**Example 3.2.** (1) Assume that  $(\Theta, \mu_\Theta, \mathbf{p}, \mathcal{X})$  is a Bayesian statistical model. Then  $(\mathcal{P}(\mathcal{X}), \mathbf{p}_*(\mu_\Theta), Id_{\mathcal{P}}, \mathcal{X})$  is a Bayesian statistical model.

(2) Let  $\mathcal{S}(\mathcal{X})_{TV}$  denote the Banach space  $\mathcal{S}(\mathcal{X})$  endowed with the total variation norm and  $d_H$  denote the Hellinger distance on  $\mathcal{P}(\mathcal{X})$ , so the topology on  $\mathcal{P}(\mathcal{X})$  induced by  $d_H$  coincides with the topology on  $\mathcal{P}(\mathcal{X})$  induced from the natural inclusion of  $i : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{S}(\mathcal{X})_{TV}$ . Assume that  $(M, \mu_M)$  is a smooth finite dimensional manifold endowed with a volume element  $\mu_M$  and  $\mathbf{p} : M \rightarrow (\mathcal{P}(\mathcal{X}), d_H)$  is a continuous map, for instance, if  $\mathbf{p}$  is a smooth map, i.e., the composition  $i \circ \mathbf{p} : M \rightarrow \mathcal{S}(\mathcal{X})_{TV}$  is a smooth map, see e.g., [AJLS2017, p. 168], [Bogachev2010, p. 380]. We shall show that  $(M, \mu_M, \mathbf{p}, \mathcal{X})$  is a Bayesian statistical model, i.e., we shall show that the map  $\mathbf{p} : (M, \mathcal{B}(M)) \rightarrow (\mathcal{P}(\mathcal{X}), \Sigma_w)$  is measurable. Denote by  $\tau_w$  the smallest topology on  $\mathcal{S}(\mathcal{X})$  such that for all  $f \in \mathcal{F}_s(\mathcal{X})$  the map  $I_f : \mathcal{S}(\mathcal{X}) \rightarrow \mathbb{R}$  is continuous. Clearly the topology  $\tau_w$  is weaker than the strong topology generated by the total variation norm. Let  $\tau_w$  also denote the induced topology on  $\mathcal{P}(\mathcal{X})$ . It follows that the map  $\mathbf{p} : M \rightarrow \mathcal{P}(\mathcal{X})$  is  $\tau_w$ -continuous. Hence the map  $\mathbf{p} : (M, \mathcal{B}(M)) \rightarrow (\mathcal{P}(\mathcal{X}), \mathcal{B}(\tau_w))$  is measurable. It is not hard to check that  $\Sigma_w \subset \mathcal{B}(\tau_w)$ , and moreover  $\mathcal{B}(\tau_v) \subset \mathcal{B}(\tau_w)$  with the equality  $\mathcal{B}(\tau_v) = \mathcal{B}(\tau_w)$  if and only if  $\mathcal{X}$  is countable [GH1989]. We conclude that  $\mathbf{p} : (M, \mathcal{B}(M)) \rightarrow (\mathcal{P}(\mathcal{X}), \Sigma_w)$  is a measurable mapping.

(3) Assume that  $(\Theta, \mu_\Theta, \mathbf{p}, \mathcal{X})$  is a Bayesian statistical model and  $T : \mathcal{X} \rightsquigarrow \mathcal{Y}$  is a probabilistic mapping. Then  $(\Theta, \mu_\Theta, T_* \circ \mathbf{p}, \mathcal{Y})$  is a Bayesian statistical model. If  $T' : \Theta' \rightsquigarrow \Theta$  is a probabilistic mapping, then  $(\Theta', \mu_{\Theta'}, \overline{(\mathbf{p} \circ T')}, \mathcal{X})$  is a Bayesian statistical model for any  $\mu_{\Theta'} \in \mathcal{P}(\Theta')$ .

(4) Assume that  $(\Theta, \mu_\Theta, \mathbf{p}, \mathcal{X})$  is a Bayesian statistical model. Then  $(\Theta, \mu_\Theta, \delta \circ \mathbf{p}, \mathcal{P}(\mathcal{X}))$  is a Bayesian statistical model.

(5) For any  $\mu_{\mathcal{P}} \in \mathcal{P}^2(\mathcal{X})$  the quadruple  $(\mathcal{P}(\mathcal{X}), \mu_{\mathcal{P}}, Id_{\mathcal{P}}, \mathcal{X})$  is a Bayesian statistical model. In particular, for any  $\mu \in \mathcal{P}(\mathcal{X})$  (resp. for any  $\nu \in \mathcal{P}^3(\mathcal{X})$ ) the quadruples  $(\mathcal{P}(\mathcal{X}), \delta(\mu), Id_{\mathcal{P}}, \mathcal{X})$ ,  $(\mathcal{P}(\mathcal{X}), P_*(\delta)(\mu), Id_{\mathcal{P}}, \mathcal{X})$  are Bayesian statistical models (resp. the quadruple  $(\mathcal{P}(\mathcal{X}), ev_{\mathcal{P}^2}(\nu), Id_{\mathcal{P}}, \mathcal{X})$  is a Bayesian statistical model).

(6) Assume that  $(\Theta_1, \mu_1, \mathbf{p}_1, \mathcal{P}(\mathcal{X}))$  and  $(\Theta_2, \mu_2, \mathbf{p}_2, \mathcal{P}(\mathcal{X}))$  are Bayesian statistical models. Then  $(\Theta_1 \times \Theta_2, \mu_1 \times \mu_2, \mathbf{p}_1 \star \mathbf{p}_2, \mathcal{P}(\mathcal{X}))$  is a Bayesian statistical model, where

$$(3.1) \quad \mathbf{p}_1 \star \mathbf{p}_2(\theta_1, \theta_2) := P_*(\pi_{\mathcal{X}}) \circ P_*(\mathbf{a})((i_{p,m})_* \mathbf{p}_1(\theta_1), (i_{p,m})_* \mathbf{p}_2(\theta_2)).$$

Similarly,  $(\Theta_1, \mu_1, \mathbf{p}_1^2, \mathcal{P}(\mathcal{X}))$  is a Bayesian statistical model, where  $\mathbf{p}_1^2(\theta) := \mathbf{p}_1 \star \mathbf{p}_1(\theta, \theta)$ .

(7) Let  $(\Theta, \mu_\Theta, \mathbf{p}, \Omega_\infty)$  be a Bayesian statistical model, where  $\Omega_\infty := \{\omega_1, \dots, \omega_\infty\}$ . Then the map  $\underline{\mathbf{p}} : \Theta \rightsquigarrow \Omega_\infty$  is a *random partition* of  $\Theta$ . If  $\underline{\mathbf{p}}$  is a measurable mapping then it is also called a *measurable partition*. In the later case we compute easily the conditional probability distribution for any  $A \in \Sigma_\Theta$  [Bogachev2007, p.345, vol.2]

$$(3.2) \quad \mu_i(A) := \mu_\Theta(A | \underline{\mathbf{p}} = \omega_i) = \frac{\mu(A \cap \{\underline{\mathbf{p}} = \omega_i\})}{\mu(\underline{\mathbf{p}} = \omega_i)}$$

if  $\mu(\underline{\mathbf{p}} = \omega_i) \neq 0$ , otherwise we set  $\mu_i(A) = 0$ . Therefore  $\mu$  can be regarded as a mixture distribution:

$$\mu(A) = \sum_{i=1}^{\infty} \nu(\omega_i) \cdot \mu_i(A)$$

where we abbreviate  $\underline{\mathbf{p}}_*(\mu)$  as  $\nu$ . Let  $\kappa : \Omega_k \rightarrow \mathcal{P}(\mathcal{X})$ ,  $\kappa(\omega_i) := \mu_i \in \mathcal{P}(\Theta)$ , where  $\mu_i \neq 0$  is defined by the LHS of (3.2). Then  $(\Omega_k, \nu, \kappa, \Theta)$  also encodes a measurable partition of  $(\Theta, \mu_\Theta)$ .

**Remark 3.3.** (1) We recover the classical (frequentist) definition of a (parameterized) statistical model from Definition 3.1 by applying the “forgetful functor” from the category of measurable spaces to the category of sets.

(2) Definition 3.1 encompasses both parametric Bayesian models, which usually consist of conditional density functions  $f(x|\theta)$ , see e.g. [Berger1993, p. 4], and nonparametric Bayesian models. It is essentially equivalent to the concept of a Bayesian parameter space defined in [GR2003, p. 16] but we use the more compact functorial language of probabilistic mappings.

**3.2. A formula for the posterior distribution.** Let  $(\Theta, \mu_\Theta, \mathbf{p}, \mathcal{X})$  be a Bayesian statistical model. Then we define the joint distribution  $\mu$  on the measurable space  $(\Theta \times \mathcal{X}, \Sigma_\Theta \otimes \Sigma_{\mathcal{X}})$  as follows

$$(3.3) \quad \mu(B \times A) := \int_B \mathbf{p}_\theta(A) d\mu_\Theta \text{ for all } B \in \Sigma_\Theta, A \in \Sigma_{\mathcal{X}},$$

where we re-denote  $\mathbf{p}(\theta)$  by  $\mathbf{p}_\theta$  for interpreting  $\theta$  as parameter of the distribution. Let  $\Pi_\Theta$  and  $\Pi_{\mathcal{X}}$  denote the projections  $(\Theta \times \mathcal{X}, \Sigma_\Theta \otimes \Sigma_{\mathcal{X}}) \rightarrow (\Theta, \Sigma_\Theta)$  and  $(\Theta \times \mathcal{X}, \Sigma_\Theta \otimes \Sigma_{\mathcal{X}}) \rightarrow (\mathcal{X}, \Sigma_{\mathcal{X}})$  respectively. Clearly  $\Pi_\Theta$  and  $\Pi_{\mathcal{X}}$  are measurable mappings. Using  $\Pi_{\mathcal{X}}$  we express the marginal probability measure  $\mu_{\mathcal{X}}$  on  $\mathcal{X}$  as follows:  $\mu_{\mathcal{X}} = (\Pi_{\mathcal{X}})_*(\mu)$ . In other words  $\mu_{\mathcal{X}}(A) := \mu(\Theta \times A)$  for all  $A \in \Sigma_{\mathcal{X}}$ . Clearly  $\mu_\Theta = (\Pi_\Theta)_*(\mu)$ . We note that  $\mu_{\mathcal{X}|\Theta}(A|\theta) := p_\theta(A)$  is the conditional probability of the measure  $\mu$  on  $(\Theta \times \mathcal{X}, \Sigma_\Theta \otimes \Sigma_{\mathcal{X}})$  w.r.t. the projection  $\Pi_\Theta$ , i.e., we have the following equality

$$(3.4) \quad \mu_{\mathcal{X}|\Theta}(A|\theta) = \frac{d(\Pi_\Theta)_*(1_{\Theta \times A} \mu)}{d(\Pi_\Theta)_*\mu}(\theta),$$

where the equality should be understood as equivalence class of functions in  $L^1(\Theta, \mu_\Theta)$ , since (3.4) is equivalent to the identity (3.3) for  $B = \Theta$ .

Formula (3.4) motivates the following definition.

**Definition 3.4.** A family of probability measures  $\mu_{\Theta|\mathcal{X}}(\cdot|x) \in \mathcal{P}(\Theta)$ ,  $x \in \mathcal{X}$ , is called a family of *posterior distributions of  $\mu_\Theta$  after seeing the data  $x$*  if for all  $B \in \Sigma_\Theta$  we have

$$(3.5) \quad \mu_{\Theta|\mathcal{X}}(B|x) = \frac{d(\Pi_{\mathcal{X}})_*(1_{B \times \mathcal{X}} \mu)}{d(\Pi_{\mathcal{X}})_*\mu}(x),$$

where (3.5) should be understood as an equivalence class of functions in  $L^1(\mathcal{X}, \mu_{\mathcal{X}})$ .

**Remark 3.5.** Definition 3.4 coincides with the definition of a posterior distribution in classical Bayesian statistics see e.g. [Berger1993, §4.2.1, p. 126], [Schervish1997, p. 16]. Note that in the both definitions of the mentioned books the authors did not explicitly require that  $\mu_{\Theta|\mathcal{X}}(\cdot|x)$  must be a  $\sigma$ -additive function on  $\Sigma_\Theta$ . The last requirement is trivially satisfied when one considers only Bayesian statistical models of dominated measures, i.e., there exists a measure  $\nu_0 \in \mathcal{P}(\mathcal{X})$  such that for all  $\theta \in \Theta$  we have  $\mathbf{p}(\theta) \ll \nu_0$ . In other papers and books, e.g., in [Ferguson1973] and [GV2017], statisticians also think of posterior distributions as regular conditional distributions, which is equivalent to Definition 3.4. The existence of posterior distributions is therefore equivalent to the existence of regular conditional distributions.

**Theorem 3.6.** *Suppose that  $\mathcal{X}$  is a subset of a connected finite dimensional complete Riemannian manifold  $(M^n, g)$  with the induced metric generated by  $g$  on  $M^n$  and  $(\Theta, \mu_\Theta, \mathbf{p}, \mathcal{X})$  is a Bayesian statistical model, where  $\Theta$  is a Souslin space. Let  $D_r(x)$  denote the open ball of radius  $r$  centered at  $x \in \mathcal{X}$ . Then there exists a measurable subset  $S \subset \mathcal{X}$  of zero  $\mu_{\mathcal{X}}$ -measure, and a family of posterior distributions  $\mu_{\Theta|\mathcal{X}}(\cdot|x)$  on  $\Theta$  after seeing data  $x \in \mathcal{X}$  such that*

$$(3.6) \quad \mu_{\Theta|\mathcal{X}}(B|x) = \lim_{r \rightarrow 0} \frac{\int_B p_\theta(D_r(x)) d\mu_\Theta}{\int_\Theta p_\theta(D_r(x)) d\mu_\Theta}$$

for any  $B \in \mathcal{B}(\Theta) = \Sigma_\Theta$ , and for any  $x \in \mathcal{X} \setminus S$ . For  $x \in S$  we have  $\mu_{\Theta|\mathcal{X}}(B|x) := 0$  for any  $B \in \Sigma_\Theta$ .

**Remark 3.7.** Applying (3.3) we conclude that  $S$  is also of  $\mu'_{\mathcal{X}}$ -zero measure for any marginal measure  $\mu'_{\mathcal{X}}$  defined by a posterior distribution  $\mu_{\Theta|\mathcal{X}}(\cdot|x)$  computed by the recipe in Theorem 3.6. Hence the Bayes rule using the recipe in Theorem 3.6 is well-defined, see also [GV2017, p. 6] for a related discussion on requirements on posterior distributions.

*Proof of Theorem 3.6.* We shall prove Theorem 3.6 using the arguments in the proof for the existence of regular conditional distributions in [Bogachev2007, Theorem 10.4.5, p. 359, Corollary 10.4.15, p. 366, vol.2], combined with a formula for differentiation of measures on  $\mathcal{X}$  (Proposition 3.8).

In what follows we use shorthand notations  $\nu_1$  for the measure  $\mu_{\mathcal{X}} = (\Pi_{\mathcal{X}})_*\mu$  and  $\nu_2$  for  $(\Pi_{\mathcal{X}})_*(1_{(\Theta \times B)}\mu)$ . Then  $\nu_2 \ll \nu_1$ . We recall that

$$(3.7) \quad \nu_1(D_r(x)) = \int_{\Theta} \mathbf{p}_\theta(D_r(x)) d\mu_\Theta,$$

$$(3.8) \quad \nu_2(D_r(x)) = \int_B \mathbf{p}_\theta(D_r(x)) d\mu_\Theta.$$

For any  $x \in \mathcal{X}$  we set

$$\overline{D}_{\nu_1} \nu_2(x) := \limsup_{r \rightarrow 0} \frac{\nu_2(D_r(x))}{\nu_1(D_r(x))} \quad \text{and} \quad \underline{D}_{\nu_1} \nu_2(x) := \liminf_{r \rightarrow 0} \frac{\nu_2(D_r(x))}{\nu_1(D_r(x))}$$

where we set  $\overline{D}_{\nu_1} \nu_2(x) = \underline{D}_{\nu_1} \nu_2(x) = +\infty$  if  $\nu_1(D_r(x)) = 0$  for some  $r > 0$ .

Furthermore if  $\overline{D}_{\nu_1} \nu_2(x) = \underline{D}_{\nu_1} \nu_2(x)$  then we set

$$D_{\nu_1} \nu_2(x) := \overline{D}_{\nu_1} \nu_2(x) = \underline{D}_{\nu_1} \nu_2(x)$$

which is called *the derivative of  $\nu_2$  with respect to  $\nu_1$  at  $x$* .

**Proposition 3.8.** *There is a measurable subset  $S(\nu_1, \nu_2) \subset \mathcal{X}$  of zero  $\nu_1$ -measure such that for any  $x \in \mathcal{X} \setminus S(\nu_1, \nu_2)$  the function  $D_{\nu_1} \nu_2$  is well-defined. Setting  $\tilde{D}_{\nu_1} \nu_2(x) := 0$  for  $x \in S(\nu_1, \nu_2)$  and  $\tilde{D}_{\nu_1} \nu_2(x) = D_{\nu_1} \nu_2(x)$  for  $x \in \mathcal{X} \setminus S(\nu_1, \nu_2)$ . Then the function  $\tilde{D}_{\nu_1} \nu_2$  is measurable and serves as the Radon-Nikodym density of the measure  $\nu_2$  with respect to  $\nu_1$ .*

*Proof.* We note that the inclusion map  $\mathcal{X} \rightarrow (M^n, g)$  is continuous and hence measurable. Hence the measures  $i_*(\nu_1)$  and  $i_*(\nu_2)$  are Borel measures on  $(M^n, g)$  and  $i_*(\nu_2) \ll i_*(\nu_1)$ , since  $\nu_2 \ll \nu_1$ . Now Proposition 3.8 follows immediately from Theorem A.1 in the Appendix.  $\square$

Proposition 3.8 implies that for each  $B \in \Sigma_{\mathcal{X}}$  there exists a measurable subset  $S_B \subset \mathcal{X}$  of zero  $\mu_{\mathcal{X}}$ -measure such that the RHS of (3.6) is well defined for all  $x \in \mathcal{X} \setminus S_B$  and moreover it coincides with the RHS of (3.5) as functions in  $L^1(\mathcal{X}, \mu_{\mathcal{X}})$ . We express this fact in the following equivalent statement.

**Corollary 3.9.** *Let  $B \in \Sigma_\Theta$ . For  $x \in \mathcal{X} \setminus S_B$  let  $p(B, x) := \mu_{\Theta|\mathcal{X}}(B|x)$  and for  $x \in S_B$  we let  $p(B, x) := 0$ . Then the function  $P(B, x) : \mathcal{X} \rightarrow \mathbb{R}$  is measurable. Furthermore, for any  $A \in \Sigma_{\mathcal{X}}$  we have*

$$(3.9) \quad \mu(B \times A) = \int_A p(B, x) d\mu_{\mathcal{X}}(x).$$

To complete the proof of Theorem 3.6 we shall show that there exist a countable algebra  $\mathcal{U}_\Theta$  generating  $\mathcal{B}(\Theta)$ , a subset  $S \subset \mathcal{X}$  of zero  $\mu_{\mathcal{X}}$ -measure such that for each  $x \in \mathcal{X} \setminus S$  the RHS of (3.6) is well-defined for any  $B \in \mathcal{U}_\Theta$ , moreover it is a  $\sigma$ -additive set function on  $\mathcal{U}_\Theta$ . To prove this statement we need the assumption that  $\Theta$  is a Souslin space, namely we shall use the following

**Proposition 3.10.** ([Bogachev2007, Theorem 7.4.3, p. 85, vol. 2]) *Every Borel finite measure on a Souslin space  $\Theta$  is Radon and concentrated on a countable union of metrizable compact sets. In addition, for every  $B \in \mathcal{B}(\Theta)$  and every  $\varepsilon > 0$  there exists a metrizable compact set  $K_\varepsilon$  such that  $\mu(B \setminus K_\varepsilon) < \varepsilon$ .*

The strategy of the remainder of the proof of Theorem 3.6 is as follows. By [Bogachev2007, Corollary 6.7.5, p. 25, vol.2], there exists a countable algebra  $\mathcal{U}_\Theta$  generating  $\mathcal{B}(\Theta)$ . In Lemma 3.12 we shall show the existence of a subset  $S_0 \subset \mathcal{X}$  of zero  $\mu_{\mathcal{X}}$ -measure such that for any  $x \in \mathcal{X} \setminus S_0$  and any  $B \in \mathcal{U}_\Theta$  the RHS of (3.6) is well-defined and  $\mu_{\Theta|\mathcal{X}}(\cdot|x) = p(B, x)$ , moreover  $\mu_{\Theta|\mathcal{X}}(\cdot|x)$  is an additive function on  $\mathcal{U}_\Theta$ . Then in Lemma 3.13 we shall apply Proposition 3.11 below to show that there is a subset  $S_1 \subset (\mathcal{X} \setminus S_0)$  of zero  $\mu_{\mathcal{X}}$ -measure such that for all  $x \in \mathcal{X} \setminus (S_0 \cup S_1)$  the function  $p(B, x) = \mu_{\Theta|\mathcal{X}}(\cdot|x)$  is  $\sigma$ -additive on  $\mathcal{U}_\Theta$ , and hence it defines a probability measure on  $\Theta$ , since  $p(\Theta, x) = 1$  for all  $x \in \mathcal{X}$ . Since on  $\mathcal{X} \setminus (S_0 \cup S_1)$  the function  $p(B, x)$  coincides with  $\mu_{\Theta|\mathcal{X}}(B, x)$  this shall complete the proof of Theorem 3.6.

We recall that a family  $\mathcal{K}$  of subsets of a set  $X$  is called a *compact class*, if for any sequence  $K_n$  of its elements with  $\bigcup_{n=1}^{\infty} K_n = \emptyset$ , there exists  $N$  such that  $\bigcup_{i=1}^N K_n = \emptyset$  [Bogachev2007, Definition 1.4.1, p. 13, vol.1].

**Proposition 3.11.** [Bogachev2007, Theorem 1.4.3, p. 13, vol. 1] *Let  $\mu$  be a nonnegative additive set function on an algebra  $\mathcal{A}$ . Suppose that there exists a compact class  $\mathcal{K}$  approximating  $\mu$  in the following sense: for every  $A \in \mathcal{A}$  and every  $\varepsilon > 0$ , there exist  $K_\varepsilon \in \mathcal{K}$  and  $A_\varepsilon \in \mathcal{A}$  such that  $A_\varepsilon \subset K_\varepsilon \subset A$  and  $\mu(A \setminus A_\varepsilon) < \varepsilon$ . Then  $\mu$  is countably additive. In particular this is true if the compact class  $\mathcal{K}$  is contained in  $\mathcal{A}$  and for any  $A \in \mathcal{A}$  one has the equality*

$$\mu(A) = \sup_{K \subset A, K \in \mathcal{K}} \mu(K).$$

*Continuation of the proof of Theorem 3.6.* By Proposition 3.10 any family  $\mathcal{K}$  of compact sets in  $\Theta$  is a *compact class*. Propositions 3.10 and 3.11 imply that there exists a countable algebra  $\mathcal{U}_\Theta$  generating  $\mathcal{B}(\Theta)$  such that  $\mathcal{U}_\Theta$  contains a countable union  $\mathcal{K}_\Theta$  of metrizable compact sets on which the

measure  $\mu_\Theta$  is concentrated, and for every  $B \in \mathcal{B}(\Theta)$  and every  $\varepsilon > 0$  there exists a metrizable compact set  $K_\varepsilon$  such that  $\mu(B \setminus K_\varepsilon) < \varepsilon$ . Thus the condition of [Bogachev2007, Theorem 10.4.5 (ii), p. 359, vol.2] is satisfied for  $\mathcal{A} = \mathcal{U}_\Theta$ .

**Lemma 3.12.** *There exists a measurable subset  $S_0 \subset \mathcal{X}$  of zero  $\mu_\mathcal{X}$ -measure such that for all  $x \in \mathcal{X} \setminus S_0$  the function  $\mu_{\Theta|\mathcal{X}}(\cdot|x)$  is additive on  $\mathcal{U}_\Theta$ , moreover  $\mu_{\Theta|\mathcal{X}}(\cdot|x) = p(B, x)$ .*

*Proof.* Let  $\mathcal{U}_\Theta$  consist of countably many sets  $B_n$ . Let  $S_0 = \cup_i S_{B_i}$ . Then  $S_0$  is measurable and it has zero measure set. By Corollary 3.9,  $\mu_{\Theta|\mathcal{X}x}(\cdot|x) = p(B, x)$  for any  $x \in \mathcal{X} \setminus S_0$ . Clearly the RHS of (3.6) is additive for on  $\mathcal{U}_\Theta$  for any  $x \in \mathcal{X} \setminus S_0$ . This completes the proof of Lemma 3.12  $\square$

Now for all  $x \in \mathcal{X}$  we define a set function  $\mu_{\Theta|\mathcal{X}}(\cdot|x)$  on  $\mathcal{U}_\Theta$  by setting

$$(3.10) \quad \mu_{\Theta|\mathcal{X}}(\cdot|x) := \begin{cases} 0 & \text{for } x \in S_0, \\ p(\cdot|x) & \text{for } x \in \mathcal{X} \setminus S_0. \end{cases}$$

By Corollary 3.9 and Lemma 3.12, for any  $B \in \mathcal{U}_\Theta$  the function  $\mu_{\Theta|\mathcal{X}}(B|\cdot) : \mathcal{X} \rightarrow \mathbb{R}$  is measurable.

**Lemma 3.13.** *There exists a measurable subset  $S_1 \subset \mathcal{X} \setminus S_0$  of zero  $\mu_\mathcal{X}$ -measure such that for any  $x \in \mathcal{X} \setminus (S_0 \cup S_1)$  the set function  $\mu_{\Theta|\mathcal{X}}(\cdot|x)$  is  $\sigma$ -additive on  $\mathcal{U}_\Theta$ .*

*Proof.* Taking into account Lemma 3.12 and Proposition 3.11, to prove Lemma 3.13, it suffices to show the existence of a measurable subset  $S_1 \subset \mathcal{X} \setminus S_0$  of zero  $\mu_\mathcal{X}$ -measure such that for any  $x \in \mathcal{X} \setminus (S_0 \cup S_1)$  the compact class  $K_\Theta$  approximates the additive set function  $\mu_{\Theta|\mathcal{X}}(\cdot|x)$  on  $\mathcal{U}_\Theta$ . This will be proved using arguments in [Bogachev2007, p. 360, vol. 2]. By Proposition 3.11 for every  $n$  there are sets  $C_{n,k} \in \mathcal{K}_\Theta$  such that

$$(3.11) \quad C_{n,k} \subset B_n \text{ and } \mu(B_n \setminus C_{n,k}) < 1/k.$$

*Claim.* There exists a measurable subset  $S_1$  of zero  $\mu_\mathcal{X}$ -measure such that for all  $x \in \mathcal{X} \setminus S_1$  we have

$$(3.12) \quad \mu_{\Theta|\mathcal{X}}(B_n, x) = \sup_k \mu_{\Theta|\mathcal{X}}(C_{n,k}, x) \text{ for all } n \in \mathbb{N}.$$

*Proof of Claim (3.12).* We define a function  $q_n : \mathcal{X} \rightarrow \mathbb{R}$  by setting  $q_n(x)$  equal to the RHS of (3.11), if  $x \in \mathcal{X} \setminus S_0$  and  $q_n(x) = 0$  otherwise. Then  $q_n : \mathcal{X} \rightarrow \mathbb{R}$  is measurable, since  $\mu_{\Theta|\mathcal{X}}(C_{n,k}, \cdot) : \mathcal{X} \rightarrow \mathbb{R}$  is measurable. Since  $C_{n,k} \subset B_n$  for all  $k$  and for all  $x \in \mathcal{X} \setminus S_0$  we have

$$(3.13) \quad q_n(x) \leq \mu_{\Theta|\mathcal{X}}(B_n, x) \text{ for all } x \in \mathcal{X} \setminus S_0.$$

Since  $\mu_{\Theta|\mathcal{X}}(C_{n,k}, x) \leq q_n(x)$  for  $x \in \mathcal{X} \setminus S_0$ , and  $S_0$  is a measurable subset of zero  $\mu$ -measure, taking into account (3.9) and (3.10), we have

$$(3.14) \quad \mu(C_{n,k} \times \mathcal{X}) = \int_{\mathcal{X}} \mu_{\Theta|\mathcal{X}}(C_{n,k}, x) d\mu \leq \int_{\mathcal{X}} q_n(x) d\mu.$$

Taking into account (3.13) and (3.11), we obtain from (3.14)

$$(3.15) \quad \sup_k \mu(C_{n,k} \times \mathcal{X}) \leq \int_{\mathcal{X}} q_n(x) d\mu \leq \int_{\mathcal{X}} \mu_{\Theta|\mathcal{X}}(B_n, x) d\mu = \mu(B_n \times \mathcal{X}).$$

By (3.11), the LHS of (3.15) is equal to the RHS of (3.15). Taking into account (3.13), we conclude that there exists a subset  $S_1 \subset \mathcal{X} \setminus S_0$  such that  $q_n(x) = \mu_{\Theta|\mathcal{X}}(x)$  for all  $x \in \mathcal{X} \setminus (S_0 \cup S_1)$ . Since both the functions  $q_n$  and  $\mu_{\Theta|\mathcal{X}}$  are measurable, the subset  $S_1$  is measurable. This proves Claim (3.12).

Claim (3.12) implies that for all  $x \in \mathcal{X} \setminus (S_0 \cup S_1)$  the additive function  $\mu_{\Theta|\mathcal{X}}(\cdot|x)$  on the algebra  $\mathcal{U}_{\Theta}$  has the property that the compact class  $\mathcal{K}_{\Theta}$  approximates  $p(\cdot, x)$  on  $\mathcal{U}_{\Theta}$ . This completes the proof of Lemma 3.13.  $\square$

Since  $\mu_{\Theta|\mathcal{X}}(\cdot|x)$  is  $\sigma$ -additive on  $\mathcal{U}_{\Theta}$ , it extends to a measure on  $\Theta$ . Since  $\mu_{\Theta|\mathcal{X}}(\Theta, x) = 1$  for any  $x \in \mathcal{X}$  this completes the proof of Theorem 3.6.  $\square$

**Corollary 3.14.** *Assume the conditions of Theorem 3.6. Given a point  $x_0 \in \mathcal{X} \setminus S$  assume that  $p_{\theta}(x_0) = 0$  for all  $\theta \in \Theta$ . If the condition for differentiation w.r.t.  $r$  at 0 under the integral  $\int_C p_{\theta}(D_r(x_0)) d\mu_{\Theta}$  holds for  $C \in \mathcal{U}_{\Theta} \cup \{\Theta\}$ , then for any  $B \in \mathcal{U}_{\Theta}$  we have*

$$(3.16) \quad \mu_{\Theta|\mathcal{X}}(B|x_0) = \frac{\int_B \frac{d}{dr}|_{r=0} p_{\theta}(D_r(x_0)) d\mu_{\Theta}}{\int_{\Theta} \frac{d}{dr}|_{r=0} p_{\theta}(D_r(x_0)) d\mu_{\Theta}},$$

if the dominator in the RHS of (3.16) does not vanish.

**Remark 3.15.** If  $\{p_{\theta} | \theta \in \Theta\}$  is a family of dominated measures, then

$$(3.17) \quad \mu(B|x) = \int_B \frac{d\mu_{\Theta|\mathcal{X}}}{d\mu_{\Theta}}(\theta|x) d\mu_{\Theta}$$

where, by Bayes' formula we have [Schervish1997, Theorem 1.31, p.16]

$$(3.18) \quad \frac{d\mu_{\Theta|\mathcal{X}}}{d\mu_{\Theta}}(\theta|x) = \frac{f_{\mathcal{X}|\Theta}(x|\theta)}{\int_{\Theta} f_{\mathcal{X}|\Theta}(x|t) d\mu_{\Theta}(t)}$$

for  $\mu_{\mathcal{X}}$ -a.e.  $x \in \mathcal{X}$ . We regard both (3.6) and (3.18) as recipes for computing the posterior distribution  $\mu(B|x)$  under different assumptions.

**Remark 3.16.** Assume the conditions of Theorem 3.6. Given  $x_0 \in \mathcal{X} \setminus S$ , assuming that  $p_{\theta}(x_0) = 0$  for all  $\theta \in \Theta$ , a sufficient condition for differentiation under the integral sign for any  $C \in \mathcal{U}_{\Theta} \cup \Theta$

$$(3.19) \quad \lim_{r \rightarrow 0} \frac{1}{r} \int_C p_{\theta}(D_r(x_0)) d\mu_{\Theta} = \int_C \frac{d}{dr}|_{r=0} p_{\theta}(D_r(x_0)) d\mu_{\Theta}$$

is the differentiability in  $r$  of the function  $p_{\theta}(D_r(x_0))$  of the variable  $r$  in a neighborhood of  $0 \in [0, 1]$  and the existence of a  $\mu_{\Theta}$ -measurable function  $F(\theta, x_0)$  of the variable  $\theta$  such that  $|\frac{d}{dr} p_{\theta}(D_r(x_0))| \leq F(\theta, x_0)$  in this neighborhood, see e.g. [Jost2005, Theorem 16.11, p. 213], which is also valid for an arbitrary measurable space  $\Theta$ .

**Remark 3.17.** Formula (3.6) for the (regular) posterior distribution  $\mu_{\Theta|\mathcal{X}}(B|x)$  is obtained from Proposition 3.8, which gives a formula for the Radon-Nykodym derivative  $d\nu/d\mu$ . As a formula for a (not necessarily regular) conditional distribution, (3.6) is also valid  $\mu_{\mathcal{X}}$ -a.e. without any assumption on  $\Theta$ . There is also another formula for the Radon-Nykodym derivative  $d\nu/d\mu$  using the lifting  $L : \Sigma_{\mathcal{X}} \rightarrow \Sigma_{\mathcal{X}}$  for constructing a differentiation base, see [BBT2008, §8.6.3, p. 333]. Since the construction of a lifting is not simple, see [Bogachev2007, §10.5, p. 371-373, vol. 2], a formula for  $\mu_{\Theta|\mathcal{X}}(B|x)$  using the lifting construction is not easy to implement.

**Remark 3.18.** We can slightly generalize Theorem 3.6 to the case of a finite dimensional complete Riemannian manifold with a countable number of connected components by setting the distance between two points from different connected components to be  $\infty$ .

**Example 3.19.** Assume that  $\mathcal{X} \subset (M, g)$  is a closed (resp. open) subset of  $(M, g)$ . Then  $\mathcal{X}$  is a Polish space and  $\mathcal{P}(\mathcal{X})$  is a Polish space and hence a Souslin space. Theorem 3.6 implies that for any (prior) measure  $\mu_{\Theta} \in \mathcal{P}^2(\mathcal{X})$  our formula (3.6) gives a posterior distribution of  $\mu_{\Theta}$  on  $\Theta := \mathcal{P}(\mathcal{X})$  after seeing data  $x \in \mathcal{X}$ . If  $\dim \mathcal{X} \geq 1$  then  $\mathcal{P}(\mathcal{X})$  is not a family of dominated measures and therefore we cannot apply Bayes' formula for computing the posterior distribution of  $\mu_{\Theta}$  after seeing data  $x$ .

In some cases, e.g., if  $\mathbf{p} : \Theta \rightarrow \mathcal{P}(\mathcal{X})$  is an exponential family, we can compute the posterior distribution  $\mu_{\Theta|\mathcal{X}}(B|x)$  without using Bayes' formula, see e.g., [Amari2016, §11.5. 1, p. 266] for a geometric explanation, or without using formula (3.6) e.g., for Dirichlet prior nonparametric distributions on  $\mathcal{P}(\mathcal{X})$ , which we shall revisit in the last section.

#### 4. DIRICHLET MEASURES REVISITED

In this section, first we discuss some functorial methods of generating probability measures on  $\mathcal{P}(\mathcal{X})$ . Then, using the functorial language of probabilistic mappings, we revisit Dirichlet distributions, which are Dirichlet measures on  $\mathcal{P}(\mathcal{X})$  where the  $\mathcal{X}$  are finite sample spaces, see Definition 4.5 and the remark thereafter. Finally we give a new proof of the existence of Dirichlet measures over any measurable space using a functorial property of the Dirichlet map constructed by Sethuraman (Theorem 4.6).

**4.1. Probability measures on  $\mathcal{P}(\mathcal{X})$ .** There are several known techniques for construction of random measures, i.e., measures on  $\mathcal{P}(\mathcal{X})$ , see [GV2017, Chapter 3] for an extensive account. In this section we shall use probabilistic mappings for construction of a probability measure on  $\mathcal{P}(\mathcal{X})$ . The most natural way is to look at a “simpler” measurable space  $\mathcal{X}_s$  and construct a probabilistic mapping  $T : \mathcal{X}_s \rightsquigarrow \mathcal{X}$  together with a measure  $\mu \in \mathcal{P}(\mathcal{X}_s)$ , a measure  $\mu_{\mathcal{P}} \in \mathcal{P}^2(\mathcal{X}_s)$ , and examine if the measures  $P_*(\overline{T})(\mu)$ ,  $ev_{\mathcal{P}} \circ$

$P_*^2(\bar{T})(\mu_{\mathcal{P}})$ ,  $P_*^2(T)(\mu_{\mathcal{P}}) \in \mathcal{P}^2(\mathcal{X})$  satisfy our requirement.

$$\begin{array}{ccccc}
 \mathcal{X}_s & \xrightleftharpoons[\text{ev}]{\delta} & \mathcal{P}(\mathcal{X}_s) & \xrightleftharpoons[\text{ev}_P]{\delta} & \mathcal{P}^2(\mathcal{X}_s) \\
 \downarrow \left. \begin{array}{l} T \\ \downarrow \end{array} \right\} & \searrow \bar{T} & \downarrow P_*(T) & \searrow P_*(\bar{T}) & \downarrow P_*^2(T) \\
 \mathcal{X} & \xrightleftharpoons[\text{ev}]{\delta} & \mathcal{P}(\mathcal{X}) & \xrightleftharpoons[\text{ev}_P]{\delta} & \mathcal{P}^2(\mathcal{X}) & \xrightleftharpoons[\text{ev}_P]{\delta} & \mathcal{P}^3(\mathcal{X}) \\
 & & & & & \searrow P_*^2(\bar{T}) & \\
 & & & & & & \mathcal{P}^3(\mathcal{X})
 \end{array}$$

By Lemma 2.12 (3),  $\text{ev}_P \circ P_*^2(\bar{T})(\mu_{\mathcal{P}}) = P_*(\bar{T})(\text{ev}_P(\mu_{\mathcal{P}}))$ , hence we need only to look at  $P_*(\bar{T})(\mu)$  and  $P_*^2(T)(\mu_{\mathcal{P}})$ . Both these constructions are utilized in the proof of Theorem 4.6, where we give an alternative proof of the Sethuraman theorem.

Let us recall that the  $\sigma$ -algebra on  $\mathcal{P}(\mathcal{X})$  is generated by the subsets  $\langle A, B^* \rangle_P := \langle A, B^* \rangle \cap \mathcal{P}(\mathcal{X})$  where  $A \in \Sigma_{\mathcal{X}}$  and  $B^* \in \mathcal{B}([0, 1])$  (see the proof of Prop 2.3). Hence we obtain the following easy Lemma whose proof is omitted.

**Lemma 4.1.** *Any  $\mu_{\mathcal{P}} \in \mathcal{P}^2(\mathcal{X})$  is determined uniquely by its value on the measurable subsets  $\langle A_1, B_1^* \rangle_P \cap \dots \cap \langle A_n, B_n^* \rangle_P$  where  $A_i \in \Sigma_{\mathcal{X}}$ ,  $B_i^* \in \mathcal{B}([0, 1])$  and  $n \in \mathbb{N}$ .*

Note that the collection  $\{(A_1 \times B_1^*) \times \dots \times (A_n \times B_n^*)\}$  of measurable subsets generates the  $\sigma$ -algebra in  $\mathcal{X}_{univ} := ((\mathcal{X} \times [0, 1])^\infty, \Sigma_{\mathcal{X}_{univ}} := (\Sigma_{\mathcal{X}} \otimes \mathcal{B}([0, 1]))^\infty)$ , moreover any measure  $\nu \in \mathcal{P}(\mathcal{X}_{univ})$  is determined uniquely by its values on the subsets in this collection. This suggests that we could take  $\mathcal{X}_s := \mathcal{X}_{univ}$  to define a prior measure on  $\mathcal{P}(\mathcal{X})$  such that the obtained prior measure on  $\mathcal{P}(\mathcal{X})$  satisfies Ferguson's required properties: the support of the prior distribution should be large and the posterior distributions should be manageable analytically. A required probabilistic mapping  $\mathcal{X}_{univ} \rightarrow \mathcal{X}$  has been constructed in Sethuraman's proof of the existence of Dirichlet measures [Sethuraman1994] which we shall revisit at the end of this section.

**4.2. Dirichlet distributions.** Dirichlet distributions are most commonly used as the prior distribution over finite sample spaces in Bayesian statistics. In this subsection, following [GR2003, §3.1.1] and [Ferguson1973], we recall the notion of a Dirichlet distribution  $Dir(\alpha_1, \dots, \alpha_k)$  on  $\Delta_k = \mathcal{P}(\Omega_k)$ , where  $\alpha := (\alpha_1, \dots, \alpha_k) \in \mathbb{R}_{\geq 0}^k \setminus \{0\}$  is a parameter of the distribution. Classically, Dirichlet distributions  $Dir(\alpha)$  are defined for  $\alpha \in \mathbb{R}_{> 0}^k$  but Ferguson's definition of a Dirichlet distribution (Definition 4.2) extends naturally to a definition of a Dirichlet measure in general case (Definition 4.5) when  $\mathcal{X}$  need not be finite, while the classical definition doesn't.

**Definition 4.2.** (cf. [GR2003, Definition 3.1.1, p. 89], [Ferguson1973, p. 211]) Given  $\alpha \in \mathcal{M}^*(\Omega_k)$  let  $\Omega(\alpha) := \{\omega_j \in \Omega_k \mid \alpha(\omega_j) \neq 0\}$ . Let  $\pi_\alpha : \Omega(\alpha) \rightarrow \Omega_k$  denote the natural inclusion. Let  $l(\alpha) := \#\Omega(\alpha)$ . The Dirichlet distribution  $Dir(\alpha) \in \mathcal{P}^2(\Omega_k) = \mathcal{P}(\Delta_k)$  is the measure  $P_*^2(\pi_\alpha)Dir(\alpha|_{\Omega(\alpha)})$ ,

where  $Dir(\alpha|_{\Omega(\alpha)}) \in \mathcal{P}^2(\Omega(\alpha))$  is the classical  $(l(\alpha) - 1)$ -dimensional Dirichlet distribution on  $\Delta_{l(\alpha)}$  with the following density function with respect to the  $(l(\alpha) - 1)$ -dimensional Lebesgue measure on  $\Delta_{l(\alpha)}$ :

$$(4.1) \quad Dir(x_{i_1}, \dots, x_{i_{l(\alpha)}})|_{\alpha|_{\Omega(\alpha)}} := \frac{\Gamma(\alpha_{i_1} + \dots + \alpha_{i_{l(\alpha)}})}{\Gamma(\alpha_{i_1}) \dots \Gamma(\alpha_{i_{l(\alpha)}})} \prod_{j=1}^{l(\alpha)} x_{i_j}^{\alpha_{i_j} - 1}.$$

We summarize important known properties of Dirichlet distributions, which shall be needed later, in the following

**Proposition 4.3.** (1) *The map  $Dir_k : (\mathcal{M}^*(\Omega_k), \tau_v) \rightarrow (\mathcal{P}^2(\Omega_k), \tau_v) : \alpha \mapsto Dir(\alpha)$  is continuous [GR2003, 11(a), p. 93].*

(2) *If  $p \in \Delta_k$  is distributed by  $Dir(\alpha_1, \dots, \alpha_k)$  then for any partition  $A_1, \dots, A_l$  of  $\Omega_k$  the vector  $(\sum_{\omega_i \in A_1} p_i, \sum_{\omega_i \in A_2} p_i, \dots, \sum_{\omega_i \in A_l} p_i)$  is distributed by  $Dir(\alpha(A_1), \dots, \alpha(A_l))$  [GR2003, p. 90].*

(3) *Let  $\alpha \in \mathcal{M}^*(\Omega_k)$  and  $(\mathcal{P}(\Omega_k), Dir(\alpha), Id_{\mathcal{P}}, \Omega_k)$  a Bayesian statistical model. Then the posterior distribution of the prior distribution  $Dir(\alpha)$  after seeing the data  $x \in \Omega_k$  is  $Dir(\alpha + \delta_x)$  [Ferguson1973, p. 212], [GR2003, p. 92].*

**Remark 4.4.** (1) Let  $\mathcal{X} = A_1 \dot{\cup} \dots \dot{\cup} A_n$  be a measurable partition of  $\mathcal{X}$  into  $n$  disjoint measurable subsets. This partition induces a measurable map  $\pi : \mathcal{X} \rightarrow \Omega_n := \{A_1, \dots, A_n\}$ ,  $\pi(x) := A_i$  if  $x \in A_i$ . We have  $M_*(\pi)(\alpha) = (\alpha(A_1), \dots, \alpha(A_n))$  for any  $\alpha \in \mathcal{M}(\mathcal{X})$ . Hence the assertion (2) in Proposition 4.3 is equivalent to the commutativity of the following diagram for any  $n \geq k$  and any surjective mapping  $\pi_{nk} : \Omega_n \rightarrow \Omega_k$

$$\begin{array}{ccc} \mathcal{M}^*(\Omega_n) & \xrightarrow{Dir_n} & \mathcal{P}^2(\Omega_n) \\ \downarrow M_*(\pi_{nk}) & & \downarrow P_*^2(\pi_{nk}) \\ \mathcal{M}^*(\Omega_k) & \xrightarrow{Dir_k} & \mathcal{P}^2(\Omega_k). \end{array}$$

(2) Let  $k < l$  and  $\pi_{kl} : \Omega_k \rightarrow \Omega_l$  is an injection. Let  $\pi_{lk} : \Omega_l \rightarrow \Omega_k$  be a left inverse of  $\pi_{kl}$ . By the definition of  $Dir(\alpha)$  it is not hard to see that the following diagram is commutative

$$\begin{array}{ccccc} \mathcal{M}^*(\Omega_k) & \xrightarrow{M_*(\pi_{kl})} & \mathcal{M}^*(\Omega_l) & \xrightarrow{(\pi_{lk})_*} & \mathcal{M}^*(\Omega_k) \\ \downarrow Dir_k & & \downarrow Dir_l & & \downarrow Dir_k \\ \mathcal{P}^2(\Omega_k) & \xrightarrow{P_*^2(\pi_{kl})} & \mathcal{P}^2(\Omega_l) & \xrightarrow{P_*^2(\pi_{lk})} & \mathcal{P}^2(\Omega_k). \end{array}$$

It follows that for any map  $\pi_{kl} : \Omega_k \rightarrow \Omega_l$  we have

$$P_*^2(\pi_{kl}) \circ Dir_k = Dir_l \circ M_*(\pi_{kl}).$$

Hence  $Dir_k : \mathcal{M}^*(\Omega_k) \rightarrow \mathcal{P}^2(\Omega_k)$  is a natural transformation of the functor  $M_*$  to the functor  $P_*^2$  in the category of finite sample spaces  $\Omega_k$  whose morphisms are (measurable) mappings.

(3) The inverse statement of Remark 4.4(2) is also valid, assuming a normalization condition. Namely for each  $n \in \mathbb{N}^+$  there exists a unique mapping  $Dir_n : \mathcal{M}^*(\Omega_n) \rightarrow \mathcal{P}^2(\Omega_n)$  satisfying the following properties.

(i) Normalization: for  $n = 2$  we have  $Dir_2(\alpha) = Dir(\alpha)$ .

(ii) Naturality: for any  $n, k \geq 1$  and any mapping  $\pi_{nk} : \Omega_n \rightarrow \Omega_k$  the following diagram is commutative

$$\begin{array}{ccc} \mathcal{M}^*(\Omega_n) & \xrightarrow{Dir_n} & \mathcal{P}^2(\Omega_n) \\ \downarrow (\pi_{nk})^* & & \downarrow P_*^2(\pi_{nk}) \\ \mathcal{M}^*(\Omega_k) & \xrightarrow{Dir_k} & \mathcal{P}^2(\Omega_k). \end{array}$$

### 4.3. Dirichlet measures.

**Definition 4.5.** (cf. [Ferguson1973, Definition 1, p. 214]) Let  $\mathcal{X}$  be a measurable space and  $\alpha \in \mathcal{M}^*(\mathcal{X})$ . An element  $\mathcal{D}(\alpha) \in \mathcal{P}^2(\mathcal{X})$  is called a *Dirichlet measure on  $\mathcal{P}(\mathcal{X})$  parameterized by  $\alpha$* , if for all surjective measurable mappings  $\pi_k : \mathcal{X} \rightarrow \Omega_k$  we have

(4.2)

$$P_*^2(\pi_k)(\mathcal{D}(\alpha)) = Dir(\alpha(\pi_k^{-1}(\omega_1), \dots, \alpha(\pi_k^{-1}(\omega_k))) \in \mathcal{P}^2(\Omega_k) = \mathcal{P}(\Delta_k),$$

where  $Dir(\alpha_1, \dots, \alpha_k)$  is the Dirichlet distribution with the parameter  $(\alpha_1, \dots, \alpha_k)$  on  $\Delta_k$  defined in Definition 4.2. If  $\mathcal{D}(\alpha)$  is defined for all  $\alpha \in \mathcal{M}^*$  we shall call  $\mathcal{D}$  a *Dirichlet map*.

Proposition 4.3 (2) implies that  $Dir(\alpha_1, \dots, \alpha_n)$  is a Dirichlet measure on  $\mathcal{P}(\Omega_n)$ .

**Theorem 4.6.** *For any measurable space  $\mathcal{X}$  there exists a measurable mapping  $\mathcal{D} : \mathcal{M}^*(\mathcal{X}) \rightarrow \mathcal{P}^2(\mathcal{X})$  such that  $\mathcal{D}(\alpha)$  is a Dirichlet measure parameterized by  $\alpha$ . Moreover the mapping  $\mathcal{D}$  is a natural transformation of the functor  $M_*$  to the functor  $P_*^2$  in the category of measurable spaces whose morphisms are measurable mappings.*

*Proof.* We shall show that the Dirichlet mapping  $\mathcal{D} : \mathcal{M}^*(\mathcal{X}) \rightarrow \mathcal{P}^2(\mathcal{X})$  constructed by Sethuraman [Sethuraman1994] satisfies the condition in Theorem 4.6(1). Let us recall the construction of  $\mathcal{D}$ . First we define a mapping  $T : \mathcal{M}^*(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X}_{univ})$  as follows

$$T(\alpha) := (\pi_{\mathcal{X}}(\alpha) \times beta(1, \alpha(\mathcal{X})))^\infty \in \mathcal{P}((\mathcal{X} \times [0, 1])^\infty).$$

Here,  $beta$  is the usual beta distribution. Since  $\alpha(\mathcal{X}) > 0$ , the map  $T_1 : \mathcal{M}^*(\mathcal{X}) \rightarrow \mathcal{P}^*(\mathcal{X} \times [0, 1]), \alpha \mapsto beta(1, \alpha(\mathcal{X}))$  is a measurable map. Since  $\pi_{\mathcal{X}}$  is a measurable map by Proposition 2.11, the map  $T$  is measurable.

Let  $\pi_i : \mathcal{X}_{univ} \rightarrow \mathcal{X} \times [0, 1]$  denote the projection on the  $i$ -factor of  $\mathcal{X}_{univ}$ . Denote by  $j_1$  (resp.  $j_2$ ) the projection from  $\mathcal{X} \times [0, 1]$  to  $\mathcal{X}$  (resp. to  $[0, 1]$ ). Then we define for  $i \in \mathbb{N}$  and  $n \in \mathbb{N}$  the following measurable mappings:

- $\theta_i : \mathcal{X}_{univ} \rightarrow \mathbb{R}, \theta_i(x_{univ}) := j_2 \circ \pi_i(x_{univ}),$
- $q_i : \mathcal{X}_{univ} \rightarrow \mathcal{X}, q_i(x_{univ}) := j_1 \circ \pi_i(x_{univ}),$

- $p_1 : \mathcal{X}_{univ} \rightarrow \mathbb{R}$ ,  $p_1(x_{univ}) := \theta_1(x_{univ})$ ,
- $p_n : \mathcal{X}_{univ} \rightarrow \mathbb{R}$ ,  $p_n(x_{univ}) := \theta_n(x_{univ}) \prod_{i=1}^{n-1} (1 - \theta_i(x_{univ}))$ , if  $n \geq 2$ ,
- $p_{univ} : \mathcal{X}_{univ} \rightarrow \mathcal{P}(\mathcal{X})$ ,  $p_{univ}(x_{univ}) := \sum_{i=1}^{\infty} p_i(x_{univ}) \cdot \delta_{q_i(x_{univ})}$
- $\mathcal{D} : \mathcal{M}^*(\mathcal{X}) \rightarrow \mathcal{P}^2(\mathcal{X})$ ,  $\mathcal{D}(\alpha) := (p_{univ})_*(T_{univ}(\alpha))$ .

**Remark 4.7.** It is not hard to see that for any  $x_{univ} \in \mathcal{X}_{univ}$  we have  $\sum_{i=1}^{\infty} p_i(x_{univ}) \in [0, 1]$ . Therefore the image of  $p_{univ}(\mathcal{X}_{univ}) \subset \mathcal{M}(\mathcal{X})$ . Furthermore it is known that there exists a measurable subset  $\mathcal{X}_{univ}^{reg}$  of full  $\mu_{univ}$ -measure for any  $\mu_{univ} = \otimes^{\infty}(\mu_{\mathcal{X}} \times \nu) \in \mathcal{P}(\mathcal{X}_{univ})$ , where  $\mu_{\mathcal{X}} \in \mathcal{P}(\mathcal{X})$  and  $\nu \in \mathcal{P}([0, 1])$  is dominated by the Lebesgue measure on  $[0, 1]$ , such that the map  $p_{univ}(\mathcal{X}_{univ}^{reg}) \subset \mathcal{P}(\mathcal{X})$ , see e.g. [GV2017, Lemma 3.4, p. 31]. Thus the expression  $(p_{univ})_*(T(\alpha))$  should be understood as  $(p_{univ})_* \circ r(\alpha_{univ}) \in \mathcal{P}^2(\mathcal{X})$  where  $r : \mathcal{P}(\mathcal{X}_{univ}) \rightarrow \mathcal{P}(\mathcal{X}_{univ}^{reg})$  is the restriction and  $\alpha_{univ} := T(\alpha)$ .

Since the mappings  $p_i$  are measurable, and  $\sum_{i=1}^{\infty} p_i(x_{univ}) = 1$  for  $x_{univ} \in \mathcal{X}_{univ}^{reg}$ , the map  $\mathcal{D}(\alpha) : \mathcal{M}^*(\mathcal{X}) \rightarrow \mathcal{P}_*^2(\mathcal{X})$  is a measurable map.

Now we shall show that Sethuraman's Dirichlet map  $\mathcal{D}$  is a natural transformation of the functor  $M_*$  to the functor  $P_*^2$ .

Let  $\kappa : \mathcal{X} \rightarrow \mathcal{Y}$  be a measurable mapping. Denote by  $\kappa_{\infty} : \mathcal{X}_{univ} \rightarrow \mathcal{Y}_{univ}$  the induced measurable mapping:

$$\kappa_{\infty}((x_1, \theta_1), \dots, (x_{\infty}, \theta_{\infty})) := ((\kappa(x_1), \theta_1), \dots, (\kappa(x_{\infty}), \theta_{\infty})).$$

Clearly we have  $\kappa_{\infty}(\mathcal{X}_{univ}^{reg}) \subset \mathcal{Y}_{univ}^{reg}$ . Let us consider the following diagram

$$(4.3) \quad \begin{array}{ccccc} \mathcal{M}^*(\mathcal{X}) & \xrightarrow{r \circ T} & \mathcal{P}(\mathcal{X}_{univ}^{reg}) & \xrightarrow{(p_{univ})_*} & \mathcal{P}^2(\mathcal{X}) \\ \downarrow \kappa_* & & \downarrow (\kappa_{\infty})_* & & \downarrow P_*^2(\kappa) \\ \mathcal{M}^*(\mathcal{Y}) & \xrightarrow{r \circ T} & \mathcal{P}(\mathcal{Y}_{univ}^{reg}) & \xrightarrow{(p_{univ})_*} & \mathcal{P}^2(\mathcal{Y}). \end{array}$$

Assume that  $\alpha \in \mathcal{M}^*(\mathcal{X})$ . Since  $\kappa_*(\alpha)(\mathcal{Y}) = \alpha(\mathcal{X})$  we obtain

$$(4.4) \quad \begin{aligned} r \circ T \circ \kappa_*(\alpha) &= r \circ ((\pi_{\mathcal{Y}}(\kappa_* \alpha) \times \beta(1, \alpha(\mathcal{X})))^{\infty}) \\ &= (\kappa_{\infty})_* r \circ T(\alpha) \in \mathcal{P}(\mathcal{Y}_{univ}). \end{aligned}$$

Let  $\langle A_i, B_i^* \rangle_P \in \Sigma_{\mathcal{P}(\mathcal{Y})}$ . Then

$$(4.5) \quad \begin{aligned} (p_{univ})_* (\kappa_{\infty})_* (r(\alpha_{univ})) (\cap_i \langle A_i, B_i^* \rangle_P) &= r(\alpha_{univ}) (\kappa_{\infty}^{-1} (p_{univ}^{-1} (\cap_i \langle A_i, B_i^* \rangle_P))), \\ P_*^2(\kappa) (p_{univ})_* (r(\alpha_{univ})) (\cap_i \langle A_i, B_i^* \rangle_P) &= r(\alpha_{univ}) (p_{univ}^{-1} (\kappa_*)^{-1} (\cap_i \langle A_i, B_i^* \rangle_P)) \\ (4.6) \quad &= r(\alpha_{univ}) (p_{univ}^{-1} (\cap_i \langle \kappa^{-1}(A_i), B^* \rangle_P)). \end{aligned}$$

Let  $\langle A, B^* \rangle_P \in \Sigma_{\mathcal{P}(\mathcal{Y})}$ . Then we have

$$(4.7) \quad p_{univ}^{-1}(\langle A, B^* \rangle_P) = \{y_{univ} \in \mathcal{Y}_{univ} \mid \sum_{i=1}^{\infty} p_i(y_{univ}) \delta_{q_i(y_{univ})}(A) \in B^*\}.$$

Hence

$$\begin{aligned}
 \kappa_\infty^{-1}(p_{univ}^{-1}(\langle A, B^* \rangle_P)) &= \{x_{univ} \in \mathcal{X}_{univ} \mid \sum_{i=1}^{\infty} p_i(\kappa_\infty(x_{univ})) \delta_{q_i(\kappa_\infty(x_{univ}))} \in A\} \\
 &= \{x_{univ} \in \mathcal{X}_{univ} \mid \sum_{i=1}^{\infty} p_i(x_{univ}) \delta_{q_i(x_{univ})} \in \kappa^{-1}(A)\} \\
 (4.8) \qquad \qquad \qquad &= p_{univ}^{-1}(\langle \kappa^{-1}(A), B \rangle_P).
 \end{aligned}$$

It follows from (4.8), (4.7) and (4.6) that

$$(4.9) \quad \kappa_\infty^{-1}(p_{univ}^{-1}(\cap_i \langle A_i, B_i^* \rangle_P)) = p_{univ}^{-1}(\kappa_*)^{-1}(\cap_i \langle A_i, B_i^* \rangle_P).$$

Taking into account Lemma 4.1, we obtain from (4.9) the following identity

$$(4.10) \quad (p_{univ})_*(\kappa_\infty)_*(r(\alpha_{univ})) = P_*^2(\kappa)(p_{univ})_*(r(\alpha_{univ})).$$

Hence we deduce from (4.4) and (4.10) the naturality of the transformation  $\mathcal{D}$  which is expressed in the following identity

$$(4.11) \quad \mathcal{D} \circ M_*(\kappa)(\alpha) = P_*^2(\kappa) \circ \mathcal{D}(\alpha).$$

As in Remark 4.4, we observe that the functoriality of the map  $\mathcal{D}$  implies that  $\mathcal{D}(\alpha)$  is a Dirichlet measure, since it is known that the restriction of the map  $\mathcal{D}$  to the category of finite sample spaces is the Dirichlet map, see e.g. [GV2017, §3.3.3]. (By Remark 4.4(3) we need only to show that the Sethuraman map defined on  $M_*(\Omega_2)$  is the Dirichlet map). This completes the proof of Theorem 4.6.  $\square$

**Remark 4.8.** 1. Let  $\Theta$  denote  $\mathcal{P}(\mathcal{X})$ . Assume that  $\mathcal{D} : \mathcal{M}^*(\mathcal{X}) \rightarrow \mathcal{P}(\Theta)$  is a Dirichlet map. In [Ferguson1973, Theorem 1, p. 217] Ferguson proved that the posterior (conditional) distribution  $\mathcal{D}_{\Theta|\mathcal{X}}(\cdot|x)$  with prior  $\mathcal{D}(\alpha)$  is equal to  $\mathcal{D}(\alpha + \delta_x)$ .

2. In [Ferguson1973] Ferguson made use of Kolmogorov's consistency theorem. Since  $\mathcal{P}(\mathbb{R}, \Sigma_w)$  is not a measurable subset of  $[0, 1]^{\mathcal{B}(\mathbb{R})}$ , see e.g., [GR2003, p. 64], the Kolmogorov theorem does not apply directly and we need a more refined technique [GV2017, Theorem 3.12, p.28]. Ferguson also suggested a second proof of the existence of a Dirichlet map, which is close to Sethuraman's proof [Sethuraman1994].

3. We have shown the benefit of the functorial language of probabilistic mappings in this paper. In particular, using the compact precise functorial language of probabilistic mappings, we don't need to use abstract generators of random variables.

#### APPENDIX A. DIFFERENTIATION OF MEASURES ON COMPLETE RIEMANNIAN MANIFOLDS

Let  $(M, g)$  be a Riemannian manifold. By the Nash embedding theorem, there exists an isometric embedding  $f : (M, g) \rightarrow (\mathbb{R}^n, g_0)$  where  $g_0$  denotes

the Euclidean metric. In general the embedding  $f$  is not easy to find explicitly. Observe that the distance  $\rho_g$  on  $(M, g)$  induced by  $g$  is larger or equal to the distance  $\rho_{g_0}$  induced by the embedding  $f$ , moreover  $\rho_g = \rho_{g_0}$  if and only if  $f(M)$  is an affine subset of  $\mathbb{R}^n$ . If  $(M, g)$  is complete the distances  $\rho_g$  and  $\rho_{g_0}$  generate the same topology on  $M$ .

In this section we extend Bogachev' theorem on differentiation of Borel measures on  $\mathbb{R}^n$  ([Bogachev2007, Theorem 5.8.8, p. 368, vol. 1]) to the case of Borel measures on a complete Riemannian manifold  $(M^n, g)$  where the distance on  $M^n$  is induced by  $g$  (Theorem A.1). First we need some notations.

Let  $\nu_1$  and  $\nu_2$  be locally finite Borel measures on  $(M^n, g)$  such that  $\nu_2 \ll \nu_1$ . For  $x \in M^n$  we denote by  $D_r(x)$  the open geodesic ball of radius  $r$  in  $M$  with center in  $x$  and we set

$$\begin{aligned}\overline{D}_{\nu_1}\nu_2(x) &:= \limsup_{r \rightarrow 0} \frac{\nu_2(D_r(x))}{\nu_1(D_r(x))}, \\ \underline{D}_{\nu_1}\nu_2(x) &:= \liminf_{r \rightarrow 0} \frac{\nu_2(D_r(x))}{\nu_1(D_r(x))},\end{aligned}$$

where we set  $\overline{D}_{\nu_1}\nu_2(x) = \underline{D}_{\nu_1}\nu_2(x) = +\infty$  if  $\nu_1(D_r(x)) = 0$  for some  $r > 0$ .

Furthermore if  $\overline{D}_{\nu_1}\nu_2(x) = \underline{D}_{\nu_1}\nu_2(x)$  then we set

$$D_{\nu_1}\nu_2(x) := \overline{D}_{\nu_1}\nu_2(x) = \underline{D}_{\nu_1}\nu_2(x)$$

which is called *the derivative of  $\nu_2$  with respect to  $\nu_1$  at  $x$* .

**Theorem A.1.** *Let  $\nu_1$  and  $\nu_2$  be two nonnegative locally finite Borel measures on a complete Riemannian manifold  $(M^n, g)$  such that  $\nu_2 \ll \nu_1$ . Then there is a measurable subset  $S_0 \subset M^n$  of zero  $\nu_1$ -measure such that the function  $D_{\nu_1}\nu_2$  is defined and finite on  $M \setminus S_0$ . Setting  $\tilde{D}_{\nu_1}\nu_2(x) := 0$  for  $x \in S_0$  and  $\tilde{D}_{\nu_1}\nu_2(x) := D_{\nu_1}\nu_2(x)$  for  $x \in M \setminus S_0$ , the function  $\tilde{D}_{\nu_1}\nu_2 : M \rightarrow \mathbb{R}$  is measurable and serves as the Radon-Nikodym density of the measure  $\nu_2$  with respect to  $\nu_1$ .*

*Proof.* The proof of Theorem A.1 uses the argument in the proof of [Bogachev2007, Theorem 5.8.8, p. 368, vol. 1] with a slight modification to deal with a general complete Riemannian metric  $g$  and, unlike Theorem 5.8.8 *ibid.*, we modify  $D_{\nu_1}\nu_2$  a bit to get a "better" function  $\tilde{D}_{\nu_1}\nu_2$  on  $M^n$ .

First we shall show that  $D_{\nu_1}\nu_2(x)$  exists and is finite for  $\nu_1$ -a.e.  $x$ . Let  $S := \{x : \overline{D}_{\nu_1}\nu_2(x) = +\infty\}$ . To show  $\nu_1(S) = 0$  we need the following

**Proposition A.2.** *Let  $0 < c < \infty$ .*

- (i) *If  $A \subset \{x : \underline{D}_{\nu_1}\nu_2(x) \leq c\}$  then  $\nu_2^*(A) \leq c\nu_1^*(A)$ .*
- (ii) *If  $A \subset \{x : \overline{D}_{\nu_1}\nu_2(x) \geq c\}$  then  $\nu_2^*(A) \geq c\nu_1^*(A)$ .*

Proposition A.2 is an extension of Lemma 5.8.7 *ibid.* and will be proved in a similar way based on Lemmas A.3 and A.4 below. We shall say that

an open geodesic ball  $D_r(x) \subset (M^n, g)$  is  $k$ -proper, if  $kr$  is at most the injectivity radius of  $(M, g)$  at  $x$ .

**Lemma A.3.** *Assume that  $\mathcal{F}$  is a collection of open 4-proper geodesic balls in a complete Riemannian manifold  $(M^n, g)$  such that the set  $A$  of the centers of the balls in  $\mathcal{F}$  is bounded. Then for some finite number  $N$  one can find subcollections  $\mathcal{F}_1, \dots, \mathcal{F}_N \subset \mathcal{F}$  each of which consist of at most countably many disjoint balls such that  $A$  is covered by the balls from  $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_N$ .*

**Lemma A.4.** *Let  $\mu$  be a locally finite Borel measure on a complete manifold  $(M^n, g)$ . Suppose that  $\mathcal{F}$  is a collection of open 4-proper geodesic balls in  $(M^n, g)$  the set of centers of which is denoted by  $A$ , and for every  $a \in A$  and every  $\varepsilon > 0$ ,  $\mathcal{F}$  contains an open 4-proper geodesic ball  $D_r(a)$  with  $r < \varepsilon$ . If  $A$  is bounded then for every nonempty open set  $U \subset M^n$  one can find an at most countable collection of disjoint balls  $D_j \in \mathcal{F}$  such that*

$$\bigcup_{j=1}^{\infty} D_j \subset U \text{ and } \mu^*((A \cap U) \setminus \bigcup_{j=1}^{\infty} D_j) = 0.$$

*Proof of Lemma A.3.* Lemma A.3 is a version of the Besicovitch covering theorem for  $(\mathbb{R}^n, g_0)$  [Besicovitch1945], which has been formulated as Theorem 5.8.1 in [Bogachev2007, p. 361, vol. 1]. There are three differences between Lemma A.3 and Theorem 5.8.1 *ibid.*: firstly we make the assumption that  $A$  is bounded, secondly, the geodesic balls are 4-proper, and thirdly, the balls are open instead of nondegenerate closed as in Theorem 5.8.1 *ibid.*<sup>4</sup>.

Denote by  $A$  the set of the centers of the balls in  $\mathcal{F}$ . Let  $R := \sup\{r : D_r(a) \in \mathcal{F}\}$ . We can find  $D_1 = D_{r_1}(a_1) \in \mathcal{F}$  with  $r_1 > 3R/4$ . The balls  $D_j, j > 1$ , are chosen inductively as follows. Let  $A_j := A \setminus \bigcup_{i=1}^{j-1} D_i$ . If the set  $A_j$  is empty, then our construction is completed and, letting  $J = j - 1$  we obtain  $J$  balls  $D_1, \dots, D_J$ . If  $A_j$  is nonempty, then we choose  $D_j : D_{r_j}(a_j) \in \mathcal{F}$  such that

$$a_j \in A_j \text{ and } r_j > \frac{3}{4} \sup\{r : D_r(a) \in \mathcal{F}, a \in A_j\}.$$

In the case of an infinite sequence of balls  $D_j$  we set  $J = \infty$ .

*Claim 1.* The balls  $D_r(a_j)$  satisfy the following properties

- (a) if  $j > i$  then  $r_j \leq 4r_i/3$ ,
- (b) the balls  $D_{r_j/3}(a_j)$  are disjoint and if  $J = \infty$  then  $r_j \rightarrow 0$ ,
- (c)  $A \subset \bigcup_{j=1}^J D_{r_j}(a_j)$ .

*Proof of Claim 1.* Property (a) follows by the definition of  $r_i$  and the inclusion  $a_j \in A_j \subset A_i$ .

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<sup>4</sup>The version of Besicovich's theorem for open balls seems known [Bogachev2007, p. 344, vol2].

Property (b) is a consequence of the following observation. If  $j > i$  then  $a_j \notin D_j$  and hence by (a) we have

$$(A.1) \quad \rho_g(a_i, a_j) \geq r_i > \frac{r_i}{3} + \frac{r_j}{3}.$$

Since  $A$  is bounded,  $r_j$  goes to 0 if  $J = \infty$ .

Finally (c) is obvious if  $J < \infty$ . If  $J = \infty$  and  $D_r(a) \in \mathcal{F}$  then there exists  $r_j$  with  $r_j < 3r/4$  by (b). Hence  $a \in \cup_{i=1}^{j-1} D_i$  by our construction of  $r_j$ . This completes the proof of Claim 1.

We fix  $k > 1$  and let

$$(A.2) \quad I_k := \{j : j < k, D_j \cap D_k \neq \emptyset\}, M_k := I_k \cap \{j : r_j \leq 3r_k\}.$$

*Claim 2.* There is a number  $c(A)$  independent of  $k$  such that  $\#(M_k) \leq c(A)$ .

*Proof of Claim 2.* If  $j \in M_k$  and  $x \in B(a_j, r_j/3)$  then the balls  $D_j$  and  $D_k$  have nonempty intersection and  $r_j \leq 3r_k$ , hence

$$\rho_g(x, a_k) \leq \rho_g(x, a_j) + \rho_g(a_j, a_k) < \frac{r_j}{3} + r_j + r_k < 5r_k.$$

It follows that  $D_{r_j/3}(a_j) \subset D_{5r_k}(a_k)$ . Denote by  $\text{vol}_g$  the Riemannian volume on  $(M^n, g)$ . By the disjointness of  $B(a_j, r_j/3)$  and the boundedness of  $A$ , taking into account the Bishop volume comparison theorem [BC1964, Theorem 15, §11.10], see also [Le1993] for a generalization, there exists a number  $c_1(A)$  such that

$$(A.3) \quad \text{vol}_g(D_{5r_k}(a_k)) \geq \sum_{j \in M_k} \text{vol}_g(D_{r_j/3}(a_j)) \geq c_1(A) \sum_{j \in M_k} \left(\frac{r_j}{3}\right)^n.$$

Using property (a) in Claim 1, we obtain from (A.3)

$$(A.4) \quad \text{vol}_g(D_{5r_k}(a_k)) \geq \sum_{j \in M_k} c_1(A) \left(\frac{r_k}{4}\right)^n = \#(M_k) c_1(A) \left(\frac{r_k}{4}\right)^n.$$

By the Bishop comparison theorem there exists a number  $c_2(A)$  such that  $\text{vol}_g(D_{5r_k}(a_k)) \leq c_2(A) \cdot (5r_k)^n$ . Combining with (A.4) we obtain

$$(A.5) \quad \#(M_k) \leq \frac{c_2(A)}{c_1(A)} 20^n.$$

This completes the proof of Claim 2.

*Claim 3.* There exists a number  $d(A)$  independent of  $k$  such that  $\#(I_k \setminus M_k) \leq d(A)$ .

*Proof of Claim 3.* Let us consider two distinct elements  $i, j \in I_k \setminus M_k$ . By (A.2) we have

$$(A.6) \quad 1 < i, j < k, D_i \cap D_k \neq \emptyset, D_j \cap D_k \neq \emptyset, r_i > 3r_k, r_j > 3r_k.$$

For notational simplicity we shall redenote  $\rho_g(a_k, a_i)$  by  $|a_i|$ . Then (A.6) implies

$$(A.7) \quad |a_i| < r_i + r_k \text{ and } |a_j| < r_j + r_k.$$

Let  $\theta_{def}(a_i, a_j) \in [0, \pi]$  be the deformed angle between the two geodesic rays  $(a_k, a_i)$  and  $(a_k, a_j)$ , connecting  $a_k$  with  $a_i$  and  $a_j$  respectively, which is defined as follows

$$\theta_{def}(a_i, a_j) := \arccos \frac{|a_i|^2 + |a_j|^2 - \rho_g(a_i, a_j)^2}{2|a_i||a_j|}.$$

We shall prove the estimate

$$(A.8) \quad \theta_{def}(a_i, a_j) \geq \theta_0 := \arccos 61/64 > 0.$$

By the construction, see also (A.1), we have  $0 = a_k \notin B_i \cup B_j$  and  $r_i \leq |a_i|$ ,  $r_j \leq |a_j|$ . W.l.o.g. we assume that  $|a_i| \leq |a_j|$ . By (A.2) and (A.7) we obtain

$$(A.9) \quad 3r_k < r_i \leq |a_i| < r_i + r_k, \quad 3r_k < r_j \leq |a_j| < r_j + r_k, \quad |a_i| \leq |a_j|.$$

We need two more claims for the proof of (A.8).

*Claim 4.* If  $\cos \theta_{def}(a_i, a_j) > 5/6$  then  $a_i \in B_j$ .

*Proof of Claim 4.* It suffices to show that if  $a_i \notin B_j$  then  $\cos \theta_{def}(a_i, a_j) \leq 5/6$ . Assume that  $a_i \notin B_j$ . We shall consider two possibilities, first assume that  $\rho_g(a_i, a_j) \geq |a_j|$ . Then our assertion follows from the following estimates

$$(A.10) \quad \cos \theta_{def}(a_i, a_j) = \frac{|a_i|^2 + |a_j|^2 - \rho_g(a_i, a_j)^2}{2|a_i||a_j|} \leq \frac{|a_i|}{2|a_j|} \leq \frac{1}{2} < \frac{5}{6}.$$

Now assume that  $\rho_g(a_i, a_j) \leq |a_j|$ . Then

$$(A.11) \quad \begin{aligned} \cos \theta_{def}(a_i, a_j) &= \frac{|a_i|^2 + |a_j|^2 - \rho_g(a_i, a_j)^2}{2|a_i||a_j|} \leq \\ &= \frac{|a_i|}{2|a_j|} + \frac{(|a_j| - \rho_g(a_i, a_j))(|a_j| + \rho_g(a_i, a_j))}{2|a_i||a_j|} \\ &\leq \frac{1}{2} + \frac{|a_j| - \rho_g(a_i, a_j)}{|a_j|} \\ &\leq \frac{1}{2} + \frac{r_j + r_k - r_j}{r_i} \leq \frac{5}{6} \end{aligned}$$

where in the second inequality we use the assumption  $|a_j| + \rho_g(a_i, a_j) \leq 2|a_j|$ , in the third inequality we use  $|a_j| \leq r_j + r_k$  and taking into account  $a_i \notin B_j$  we have  $r_j \leq \rho_g(a_i, a_j)$ , we also use  $r_j \leq |a_j|$  from (A.9), and in the last inequality we use  $3r_k \leq r_i$  from (A.6). This completes the proof of Claim 4.  $\square$

*Claim 5.* If  $a_i \in B_j$  then

$$(A.12) \quad 0 \leq \rho_g(a_i, a_j) + |a_i| - |a_j| \leq \frac{8}{3}(1 - \cos \theta_{def}(a_i, a_j))|a_j|.$$

*Proof of Claim 5.* We utilize the proof of [Bogachev2007, (5.8.3), p. 363, vol. 2]. Since  $a_i \in B_j$  we have  $i < j$ . Hence  $a_j \notin B_i$  and therefore  $\rho_g(a_i, a_j) \geq r_i$ . Keeping our convention that  $|a_i| \leq |a_j|$  we have

$$\begin{aligned} 0 &\leq \frac{\rho_g(a_i, a_j) + |a_i| - |a_j|}{|a_j|} \leq \frac{\rho_g(a_i, a_j) + |a_i| - |a_j|}{|a_j|} \frac{\rho_g(a_i, a_j) - |a_i| + |a_j|}{\rho_g(a_i, a_j)} \\ &= \frac{\rho_g(a_i, a_j)^2 - (|a_j| - |a_i|)^2}{|a_j|\rho_g(a_i, a_j)} = \frac{2|a_i|(1 - \cos \theta_{def}(a_i, a_j))}{\rho_g(a_i, a_j)} \\ &\leq \frac{2(r_i + r_k)(1 - \cos \theta_{def}(a_i, a_j))}{r_i} \leq \frac{8}{3}(1 - \cos \theta_{def}(a_i, a_j)). \end{aligned}$$

Here in the inequality before the last we use the above inequality  $r_i < \rho_g(a_i, a_j)$  and  $|a_i| < r_i + r_k$  from (A.9). This completes the proof of Claim 5.

*Continuation of the proof of (A.8).* If  $\cos \theta_{def}(a_i, a_j) \leq 5/6$  then  $\cos \theta_{def}(a_i, a_j) < 61/64$ . If  $\cos \theta_{def}(a_i, a_j) > 5/6$  then  $a_i \in B_j$  by Claim 4. Then  $i < j$  and hence  $a_j \in B_i$ . It follows that  $r_i \leq \rho_g(a_i, a_j) < r_j$ . Recall by Claim 1 (a)  $r_j \leq 4r_i/3$ . Taking into account  $r_j > 3r_k$  from (A.2) we obtain

$$\rho_g(a_i, a_j) + |a_i| - |a_j| \stackrel{(A.9)}{\geq} r_i + r_i - r_j - r_k \geq \frac{r_j}{2} - r_k \geq \frac{1}{8}(r_j + r_k) \geq \frac{1}{8}|a_j|$$

which in combination with (A.12) yields  $|a_j|/8 < 8(1 - \cos \theta_{def}(a_i, a_j))|a_j|/3$ . Hence  $\cos \theta_{def}(a_i, a_j) \leq 61/64$ . This completes the proof of estimate (A.8).

In the next step we shall prove the existence of a lower bound for the angle  $\theta_{a_k}(a_i, a_j)$  between the two geodesic rays  $(a_k, a_i)$  and  $(a_k, a_j)$ , namely  $\theta_{a_k}(a_i, a_j)$  is the angle between two vectors  $\vec{a}_i$  and  $\vec{a}_j$  on the tangent space  $T_{a_k}M^n$  provided with the restriction of the metric  $g$  to  $T_{a_k}M^n$ , where  $\vec{a}_i$  (resp.  $\vec{a}_j$ ) is the tangent vector at  $a_k$  of the geodesic  $(a_k, a_i)$  (resp. of the geodesic  $(a_k, a_j)$ ).

*Claim 6.* There exists a positive number  $\alpha(A)$  independent of  $k, i, j$  such that  $\theta_{a_k}(a_i, a_j) \geq \alpha(A)$ .

*Proof of Claim 6.* Since  $A$  is bounded, by the Bishop comparison theorem, there exists a constant  $b(A)$  independent of  $a_i, a_j, a_k$  such that  $\theta_{a_k}(a_i, a_j) < b(A) \cdot \theta_{def}(a_i, a_j)$ . Combining this with (A.8) this implies Claim 6.

*Continuation of the proof of Claim 3.*

- Let  $\delta(A)$  be the largest positive number such that:
  - (i)  $\delta(A) \leq r_A/8$ ,

(ii) For any  $x \neq y \neq z \neq x \in A$  satisfying the following relations

$$\rho_g(x, y) \leq \frac{r_A}{4} \text{ and } \rho_g(y, z) \leq \rho_g(x, y) \cdot \delta(A)$$

we have  $\theta_x(y, z) \leq \alpha(A)$ .

The existence of  $\delta(A)$  follows from the boundedness of  $A$  and Bishop's comparison theorem.

• Let  $d(A)$  be the smallest natural number such that for any  $x \in A$  and any  $r \in (0, r_A/4)$  we can cover the geodesic sphere  $S(x, r)$  of radius  $r$  centered at  $x$  by at most  $d(A)$  balls of radius  $r \cdot \delta(A)$ . The existence of  $d(A)$  follows from the boundedness of  $A$  and Bishop's comparison theorem.

Claim 6 implies that that  $\#(I_k \setminus M_k) \leq d(A)$ . This completes the proof of Claim 3.

*Completion of the proof of Lemma A.3.* Claims 2 and 3 imply that  $\#(I_k) \leq c(A) + d(A)$ . Now we make a choice of  $\mathcal{F}_i$  in the same way as in the proof of Theorem 5.8.1 *ibid.* so we omit the details and refer the reader to [Bogachev2007, p. 363, vol. 1].  $\square$

*Proof of Lemma A.4.* The proof of Lemma A.4 is based on Lemma A.3 and can be produced by repeating the proof of Corollary 5.8.2 *ibid.* word for word so we omit it here.  $\square$

*Proof of Proposition A.2.* By the property of outer measures it suffices to prove Proposition A.2 for bounded sets  $A$ . Now we derive Proposition A.2 from Lemma A.4 as in the proof of Lemma 5.8.7 *ibid.* and we leave the details to the reader.  $\square$

*Completion of the proof of Theorem A.1.* Proposition A.2 implies that  $\nu_1^*(S) = 0$ .

Next let  $0 < a < b$  and set

$$S(a, b) : \{x : \underline{D}_{\nu_1} \nu_2(x) < a < b < \overline{D}_{\nu_1} \nu_2(x) < +\infty\}.$$

Proposition A.2 implies that

$$b \nu_1^*(S(a, b)) \leq \nu_2(S(a, b)) \leq a \nu_1^*(S(a, b)).$$

Hence  $\nu_1^*(S(a, b)) = 0$  because  $a < b$ . The union  $S_1$  of  $S(a, b)$  over all positive rational numbers  $a, b$  also has zero  $\nu_1^*$ -measure. Hence there exists a measurable subset  $S_0 \subset M$  of zero  $\nu_1$ -measure such that  $S \cup S_1 \subset S_0$ . This proves the first assertion of Theorem A.1.

Now let us show that  $\tilde{D}_{\nu_1} \nu_2(x)$  is measurable. Clearly, it suffices to show that  $D_{\nu_1} \nu_2 : M^n \setminus S_0 \rightarrow \mathbb{R}$  is measurable.

**Lemma A.5.** *For each  $r > 0$  the function  $f_r(x) := \nu_1(D_r(x)) : M^n \rightarrow \mathbb{R}$  is lower-semi continuous and hence measurable.*

*Proof.* Since  $\lim_{k \rightarrow \infty} \nu_1(D_{r-1/k}(x)) = \nu_1(D_r(x))$ , taking into account that  $D_{r-1/k}(x) \subset D_r(y)$  if  $|x - y| < 1/k$ , we obtain

$$\liminf_{y \rightarrow x} \nu_1(D_r(y)) \geq \nu_1(D_r(x))$$

which we needed to prove.  $\square$

Since  $S_0$  is measurable, we obtain immediately from Lemma A.5 the following

**Corollary A.6.** *For each  $r > 0$  the restriction  $f_r|_{M \setminus S_0}$  is a measurable function.*

In the same way, the restriction of function  $f'_r(x) := \nu_2(D_r(x))$  to  $M \setminus S_0$  is measurable. For  $k \in \mathbb{N}^+$  and  $x \in M \setminus S_0$  we set

$$\tau_k(x) := \frac{\nu_2(D_{1/k}(x))}{\nu_1(D_{1/k}(x))}.$$

It follows that the function  $\tau_k : M \setminus S_0 \rightarrow \mathbb{R}$  is measurable. Hence the function  $D_{\nu_1} \nu_2(x) : M \setminus S_0 \rightarrow \mathbb{R}$  is measurable, which we had to prove.

Finally we prove that  $\tilde{D}_{\nu_1} \nu_2$  serves as the Radon-Nikodym derivative of  $\nu_2$  w.r.t.  $\nu_1$  by the same argument as for the case  $(M, g) = (\mathbb{R}^n, g_0)$  in [Bogachev2007, p. 368, 369, vol. 1], noting that our function  $\tilde{D}_{\nu_1} \nu_2$  coincides with the function  $D_{\nu_1} \nu_2$  defined in [Bogachev2007, p. 368, vol.2] up to a zero  $\nu_1^*$ -measure set, and hence we omit the details. This completes the proof of Theorem A.1.

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