Asymptotic dynamics of Young differential equations: a unified approach

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Abstract

We provide a unified approach to study the asymptotic behavior of Young differential equations, which consists of two steps of applying the continuous and discrete Gronwall lemmas. Our method helps to generalize the result on the existence, and on the diameter estimate, of the global pullback attractor for the generated random dynamical system.

Keywords: stochastic differential equations (SDE), Young integral, rough path theory, rough differential equations, exponential stability.

1 Introduction

This paper studies the asymptotic behavior of the Young differential equation

$$dy_t = [Ay_t + f(y_t)]dt + g(y_t)dx_t, t \in \mathbb{R},\ y(0) = y_0 \in \mathbb{R}^d, \tag{1.1}$$

where we assume for simplicity that $A \in \mathbb{R}^{d\times d}$, $f : \mathbb{R}^d \to \mathbb{R}^d, g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$, are globally Lipschitz continuous, $g \in C^1$ such that D_g is also globally Lipschitz continuous with respectively Lipschitz coefficients C_f, C_g of f and g. We also assume that $x \in C^{p-\rm var}(\mathbb{R}, \mathbb{R}^m)$ is a realization of a stationary stochastic process $Z_t(\omega)$ with almost sure all realizations in the space $C^{p-\rm var}(\mathbb{R}, \mathbb{R}^m)$, such that

$$\Big(E\, \|Z\|^p_{p-\rm var,[-1,1]}\Big)^{\frac{1}{p}} < \infty.$$

Such system is often understood in the sense of a Young differential equation [18], as a pathwise approach of a stochastic differential equation with a Hölder continuous stochastic noise. Our aim is to investigate the role of the driving noise in the longterm behavior of system (1.1).
Although no deterministic equilibrium such as the zero solution can in general be found, system (1.1) is expected to possess a pathwise attractor. The reader is refered to [10], [11], [8], [9] and references therein for recent development in studying the asymptotic behavior of Young differential equations and rough differential equation in general. In particular, the existence of random attractor for the generated random dynamical system is studied in [13] and [10] for Young differential equations with small noise in the sense that the Hölder seminorm of its realization is integrable and can be controlled to be small. Our method, by contrast, works for a general source of noise, and produces a stability criterion which matches the classical one for ordinary differential equations when the driving noise is deleted. Moreover, as discussed in details in Remark 3.8, the method could also be applied to study the attractor of rough differential equations, although the estimates for rough integrals are expected to be quite technical and would be studied separately in a coming project.

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The paper is organized as follows. Section 2 is devoted to present the existence, uniqueness and the norm estimates of the solution. In subsection 3.1, we introduce the generation of random dynamical system by the equation (1.1). Using Lemma 3.4, we prove the existence of a global random pullback attractor and estimate its diameter in Theorem 3.6 and Theorem 3.10, and derive an exponential stability criterion for the trivial solution in Corollary 3.7. At the end of this section, we discuss particular cases in which we could prove that the attractor is an one point set.

2 Young differential equations

In this section, we briefly make a survey on Young integrals and Young differential equations. Let $C([a,b],\mathbb{R}^r)$ denote the space of all continuous paths $x: [a,b] \to \mathbb{R}^r$ equipped with sup norm $\|\cdot\|_{\infty,[a,b]}$ given by $\|x\|_{\infty,[a,b]} = \sup_{t\in[a,b]} \|x_t\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^r$. For $p \geq 1$ and $[a,b] \subset \mathbb{R}$, $\mathcal{C}^{p-\mathrm{var}}([a,b],\mathbb{R}^r) \subset C([a,b],\mathbb{R}^r)$ denotes the space of all continuous paths $x:[a,b] \to \mathbb{R}^r$ which is of finite $p-$variation

$$\left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} := \left(\sup_{\Pi(a,b)} \sum_{i=1}^{n} \|x_{t_{i+1}} - x_{t_i}\|^p \right)^{1/p} < \infty, \tag{2.1}$$

where the supremum is taken over the whole class of finite partitions of $[a,b]$. $\mathcal{C}^{p-\mathrm{var}}([a,b],\mathbb{R}^r)$ equipped with the $p-$var norm

$$\|x\|_{p-\mathrm{var},[a,b]} \quad := \quad \|x(a)\| + \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]},$$

is a nonseparable Banach space [12, Theorem 5.25, p. 92]. Also for each $0<\alpha<1$, we denote by $C^{\alpha-\mathrm{Hol}}([a,b],\mathbb{R}^r)$ the space of Hölder continuous functions with exponent α on $[a,b]$ equipped with the norm

$$\|x\|_{\alpha-\mathrm{Hol},[a,b]} := \|x_a\| + \sup_{a\leq s<t\leq b} \frac{\|x_t - x_s\|}{(t-s)^\alpha}.$$

Given a simplex $\Delta[a,b] := \{(s,t) |\ a \leq s \leq t \leq b\}$, a continuous map $\overline{\omega} : \Delta[a,b] \longrightarrow \mathbb{R}^+$ is called a *control* (see e.g. [12]) if it is zero on the diagonal and superadditive, i.e

(i), For all $t\in[a,b]$, $\overline{\omega}_{t,t} = 0$,

(ii), For all $s \leq t \leq u$ in $[a,b]$, $\overline{\omega}_{s,t} + \overline{\omega}_{t,u} \leq \overline{\omega}_{s,u}$.

Now, consider $y \in \mathcal{C}^{q-\mathrm{var}}([a,b],\mathbb{R}^{d\times m})$ and $x\in \mathcal{C}^{p-\mathrm{var}}([a,b],\mathbb{R}^m)$ with $\frac{1}{p}+\frac{1}{q}>1$, the Young integral $\int_a^b y_t dx_t$ can be defined as

$$\int_a^b y_s dx_s := \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} y_u(x_v - x_u),$$

where the limit is taken on all the finite partitions $\Pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$ with $|\Pi| := \max\limits_{[u,v]\in\Pi} |v-u|$ (see [18, p. 264–265]). This integral satisfies additive property by the construction, and the so-called Young-Loeve estimate [12, Theorem 6.8, p. 116]

$$\left\| \int_s^t y_u dx_u - y_s[x_t - x_s] \right\| \leq (1-2^{1-\theta})^{-1} \left\vert\!\left\vert\!\left\vert y \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[s,t]} \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]}, \ \forall [s,t]\subset[a,b], \tag{2.2}$$

where $\theta = \frac{1}{p}+\frac{1}{q}$.

From now on, we only consider $q=p$ for convenience and set

$$K := (1-2^{1-\frac{2}{p}})^{-1}. \tag{2.3}$$

In addition, we would like to construct, for any $\gamma > 0$ and any given interval $[a,b]$, a sequence of greedy times $\{\tau_k(\gamma)\}_{k\in\mathbb{N}}$ as follows

$$\tau_0 = a, \tau_{k+1}(\gamma) := \inf\{t > \tau_k(\gamma) : \|x\|_{p-\mathrm{var},[\tau_k(\gamma),t]} = \gamma\} \wedge b. \tag{2.4}$$

Define

$$N = N_{\gamma,p,[a,b]}(x) := \sup\{k \in \mathbb{N}, \tau_k(\gamma) \le b\}, \tag{2.5}$$

then due to the superadditivity of $\|x\|^p_{p-\mathrm{var},[s,t]}$

$$\begin{aligned} N-1 &\le \sum_{k=0}^{N-2} \gamma^{-p} \|x\|^p_{p-\mathrm{var},[\tau_k,\tau_{k+1}]} \le \gamma^{-p} \|x\|^p_{p-\mathrm{var},[\tau_0,\tau_{N-1}]} \le \gamma^{-p} \|x\|^p_{p-\mathrm{var},[a,b]}, \\ \text{which yields } \ N &\le 1 + \gamma^{-p} \|x\|^p_{p-\mathrm{var},[a,b]}. \end{aligned} \tag{2.6}$$

In this paper, we fix $p \in (1,2)$ and $\gamma := \frac{1}{2(K+1)C_g}$, and write in short $N_{[a,b]}(x)$ to specify the dependence of N on x and the interval $[a,b]$.
The following theorem shows a standard method to estimate the variation and the supremum norms of the solution of (1.1), by using Gronwall lemma and discretization scheme with the greedy times.

Theorem 2.1 *There exists a unique solution to* (1.1) *for any initial value, whose supremum and* $p-$*variation norms are estimated as follows*

$$\|y\|_{\infty,[a,b]} \le \Big[\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big) N_{[a,b]}(x)\Big] e^{\alpha N_{[a,b]}(x) + 2L(b-a)}, \tag{2.7}$$

$$\|y\|_{p-\mathrm{var},[a,b]} \le \Big[\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big) N_{[a,b]}(x)\Big] e^{\alpha N_{[a,b]}(x) + 2L(b-a)} N^{\frac{p-1}{p}}_{[a,b]}(x), \tag{2.8}$$

where $L = \|A\| + C_f, \alpha = \log(1 + \frac{1}{K+1})$.

Proof: Write in short $L = \|A\| + C_f$. The existence and uniqueness theorem is proved in [5]. To prove (2.7), we use the fact that $\|g(y)\|_{p-\mathrm{var},[s,t]} \le C_g \|y\|_{p-\mathrm{var},[s,t]}$ to derive

$$\|y_t - y_s\| \le \int_s^t (L\|y_u\| + \|f(0)\|)du + \|x\|_{p-\mathrm{var},[s,t]} \Big(\|g(y_s)\| + KC_g \|y\|_{q-\mathrm{var},[s,t]}\Big)$$

which yields

$$\begin{aligned} \|y\|_{p-\mathrm{var},[s,t]} &\le \int_s^t L \|y\|_{q-\mathrm{var},[s,u]} du + (\|f(0)\| + L\|y_s\|)(t-s) \\ &\quad + \|x\|_{p-\mathrm{var},[s,t]} \Big(\|g(y)\|_{\infty,[s,t]} + KC_g \|y\|_{p-\mathrm{var},[s,t]}\Big) \\ &\le \int_s^t L \|y\|_{q-\mathrm{var},[s,u]} du + (\|f(0)\| + L\|y_s\|)(t-s) \\ &\quad + \|x\|_{p-\mathrm{var},[s,t]} \Big(\|g(y_s)\| + (K+1)C_g \|y\|_{p-\mathrm{var},[s,t]}\Big). \end{aligned}$$

As a result, we obtain

$$\begin{aligned} \|y\|_{p-\mathrm{var},[s,t]} \Big(1 - (K+1)C_g \|x\|_{p-\mathrm{var},[s,t]}\Big) &\le \int_s^t L \|y\|_{p-\mathrm{var},[s,u]} du + (\|f(0)\| + L\|y_s\|)(t-s) \\ &\quad + \|x\|_{p-\mathrm{var},[s,t]} \|g(y_s)\|, \end{aligned}$$

which derives

$$\left\| y \right\|_{p-\text{var},[s,t]} \leq \int_s^t 2L \left\| y \right\|_{p-\text{var},[s,u]} du + 2(\|f(0)\| + L\|y_s\|)(t-s) + 2\left\| x \right\|_{p-\text{var},[s,t]} \|g(y_s)\|.$$

whenever $(K+1)C_g \left\| x \right\|_{p-\text{var},[s,t]} \leq \frac{1}{2}$. Applying the continuous Gronwall lemma 4.1, we obtain

$$\begin{aligned}
\left\| y \right\|_{p-\text{var},[s,t]} &\leq 2(\|f(0)\| + L\|y_s\|)(t-s) + 2\left\| x \right\|_{p-\text{var},[s,t]} \|g(y_s)\| \\
&\quad + \int_s^t 2Le^{2L(t-u)}\Big[2(\|f(0)\| + L\|y_s\|)(u-s) + 2\left\| x \right\|_{p-\text{var},[s,u]} \|g(y_s)\|\Big] du \\
&\leq \Big(\frac{\|f(0)\|}{L} + \|y_s\| + 2\left\| x \right\|_{p-\text{var},[s,t]} \|g(y_s)\|\Big)e^{2L(t-s)} - \|y_s\| \\
&\leq \Big(\frac{\|f(0)\|}{L} + 2\left\| x \right\|_{p-\text{var},[a,b]} \|g(0)\| + \|y_s\|(1 + 2C_g \left\| x \right\|_{p-\text{var},[s,t]} \|)\Big)e^{2L(t-s)} - \|y_s\| \\
&\leq \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g} + (1 + \frac{1}{K+1})\|y_s\|\Big)e^{2L(t-s)} - \|y_s\| \qquad (2.9)
\end{aligned}$$

whenever $\left\| x \right\|_{p-\text{var},[s,t]} \leq \gamma$. By constructing the sequence of greedy times $\{\tau_k = \tau_k(\gamma)\}_{k\in\mathbb{N}}$ on interval $[a,b]$, it follows from induction that

$$\begin{aligned}
&\|y_{\tau_{k+1}}\| \leq \|y\|_{\infty,[\tau_k,\tau_{k+1}]} \leq \|y\|_{p-\text{var},[\tau_k,\tau_{k+1}]} \\
&\leq \Big(\frac{K+2}{K+1}\|y_{\tau_k}\| + \frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big)e^{2L(\tau_{k+1}-\tau_k)} \\
&\leq \Big(\frac{K+2}{K+1}\Big)^{k+1} e^{2L(\tau_{k+1}-\tau_0)}\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big)\sum_{j=0}^k \Big(\frac{K+2}{K+1}\Big)^{k-j} e^{2L(\tau_{k+1}-\tau_j)} \\
&\leq \Big[e^{\alpha(k+1)}\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big)e^{\alpha k}(k+1)\Big]e^{2L(\tau_{k+1}-\tau_0)} \\
&\leq \Big[\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big)(k+1)\Big]e^{\alpha(k+1)+2L(\tau_{k+1}-\tau_0)}, \quad \forall k = 0,\ldots,N_{[a,b]}(x)-1,
\end{aligned}$$

which proves (2.7) since $\tau_{N_{[a,b]}(x)} = b$. On the other hand,

$$\begin{aligned}
\left\| y \right\|_{p-\text{var},[\tau_k,\tau_{k+1}]} &\leq \|y_{\tau_k}\|\Big(e^{\alpha+2L(\tau_{k+1}-\tau_k)} - 1\Big) + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big)e^{2L(\tau_{k+1}-\tau_k)} \\
&\leq \|y_a\|\Big(e^{\alpha(k+1)+2L(\tau_{k+1}-\tau_0)} - e^{\alpha k+2L(\tau_k-\tau_0)}\Big) \\
&\quad + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big)\Big(e^{\alpha(k+1)+2L(\tau_{k+1}-\tau_0)} - e^{\alpha k+2L(\tau_k-\tau_0)}\Big) \\
&\quad + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big)e^{2L(\tau_{k+1}-\tau_k)}, \quad \forall k = 0,\ldots,N_{[a,b]}(x)-1,
\end{aligned}$$

It then follows from inequality of p-variation seminorm in [9] that

$$\begin{aligned}
&\left\| y \right\|_{p-\text{var},[a,b]} \\
&\leq N_{[a,b]}^{\frac{p-1}{p}}(x) \sum_{k=0}^{N_{[a,b]}(x)-1} \left\| y \right\|_{q-\text{var},[\tau_k,\tau_{k+1}]} \\
&\leq \|y_a\| N_{[a,b]}^{\frac{p-1}{p}}(x) \sum_{k=0}^{N_{[a,b]}(x)-1} \Big(e^{\alpha(k+1)+2L(\tau_{k+1}-\tau_0)} - e^{\alpha k+2L(\tau_k-\tau_0)}\Big)
\end{aligned}$$

$$+\Big(\frac{\|f(0)\|}{L}+\frac{\|g(0)\|}{(K+1)C_g}\Big)N_{[a,b]}^{\frac{p-1}{p}}(x)\sum_{k=0}^{N_{[a,b]}(x)-1}\Big(\sum_{j=0}^{k+1}e^{\alpha(k+1-j)+2L(\tau_{k+1}-\tau_j)}-\sum_{j=0}^{k}e^{\alpha(k-j)+2L(\tau_k-\tau_j)}\Big)$$

$$\leq N_{[a,b]}^{\frac{p-1}{p}}(x)\Big\{\|y_a\|\Big(e^{\alpha N+2L(b-a)}-1\Big)+\Big(\frac{\|f(0)\|}{L}+\frac{\|g(0)\|}{(K+1)C_g}\Big)\Big(\sum_{j=0}^{N_{[a,b]}(x)}e^{\alpha(N_{[a,b]}(x)-j)+2L(b-\tau_j)}-1\Big)\Big\}$$

$$\leq N_{[a,b]}^{\frac{p-1}{p}}(x)\Big[\|y_a\|+\Big(\frac{\|f(0)\|}{L}+\frac{\|g(0)\|}{(K+1)C_g}\Big)N_{[a,b]}(x)\Big]e^{\alpha N_{[a,b]}(x)+2L(b-a)}-\|y_a\|$$

which proves (2.8).

□

The following corollary give an another estimate for the solution of (1.1).

Corollary 2.2 *The following estimate holds*

$$\|y\|_{p-\mathrm{var},[a,b]} \leq \Big[\|y_a\|+\max\Big\{\frac{\|f(0)\|}{L},2\|g(0)\|\Big\}(1+\|\!|x|\!\|_{p-\mathrm{var},[a,b]})N_{[a,b]}(x)\Big]\times$$
$$\times e^{\alpha N_{[a,b]}(x)+2L(b-a)}N_{[a,b]}^{\frac{p-1}{p}}(x) \tag{2.10}$$

Proof: Prove similar to Theorem (2.1) we have

$$\|\!|y|\!\|_{p-\mathrm{var},[s,t]} \leq \Big(\frac{\|f(0)\|}{L}+2\,\|\!|x|\!\|_{p-\mathrm{var},[a,b]}\,\|g(0)\|+\|y_s\|(1+2C_g\,\|\!|x|\!\|_{p-\mathrm{var},[s,t]}\,\|)\Big)e^{2L(t-s)}-\|y_s\|$$
$$\leq \Big(\max\Big\{\frac{\|f(0)\|}{L},2\|g(0)\|\Big\}(1+\|\!|x|\!\|_{p-\mathrm{var},[a,b]})+(1+\frac{1}{K+1})\|y_s\|\Big)e^{2L(t-s)}-\|y_s\|$$

whenever $(K+1)C_g\,\|\!|x|\!\|_{p-\mathrm{var},[s,t]}\leq\frac{1}{2}$ and $s,t\in[a,b]$, which finally leads to (2.10). □

3 Random attractors

3.1 Generation of random dynamical systems

In this subsection we would like to present the generation of a random dynamical system from Young equation (1.1). Recall that $\mathcal{C}^{0,p-\mathrm{var}}([a,b],\mathbb{R}^m)$ is the closure of $\mathcal{C}^\infty([a,b],\mathbb{R}^m)$ in $\mathcal{C}^{p-\mathrm{var}}([a,b],\mathbb{R}^m)$ and $\mathcal{C}^{0,p-\mathrm{var}}(\mathbb{R},\mathbb{R}^m)$ is the space of all $x:\mathbb{R}\to\mathbb{R}^m$ such that $x|_I\in\mathcal{C}^{0,p-\mathrm{var}}(I,\mathbb{R}^m)$ for each compact interval $I\subset\mathbb{R}$. Then equip $\mathcal{C}^{0,p-\mathrm{var}}(\mathbb{R},\mathbb{R}^m)$ with the compact open topology given by the $p-$variation norm, i.e the topology generated by the metric:

$$d_p(x_1,x_2):=\sum_{k\geq1}\frac{1}{2^k}(\|x_1-x_2\|_{p-\mathrm{var},[-k,k]}\wedge 1).$$

Assign
$$\Omega:=\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R},\mathbb{R}^m):=\{x\in\mathcal{C}^{0,p-\mathrm{var}}(\mathbb{R},\mathbb{R}^m)|\ x_0=0\},$$
and equip with the Borel $\sigma-$ algebra $\mathcal{F}$. Note that for $x\in\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R},\mathbb{R}^m)$, $\|\!|x|\!\|_{p-\mathrm{var},I}$ and $\|x\|_{p-\mathrm{var},I}$ are equivalent norms for every compact interval I containing 0.

Let us consider a stochastic process $\bar{Z}$ defined on a probability space $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})$ with realizations in $(\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R},\mathbb{R}^m),\mathcal{F})$. Assume further that $\bar{Z}$ has stationary increments. Denote by θ the *Wiener shift*
$$(\theta_t x)_\cdot=x_{t+\cdot}-x_t,\forall t\in\mathbb{R},x\in\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R},\mathbb{R}^m).$$

It is easy to check that θ forms a continuous (and thus measurable) dynamical system $(\theta_t)_{t\in\mathbb{R}}$ on $(\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R},\mathbb{R}),\mathcal{F})$. Moreover, the Young integral satisfies the shift property with respect to θ, i.e.

$$\int_a^b y_u dx_u = \int_{a-r}^{b-r} y_{r+u} d(\theta_r x)_u \tag{3.1}$$

(see details in [5]). It follows, as the simplest version for rough cocycle in [3, Theorem 5] w.r.t. Young integrals that, there exists a probability $\mathbb{P}$ on $(\Omega,\mathcal{F}) = (\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R},\mathbb{R}^m),\mathcal{F})$ that is invariant under θ, and the so-called *diagonal process* $Z : \mathbb{R}\times\Omega \to \mathbb{R}^m, Z(t,x) = x_t$ for all $t\in\mathbb{R}, x\in\Omega$, such that Z has the same law with $\bar{Z}$ and satisfies the *helix property*:

$$Z_{t+s}(x) = Z_s(x) + Z_t(\theta_s x), \forall x\in\Omega, t,s\in\mathbb{R}.$$

Such stochastic process Z has also stationary increments and almost all of its realization belongs to $\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R},\mathbb{R}^m)$. It is important to note that the existence of $\bar{Z}$ is necessary to construct the diagonal process Z. We assume additionally that $(\Omega,\mathcal{F},\mathbb{P},\theta)$ is ergodic.
It is important to note that, when dealling with fractional Brownian motion [16], we can start with the space $\mathcal{C}_0(\mathbb{R},\mathbb{R}^m)$ of continuous functions on $\mathbb{R}$ vanishing at zero, with the Borel $\sigma-$algebra $\mathcal{F}$, and the Wiener shift and the Wiener probability $\mathbb{P}$, and then follow [14, Theorem 1] to construct an invariant probability measure $\mathbb{P}^H = B^H\mathbb{P}$ on the subspace $\mathcal{C}^\nu$ such that $B^H\circ\theta = \theta\circ B^H$. It can be proved that θ is ergodic.
Under this circumstance, if we assume further that

$$\Big(E\,\|Z\|^p_{p-\mathrm{var},[-1,1]}\Big)^{\frac{1}{p}} = \Gamma(p) < \infty$$

then it follows from Birkhorff ergodic theorem that

$$\Gamma(x,p) := \limsup_{n\to\infty}\Big(\frac{1}{n}\sum_{k=1}^n \|\theta_{-k}x\|^p_{p-\mathrm{var},[-1,1]}\Big)^{\frac{1}{p}} = \Gamma(p) \tag{3.2}$$

for almost all realizations $x_t = Z_t(\omega)$ of Z.

Proposition 3.1 *System*

$$dy_t = [Ay_t + f(y_t)]dt + g(y_t)dZ_t(\omega) \tag{3.3}$$

generates a random dynamical system.

Proof: The proof follows directly from [3] and [5, Section 4.2]. Specifically, the solution generates a so-called *random dynamical system* defined by $\varphi(t,\omega)y_0 := y(t,\omega,y_0)$ on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ equipped with a metric dynamical system θ, i.e. $\varphi:\mathbb{R}\times\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ is a measurable mapping which is also continuous in (t,x_0) such that the cocycle property

$$\varphi(t+s,\omega) = \varphi(t,\theta_s\omega)\circ\varphi(s,\omega), \quad \forall t,s\in\mathbb{R},$$

is satisfied. □

3.2 Existence of pullback attractors

Given a random dynamical system φ on $\mathbb{R}^d$, we follow [6], [2, Chapter 9] to present the notion of random pullback attractor. Recall that a set $\hat{M} = \{M(\omega)\}_{\omega\in\Omega}$ a *random set*, if $\omega\mapsto d(x|M(\omega))$ is $\mathcal{F}$-measurable for each $x\in\mathbb{R}^d$, where $d(E|F) = \sup\{\inf\{d(x,y)|y\in F\}|x\in E\}$ for E,F are nonempty subset of $\mathbb{R}^d$ and $d(x|E) = d(\{x\}|E)$. An *universe* $\mathcal{D}$ is a family of random sets which is closed w.r.t. inclusions (i.e. if $\hat{D}_1\in\mathcal{D}$ and $\hat{D}_2\subset\hat{D}_1$ then $\hat{D}_2\in\mathcal{D}$). In our setting, we define the universe $\mathcal{D}$ to be a family of random sets $D(\omega)$ which is *tempered* (see e.g. [2, pp. 164, 386]), namely $D(\omega)$ belongs to the ball $B(0,\rho(\omega))$ for all $\omega \in \Omega$ where the radius $\rho(\omega) > 0$ is a *tempered random varible*, i.e.

$$\lim_{t\to\pm\infty} \frac{1}{t}\log\rho(\theta_t\omega) = 0. \tag{3.4}$$

An invariant random compact set $\mathcal{A} \in \mathcal{D}$ is called a *pullback random attractor* in $\mathcal{D}$, if $\mathcal{A}$ attracts any closed random set $\hat{D} \in \mathcal{D}$ in the pullback sense, i.e.

$$\lim_{t\to\infty} d(\varphi(t,\theta_{-t}\omega)\hat{D}(\theta_{-t}\omega)|\mathcal{A}(\omega)) = 0. \tag{3.5}$$

The existence of a random pullback attractor follows from the existence of a random pullback absorbing set (see [6, Theorem 3]). A random set $\mathcal{B} \in \mathcal{D}$ is called *pullback absorbing* in a universe $\mathcal{D}$ if $\mathcal{B}$ absorbs all sets in $\mathcal{D}$, i.e. for any $\hat{D} \in \mathcal{D}$, there exists a time $t_0 = t_0(\omega,\hat{D})$ such that

$$\varphi(t,\theta_{-t}\omega)\hat{D}(\theta_{-t}\omega) \subset \mathcal{B}(\omega), \text{ for all } t \geq t_0. \tag{3.6}$$

Given a universe $\mathcal{D}$ and a random compact pullback absorbing set $\mathcal{B} \in \mathcal{D}$, there exists a unique random pullback attractor (which is then a weak attractor) in $\mathcal{D}$, given by

$$\mathcal{A}(\omega) = \cap_{s\geq 0}\overline{\cup_{t\geq s}\varphi(t,\theta_{-t}\omega)\mathcal{B}(\theta_{-t}\omega)}. \tag{3.7}$$

We need the following auxiliary results.

Proposition 3.2 *Assume that A has all eigenvalues of negative real parts. Then there exist constant $C_A \geq 1, \lambda_A > 0$ such that*

$$\|\Phi\|_{\infty,[a,b]} \leq C_A e^{-\lambda_A a}, \tag{3.8}$$

$$\|\Phi\|_{p-\mathrm{var},[a,b]} \leq \|A\|C_A e^{-\lambda_A a}(b-a), \quad \forall\, 0 \leq a < b, \tag{3.9}$$

where $\Phi(t) = e^{At}$.

Proof: Denote by $\lambda_1,...,\lambda_d$ all the eigen values of A then for every $x_0 \in \mathbb{R}^d$, $\Phi(t)x_0 = \sum_{k=1}^d e^{\lambda_j t}P_j(t)x_0$, where P_j are matrices with polynomial entities (see [7, p. 89]). Fix an $\varepsilon \in (0, -\max_i \mathrm{Re}\lambda_i)$, and define $C_A := \sum_{j=1}^d \sup_{t>0} e^{-\varepsilon t}\|P_j(t)\|$. By choosing $\lambda_A := -\max \mathrm{Re}\lambda_i - \varepsilon$, we obtain

$$\|\Phi(t)\| \leq e^{-\lambda_A t}\sum_{k=1}^m e^{-\varepsilon t}\|P_j(t)\| \leq C_A e^{-\lambda_A t} \leq C_A e^{-\lambda_A a}, \quad \forall [a,b] \subset \mathbb{R}^+,$$

which yields (3.8). On the other hand, since

$$\|\Phi(u)-\Phi(v)\| = \left\|\int_u^v A\Phi(s)ds\right\| \leq \int_u^v \|A\|C_A e^{-\lambda_A s}ds \leq \frac{\|A\|C_A}{\lambda_A}\Big(e^{-\lambda_A u} - e^{-\lambda_A v}\Big)$$

for any $u < v$ in $[a,b]$ and $e^{-\lambda_A \cdot}$ is a decreasing function, it follows that

$$\|\Phi\|_{p-\mathrm{var},[a,b]} \leq \frac{\|A\|C_A}{\lambda_A}\left\|e^{-\lambda_A\cdot}\right\|_{p-\mathrm{var},[a,b]} \leq \frac{\|A\|C_A}{\lambda_A}\Big(e^{-\lambda_A a} - e^{-\lambda_A b}\Big) \leq \|A\|C_A e^{-\lambda_A a}(b-a),$$

which proves (3.9).

□

Proposition 3.3 *Given (3.8) and (3.9), the following estimate holds: for any $0 \leq a < b \leq c$*

$$\Big\| \int_a^b \Phi(c-s) g(y_s) dx_s \Big\| \leq KC_A \Big[1 + \|A\|(b-a)\Big] \left\| x \right\|_{p-\text{var},[a,b]} e^{-\lambda_A (c-b)} \Big[C_g \|y\|_{p-\text{var},[a,b]} + \|g(0)\| \Big]. \tag{3.10}$$

Proof: Since g is Lipchitz continuous, it follows that

$$\|g(y_a) - g(y_b)\| \leq C_g \|y_a - y_b\| \leq C_g \left\| y \right\|_{p-\text{var},[a,b]},$$

which yields $\left\| g(y) \right\|_{p-\text{var},[a,b]} \leq C_g \left\| y \right\|_{p-\text{var},[a,b]}$. Then (3.8) and (3.9) derive

$$\begin{aligned}
& \Big\| \int_a^b \Phi(c-s) g(y_s) dx_s \Big\| \\
\leq \;& \left\| x \right\|_{p-\text{var},[\text{a,b}]} \Big(\|\Phi(c-a) g(y_a)\| + K \left\| \Phi(c-\cdot) g(y_\cdot) \right\|_{p-\text{var},[a,b]} \Big) \\
\leq \;& \left\| x \right\|_{p-\text{var},[a,b]} \Big\{ \|\Phi(c-a)\| \|g(y_a)\| \\
& + K \Big(\left\| \Phi(c-\cdot) \right\|_{p-\text{var},[a,b]} \|g(y)\|_{\infty,[a,b]} + \|\Phi(c-\cdot)\|_{\infty,[a,b]} \left\| g(y) \right\|_{p-\text{var},[a,b]} \Big) \Big\} \\
\leq \;& KC_A \left\| x \right\|_{p-\text{var},[a,b]} e^{-\lambda_A (c-b)} \times \\
& \times \Big[C_g \|y_a\| + \|g(0)\| + \|A\|(b-a)\big(C_g \|y\|_{\infty,[a,b]} + \|g(0)\|\big) + C_g \left\| y \right\|_{p-\text{var},[a,b]} \Big] \\
\leq \;& KC_A \Big[1 + \|A\|(b-a)\Big] \left\| x \right\|_{p-\text{var},[a,b]} e^{-\lambda_A (c-b)} \Big[C_g \|y\|_{p-\text{var},[a,b]} + \|g(0)\| \Big].
\end{aligned}$$

□

The following lemma is the crucial technique of this paper.

Lemma 3.4 *Assume that y_t satisfies*

$$y_t = \Phi(t) y_0 + \int_0^t \Phi(t-s) f(y_s) ds + \int_0^t \Phi(t-s) g(y_s) dx_s, \quad \forall t \geq 0. \tag{3.11}$$

Then for any $r > 0$ given and $n \geq 0$,

$$\begin{aligned}
\|y_t\| e^{\lambda t} \leq \;& C_A \|y_0\| + \frac{C_A}{\lambda_A - L_f} \|f(0)\| \Big(e^{\lambda t} - 1 \Big) \\
& + \sum_{k=0}^n e^{\lambda_A r} KC_A (1 + \|A\| r) \left\| x \right\|_{p-\text{var},\Delta_k^r} e^{\lambda k r} \Big[C_g \|y\|_{p-\text{var},\Delta_k^r} + \|g(0)\| \Big], \forall t \in \Delta_n^r,
\end{aligned} \tag{3.12}$$

where $\Delta_k^r := [kr, (k+1)r]$, $L_f := C_A C_f, \lambda := \lambda_A - L_f$.

Proof: First, for any $t \in [nr, (n+1)r)$, it follows from (3.8) and the global Lipschitz continuity of f that

$$\begin{aligned}
\|y_t\| \leq \;& \|\Phi(t) y_0\| + \int_0^t \|\Phi(t-s) f(y_s)\| ds + \Big\| \int_0^t \Phi(t-s) g(y_s) dx_s \Big\| \\
\leq \;& C_A e^{-\lambda_A t} \|y_0\| + \int_0^t C_A e^{-\lambda_A (t-s)} \Big(C_f \|y_s\| + \|f(0)\| \Big) ds + \Big\| \int_0^t \Phi(t-s) g(y_s) dx_s \Big\| \\
\leq \;& C_A e^{-\lambda_A t} \|y_0\| + \frac{C_A}{\lambda_A} \|f(0)\| (1 - e^{-\lambda_A t}) + \beta_t + C_A C_f \int_0^t e^{-\lambda_A (t-s)} \|y_s\| ds,
\end{aligned}$$

where $\beta_t := \Big\| \int_0^t \Phi(t-s)g(y_s)dx_s \Big\|$. Multiplying both sides with $e^{\lambda_A t}$ yields

$$\|y_t\| e^{\lambda_A t} \leq C_A \|y_0\| + \frac{C_A}{\lambda_A} \|f(0)\| (e^{\lambda_A t} - 1) + \beta_t e^{\lambda_A t} + C_A C_f \int_0^t e^{\lambda_A s} \|y_s\| ds.$$

By applying the continuous Gronwall lemma 4.1, we obtain

$$\begin{aligned}
\|y_t\| e^{\lambda_A t} &\leq C_A \|y_0\| + \frac{C_A}{\lambda_A} \|f(0)\| (e^{\lambda_A t} - 1) + \beta_t e^{\lambda_A t} \\
&\quad + \int_0^t L_f e^{L_f (t-s)} \Big[C_A \|y_0\| + \frac{C_A}{\lambda_A} \|f(0)\| (e^{\lambda_A s} - 1) + \beta_s e^{\lambda_A s} \Big] ds.
\end{aligned}$$

Multiplying both sides with $e^{-L_f t}$ yields

$$\begin{aligned}
\|y_t\| e^{(\lambda_A - L_f)t} &\leq C_A \|y_0\| e^{-L_f t} + \frac{C_A}{\lambda_A} \|f(0)\| \Big(e^{(\lambda_A - L_f)t} - e^{-L_f t} \Big) + \beta_t e^{(\lambda_A - L_f)t} \\
&\quad + \int_0^t L_f e^{-L_f s} \Big[C_A \|y_0\| + \frac{C_A}{\lambda_A} \|f(0)\| (e^{\lambda_A s} - 1) + \beta_s e^{\lambda_A s} \Big] ds \\
&\leq C_A \|y_0\| + \frac{C_A}{\lambda_A - L_f} \|f(0)\| \Big(e^{(\lambda_A - L_f)t} - 1 \Big) + \beta_t e^{(\lambda_A - L_f)t} + \int_0^t L_f \beta_s e^{(\lambda_A - L_f)s} ds.
\end{aligned} \tag{3.13}$$

Next, observe from (3.10) that for all $s \leq t$

$$\begin{aligned}
&\beta_s e^{(\lambda_A - L_f)s} \\
= \;& e^{(\lambda_A - L_f)s} \Big\| \int_0^s \Phi(s-u) g(y_u) dx_u \Big\| \\
\leq \;& e^{(\lambda_A - L_f)s} \sum_{k=0}^{\lfloor \frac{s}{r} \rfloor - 1} \Big\| \int_{\Delta_k^r} \Phi(s-u) g(y_u) dx_u \Big\| + \Big\| \int_{r\lfloor s/r \rfloor}^s \Phi(s-u) g(y_u) dx_u \Big\| \\
\leq \;& e^{(\lambda_A - L_f)s} \sum_{k=0}^{\lfloor \frac{s}{r} \rfloor - 1} K C_A (1 + \|A\| r) \, |\!|\!| x |\!|\!|_{p-\text{var}, \Delta_k^r} \, e^{-\lambda_A (s - kr - r)} \Big[C_g \|y\|_{p-\text{var}, \Delta_k^r} + \|g(0)\| \Big] \\
& + e^{(\lambda_A - L_f)s} K C_A \Big[1 + \|A\| (s - r \lfloor \frac{s}{r} \rfloor) \Big] \, |\!|\!| x |\!|\!|_{p-\text{var}, [r\lfloor \frac{s}{r} \rfloor, s]} \, \Big[C_g \|y\|_{p-\text{var}, [r\lfloor \frac{s}{r} \rfloor, s]} + \|g(0)\| \Big] \\
\leq \;& \sum_{k=0}^{\lfloor \frac{s}{r} \rfloor} K C_A (1 + \|A\| r) \, |\!|\!| x |\!|\!|_{p-\text{var}, \Delta_k^r} \, e^{(\lambda_A - L_f)s} e^{-\lambda_A (s - kr - r)} \Big[C_g \|y\|_{p-\text{var}, \Delta_k^r} + \|g(0)\| \Big] \\
\leq \;& \sum_{k=0}^{\lfloor \frac{s}{r} \rfloor} e^{\lambda_A r} K C_A (1 + \|A\| r) \, |\!|\!| x |\!|\!|_{p-\text{var}, \Delta_k^r} \, e^{(\lambda_A - L_f)kr} e^{-L_f (s - kr)} \Big[C_g \|y\|_{p-\text{var}, \Delta_k^r} + \|g(0)\| \Big]
\end{aligned} \tag{3.14}$$

Replacing (3.14) into (3.13) yields

$$\begin{aligned}
&\|y_t\| e^{(\lambda_A - L_f)t} \\
\leq \;& C_A \|y_0\| + \frac{C_A}{\lambda_A - L_f} \|f(0)\| \Big(e^{(\lambda_A - L_f)t} - 1 \Big) \\
& + e^{\lambda_A r} K C_A (1 + \|A\|) \sum_{k=0}^{n} |\!|\!| x |\!|\!|_{p-\text{var}, \Delta_k^r} \, e^{(\lambda_A - L_f)kr} e^{-L_f (t - kr)} \Big[C_g \|y\|_{p-\text{var}, \Delta_k^r} + \|g(0)\| \Big] \\
& + L_f K C_A (1 + \|A\| r) \int_0^t \sum_{k=0}^{\lfloor \frac{s}{r} \rfloor} e^{\lambda_A r} \, |\!|\!| x |\!|\!|_{p-\text{var}, \Delta_k^r} \, e^{(\lambda_A - L_f)kr} e^{-L_f (s - kr)} \Big[C_g \|y\|_{p-\text{var}, \Delta_k^r} + \|g(0)\| \Big] ds
\end{aligned}$$

$$
\begin{aligned}
&\le \; C_A\|y_0\| + \frac{C_A}{\lambda_A - L_f}\|f(0)\|\Big(e^{(\lambda_A - L_f)t} - 1\Big) \\
&\quad + \sum_{k=0}^{n} e^{\lambda_A} K C_A (1+\|A\|)\, \|\!|x|\!\|_{p-\mathrm{var},\Delta_k^r}\, e^{(\lambda_A - L_f)kr}\Big[C_g\|y\|_{p-\mathrm{var},[kr,(k+1)r]} + \|g(0)\|\Big] \times \\
&\quad \times \Big(e^{-L_f(t-kr)} + \int_{kr}^{t} L_f e^{-L_f(s-kr)} ds\Big) \\
&\le \; C_A\|y_0\| + \frac{C_A}{\lambda_A - L_f}\|f(0)\|\Big(e^{(\lambda_A - L_f)t} - 1\Big) \\
&\quad + \sum_{k=0}^{n} e^{\lambda_A r} K C_A (1+\|A\|r)\, \|\!|x|\!\|_{p-\mathrm{var},\Delta_k^r}\, e^{(\lambda_A - L_f)kr}\Big[C_g\|y\|_{p-\mathrm{var},\Delta_k^r} + \|g(0)\|\Big],
\end{aligned}
$$

where we use the fact that $e^{-L_f(t-kr)} + \int_{kr}^t L_f e^{-L_f(s-kr)} ds = 1$ for all $t \ge kr$. Hence, for $t \in [nr,(n+1)r)$,

$$
\begin{aligned}
\|y_t\|e^{(\lambda_A - L_f)t} \;\le\; & C_A\|y_0\| + \frac{C_A}{\lambda_A - L_f}\|f(0)\|\Big(e^{(\lambda_A - L_f)t} - 1\Big) \\
& + \sum_{k=0}^{n-1} e^{\lambda_A r} K C_A (1+\|A\|r)\, \|\!|x|\!\|_{p-\mathrm{var},\Delta_k^r}\, e^{(\lambda_A - L_f)kr}\Big[C_g\|y\|_{p-\mathrm{var},\Delta_k^r} + \|g(0)\|\Big].
\end{aligned}
$$

The continuity of y at $t = (n+1)r$ then proves (3.12).

□

Using (2.6), we use from now on the following estimate

$$
\begin{aligned}
1 < F(x,[a,b]) \;:=\; & \exp\Big\{\log\frac{K+2}{K+1} N_{[a,b]}(x) + 2L(b-a)\Big\} \\
\le\; & \frac{K+2}{K+1}\exp\Big\{\frac{1}{K+1}[2(K+1)C_g]^p\, \|\!|x|\!\|^p_{p-\mathrm{var},[a,b]} + 2L(b-a)\Big\}. \qquad (3.15)
\end{aligned}
$$

Proposition 3.5 *Define*

$$
G(x,[a,b]) \;:=\; \|\!|x|\!\|_{p-\mathrm{var},[a,b]}\, F(x,[a,b]) N_{[a,b]}^{\frac{p-1}{p}}(x), \qquad (3.16)
$$

$$
H(x,[a,b]) \;:=\; 1 + \|\!|x|\!\|_{p-\mathrm{var},[a,b]}\Big[1 + F(x,[a,b]) N_{[a,b]}^{\frac{2p-1}{p}}(x)\Big], \qquad (3.17)
$$

and

$$
b(x) := \sum_{k=1}^{\infty} e^{-\lambda k} H(\theta_{-k}x,[-1,1]) \prod_{j=1}^{k-1}\Big[1 + M_1 C_g G(\theta_{-j}x,[-1,1])\Big] \qquad (3.18)
$$

(which can be infinity), where $\lambda > 0$, $M_1 := C_A e^{\lambda_A}(1+\|A\|)K$ *and* F *is given by* (3.15)*. Assume further that*

$$
\lambda > \hat{G} := C_A e^{\lambda_A + 4L}(1+\|A\|)\Big\{\Big[2(K+1)C_g\Gamma(p)\Big]^p + \Big[2(K+1)C_g\Gamma(p)\Big]\Big\}. \qquad (3.19)
$$

Then $b(x)$ *is finite and tempered a.s., i.e.*

$$
\lim_{t\to\pm\infty}\frac{1}{t}\log b(\theta_t x) = 0. \qquad (3.20)
$$

Proof: Assign $\Delta_k = [k, k+1]$ and $N_k(x) := N_{\Delta_k}(x)$. Observe from (2.6) that

$$\begin{aligned}
N_k^{\frac{p-1}{p}}(x) &\leq \Big(1 + [2(K+1)C_g]^p \, \|x\|^p_{p-\text{var},\Delta_k}\Big)^{\frac{p-1}{p}} \leq 1 + [2(K+1)C_g]^{p-1} \, \|x\|^{p-1}_{p-\text{var},\Delta_k}, \\
N_k^{\frac{2p-1}{p}}(x) &\leq \Big(1 + [2(K+1)C_g]^p \, \|x\|^p_{p-\text{var},\Delta_k}\Big)^{\frac{2p-1}{p}} \leq 2^{\frac{p-1}{p}} \Big(1 + [2(K+1)C_g]^{2p-1} \, \|x\|^{2p-1}_{p-\text{var},\Delta_k}\Big),
\end{aligned}$$

As a result, a direct computation shows that

$$G(x,[a,b]) \leq \Big[\|x\|_{p-\text{var},[a,b]} + [2(K+1)C_g]^{p-1} \|x\|^p_{p-\text{var},[a,b]}\Big] F(x,[a,b]), \tag{3.21}$$

$$H(x,[a,b]) \leq 1 + \|x\|_{p-\text{var},[a,b]} + 2^{\frac{p-1}{p}} F(x,[a,b]) \Big(\|x\|_{p-\text{var},[a,b]} + [2(K+1)C_g]^{2p-1} \|x\|^{2p}_{p-\text{var},[a,b]}\Big) \tag{3.22}$$

Due to the inequality $\log(1 + ae^b) \leq a + b$ for $a, b \geq 0$, (3.21) yields

$$\begin{aligned}
&\log\Big(1 + M_1 C_g G(x,[-1,1])\Big) \\
\leq\ & M_1 C_g \frac{K+2}{K+1} e^{4L} \Big[\|x\|_{p-\text{var},[-1,1]} + [2(K+1)C_g]^{p-1} \|x\|^p_{p-\text{var},[-1,1]}\Big] + \\
&+ \frac{1}{K+1} [2(K+1)C_g]^p \|x\|^p_{p-\text{var},[-1,1]} \\
\leq\ & \Big[M_1 e^{4L} \frac{K+2}{K+1} + 2\Big] [2(K+1)]^{p-1} C_g^p \|x\|^p_{p-\text{var},[-1,1]} + M_1 e^{4L} \frac{K+2}{K+1} C_g \|x\|_{p-\text{var},[-1,1]}.
\end{aligned}$$

It follows that for a.s. all x,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \prod_{k=0}^{n-1} \Big[1 + M_1 K C_g G(\theta_{-k}x,[-1,1])\Big] = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log\Big[1 + M_1 C_g G(\theta_k x,[-1,1])\Big] \\
\leq\ & \Big[M_1 e^{4L} \frac{K+2}{2(K+1)^2} + \frac{1}{K+1}\Big] [2(K+1)C_g]^p \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \|\theta_{-k}x\|^p_{p-\text{var},[-1,1]} \\
&+ M_1 e^{4L} \frac{K+2}{2(K+1)^2} [2(K+1)C_g] \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \|\theta_{-k}x\|_{p-\text{var},[-1,1]} \\
\leq\ & \Big[M_1 e^{4L} \frac{K+2}{2(K+1)^2} + \frac{1}{K+1}\Big] \Big\{\Big[2(K+1)C_g\Gamma(p)\Big]^p + \Big[2(K+1)C_g\Gamma(p)\Big]\Big\} \\
\leq\ & C_A e^{\lambda_A} (1 + \|A\|) e^{4L} \Big\{\Big[2(K+1)C_g\Gamma(p)\Big]^p + \Big[2(K+1)C_g\Gamma(p)\Big]\Big\} = \hat{G}.
\end{aligned}$$

Meanwhile, (3.22) and (3.15) yield

$$\begin{aligned}
\log H(x,[-1,1]) \leq\ & \log(1 + \|x\|_{p-\text{var},[-1,1]}) + \log\Big[2^{\frac{p-1}{p}} F(x,[a,b])\Big] \\
&+ \log\Big(1 + \|x\|_{p-\text{var},[a,b]} + [2(K+1)C_g]^{2p-1} \|x\|^{2p}_{p-\text{var},[a,b]}\Big) \\
\leq\ & 2\log(1 + \|x\|_{p-\text{var},[-1,1]}) + [2(K+1)C_g]^{2p-1} + 2p\log(1 + \|x\|_{p-\text{var},[-1,1]}) \\
&+ \Big(\log 2^{\frac{p-1}{p}} + \log\frac{K+2}{K+1} + \frac{1}{K+1} [2(K+1)C_g]^p \|x\|^p_{p-\text{var},[-1,1]} + 4L \\
\leq\ & (2+2p) \|x\|_{p-\text{var},[-1,1]} + \frac{1}{K+1} [2(K+1)C_g]^p \|x\|^p_{p-\text{var},[-1,1]} \\
&+ \Big[4L + [2(K+1)C_g]^{2p-1} + \log 2^{\frac{p-1}{p}} + \log\frac{K+2}{K+1}\Big],
\end{aligned}$$

where we use the inequalities $\log(1+a+b) \leq \log(1+a)+\log(1+b), \forall a,b\geq 0$ and $\log(1+ab) \leq \log(1+a)+\log b, \forall a \geq 0, b\geq 1$. As a result,

$$\limsup_{n\to\infty} \frac{\log H(\theta_n x,[-1,1])}{n} = \limsup_{n\to\infty} \frac{\log H(\theta_{-n} x,[-1,1])}{n} = 0.$$

Hence, there exists for each $0 < 2\delta < \lambda - \hat{G}$ an $n_0 = n_0(\delta, x)$ such that for all $n \geq n_0$,

$$e^{(-\delta+\hat{G})n} \leq \prod_{k=0}^{n-1} \Big[1+G(\theta_{-k}x,[-1,1])\Big], \quad \prod_{k=0}^{n-1} \Big[1+G(\theta_{k}x,[-1,1])\Big] \leq e^{(\delta+\hat{G})n}$$

and

$$e^{-\delta n} \leq H(\theta_{-n}x,[-1,1]), \quad H(\theta_n x,[-1,1]) \leq e^{\delta n}.$$

Consequently,

$$\begin{aligned} b(x) &\leq \sum_{k=1}^{n_0-1} e^{-\lambda k} H(\theta_{-k}x,[-1,1]) \prod_{j=1}^{k-1}\Big(1+M_1 C_g G(\theta_{-j}x,[-1,1])\Big) + \sum_{k=n_0}^{\infty} e^{-(\lambda-2\delta-\hat{G})k} \\ &\leq \sum_{k=1}^{n_0-1} e^{-\lambda k} H(\theta_{-k}x,[-1,1]) \prod_{j=1}^{k-1}\Big(1+M_1 C_g G(\theta_{-j}x,[-1,1])\Big) + \frac{e^{-(\lambda-2\delta-\hat{G})n_0}}{1-e^{-(\lambda-2\delta-\hat{G})}} \end{aligned}$$

which is finite. The proof on the temperedness of b is quite lengthy and will be provided in the appendix.

□

We are now able to formulate the first main result of the paper.

Theorem 3.6 *Assume that A has all eigenvalues of negative real parts with λ_A satisfying* (3.8) *and* (3.9)*, and f is globally Lipschitz continuous such that $\lambda_A > C_f C_A$. Assume further that the driving path x satisfies* (3.2)*. Then under the condition*

$$\lambda_A - C_A C_f > C_A(1+\|A\|)e^{\lambda_A+4(\|A\|+C_f)}\Big\{\Big[2(K+1)C_g\Gamma(p)\Big]^p + \Big[2(K+1)C_g\Gamma(p)\Big]\Big\}, \tag{3.23}$$

where $\Gamma(p) = \Big(E\,\|Z\|^p_{p-\mathrm{var},[-1,1]}\Big)^{\frac{1}{p}}$, the random dynamical system φ possesses a pullback attractor $\mathcal{A}(x)$.

Proof: By the variation of parameter formula for Young differential equations, it is easy to prove (see e.g. [10]) that y_t satisfies

$$y_t = \Phi(t)y_0 + \int_0^t \Phi(t-s)f(y_s)ds + \int_0^t \Phi(t-s)g(y_s)dx_s. \tag{3.24}$$

Then by applying Proposition 3.4 and using the estimate in (2.7)

$$\|y\|_{p-\mathrm{var},\Delta_k} \leq \Big[\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big)N_k(x)\Big]N_k^{\frac{p-1}{p}}(x)F(x,\Delta_k),$$

where $N_k(x) := N_{\Delta_k}(x)$, we obtain

$$\|y_n\|e^{\lambda n} \leq C_A\|y_0\| + (e^{\lambda n}-1)\frac{C_A\|f(0)\|}{\lambda} + e^{\lambda_A}KC_A(1+\|A\|)\sum_{k=0}^{n-1} e^{\lambda k}\,\|x\|_{p-\mathrm{var},\Delta_k} \times$$

$$
\begin{aligned}
& \times \Big\{ C_g \Big[\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g} \Big) N_k(x) \Big] N_k^{\frac{p-1}{p}}(x) F(x,\Delta_k) + \|g(0)\| \Big\} \\
\leq \;& C_A\|y_0\| + \frac{C_A\|f(0)\|(e^\lambda - 1)}{\lambda} \sum_{k=0}^{n-1} e^{\lambda k} + e^{\lambda_A} K C_A (1+\|A\|) \sum_{k=0}^{n-1} e^{\lambda k} \left|\!\left|\!\left| x \right|\!\right|\!\right|_{p-\mathrm{var},\Delta_k} \times \\
& \times \Big\{ C_g \Big[\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g} \Big) N_k(x) \Big] N_k^{\frac{p-1}{p}}(x) F(x,\Delta_k) + \|g(0)\| \Big\} \\
\leq \;& C_A\|y_0\| + M_1 C_g \sum_{k=0}^{n-1} e^{\lambda k} \left|\!\left|\!\left| x \right|\!\right|\!\right|_{p-\mathrm{var},\Delta_k} N_k^{\frac{p-1}{p}}(x) F(x,\Delta_k) \|y_k\| \\
& + M_2 \sum_{k=0}^{n-1} e^{\lambda k} \Big[1 + \left|\!\left|\!\left| x \right|\!\right|\!\right|_{p-\mathrm{var},\Delta_k} \Big(1 + N_k^{\frac{2p-1}{p}}(x) F(x,\Delta_k) \Big) \Big]
\end{aligned}
\tag{3.25}
$$

where

$$
\begin{aligned}
\lambda &:= \lambda_A - L_f, \\
M_1 &:= C_A e^{\lambda_A}(1+\|A\|)K, \\
M_2 &:= \max\Big\{ C_A \frac{e^\lambda - 1}{\lambda}, M_1 C_g \Big(\frac{1}{L} + \frac{1}{(K+1)C_g} \Big), M_1 C_g \Big\} \max\{\|f(0)\|, \|g(0)\|\}.
\end{aligned}
\tag{3.26}
$$

By assigning $a := C_A\|y_0\|$, $u_k := \|y_k\| e^{\lambda k},\ k \geq 0$ and using (3.16), (3.17), we obtain

$$
u_n \leq a + M_1 C_g \sum_{k=0}^{n-1} G(x,\Delta_k) u_k + M_2 \sum_{k=0}^{n-1} e^{\lambda k} H(x,\Delta_k). \tag{3.27}
$$

We are now in the position to apply Lemma 4.2, so that

$$
\begin{aligned}
\|y_n(x,y_0)\| \;\leq\;& C_A\|y_0\| e^{-\lambda n} \prod_{k=0}^{n-1} \Big[1 + M_1 C_g G(\theta_k x,[0,1]) \Big] \\
& + M_2 \sum_{k=0}^{n-1} e^{-\lambda(n-k)} H(\theta_k x,[0,1]) \prod_{j=k+1}^{n-1} \Big[1 + M_1 C_g G(\theta_j x,[0,1]) \Big].
\end{aligned}
\tag{3.28}
$$

Now using (2.7) and (2.6), it follows that for any $t \in [n,n+1]$

$$
\begin{aligned}
\|y_t(x,y_0)\| \;\leq\;& \Big[\|y_n(x,y_0)\| + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g} \Big) N_n(x) \Big] F(x,\Delta_n) \\
\leq\;& C_A\|y_0\| e^{-\lambda n} F(x,\Delta_n) \prod_{k=0}^{n-1} \Big[1 + M_1 C_g G(\theta_k x,[0,1]) \Big] \\
& + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g} \Big) N_n(x) F(x,\Delta_n) \\
& + M_2 \sum_{k=0}^{n-1} e^{-\lambda(n-k)} F(x,\Delta_n) H(\theta_k x,[0,1]) \prod_{j=k+1}^{n-1} \Big[1 + M_1 C_g G(\theta_j x,[0,1]) \Big].
\end{aligned}
\tag{3.29}
$$

Consequently, by assigning x with $\theta_{-t}x$ in (3.29), we obtain

$$
\begin{aligned}
& \|y_t(\theta_{-t}x, y_0(\theta_{-t}x)) \\
\leq\;& C_A\|y_0(\theta_{-t}x)\| e^{-\lambda n} F(\theta_{-t}x,\Delta_n) \prod_{k=0}^{n-1} \Big[1 + M_1 C_g G(\theta_{k-t}x,[0,1]) \Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\frac{\|f(0)\|}{L}+\frac{\|g(0)\|}{(K+1)C_g}\Big)N_n(\theta_{-t}x)F(x,\Delta_n)\\
&+M_2\sum_{k=0}^{n-1}e^{-\lambda(n-k)}F(\theta_{-t}x,\Delta_n)H(\theta_{k-t}x,[0,1])\prod_{j=k+1}^{n-1}\Big[1+M_1C_gG(\theta_{j-t}x,[0,1])\Big]\\
\le\ & C_A\|y_0(\theta_{-t}x)\|e^{-\lambda n}F(x,[n-t,n-t+1])\prod_{k=0}^{n-1}\Big[1+M_1C_gG(\theta_{k-n}x,[-1,1])\Big]\\
&+\Big(\frac{\|f(0)\|}{L}+\frac{\|g(0)\|}{(K+1)C_g}\Big)N_{[n-t,n-t+1]}(x)F(x,[n-t,n-t+1])\\
&+M_2\sum_{k=0}^{n-1}e^{-\lambda(n-k)}F(x,[n-t,n-t+1])H(\theta_{k-n}x,[-1,1])\prod_{j=k+1}^{n-1}\Big[1+M_1C_gG(\theta_{j-n}x,[-1,1])\Big]\\
\le\ & C_AF(x,[-1,1])\|y_0(\theta_{-t}x)\|e^{-\lambda n}\prod_{k=0}^{n-1}\Big[1+M_1C_gG(\theta_{k-n}x,[-1,1])\Big]\\
&+\Big(\frac{\|f(0)\|}{L}+\frac{\|g(0)\|}{(K+1)C_g}\Big)N_{[-1,1]}(x)F(x,[-1,1])\\
&+M_2F(x,[-1,1])\sum_{k=0}^{n-1}e^{-\lambda(n-k)}H(\theta_{k-n}x,[-1,1])\prod_{j=k+1}^{n-1}\Big[1+M_1C_gG(\theta_{j-n}x,[-1,1])\Big]\\
\le\ & C_AF(x,[-1,1])\|y_0(\theta_{-t}x)\|e^{-\lambda n}\prod_{k=1}^{n}\Big[1+M_1C_gG(\theta_{-k}x,[-1,1])\Big]\\
&+\Big(\frac{\|f(0)\|}{L}+\frac{\|g(0)\|}{(K+1)C_g}\Big)N_{[-1,1]}(x)F(x,[-1,1])\\
&+M_2F(x,[-1,1])\sum_{k=1}^{n}e^{-\lambda k}H(\theta_{-k}x,[-1,1])\prod_{j=1}^{k-1}\Big[1+M_1C_gG(\theta_{-j}x,[-1,1])\Big] \qquad (3.30)
\end{aligned}
$$

We are now in the position to apply Proposition 3.5 into (3.30) so that for $t\in\Delta_n$ with $0<\delta<\frac{1}{2}(\lambda-\hat{G})$ and n large enough

$$
\begin{aligned}
\|y_t(\theta_{-t}x,y_0)\| \le\ & C_Ae^{\lambda_A}\|y_0(\theta_{-t}x)\|F(x,[-1,1])\exp\Big\{-\Big(\lambda-\hat{G}-\delta\Big)n\Big\}+\\
&+\Big(\frac{\|f(0)\|}{L}+\frac{\|g(0)\|}{(K+1)C_g}\Big)N_{[-1,1]}(x)F(x,[-1,1])\\
&+M_2F(x,[-1,1])\sum_{k=1}^{\infty}e^{-\lambda k}H(\theta_{-k}x,[-1,1])\prod_{j=1}^{k-1}\Big[1+M_1C_gG(\theta_{-j}x,[-1,1])\Big]\\
\le\ & C_Ae^{\lambda_A}F(x,[-1,1])\|y_0(\theta_{-t}x)\|\exp\Big\{-\Big(\lambda-\hat{G}-\delta\Big)n\Big\}+M_2b(x)\\
&+\Big(\frac{\|f(0)\|}{L}+\frac{\|g(0)\|}{(K+1)C_g}\Big)N_{[-1,1]}(x)F(x,[-1,1]), \qquad (3.31)
\end{aligned}
$$

where $b(x)$ is given by (3.18). This implies that, starting from any point $y_0(\theta_{-t}x)\in D(\theta_{-t}x)$ which is tempered due to (3.4), there exists n large enough such that for $t\in[n,n+1]$

$$
\|y_t(\theta_{-t}x,y_0)\|\le 1+M_2b(x)+\Big(\frac{\|f(0)\|}{L}+\frac{\|g(0)\|}{(K+1)C_g}\Big)N_{[-1,1]}(x)F(x,[-1,1])=:\hat{b}(x). \qquad (3.32)
$$

In addition, it follows, using (2.6) and the inequality $\log(1+ab)\le\log(1+a)+\log b$ for all $a\ge0, b\ge1$, that

$$\begin{aligned}
\log \hat{b}(x) &\leq \log(1+M_2) + \log[1+b(x)] \\
&\quad + \log\Big\{1 + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big)\Big[1 + [2(K+1)C_g]^p \left\| x \right\|^p_{p-\mathrm{var},[-1,1]}\Big] F(x,[-1,1])\Big\} \\
&\leq \log(1+M_2) + \log[1+b(x)] + \log\Big[1 + \frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big] \\
&\quad + [2(K+1)C_g]^p \left\| x \right\|^p_{p-\mathrm{var},[-1,1]} + \log F(x,[-1,1]) \\
&\leq \log(1+M_2) + \log[1+b(x)] + \log\Big[1 + \frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big] + [2(K+1)C_g]^p \left\| x \right\|^p_{p-\mathrm{var},[-1,1]} \\
&\quad \log\frac{K+2}{K+1} + \frac{1}{K+1}[2(K+1)C_g]^p \left\| x \right\|^p_{p-\mathrm{var},[-1,1]} + 4L.
\end{aligned}$$

Hence the temperedness of $\hat{b}(x)$ follows from the temperedness (3.20) of $b(x)$ and of $\left\| x \right\|^p_{p-\mathrm{var},[-1,1]}$. Therefore, there exists a compact absorbing set $\mathcal{B}(x) = \bar{B}(0,\hat{b}(x))$ and thus a pullback attractor $\mathcal{A}(x)$ for system (1.1) which is given by (3.7). □

Corollary 3.7 *Assume that $f(0) = g(0) = 0$ so that $y \equiv 0$ is a solution of* (1.1)*. Then* (3.23) *in Theorem 3.6 is the exponential stability criterion for the trivial attractor $\mathcal{A}(x) \equiv 0$.*

Proof: Using (3.31) and the fact that $M_2 = 0$ if $f(0) = g(0) = 0$, we obtain

$$\|y_t(\theta_{-t}x, y_0)\| \leq C_A e^{\lambda_A} F(x,[-1,1]) \|y_0(\theta_{-t}x)\| \exp\Big\{-\Big(\lambda - \hat{G} - \delta\Big) n\Big\} \tag{3.33}$$

for $t \in \Delta_n$. It follows that all other solutions converge exponentially in the pullback sense to the trivial solution, which plays a role as the global pullback attractor.

□

Remark 3.8 (i), In [13] and [10] the authors develop the semigroup method to estimate the Hölder norm of y on intervals τ_k, τ_{k+1} where τ_k is a sequence of stopping times

$$\tau_0 = 0, \tau_{k+1} - \tau_k + \left\| x \right\|_{\beta,[\tau_k,\tau_{k+1}]} = \mu$$

for some $\mu \in (0,1)$ and $\beta > \frac{1}{p}$, which leads to the estimate of the exponent

$$-(\lambda_A - Q e^{\lambda_A} \max\{C_f, C_g\} \frac{n}{\tau_n}) \tau_n$$

for some generic coefficient Q independent of A, f, g, x. It is then proved that there exists $\liminf\limits_{n\to\infty} \frac{\tau_n}{n} = \frac{1}{d}$, where $d = d(\mu)$ depends on the moment of the stochastic noise. As such the exponent is estimated as

$$-\Big(\lambda_A - Q e^{\lambda_A} \max\{C_f, C_g\} d\Big). \tag{3.34}$$

However, it is required from the stopping time analysis (see [10, Section 4]) that the stochastic noise has to be small in the sense that the moment of Hölder semi-norm $\left\| x \right\|_{\beta,[-1,1]}$ must be controlled as small as possible. In addition, in case the noise is diminished, i.e. $g \equiv 0$, (3.34) reduces to a very rough criterion for exponential stability of the ordinary differential equation

$$C_f \leq \frac{1}{Q} \lambda_A e^{-\lambda_A}.$$

By constrast, our method uses crucial Lemma 3.4 by applying first the continuous Gronwall lemma in (3.13) in order to clear the role of the drift coefficient f. Then by using (2.7) to give a direct estimate of y_k, we can apply the discrete Gronwall lemma without using technical stopping time analysis to control the role of driving path x. The left and the right hand sides of criterion (3.23)

$$\lambda_A - C_A C_f > C_A(1+\|A\|)e^{\lambda_A+4(\|A\|+C_f)}\Big\{\Big[2(K+1)C_g\Gamma(p)\Big]^p + \Big[2(K+1)C_g\Gamma(p)\Big]\Big\}$$

can be interpreted as, respectively, the decay rate of the drift term and the intensity of the volatility term, where the term $e^{\lambda_A+4(\|A\|+C_f)}$ is the unavoidable effect of the discretization scheme. Criterion (3.23) is therefore a better generalization of the classical criterion for stability of ordinary differential equations, and is satisfied if either C_g or $\Gamma(p)$ is sufficiently small. In particular, when $g \equiv 0$, (3.23) reduces to $\lambda_A > C_A C_f$, which matches to the classical result.

(ii), A similar proof of Theorem 3.6 using step size r with $\Delta_k = [kr,(k+1)r]$ then leads to a criterion for the existence of a global random pullback attractor

$$\lambda_A - C_A C_f > \frac{1}{r} C_A(1+\|A\|r)e^{\Big[\lambda_A+4(\|A\|+C_f)\Big]r}\Big\{\Big[2(K+1)C_g\Gamma(p,r)\Big]^p + \Big[2(K+1)C_g\Gamma(p,r)\Big]\Big\}, \quad (3.35)$$

where $\Gamma(p,r) = \lim\limits_{n\to\infty}\Big(\frac{1}{n}\sum_{k=1}^n \left\| \theta_{-kr}x\right\|^p_{p-\text{var},[-r,r]}\Big)^{\frac{1}{p}} = \Big(E \left\| Z\right\|^p_{p-\text{var},[-r,r]}\Big)^{\frac{1}{p}}$ for almost sure all realizations x. As a result, the final criterion can be optimized to

$$\lambda_A - C_A C_f > \inf_{r>0} \frac{1}{r} C_A(1+\|A\|r)e^{\Big[\lambda_A+4(\|A\|+C_f)\Big]r}\Big\{\Big[2(K+1)C_g\Gamma(p,r)\Big]^p + \Big[2(K+1)C_g\Gamma(p,r)\Big]\Big\}.$$

(iii), Theorem 3.6 still holds for the rough differential equations in the sense of Gubinelli [15], using Lemma 3.4. In that case, the estimates for Young integrals would be replaced by the estimates for rough integral, using $p-$ variation norms. Details of the proof would be provided in a coming project. The reader is also referred to a simpler version for rough differential equations in the recent works [8], [9] on exponetial stability of the trivial solution using Lyapunov function method.

As presented in the following, we could prove that the diameter of the random attractor can be controlled by parameter C_g. We first introduce a quantity.

Proposition 3.9 *Assume that x satisfies* (3.2). *Then under criterion* (3.23), *the following quantity is well defined and finite*

$$\begin{aligned}\xi(x) &:= \sum_{k=1}^\infty e^{-\lambda k} \left\| \theta_{-k}x\right\|_{p-\text{var},[0,1]}(1+\left\| \theta_{-k}x\right\|_{p-\text{var},[0,1]})\Big[\max\Big\{\frac{\|f(0)\|}{L},2\|g(0)\|\Big\}+\hat{b}(\theta_{-k}x)\Big]\times \\ &\quad \times \exp\Big\{2N_{[0,1]}(\theta_{-k}x)+2L\Big\}. \qquad (3.36)\end{aligned}$$

Proof: Observe that the existence of $\Gamma(x,p)$ implies the temperedness of $\exp\Big\{2N_{[0,1]}(x)+2L\Big\}$ and $\left\| x\right\|_{p-\text{var},[0,1]}$. On the other hand, we use the inequalities $\log(1+a+b) \leq \log(1+a)+\log(1+b)$, $\forall a,b\geq 0$ and $\log(1+ab)\leq \log(1+a)+\log b$, $\forall a\geq 0, b\geq 1$ to obtain

$$\begin{aligned}&\log[\left\| x\right\|_{p-\text{var},[0,1]}(1+\left\| x\right\|_{p-\text{var},[0,1]})] + \log\Big[\max\Big\{\frac{\|f(0)\|}{L},2\|g(0)\|\Big\}+\hat{b}(x)\Big] + \\ &+2N_{[0,1]}(x)+2L \\ \leq\ & 2\left\| x\right\|_{p-\text{var},[0,1]} + \log\hat{b}(x) + \max\Big\{\frac{\|f(0)\|}{L},2\|g(0)\|\Big\} + 2\left(N_{[0,1]}(x)+L\right).\end{aligned}$$

Then it follows from the temperedness of $\hat{b}(x)$ that the quantity

$$\left\| x\right\|_{p-\text{var},[0,1]}(1+\left\| x\right\|_{p-\text{var},[0,1]})\Big[\max\Big\{\frac{\|f(0)\|}{L},2\|g(0)\|\Big\}+\hat{b}(x)\Big]\exp\Big\{2N_{[0,1]}(x)+2L\Big\}$$

in (3.36) is tempered. The convergence of the series in (3.36) can then be proved similarly to the convergence of $b(x)$ in Proposition 3.5. □

Theorem 3.10 *Under the assumptions of Theorem 3.6, the diameter of $\mathcal{A}$ is estimated as*

$$\mathrm{diam}(\mathcal{A}(x)) \leq 2C_A e^{\lambda_A}(1+\|A\|)KC_g\xi(x) \tag{3.37}$$

where $\xi(x)$ is given in (3.36)*.*

Proof: The existence of the pullback attractor $\mathcal{A}$ is followed by Theorem 3.6. Take any two points $a_1, a_2 \in \mathcal{A}(x)$. For a given $n \in \mathbb{N}$, assign $x^* := \theta_{-n}x$ and consider the equation

$$dy_t = [Ay_t + f(y_t)]dt + g(y_t)dx^*_t. \tag{3.38}$$

Due to the invariance of $\mathcal{A}$ under the flow, there exist $b_1, b_2 \in \mathcal{A}(x^*)$ such that $a_i = y_n(x^*, b_i)$. Put $z_t = z_t(x^*) := y_t(x^*, b_1) - y_t(x^*, b_2)$ then $z_n(x^*) = a_1 - a_2$ and we have

$$dz_t = [Az_t + P(t, z_t)]dt + Q(t, z_t)dx^*_t \tag{3.39}$$

where we write in short $y^1_t = y_t(x^*, b_1)$ and

$$\begin{aligned} P(t, z_t) &= f(y(t, x^*, b_2)) - f(y(t, x^*, b_1)) = f(y^1_t + z_t) - f(y^1_t), \\ Q(t, z_t) &= g(y(t, x^*, b_2)) - g(y(t, x^*, b_1)) = g(y^1_t + z_t) - g(y^1_t). \end{aligned}$$

Observe that

$$\|P(t, z) - P(t, z')\| \leq C_f\|z - z'\|, \|Q(t, z) - Q(t, z')\| \leq C_g\|z - z'\|$$

and $P(t, 0) = Q(t, 0) \equiv 0$. Consequently,

$$\|P(t, z_t)\| \leq C_f\|z_t\|, \quad \|Q(t, z_t)\| \leq C_g\|z_t\|.$$

Since Dg is bounded by C_g, for $u, v \in \Delta_k$

$$\|Q(u) - Q(v)\| = \|g(y^1_u) - g(y^2_u) - g(y^1_v) + g(y^2_v)\| \leq C_g\|y^1_u - y^1_v\| + C_g\|y^2_u - y^2_v\|, \tag{3.40}$$

which yields

$$\|\!|Q|\!\|_{p-\mathrm{var},[u,v]} \leq C_g \left\|\!\left|y^1\right|\!\right\|_{p-\mathrm{var},[u,v]} + C_g \left\|\!\left|y^2\right|\!\right\|_{p-\mathrm{var},[u,v]}. \tag{3.41}$$

Now, repeating the estimate in the proof of Theorem 3.6 with $\beta^*_t = \| \int_0^t \Phi(t-s)Q(s, z_s)dx^*_s\|$ we obtain

$$e^{\lambda_A t}\|z_t\| \leq C_A\|z_0\| + e^{\lambda_A t}\beta^*_t + L_f \int_0^t \Big(C_A\|z_0\| + e^{\lambda_A s}\beta^*_s\Big)e^{L_f(t-s)}ds$$

and then

$$e^{\lambda t}\|z_t\| \leq C_A\|z_0\| + e^{\lambda t}\beta^*_t + L_f \int_0^t e^{\lambda s}\beta^*_s ds \tag{3.42}$$

Similarly to (3.14) we have

$$\begin{aligned} \beta^*_t e^{\lambda t} &= e^{\lambda t}\Big\| \int_0^t \Phi(t-s)Q(s, z_s)dx_s\Big\| \\ &\leq e^{\lambda t}\sum_{k=0}^{\lfloor t\rfloor} KC_AC_g(1+\|A\|)\,\|\!|x^*|\!\|_{p-\mathrm{var},\Delta_k} e^{-\lambda_A(t-k-1)}\Big(\|z_k\| + \left\|\!\left|y^1\right|\!\right\|_{p-\mathrm{var},\Delta_\mathrm{k}} + \left\|\!\left|y^2\right|\!\right\|_{p-\mathrm{var},\Delta_\mathrm{k}}\Big) \end{aligned}$$

$$\leq M_1 C_g \sum_{k=0}^{\lfloor t\rfloor} |\!|\!| x^*|\!|\!|_{p-\mathrm{var},\Delta_k} e^{\lambda k} e^{-L_f(t-k)}\Big(\|y^1_k\| + \|y^2_k\| + |\!|\!| y^1|\!|\!|_{p-\mathrm{var},\Delta_k} + |\!|\!| y^2|\!|\!|_{p-\mathrm{var},\Delta_k}\Big). \quad (3.43)$$

Therefore

$$\begin{aligned}
e^{\lambda t}\|z_t\| &\leq C_A\|z_0\| + M_1 C_g \sum_{k=0}^{\lfloor t\rfloor} |\!|\!| x^*|\!|\!|_{p-\mathrm{var},\Delta_k} e^{\lambda k} e^{-L_f(t-k)}\Big(\|y^1\|_{p-\mathrm{var},\Delta_k} + \|y^2\|_{p-\mathrm{var},\Delta_k}\Big) + \\
&\quad + L_f M_1 C_g \int_0^t \sum_{k=0}^{\lfloor s\rfloor} |\!|\!| x^*|\!|\!|_{p-\mathrm{var},\Delta_k} e^{\lambda k} e^{-L_f(s-k)}\Big(\|y^1\|_{p-\mathrm{var},\Delta_k} + \|y^2\|_{p-\mathrm{var},\Delta_k}\Big) ds \\
&\leq C_A\|z_0\| + M_1 C_g \sum_{k=0}^{\lfloor t\rfloor} |\!|\!| x^*|\!|\!|_{p-\mathrm{var},\Delta_k} e^{\lambda k} \Big(\|y^1\|_{p-\mathrm{var},\Delta_k} + \|y^2\|_{p-\mathrm{var},\Delta_k}\Big) \times \\
&\quad \times \Big(e^{-L_f(t-k)} + L_f \int_k^t e^{-L_f(s-k)} ds\Big) \\
&\leq C_A\|z_0\| + M_1 C_g \sum_{k=0}^{\lfloor t\rfloor} |\!|\!| x^*|\!|\!|_{p-\mathrm{var},\Delta_k} e^{\lambda k} \Big(\|y^1\|_{p-\mathrm{var},\Delta_k} + \|y^2\|_{p-\mathrm{var},\Delta_k}\Big) \quad (3.44)
\end{aligned}$$

Since $b_i \in \mathcal{A}(x^*)$ for $i = 1, 2$, it follows from the invariance of $\mathcal{A}$ that $y^i(k, x^*, b_i) \in \mathcal{A}(\theta_k x^*)$. Moreover, it follows from (3.7) and (3.32) that

$$\sup_{y \in \mathcal{A}(x)} \|y\| \leq \hat{b}(x). \quad (3.45)$$

Indeed, taking $y^* \in \mathcal{A}(x)$ be arbitrary, it follows from (3.7) that there exists a sequence $t_k \to \infty$ such that

$$y^* = \lim_k \varphi(t_k, \theta_{-t_k}x, y_0(\theta_{-t_k}x))$$

where $y_0(\theta_{-t_k}x) \in \mathcal{B}(\theta_{-t_k}x)$. Since $\hat{b}(x)$ is tempered, by choosing t_k large enough so that (3.32) holds, we conclude that (3.45) holds. As a consequence, (3.45) yields $\|y^1(k, x^*, b_1)\| \leq \hat{b}(\theta_k x^*)$. Similarly, $\|z_0\| \leq \|b_1\| + \|b_2\| < 2\hat{b}(x^*)$. On the other hand, due to (2.10)

$$\begin{aligned}
&\|y^i(x^*)\|_{p-\mathrm{var},\Delta_k} \\
&\leq \Big[\|y^i_k(x^*)\| + \max\Big\{\frac{\|f(0)\|}{L}, 2\|g(0)\|\Big\}\Big] e^{\alpha N_k(x^*) + 2L} N_k^{\frac{2p-1}{p}}(x^*)(1 + |\!|\!| x^*|\!|\!|_{p-\mathrm{var},\Delta_k}) \\
&\leq \Big[\|y^i_k(x^*)\| + \max\Big\{\frac{\|f(0)\|}{L}, 2\|g(0)\|\Big\}\Big] e^{(2+\alpha-\frac{1}{p})N_k(x^*) + 2L}(1 + |\!|\!| x^*|\!|\!|_{p-\mathrm{var},\Delta_k}) \\
&\leq \Big[\|y^i_k(x^*)\| + \max\Big\{\frac{\|f(0)\|}{L}, 2\|g(0)\|\Big\}\Big] e^{2N_k(x^*) + 2L}(1 + |\!|\!| x^*|\!|\!|_{p-\mathrm{var},\Delta_k}), i = 1, 2 \quad (3.46)
\end{aligned}$$

since $\alpha = \log \frac{K+2}{K+1} < \frac{1}{2} < \frac{1}{p}$. Hence (3.44) yields

$$\begin{aligned}
\|z_n\| &\leq 2C_A\hat{b}(x^*)e^{-\lambda n} + 2M_1C_g \sum_{k=0}^{n-1} |\!|\!| x^*|\!|\!|_{p-\mathrm{var},\Delta_k} (1 + |\!|\!| x^*|\!|\!|_{p-\mathrm{var},\Delta_k}) e^{-\lambda(n-k)} \times \\
&\quad \times \Big[\max\Big\{\frac{\|f(0)\|}{L}, 2\|g(0)\|\Big\} + \hat{b}(\theta_k x^*)\Big] \exp\Big\{2N_k(x^*) + 2L\Big\} \\
&\leq 2C_A\hat{b}(\theta_{-n}x)e^{-\lambda n} + 2M_1C_g \sum_{k=1}^{n} |\!|\!| \theta_{-k}x|\!|\!|_{p-\mathrm{var},[0,1]} (1 + |\!|\!| \theta_{-k}x|\!|\!|_{p-\mathrm{var},[0,1]}) e^{-\lambda k} \times \\
&\quad \times \Big[\max\Big\{\frac{\|f(0)\|}{L}, 2\|g(0)\|\Big\} + \hat{b}(\theta_{-k}x)\Big] \exp\Big\{2N_{[0,1]}(\theta_{-k}x) + 2L\Big\}. \quad (3.47)
\end{aligned}$$

Letting n tend to infinity, the first term in the last line of (3.47) tends to zero due to the temperedness of $\hat{b}(x)$. Hence it follows from (3.36) in Proposition 3.9 that

$$\|a_1 - a_2\| \leq 2M_1 C_g \xi(x)$$

which proves (3.37).

□

In the rest of the paper, we would like to discuss on sufficient conditions for the global attractor to consist of only one point, as seen, for instance, in Corrollary 3.7. The answer is affirmative for g of linear form, as proved in [11] for dissipative systems. Here we can also present a simpler version using semigroup method.

Theorem 3.11 *Assume that $g(y) = Cy$ is a linear map. Then under the condition*

$$\lambda_A - C_A C_f > C_A(1+\|A\|)e^{\lambda_A + 4(\|A\|+C_f)}\Big\{\Big[2(K+1)\|C\|\Gamma(p)\Big]^p + \Big[2(K+1)\|C\|\Gamma(p)\Big]\Big\}, \quad (3.48)$$

the attractor consists of only one point, i.e. $\mathcal{A}(x) = \{a(x)\}$.

Proof: Firstly, note that (3.48) assures the existence of the random pullback attractor of φ with $C_g = \|C\|$ now. Since $g(y) = Cy$ is linear,

$$\left\|\!\left| Q \right|\!\right\|_{p-\text{var},[u,v]} \leq C_g \left\|\!\left| z \right|\!\right\|_{p-\text{var},[u,v]}.$$

As a result, the estimates in (3.44) can be rewritten as

$$e^{\lambda n}\|z_n\| \quad \leq \quad C_A\|z_0\| + M_1 C_g \sum_{k=0}^{n-1} \left\|\!\left| x^* \right|\!\right\|_{p-\text{var},\Delta_k} e^{\lambda k}\|z\|_{p-\text{var},\Delta_k} \quad (3.49)$$

Meanwhile, using similar arguments to the proof of Theorem 2.1, we obtain

$$\|z_t - z_s\| \leq \int_s^t L\|z_u\|du + C_g \left\|\!\left| x^* \right|\!\right\|_{p-\text{var},[s,t]} \Big(\|z_s\| + K \left\|\!\left| z \right|\!\right\|_{p-\text{var},[s,t]}\Big).$$

which yields

$$\left\|\!\left| z \right|\!\right\|_{q-\text{var},[s,t]} \leq \int_s^t L\|z_u\|du + C_g \left\|\!\left| x^* \right|\!\right\|_{p-\text{var},[s,t]} \Big(\|z_s\| + (K+1) \left\|\!\left| z \right|\!\right\|_{q-\text{var},[s,t]}\Big)$$

and then

$$\|z\|_{q-\text{var},[s,t]} = \|z_s\| + \left\|\!\left| z \right|\!\right\|_{q-\text{var},[s,t]} \leq \int_s^t 2L\|z\|_{q-\text{var},[s,u]}du + \Big(1 + \frac{1}{K+1}\Big)\|z_s\|$$

whenever $(K+1)C_g \left\|\!\left| x^* \right|\!\right\|_{p-\text{var},[s,t]} \leq \frac{1}{2}$. Therefore similar arguments as in the proof of Theorem 2.1 show that

$$\begin{aligned} \|z\|_{q-\text{var},[a,b]} \quad &\leq \quad N_{[a,b]}^{\frac{p-1}{p}}(x^*)e^{\alpha N_{[a,b]}(x^*)}e^{2L(b-a)}\|z_a\|, \quad \text{with} \\ N_{[a,b]}(x^*) \quad &\leq \quad 1 + [2(K+1)C_g]^p \left\|\!\left| x^* \right|\!\right\|^p_{p-\text{var},[a,b]}. \end{aligned}$$

As a result, (3.51) has the form

$$\begin{aligned} e^{\lambda n}\|z_n\| \quad &\leq \quad C_A\|z_0\| + M_1 C_g \sum_{k=0}^{n-1} I_k(x^*)e^{\lambda k} N_{[a,b]}^{\frac{p-1}{p}}(x^*)e^{\alpha N_{[a,b]}(x^*)}e^{2L(b-a)}\|z_k\| \\ &\leq \quad C_A\|z_0\| + \sum_{k=0}^{n-1} I_k(x^*)e^{\lambda k}\|z_k\|, \end{aligned}$$

where

$$I_k = M_1 C_g \left\| x^* \right\|_{p-\mathrm{var},[0,1]} N_k^{\frac{p-1}{p}}(x^*) e^{\alpha N_k(x^*)+2L}$$

Now applying the discrete Gronwall lemma, we obtain

$$e^{\lambda n}\|z_n\| \leq C_A\|z_0\| \prod_{k=0}^{n-1}(1+I_k)$$

From the estimate of $N_k(x^*)$ we have

$$\begin{aligned} I_k &\leq \frac{M_1(K+2)}{2(K+1)^2}\Big[\Big(2(K+1)C_g \left\|\theta_{k-n}x\right\|_{p-\mathrm{var},[0,1]}\Big) + \Big(2(K+1)C_g \left\|\theta_{k-n}x\right\|_{p-\mathrm{var},[0,1]}\Big)^p\Big] \\ &\quad \times \exp\Big\{2L + \alpha\Big(2(K+1)C_g \left\|\theta_{k-n}x\right\|_{p-\mathrm{var},[0,1]}\Big)^p\Big\} \end{aligned}$$

and then

$$\begin{aligned} \log(1+I_k) &\leq \frac{M_1(K+2)}{2(K+1)^2}e^{2L}\Big(2(K+1)C_g \left\|\theta_{k-n}x\right\|_{p-\mathrm{var},[0,1]}\Big) \\ &\quad +\Big(\frac{M_1(K+2)}{2(K+1)^2}e^{2L} + \frac{1}{K+1}\Big)\Big(2(K+1)C_g \left\|\theta_{k-n}x\right\|_{p-\mathrm{var},[0,1]}\Big)^p \\ &\leq C_A e^{\lambda_A+2L}(1+\|A\|)\Big[\Big(2(K+1)C_g \left\|\theta_{k-n}x\right\|_{p-\mathrm{var},[0,1]} + \Big(2(K+1)C_g \left\|\theta_{k-n}x\right\|_{p-\mathrm{var},[0,1]}\Big)^p\Big]. \end{aligned}$$

Hence,

$$\begin{aligned} \limsup_{n\to\infty}\frac{1}{n}\log\|z_n\| &\leq -\lambda + \limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\log(1+I_k) \\ &\leq -\lambda + C_A e^{\lambda_A+2L}(1+\|A\|)\Big[2(K+1)C_g\Gamma(p) + \Big(2(K+1)C_g\Big)^p\Gamma(p)^p\Big] < 0 \end{aligned}$$

under the condition (3.23). This follows that $\lim_{n\to\infty}\|a_1 - a_2\| = 0$ or $\mathcal{A}$ is an one point set. □

Remark 3.12 We would like to discuss the technical difficulty in proving the one-point attractor result for general nonlinear g, under further assumption that Dg is also globally Lipschitz with the same coefficient C_g. Specifically, it is easy then to prove that

$$\|Q(u) - Q(v)\| \leq C_g\|z_u - z_v\| + C_g\|z\|_{\infty,[u,v]}\|y_u^1 - y_v^1\|, \ \forall u < v \in \Delta_k$$

which yields

$$\left\|Q\right\|_{p-\mathrm{var},[u,v]} \leq C_g \left\|z\right\|_{p-\mathrm{var},[u,v]} + C_g\|z\|_{\infty,[u,v]} \left\|y^1\right\|_{p-\mathrm{var},[u,v]}. \tag{3.50}$$

As a result, the estimates in (3.44) can be rewritten as

$$e^{\lambda n}\|z_n\| \leq C_A\|z_0\| + M_1C_g\sum_{k=0}^{n-1}\left\|x^*\right\|_{p-\mathrm{var},\Delta_k} e^{\lambda k}\Big(1+\left\|y^1\right\|_{p-\mathrm{var},\Delta_k}\Big)\|z\|_{p-\mathrm{var},\Delta_k} \tag{3.51}$$

Meanwhile, using similar arguments to the proof of Theorem 2.1, we are able to show that

$$\|z\|_{q-\mathrm{var},[a,b]} \leq (N')^{\frac{p-1}{p}}2^{N'}\|z_a\|e^{2L(b-a)}, \tag{3.52}$$

where $N' = N'_{[a,b]}(x^*)$ is the maximal index of the maximal greedy time of the sequence

$$\tau_0 = a, \quad \tau_{i+1} := \inf\{t > \tau_i : 2KC_g \left\|x^*\right\|_{p-\mathrm{var},[\tau_i,t]}(1+\left\|y^1\right\|_{q-\mathrm{var},[\tau_i,t]}) = \frac{1}{2}\} \wedge b$$

that lies in interval $[a,b]$. Note that $N'_{[a,b]}(x^*)$ can be estimated as

$$N'_{[a,b]}(x^*) \leq 1 + (4KC_g)^p \left\|x^*\right\|^p_{p-\mathrm{var},[a,b]} (1 + \left\|y^1\right\|_{q-\mathrm{var},[a,b]})^p.$$

Combining this with (3.51) for $t = n$ we get

$$\begin{aligned} e^{\lambda n}\|z_n\| &\leq C_A\|z_0\| + \sum_{k=0}^{n-1} M_1 C_g \left\|x^*\right\|_{p-\mathrm{var},\Delta_k} \Big(1 + \left\|y^1\right\|_{p-\mathrm{var},\Delta_k}\Big) \times \\ &\quad \times \Big(1 + N'_{\Delta_k}(x^*) e^{2L + N'_{\Delta_k}(x^*)\log 2}\Big) e^{\lambda k}\|z_k\| \\ &\leq C_A\|z_0\| + \sum_{k=0}^{n-1} I_k e^{\lambda k}\|z_k\| \end{aligned} \tag{3.53}$$

in which

$$\begin{aligned} I_k &= M_1 C_g \left\|x^*\right\|_{p-\mathrm{var},\Delta_k} (1 + \left\|y^1\right\|_{p-\mathrm{var},\Delta_k}) \Big[1 + \Big(N'_{\Delta_k}(x^*)\Big)^{\frac{p-1}{p}} e^{2L + N'_{\Delta_k}(x^*)\log 2}\Big] \\ &\leq M_1 e^{2L} \exp\Big\{(4KC_g)^p \left\|x^*\right\|^p_{p-\mathrm{var},\Delta_k} \Big(1 + \left\|y^1\right\|_{p-\mathrm{var},\Delta_k}\Big)^p\Big\} \times \\ &\quad \times \Big[4KC_g \left\|x^*\right\|_{p-\mathrm{var},\Delta_k} \Big(1 + \left\|y^1\right\|_{p-\mathrm{var},\Delta_k}\Big) + \Big(4KC_g \left\|x^*\right\|_{p-\mathrm{var},\Delta_k} \Big(1 + \left\|y^1\right\|_{p-\mathrm{var},\Delta_k}\Big)\Big)^p\Big] \end{aligned} \tag{3.54}$$

and $\left\|y^1\right\|_{p-\mathrm{var},\Delta_k}$ is bounded by

$$\begin{aligned} \|y^1\|_{p-\mathrm{var},\Delta_k} &\leq \Big[\|y_k(x^*)\| + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big) N_{\Delta_k}(x^*)\Big] F(x^*,\Delta_k) \\ &\leq \Big[\hat{b}(\theta_{k-n}x)\| + \Big(\frac{\|f(0)\|}{L} + \frac{\|g(0)\|}{(K+1)C_g}\Big) N_{[0,1]}(\theta_{k-n}x)\Big] F(\theta_{k-n}x,[0,1]) =: \hat{F}(\theta_{k-n}x). \end{aligned}$$

Applying Lemma 4.2, we finally can show that

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n}\log\|z_n\| \\ \leq\ & -\lambda + \frac{1}{n}\sum_{k=1}^{n} \log\Big\{1 + M_1 C_g \left\|\theta_{-k}x\right\|_{p-\mathrm{var},[0,1]} (1 + \hat{F}(\theta_{-k}x)) \Big[1 + N'_{[0,1]}(\theta_{-k}x) e^{2L + N'_{[0,1]}(\theta_{-k}x)\log 2}\Big]\Big\} \\ \leq\ & -\lambda + E\log\Big\{1 + M_1 C_g \left\|x\right\|_{p-\mathrm{var},[0,1]} (1 + \hat{F}(x)) \Big[1 + N'_{[0,1]}(x) e^{2L + N'_{[0,1]}(x)\log 2}\Big]\Big\} \end{aligned} \tag{3.55}$$

However, to derive from (3.55) an estimate for exponential decaying rate, it is required that $\hat{b}(x)$ and $b(x)$ have to be integrable, which is rarely satisfied even with fractional Brownian noises. We therefore leave this interesting open problem for future work.

4 Appendix

Lemma 4.1 (Continuous Gronwall Lemma) *Assume that $u_t, \alpha_t, \beta > 0$ such that*

$$u_t \leq \alpha_t + \int_a^t \beta u_s ds, \forall t \geq a.$$

Then

$$u_t \leq \alpha_t + \int_a^t \beta e^{\beta(t-s)} \alpha_s ds, \forall t \geq a.$$

Proof: See [1, Lemma 6.1, p 89]. □

Lemma 4.2 (Discrete Gronwall Lemma) *Let a be a non negative constant and u_n, α_n, β_n be nonnegative sequences satisfying*

$$u_n \leq a + \sum_{k=0}^{n-1} \alpha_k u_k + \sum_{k=0}^{n-1} \beta_k, \quad \forall n \geq 1$$

then

$$u_n \leq \max\{a, u_0\} \prod_{k=0}^{n-1} (1+\alpha_k) + \sum_{k=0}^{n-1} \beta_k \prod_{j=k+1}^{n-1} (1+\alpha_j) \tag{4.1}$$

for all $n \geq 1$.

Proof: Put

$$\begin{aligned} S_n &:= a + \sum_{k=0}^{n-1} \alpha_k u_k + \sum_{k=0}^{n-1} \beta_k \\ T_n &:= \max\{a, u_0\} \prod_{k=0}^{n-1} (1+\alpha_k) + \sum_{k=0}^{n-1} \beta_k \prod_{j=k+1}^{n-1} (1+\alpha_j). \end{aligned}$$

We will prove by induction that $S_n \leq T_n$ for all $n \geq 1$. Namely, the statement holds for $n = 1$ since $S_1 = a + \alpha_0 u_0 + \beta_0 \leq \max\{a, u_0\}(1+\alpha_0) + \beta_0 = T_1$.

We assume that $S_n \leq T_n$ for $n \geq 1$, then due to the fact that $u_n \leq S_n$ we obtain

$$\begin{aligned} S_{n+1} &= a + \sum_{k=0}^{n-1} \alpha_k u_k + \sum_{k=0}^{n-1} \beta_k + \alpha_n u_n + \beta_n \\ &= S_n + \alpha_n u_n + \beta_n \\ &\leq S_n + \alpha_n S_n + \beta_n \\ &\leq T_n(1+\alpha_n) + \beta_n \\ &\leq \left[\max\{a, u_0\} \prod_{k=0}^{n-1} (1+\alpha_k) + \sum_{k=0}^{n-1} \beta_k \prod_{j=k+1}^{n-1} (1+\alpha_j)\right] (1+\alpha_n) + \beta_n \\ &\leq \max\{a, u_0\} \prod_{k=0}^{n} (1+\alpha_k) + \sum_{k=0}^{n} \beta_k \prod_{j=k+1}^{n-1} (1+\alpha_j) = T_{n+1}. \end{aligned}$$

Since $u_n \leq S_n$, (4.1) holds.

□

Proof: [**The temperedness of** b] For $n > n_0$ we have

$$\begin{aligned} b(\theta_{-n} x) &= \sum_{k=1}^{\infty} e^{-\lambda k} H(\|\theta_{-(k+n)} x\|_{p-\mathrm{var},[-1,1]}) \prod_{j=1}^{k-1} \Big[1 + M_1 C_g G(\|\theta_{-(j+n)} x\|_{p-\mathrm{var},[-1,1]})\Big] \\ &= \sum_{k=1}^{\infty} e^{-\lambda k} H(\|\theta_{-(k+n)} x\|_{p-\mathrm{var},[-1,1]}) \prod_{j=n+1}^{n+k-1} \Big[1 + M_1 C_g G(\|\theta_{-j} x\|_{p-\mathrm{var},[-1,1]})\Big] \\ &= \frac{e^{\lambda n}}{\prod_{j=1}^{n} \Big[1 + M_1 C_g G(\|\theta_{-j} x\|_{p-\mathrm{var},[-1,1]})\Big]} \times \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&\times \sum_{k=1}^{\infty} e^{-\lambda(n+k)} H(\|\theta_{-(k+n)}x\|_{p-\mathrm{var},[-1,1]}) \prod_{j=1}^{n+k-1} \left[1 + M_1 C_g G(\|\theta_{-j}x\|_{p-\mathrm{var},[-1,1]})\right] \\
\leq\ & \frac{e^{\lambda n}}{e^{(-\delta+\hat{G})n}} \sum_{k=1}^{\infty} e^{-\lambda(n+k)} e^{(2\delta+\hat{G})(n+k)} \\
\leq\ & \frac{e^{\lambda n}}{e^{(-\delta+\hat{G})n}} \frac{e^{-(\lambda-2\delta-\hat{G})(n+1)}}{1-e^{-(\lambda-2\delta-\hat{G})}} \\
\leq\ & \frac{e^{3\delta n}}{e^{\lambda-2\delta-\hat{G}}-1},
\end{aligned}
$$

which implies that $\lim_{n\to\infty} \frac{\log b(\theta_{-n}x)}{n} \leq 3\delta$ for all $0 < \delta < \frac{1}{2}(\lambda - \hat{G})$ or $\lim_{n\to\infty} \frac{\log b(\theta_{-n}x)}{n} = 0$.
Next,

$$
\begin{aligned}
b(\theta_n x) &= \sum_{k=1}^{\infty} e^{-\lambda k} H(\|\theta_{-k+n}x\|_{p-\mathrm{var},[-1,1]}) \prod_{j=1}^{k-1} \Big(1 + M_1 C_g G(\|\theta_{-j+n}x\|_{p-\mathrm{var},[-1,1]})\Big) \\
&= \sum_{k=1}^{n} e^{-\lambda k} H(\|\theta_{-k+n}x\|_{p-\mathrm{var},[-1,1]}) \prod_{j=1}^{k-1} \Big(1 + M_1 C_g G(\|\theta_{-j+n}x\|_{p-\mathrm{var},[-1,1]})\Big) + \\
&\quad + \sum_{k=n+1}^{\infty} e^{-\lambda k} H(\|\theta_{-k+n}x\|_{p-\mathrm{var},[-1,1]}) \prod_{j=1}^{k-1} \Big(1 + M_1 C_g G(\|\theta_{-j+n}x\|_{p-\mathrm{var},[-1,1]})\Big). \quad (4.3)
\end{aligned}
$$

We now estimate the first summation in (4.3),

$$
\begin{aligned}
&\sum_{k=1}^{n} e^{-\lambda k} H(\|\theta_{-k+n}x\|_{p-\mathrm{var},[-1,1]}) \prod_{j=1}^{k-1} \Big(1 + M_1 C_g G(\|\theta_{-j+n}x\|_{p-\mathrm{var},[-1,1]})\Big) \\
=\ & \sum_{k=1}^{n} e^{-\lambda k} H(\|\theta_{-k+n}x\|_{p-\mathrm{var},[-1,1]}) \prod_{j=n-k+1}^{n-1} \Big(1 + M_1 C_g G(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big) \\
=\ & \prod_{j=1}^{n-1} \Big(1 + M_1 C_g G(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big) \sum_{k=1}^{n} \frac{e^{-\lambda k} H(\|\theta_{-k+n}x\|_{p-\mathrm{var},[-1,1]})}{\prod_{j=1}^{n-k} \Big(1 + M_1 C_g G(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big)} \\
=\ & \prod_{j=1}^{n-1} \Big(1 + M_1 C_g G(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big) \sum_{k=0}^{n-1} \frac{e^{-\lambda(n-k)} H(\|\theta_k x\|_{p-\mathrm{var},[-1,1]})}{\prod_{j=1}^{k} \Big(1 + M_1 C_g G(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big)} \\
=\ & e^{-\lambda n} \prod_{j=1}^{n-1} \Big(1 + M_1 C_g G(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big) \sum_{k=0}^{n-1} \frac{e^{\lambda k} H(\|\theta_k x\|_{p-\mathrm{var},[-1,1]})}{\prod_{j=1}^{k} \Big(1 + M_1 C_g G(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big)} \\
=\ & e^{-(\lambda-\delta-\hat{G})n} \times \\
&\times \left(\sum_{k=0}^{n_0} \frac{e^{\lambda k} H(\|\theta_k x\|_{p-\mathrm{var},[-1,1]})}{\prod_{j=1}^{k} \Big(1 + M_1 C_g G(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big)} + \sum_{k=n_0+1}^{n-1} \frac{e^{\lambda k} H(\|\theta_k x\|_{p-\mathrm{var},[-1,1]})}{\prod_{j=1}^{k} \Big(1 + M_1 C_g G(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big)} \right) \\
\leq\ & e^{-(\lambda-\delta-\hat{G})n} \left(\sum_{k=0}^{n_0} \frac{e^{\lambda k} H(\|\theta_k x\|_{p-\mathrm{var},[-1,1]})}{\prod_{j=1}^{k} \Big(1 + M_1 C_g G(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big)} + e^{-\lambda} \sum_{k=n_0+1}^{n-1} e^{(\lambda-\hat{G}+2\delta)k} \right) \\
\leq\ & e^{2\delta n} \left(\sum_{k=0}^{n_0} \frac{e^{\lambda k} H(\|\theta_k x\|_{p-\mathrm{var},[-1,1]})}{\prod_{j=1}^{k-1} \Big(1 + M_1 C_g G(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big)} + \frac{1}{e^{\lambda-\hat{G}+2\delta}-1} \right). \quad (4.4)
\end{aligned}
$$

The second term in (4.3)

$$\begin{aligned}
&\sum_{k=n+1}^{\infty} e^{-\lambda k} H(\|\theta_{-k+n}x\|_{p-\mathrm{var},[-1,1]}) \prod_{j=1}^{k-1}\Big(1+M_1C_gG(\|\theta_{-j+n}x\|_{p-\mathrm{var},[-1,1]})\Big)\\
=\ & \prod_{j=0}^{n-1}\Big(1+M_1C_gG(\|\theta_{j}x\|_{p-\mathrm{var},[-1,1]})\Big)\times\\
&\times \sum_{k=n+1}^{\infty} e^{-\lambda k} H(\|\theta_{-k+n}x\|_{p-\mathrm{var},[-1,1]}) \prod_{j=1}^{k-n-1}\Big(1+M_1C_gG(\|\theta_{-j}x\|_{p-\mathrm{var},[-1,1]})\Big)\\
=\ & e^{-\lambda n}\prod_{j=0}^{n-1}\Big(1+M_1C_gG(\|\theta_{j}x\|_{p-\mathrm{var},[-1,1]})\Big)\times\\
&\times \sum_{k=1}^{\infty} e^{-\lambda k} H(\|\theta_{-k}x\|_{p-\mathrm{var},[-1,1]}) \prod_{j=1}^{k-1}\Big(1+M_1C_gG(\|\theta_{-j}x\|_{p-\mathrm{var},[-1,1]})\Big)\\
\leq\ & b(x).
\end{aligned}\tag{4.5}$$

Hence

$$b(\theta_n x) \quad \leq \quad e^{3\delta n}\left(\sum_{k=0}^{n_0}\frac{e^{\lambda k}H(\|\theta_k x\|_{p-\mathrm{var},[-1,1]})}{\prod_{j=1}^{k-1}\Big(1+M_1C_gG(\|\theta_j x\|_{p-\mathrm{var},[-1,1]})\Big)} + \frac{1}{e^{(\lambda-\hat{G}+2\delta)}-1} + b(x)\right)$$

which point out that $\lim\limits_{n\to\infty}\dfrac{\log b(\theta_n x)}{n} = 0$. Similarly (3.20) is obtained with some modification.

□

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