Random attractors for rough differential equations

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Abstract

We apply the unified approach in [9] to study the asymptotic behavior of rough differential equations, which consists of two steps of applying the continuous and discrete Gronwall lemmas. The existence of the global pullback attractor for the generated random dynamical system is then proved. We also derive an estimate for the diameter of the global attractor, and prove that for the linear diffusion function of linear form, the pullback attractor collapses to a random point.

Keywords: stochastic differential equations (SDE), rough path theory, rough differential equations, exponential stability.

1 Introduction

This paper studies the asymptotic behavior of the rough differential equation

$$dy_t = [Ay_t + f(y_t)]dt + g(y_t)dx_t, t \in \mathbb{R},\ y(0) = y_0 \in \mathbb{R}^d, \tag{1.1}$$

where we assume for simplicity that $A \in \mathbb{R}^{d\times d}$, $f : \mathbb{R}^d \to \mathbb{R}^d, g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$; f is globally Lipschitz continuous with Lipschitz coefficient C_f; g either belongs to C^3_b such that

$$C_g := \max\Big\{\|g\|_\infty, \|Dg\|_\infty, \|D^2_g\|_\infty, \|D^3_g\|_\infty\Big\} < \infty, \tag{1.2}$$

or has a simple linear form $g(y) = Cy$, where $C \in \mathbb{R}^d \otimes \mathbb{R}^{d\times m}$. Such system is understood in the pathwise sense of a stochastic differential equation driven by a Hölder continuous stochastic process. Namely, we also assume that the driving path $x \in C^{\nu-\mathrm{Hol}}(\mathbb{R},\mathbb{R}^m) \subset C^{p-\mathrm{var}}(\mathbb{R},\mathbb{R}^m)$, with $\frac{1}{3} < \nu < \frac{1}{2}, p > \frac{1}{\nu}$ for simplicity, can be lifted into a realization $\mathbf{x}$ of a stationary stochastic process $\mathbf{X}_t(\omega)$, which has almost sure all realizations in the space $C^{\beta-\mathrm{Hol}}(\mathbb{R}, T^2_1(\mathbb{R}^m)) \subset C^{p-\mathrm{var}}(\mathbb{R}, T^2_1(\mathbb{R}^m))$, such that

$$\Big(E\, \|\mathbf{X}_\cdot(\cdot)\|^p_{p-\mathrm{var},[-1,1]}\Big)^{\frac{1}{p}} < \infty.$$

In this circumstance, the solution is often solved in the sense of Friz-Victoir [11], and the existence and uniqueness theorem is recently proved in [22]. However, it is not clear how to apply the semigroup technique, which is well developed in [8] and [9] for Young differential equations [24], to estimate the rough path integrals. Therefore, we would like to study equation (1.1) in the sense of Gubinelli [15], in order to take advantage of the concept of rough integrals for controlled rough paths. Our aim is then to investigate the role of the driving noise in the longterm behavior of rough system (1.1).

Although no deterministic equilibrium such as the zero solution can in general be found, system (1.1) is expected to possess a pathwise attractor. The reader is refered to [12], [8], [9] and the

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references therein for recent development in studying the asymptotic behavior of Young differential equations, and [14], [6], [7] for rough differential equations.

The paper is organized as follows. Section 2 is devoted to present the existence, uniqueness and the norm estimates of the solution. In subsection 3.1, we introduce the generation of random dynamical system by the equation (1.1). Using Lemma 3.5, we prove the existence of a global random pullback attractor and estimate its diameter in Theorem 3.7 and Theorem 3.10. We also prove in Theorem 3.11 that in case $g(y) = Cy$, the attractor is actually a random point.

2 Rough differential equations

We would like to give a brief introduction to Young integrals. Given any compact time interval $I \subset \mathbb{R}$, let $C(I,\mathbb{R}^d)$ denote the space of all continuous paths $y: \ I \to \mathbb{R}^d$ equipped with sup norm $\|\cdot\|_{\infty,I}$ given by $\|y\|_{\infty,I} = \sup_{t\in I}\|y_t\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$. We write $y_{s,t} := y_t - y_s$. For $p \geq 1$, denote by $\mathcal{C}^{p-\rm var}(I,\mathbb{R}^d) \subset C(I,\mathbb{R}^d)$ the space of all continuous path $y : I \to \mathbb{R}^d$ which is of finite p-variation

$$\left\| y \right\|_{p\text{-var},I} := \left(\sup_{\Pi(I)} \sum_{i=1}^n \|y_{t_i,t_{i+1}}\|^p\right)^{1/p} < \infty, \tag{2.1}$$

where the supremum is taken over the whole class of finite partition of I. $\mathcal{C}^{p-\rm var}(I,\mathbb{R}^d)$ equipped with the $p-$var norm

$$\|y\|_{p\text{-var},I} \quad := \quad \|y_{\min I}\| + \left\| y \right\|_{p-\rm var,I},$$

is a nonseparable Banach space [11, Theorem 5.25, p. 92]. Also for each $0 < \alpha < 1$, we denote by $C^\alpha(I,\mathbb{R}^d)$ the space of Hölder continuous functions with exponent α on I equipped with the norm

$$\|y\|_{\alpha,I} := \|y_{\min I}\| + \left\| y \right\|_{\alpha,I} = \|y(a)\| + \sup_{s<t\in I}\frac{\|y_{s,t}\|}{(t-s)^\alpha},$$

A continuous map $\overline{\omega} : \Delta^2(I) \longrightarrow \mathbb{R}^+, \Delta^2(I) := \{(s,t) : \min I \leq s \leq t \leq \max I\}$ is called a *control* if it is zero on the diagonal and superadditive, i.e. $\overline{\omega}_{t,t} = 0$ for all $t \in I$, and $\overline{\omega}_{s,u} + \overline{\omega}_{u,t} \leq \overline{\omega}_{s,t}$ for all $s \leq u \leq t$ in I.
Now, consider $y \in \mathcal{C}^{q-\rm var}(I,\mathcal{L}(R^m,\mathbb{R}^d))$ and $x \in \mathcal{C}^{p-\rm var}(I,\mathbb{R}^m)$ with $\frac{1}{p}+\frac{1}{q} > 1$, the Young integral $\int_I y_t dx_t$ can be defined as

$$\int_I y_s dx_s := \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} y_u x_{u,v},$$

where the limit is taken on all the finite partition $\Pi = \{\min I = t_0 < t_1 < \cdots < t_n = \max I\}$ of I with $|\Pi| := \max\limits_{[u,v]\in\Pi} |v-u|$ (see [24, p. 264–265]). This integral satisfies additive property by the construction, and the so-called Young-Loeve estimate [11, Theorem 6.8, p. 116]

$$\begin{aligned}\left\|\int_s^t y_u dx_u - y_s x_{s,t}\right\| &\leq K(p,q) \left\| y \right\|_{q\text{-var},[s,t]} \left\| x \right\|_{p\text{-var},[s,t]} \\ &\leq K(p,q)|t-s|^{\frac{1}{p}+\frac{1}{q}} \left\| y \right\|_{\frac{1}{p},[s,t]} \left\| x \right\|_{\frac{1}{q},[s,t]},\end{aligned} \tag{2.2}$$

for all $[s,t] \subset I$, where

$$K(p,q) := (1-2^{1-\frac{1}{p}-\frac{1}{q}})^{-1}. \tag{2.3}$$

We also introduce the construction of the integral using rough paths for the case $y, x \in C^\beta(I)$ when $\beta \in (\frac{1}{3},\nu)$. To do that, we need to introduce the concept of rough paths. Following [10], a couple $\mathbf{x} = (x, \mathbb{X})$, with $x \in C^\beta(I, \mathbb{R}^m)$ and $\mathbb{X} \in C_2^{2\beta}(\Delta^2(I), \mathbb{R}^m \otimes \mathbb{R}^m) := \{\mathbb{X} : \sup_{s<t} \frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{2\beta}} < \infty\}$ where the tensor product $\mathbb{R}^m \otimes \mathbb{R}^n$ can be indentified with the matrix space $\mathbb{R}^{m\times n}$, is called a *rough path* if they satisfies Chen's relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = x_{u,t} \otimes x_{s,u}, \qquad \forall \min I \le s \le u \le t \le \max I. \tag{2.4}$$

$\mathbb{X}$ is viewed as *postulating* the value of the quantity $\int_s^t x_{s,r} \otimes dx_r := \mathbb{X}_{s,t}$ where the right hand side is taken as a definition for the left hand side. Denote by $\mathcal{C}^\beta(I) \subset C^\beta \oplus C_2^{2\beta}$ the set of all rough paths in I, then $\mathcal{C}^\beta$ is a closed set but not a linear space, equipped with the rough path semi-norm

$$\|\!|\mathbf{x}|\!\|_{\beta,I} := \|\!|x|\!\|_{\beta,I} + \|\!|\mathbb{X}|\!\|_{2\beta,\Delta^2(I)}^{\frac{1}{2}} < \infty. \tag{2.5}$$

Let $3 > p > 2, \nu > \frac{1}{p}$. Throughout this paper, we will assume that $x(\omega) : I \to \mathbb{R}^m$ and $\mathbb{X}(\omega) : I \times I \to \mathbb{R}^m \otimes \mathbb{R}^m$ are random funtions that satisfy Chen's relation relation (2.4) and

$$\Big(E\|x_{s,t}\|^p\Big)^{\frac{1}{p}} \le C|t-s|^\nu, \quad \text{and} \quad \Big(E\|\mathbb{X}_{s,t}\|^{\frac{p}{2}}\Big)^{\frac{2}{p}} \le C|t-s|^{2\nu}, \forall s,t \in I \tag{2.6}$$

for some constant C. Then, due to the Kolmogorov criterion for rough paths [11, Appendix A.3] for all $\beta \in (\frac{1}{3}, \nu)$ there is a version of $\omega-$wise $(x, \mathbb{X})$ and random variables $K_\beta \in L^p, \mathbb{K}_\beta \in L^{\frac{p}{2}}$, such that, $\omega-$wise speaking, for all $s,t \in I$,

$$\|x_{s,t}\| \le K_\alpha |t-s|^\beta, \quad \|\mathbb{X}_{s,t}\| \le \mathbb{K}_\beta |t-s|^{2\beta},$$

so that $(x, \mathbb{X}) \in \mathcal{C}^\beta$. Moreover, we could choose β such that

$$x \in C^{0,\beta}(I) := \{x \in C^\beta : \lim_{\delta\to 0} \sup_{0<t-s<\delta} \frac{\|x_{s,t}\|}{|t-s|^\beta} = 0\},$$
$$\mathbb{X} \in C^{0,2\beta}(\Delta^2(I)) := \{\mathbb{X} \in C^{2\beta}(\Delta^2(I)) : \lim_{\delta\to 0} \sup_{0<t-s<\delta} \frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{2\beta}} = 0\},$$

then $\mathcal{C}^{0,\beta}(I) \subset C^{0,\beta}(I) \oplus C^{0,2\beta}(\Delta^2(I))$ is separable due to the separability of $C^{0,\beta}(I)$ and $C^{0,2\beta}(\Delta^2(I))$.

2.1 Controlled rough paths

Following [15], a path $y \in C^\beta(I, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$ is then called to be *controlled by* $x \in C^\beta(I, \mathbb{R}^m)$ if there exists a tube (y', R^y) with $y' \in C^\beta(I, \mathcal{L}(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))), R^y \in C^{2\beta}(\Delta^2(I), \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$ such that

$$y_{s,t} = y'_s x_{s,t} + R^y_{s,t}, \qquad \forall \min I \le s \le t \le \max I.$$

y' is called Gubinelli derivative of y, which is uniquely defined as long as $x \in C^\beta \setminus C^{2\beta}$ (see [10, Proposition 6.4]). The space $\mathcal{D}_x^{2\beta}(I)$ of all the couple (y, y') that is controlled by x will be a Banach space equipped with the norm

$$\begin{aligned} \|y, y'\|_{x,2\beta,I} &:= \|y_{\min I}\| + \|y'_{\min I}\| + \|\!|y, y'|\!\|_{x,2\beta,I}, \qquad \text{where} \\ \|\!|y, y'|\!\|_{x,2\beta,I} &:= \|\!|y'|\!\|_{\beta,I} + \|\!|R^y|\!\|_{2\beta,I}, \end{aligned}$$

where we omit the value space for simplicity of presentation. Now fix a rough path $(x, \mathbb{X})$, then for any $(y, y') \in \mathcal{D}_x^{2\beta}(I)$, it can be proved that the function $F \in C^\beta(\Delta^2(I), \mathbb{R}^d)$ defined by

$$F_{s,t} := y_s x_{s,t} + y'_s \mathbb{X}_{s,t}$$

belongs to the space

$$C_2^{\beta,3\beta}(I) := \Big\{F \in C^\beta(\Delta^2(I)) : F_{t,t} = 0 \quad \text{and} \\ \|\!|\delta F|\!\|_{3\beta,I} := \sup_{\min I \le s \le u \le t \le \max I} \frac{\|F_{s,t} - F_{s,u} - F_{u,t}\|}{|t-s|^{3\beta}} < \infty\Big\}.$$

Thanks to the sewing lemma [10, Lemma 4.2], the integral $\int_s^t y_u dx_u$ can be defined as

$$\int_s^t y_u dx_u := \lim_{|\Pi|\to 0} \sum_{[u,v]\in\Pi} [y_u x_{u,v} + y'_u \mathbb{X}_{u,v}]$$

where the limit is taken on all the finite partition Π of I with $|\Pi| := \max_{[u,v]\in\Pi} |v-u|$ (see [15]). Moreover, there exists a constant $C_\beta = C_{\beta,|I|} > 1$ with $|I| := \max I - \min I$, such that

$$\Big\| \int_s^t y_u dx_u - y_s x_{s,t} + y'_s \mathbb{X}_{s,t} \Big\| \le C_\beta |t-s|^{3\beta} \Big(\|\!|x|\!\|_{\beta,[s,t]} \|\!|R^y|\!\|_{2\beta,\Delta^2[s,t]} + \|\!|y'|\!\|_{\beta,[s,t]} \|\!|\mathbb{X}|\!\|_{2\beta,\Delta^2[s,t]} \Big). \quad (2.7)$$

From now on, if no other emphasis, we will simply write $\|\!|x|\!\|_\beta$ or $\|\!|\mathbb{X}|\!\|_{2\beta}$ without addressing the domain in I or $\Delta^2(I)$. In particular, for any $f \in C_b^3(\mathbb{R}^d, \mathbb{R}^d)$ we get the formula for integration by composition

$$f(x_t) = f(x_s) + \int_s^t \nabla f(x_u) dx_u + \frac{1}{2} \int_s^t \nabla^2 f(x_u) d[x]_{s,u},$$

where the last integral is understood in the Young sense and $[x]_{s,t} := x_{s,t} \otimes x_{s,t} - 2\ \mathrm{Sym}\ (\mathbb{X}_{s,t}) \in C^{2\beta}$. Notice that for geometric rough path $\mathbb{X}_{s,t} = \int_s^t x_{s,r} \otimes dx_r$, then $\mathrm{Sym}\ (\mathbb{X}_{s,t}) = \frac{1}{2} x_{s,t} \otimes x_{s,t}$, thus $[x]_{s,t} \equiv 0$.

As proved in [15], the rough integral of controlled rough paths follows the rule of integration by parts. In practice, we would use the p-var norm

$$\begin{aligned} \|y, y'\|_{x,p,I} &:= \|y_{\min I}\| + \|y'_{\min I}\| + \|\!|y, y'|\!\|_{x,p,I}, \qquad \text{where} \\ \|\!|y, y'|\!\|_{x,p,I} &:= \|\!|y'|\!\|_{p-\mathrm{var},I} + \|\!|R^y|\!\|_{q-\mathrm{var},I}. \end{aligned}$$

Thanks to the sewing lemma [4], we can use a similar version to (2.7) under $p-$var norm as follows.

$$\Big\| \int_s^t y_u dx_u - y_s x_{s,t} + y'_s \mathbb{X}_{s,t} \Big\| \le C_p \Big(\|\!|x|\!\|_{p-\mathrm{var},[s,t]} \|\!|R^y|\!\|_{q-\mathrm{var},\Delta^2[s,t]} + \|\!|y'|\!\|_{p-\mathrm{var},[s,t]} \|\!|\mathbb{X}|\!\|_{q-\mathrm{var},\Delta^2[s,t]} \Big). \quad (2.8)$$

2.2 Greedy times and integrability

In the following, we would like to construct a sequence of greedy times as presented in [3]. Given fixed $\nu \in (\frac{1}{3}, \frac{1}{2})$, $\frac{1}{p} \in (\frac{1}{3}, \nu)$ and $\beta > \frac{1}{p}$, on each compact interval I such that $|I| = \max I - \min I \le 1$, consider a rough path $\mathbf{x} = (x, \mathbb{X}) \in \mathcal{C}^{p-\mathrm{var}}(I)$ with the $p-$var norm

$$\|\!|\mathbf{x}|\!\|_{p-\mathrm{var},I} := \Big(\|\!|x|\!\|^p_{p-\mathrm{var},I} + \|\!|\mathbb{X}|\!\|^q_{q-\mathrm{var},I} \Big)^{\frac{1}{p}}, \quad \text{where } q = \frac{p}{2}. \quad (2.9)$$

Given $\frac{1}{p} \in (\frac{1}{3}, \nu)$, we construct for any fixed $\gamma \in (0,1)$ the sequence of greedy times $\{\tau_i(\gamma, I, p-\mathrm{var})\}_{i\in\mathbb{N}}$ w.r.t. Hölder norms

$$\tau_0 = \min I, \quad \tau_{i+1} := \inf \Big\{ t > \tau_i : \|\!|\mathbf{x}|\!\|_{p-\mathrm{var},[\tau_i,t]} = \gamma \Big\} \wedge \max I. \quad (2.10)$$

Denote by $N_{\gamma,I,p}(\mathbf{x}) := \sup\{i \in \mathbb{N} : \tau_i \le \max I\}$. It follows that

$$N_{\gamma,I,p}(\mathbf{x}) \le 1 + \gamma^{-p} \|\!|\mathbf{x}|\!\|^p_{p-\mathrm{var},I}. \quad (2.11)$$

From now on, we would like to fix $\gamma = \frac{1}{4C_pC_g}$ and would like to write in short $N_{[a,b]}(\mathbf{x})$ for convenience.

2.3 Existence and uniqueness theorem

In this part, we would like to prove the existence and uniqueness theorem for rough differential equation (1.1), where the rough integral is understood in the sense of Gubinelli [15] for controlled rough paths. The idea is to prove first the existence, uniqueness and the differentiability w.r.t. the initial condition, of the solution of the rough differential equation

$$dy_t = g(y_t)dx_t, \quad \forall t \in [a,b], y_a \in \mathbb{R}^d, \tag{2.12}$$

and then to apply Doss-Sussmann technique [23] to transform the system to an equivalent ordinary differential equation. Note that the existence, uniqueness and continuity of the solution of (2.12) is already provided in [15], but the differentiability of the solution $y_t(\mathbf{x}, y_a)$ w.r.t. y_a is somehow missing due to the technical complex. We will derive below the proof for this statement.

Proposition 2.1 *The solution $y_t(\mathbf{x}, y_a)$ is uniformly continuous w.r.t. y_a, i.e.*

$$\begin{aligned} \|\bar{y} - y\|_{\infty,[a,b]} &\leq \|\bar{y}_a - y_a\| e^{(\log 2)\bar{N}_{[a,b]}(\mathbf{x})}, \\ \vert\!\vert\!\vert \bar{y} - y, R \vert\!\vert\!\vert_{p-\mathrm{var},[a,b]} &\leq \|\bar{y}_a - y_a\| \bar{N}^{\frac{p-1}{p}}_{[a,b]}(\mathbf{x}) e^{(\log 2)\bar{N}_{[a,b]}(\mathbf{x})} - \|\bar{y}_a - y_a\|, \end{aligned} \tag{2.13}$$

where $\bar{N}_{[a,b]}(\mathbf{x})$ is the maximal index of the maximal greedy time in the sequence

$$\tau_0 = a, \tau_{k+1} := \inf\Big\{t > \tau_k : \vert\!\vert\!\vert \mathbf{x} \vert\!\vert\!\vert_{p-\mathrm{var},[\tau_k,t]} = \Big[8C_pC_g\Big(1 + \frac{2}{C_p} N^{\frac{2p-1}{p}}_{\frac{1}{8C_pC_g},[a,b]}(\mathbf{x})\Big)\Big]^{-1}\Big\} \wedge b, \tag{2.14}$$

that lies in the interval $[a,b]$.

Proof: The proof is lengthy and is provided in the appendix. □

Proposition 2.2 *The solution $y_t(\mathbf{x}, y_a)$ is differentiable w.r.t. initial condition y_a, moreover, its derivatives $\frac{\partial y_t}{\partial y_a}(\mathbf{x}, y_a)$ is the matrix solution of the linearized rough differential equation*

$$d\xi_t = Dg(y_t)\xi_t dx_t \tag{2.15}$$

Proof: The proof is lengthy and is provided in the appendix.

□

The following theorem shows a standard method to estimate the variation and the supremum norms of the solution of (1.1), by using Gronwall lemma and discretization scheme with the greedy times.

Theorem 2.3 *There exists a unique solution to* (1.1) *for any initial value, whose supremum and $p-$variation norms are estimated as follows*

$$\|y\|_{\infty,[a,b]} \leq \Big[\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big) N_{[a,b]}(\mathbf{x})\Big] e^{4L(b-a)}, \tag{2.16}$$

$$\vert\!\vert\!\vert y, R \vert\!\vert\!\vert_{p-\mathrm{var},[a,b]} \leq \Big[\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big) N_{[a,b]}(\mathbf{x})\Big] e^{4L(b-a)} N^{\frac{p-1}{p}}_{[a,b]}(\mathbf{x}) - \|y_a\|, \tag{2.17}$$

where $L = \|A\| + C_f$ and $\vert\!\vert\!\vert y, R \vert\!\vert\!\vert_{p-\mathrm{var},[s,t]} := \vert\!\vert\!\vert y \vert\!\vert\!\vert_{p-\mathrm{var},[s,t]} + \vert\!\vert\!\vert R^y \vert\!\vert\!\vert_{q-\mathrm{var},[s,t]}$.

Proof: Write in short $L = \|A\| + C_f$. The existence and uniqueness theorem follows [22] with the Doss-Sussmann method. Namely, using the integration by parts for the transformation $y_t = \varphi(t, x, z_t)$, it can be proved that there is a one-one corresponding between the solution of

$$dy_t = [Ay_t + f(y_t)]dt + g(y_t)dx_t = F(y_t)dt + g(y_t)dx_t. \tag{2.18}$$

and the solution of the ordinary differential equation

$$\dot{z}_t = \Big[\frac{\partial \varphi}{\partial y}(t,x,z_t)\Big]^{-1} F(\varphi(t,x,z_t)). \tag{2.19}$$

Since the right hand side of (2.19) satisfies the global Lipschitz continuity and linear growth, by similar arguments as in [22] there exists a unique solution given the initial value. That in turn proves the existence and uniqueness of system (2.18).

To prove (2.16), rewrite (1.1) in the integral form

$$y_{s,t} = \int_s^t [Ay_u + f(y_u)]du + \int_s^t g(y_u)dx_u. \tag{2.20}$$

Together with (1.2) and (2.8), we obtain

$$\begin{aligned}
&\|y_{s,t}\| \\
\leq\ & \int_s^t \Big(\|Ay_u\| + \|f(y_u)\|\Big)du + \Big\|\int_s^t g(y_u)dx_u\Big\| \\
\leq\ & \int_s^t (L\|y_u\| + \|f(0)\|)du + C_g \left\|x\right\|_{p-\mathrm{var},[s,t]} + C_g^2 \left\|\mathbb{X}\right\|_{q-\mathrm{var},[s,t]} + C_p\Big\{2C_g^2 \left\|\mathbb{X}\right\|_{q-\mathrm{var},[s,t]} \left\|y\right\|_{p-\mathrm{var},[s,t]} \\
& + \left\|x\right\|_{p-\mathrm{var},[s,t]} \Big[C_g \left\|R^y\right\|_{q-\mathrm{var},[s,t]} + \frac{1}{2}C_g^2 \left\|x\right\|_{p-\mathrm{var},[s,t]} \left\|y\right\|_{p-\mathrm{var},[s,t]}\Big]\Big\} \\
\leq\ & \int_s^t (L\|y_u\| + \|f(0)\|)du + C_g \left\|x\right\|_{p-\mathrm{var},[s,t]} + C_g^2 \left\|\mathbb{X}\right\|_{q-\mathrm{var},[s,t]} \\
& + C_p\Big\{\Big[2C_g^2 \left\|\mathbb{X}\right\|_{q-\mathrm{var},[s,t]} + \frac{1}{2}C_g^2 \left\|x\right\|^2_{p-\mathrm{var},[s,t]}\Big] \vee C_g \left\|x\right\|_{p-\mathrm{var},[s,t]}\Big\}\Big(\left\|y\right\|_{p-\mathrm{var},[s,t]} + \left\|R^y\right\|_{q-\mathrm{var},[s,t]}\Big),
\end{aligned}$$

which yields

$$\begin{aligned}
&\left\|y\right\|_{p-\mathrm{var},[s,t]} \\
\leq\ & \int_s^t (L\|y_u\| + \|f(0)\|)du + C_g \left\|x\right\|_{p-\mathrm{var},[s,t]} + C_g^2 \left\|\mathbb{X}\right\|_{q-\mathrm{var},[s,t]} \\
& + C_p\Big\{\Big[2C_g^2 \left\|\mathbb{X}\right\|_{q-\mathrm{var},[s,t]} + \frac{1}{2}C_g^2 \left\|x\right\|^2_{p-\mathrm{var},[s,t]}\Big] \vee C_g \left\|x\right\|_{p-\mathrm{var},[s,t]}\Big\}\Big(\left\|y\right\|_{p-\mathrm{var},[s,t]} + \left\|R^y\right\|_{q-\mathrm{var},[s,t]}\Big).
\end{aligned}$$

By similar arguments, we can show that

$$\begin{aligned}
&\left\|R^y\right\|_{q-\mathrm{var},[s,t]} \\
\leq\ & \int_s^t (L\|y_u\| + \|f(0)\|)du + C_g^2 \left\|\mathbb{X}\right\|_{q-\mathrm{var},[s,t]} \\
& + C_p\Big\{\Big[2C_g^2 \left\|\mathbb{X}\right\|_{q-\mathrm{var},[s,t]} + \frac{1}{2}C_g^2 \left\|x\right\|^2_{p-\mathrm{var},[s,t]}\Big] \vee C_g \left\|x\right\|_{p-\mathrm{var},[s,t]}\Big\}\Big(\left\|y\right\|_{p-\mathrm{var},[s,t]} + \left\|R^y\right\|_{q-\mathrm{var},[s,t]}\Big).
\end{aligned}$$

Therefore by assigning $\left\|y,R\right\|_{p-\mathrm{var},[s,t]} = \left\|y\right\|_{p-\mathrm{var},[s,t]} + \left\|R^y\right\|_{q-\mathrm{var},[s,t]}$, we obtain

$$\begin{aligned}
\left\|y,R\right\|_{p-\mathrm{var},[s,t]} \leq\ & 2\int_s^t (L\left\|y\right\|_{p-\mathrm{var},[s,u]} + L\|y_s\| + \|f(0)\|)du + C_g \left\|x\right\|_{p-\mathrm{var},[s,t]} + 2C_g^2 \left\|\mathbb{X}\right\|_{q-\mathrm{var},[s,t]} \\
& + C_p\Big\{\Big[2C_g^2 \left\|\mathbb{X}\right\|_{q-\mathrm{var},[s,t]} + \frac{1}{2}C_g^2 \left\|x\right\|^2_{p-\mathrm{var},[s,t]}\Big] \vee C_g \left\|x\right\|_{p-\mathrm{var},[s,t]}\Big\} \left\|y,R\right\|_{p-\mathrm{var},[s,t]}.
\end{aligned} \tag{2.21}$$

Observe that if $2C_pC_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[s,t]} < 1$ then

$$2C_pC_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[s,t]} > C_p\Big\{ \Big[2C_g^2 \left\| \mathbb{X} \right\|_{q-\mathrm{var},[s,t]} + \frac{1}{2}C_g^2 \left\| x \right\|^2_{p-\mathrm{var},[s,t]} \Big] \vee C_g \left\| x \right\|_{p-\mathrm{var},[s,t]} \Big\}.$$

This follows that

$$\left\| y, R \right\|_{p-\mathrm{var},[s,t]} \leq \int_s^t 4L \left\| y \right\|_{p-\mathrm{var},[s,u]} du + 4(\|f(0)\| + L\|y_s\|)(t-s) + \frac{1}{C_p}$$

whenever $2C_pC_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[s,t]} \leq \frac{1}{2}$. Applying the continuous Gronwall lemma 4.1, we obtain

$$\begin{aligned} \left\| y, R \right\|_{p-\mathrm{var},[s,t]} &\leq 4(\|f(0)\| + L\|y_s\|)(t-s) + \frac{1}{C_p} \\ &\quad + \int_s^t 4Le^{4L(t-u)}\Big[4(\|f(0)\| + L\|y_s\|)(u-s) + \frac{1}{C_p}\Big]du \\ &\leq \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p} + \|y_s\|\Big)e^{4L(t-s)} - \|y_s\| \end{aligned} \tag{2.22}$$

whenever $4C_pC_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[s,t]} \leq 1$. By constructing the sequence of greedy times $\{\tau_k(\frac{1}{4C_pC_g})\}_{k\in\mathbb{N}}$ on interval $[a,b]$, it follows from induction that

$$\begin{aligned} \|y_{\tau_{k+1}}\| &\leq \|y\|_{\infty,[\tau_k,\tau_{k+1}]} \leq \|y\|_{p-\mathrm{var},[\tau_k,\tau_{k+1}]} \\ &\leq \Big(\|y_{\tau_k}\| + \frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big)e^{4L(\tau_{k+1}-\tau_k)} \\ &\leq e^{4L(\tau_{k+1}-\tau_0)}\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big)\sum_{j=0}^k e^{4L(\tau_{k+1}-\tau_j)} \\ &\leq \Big[\|y_a\| + \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big)(k+1)\Big]e^{4L(\tau_{k+1}-\tau_0)}, \quad \forall k = 0,\ldots,N_{[a,b]}(\mathbf{x})-1, \end{aligned}$$

which proves (2.16) since $\tau_{N_{[a,b]}(x)} = b$. On the other hand,

$$\begin{aligned} \left\| y, R \right\|_{p-\mathrm{var},[\tau_k,\tau_{k+1}]} &\leq \|y_{\tau_k}\|\Big(e^{4L(\tau_{k+1}-\tau_k)} - 1\Big) + \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big)e^{4L(\tau_{k+1}-\tau_k)} \\ &\leq \|y_a\|\Big(e^{4L(\tau_{k+1}-\tau_0)} - e^{4L(\tau_k-\tau_0)}\Big) + \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big)\Big(e^{4L(\tau_{k+1}-\tau_0)} - e^{4L(\tau_k-\tau_0)}\Big) \\ &\quad + \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big)e^{4L(\tau_{k+1}-\tau_k)}, \quad \forall k = 0,\ldots,N_{[a,b]}(\mathbf{x})-1, \end{aligned}$$

It then follows from inequality of p-variation seminorm in [7] that

$$\begin{aligned} \left\| y, R \right\|_{p-\mathrm{var},[a,b]} &\leq N_{[a,b]}^{\frac{p-1}{p}}(\mathbf{x}) \sum_{k=0}^{N_{[a,b]}(\mathbf{x})-1} \left\| y, R \right\|_{q-\mathrm{var},[\tau_k,\tau_{k+1}]} \\ &\leq \|y_a\|N_{[a,b]}^{\frac{p-1}{p}}(\mathbf{x}) \sum_{k=0}^{N_{[a,b]}(\mathbf{x})-1} \Big(e^{4L(\tau_{k+1}-\tau_0)} - e^{4L(\tau_k-\tau_0)}\Big) \\ &\quad + \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big)N_{[a,b]}^{\frac{p-1}{p}}(\mathbf{x}) \sum_{k=0}^{N_{[a,b]}(\mathbf{x})-1} \Big(\sum_{j=0}^{k+1} e^{4L(\tau_{k+1}-\tau_j)} - \sum_{j=0}^{k} e^{4L(\tau_k-\tau_j)}\Big) \end{aligned}$$

$$
\begin{aligned}
&\leq N_{[a,b]}^{\frac{p-1}{p}}(\mathbf{x})\Big\{\|y_a\|\Big(e^{4L(b-a)}-1\Big)+\Big(\frac{\|f(0)\|}{L}+\frac{1}{C_p}\Big)\Big(\sum_{j=0}^{N_{[a,b]}(\mathbf{x})} e^{4L(b-\tau_j)}-1\Big)\Big\}\\
&\leq N_{[a,b]}^{\frac{p-1}{p}}(\mathbf{x})\Big[\|y_a\|+\Big(\frac{\|f(0)\|}{L}+\frac{1}{C_p}\Big)N_{[a,b]}(\mathbf{x})\Big]e^{4L(b-a)}-\|y_a\|
\end{aligned}
$$

which proves (2.17).

□

Following the same arguments line by line, we could prove similar estimates for $g = Cy$ as follows.

Theorem 2.4 *There exists a unique solution to the rough differential equation*

$$
dy_t = [Ay_t + f(y_t)]dt + Cy_t dx_t, t \in \mathbb{R},\ y(0) = y_0 \in \mathbb{R}^d, \tag{2.23}
$$

for any initial value, whose supremum and $p-$variation norms are estimated as follows

$$
\begin{aligned}
\|y\|_{\infty,[a,b]} &\leq \Big[\|y_a\| + \frac{\|f(0)\|}{L}N_{[a,b]}(\mathbf{x})\Big]e^{4L(b-a)+\alpha N_{[a,b]}(\mathbf{x})}, & (2.24)\\
\|\!|y,R|\!\|_{p-\mathrm{var},[a,b]} &\leq \Big[\|y_a\| + \frac{\|f(0)\|}{L}N_{[a,b]}(\mathbf{x})\Big]e^{4L(b-a)+\alpha N_{[a,b]}(\mathbf{x})}N_{[a,b]}^{\frac{p-1}{p}}(\mathbf{x}) - \|y_a\|, & (2.25)
\end{aligned}
$$

where $L = \|A\| + C_f$, $\|\!|y,R|\!\|_{p-\mathrm{var},[s,t]} := \|\!|y|\!\|_{p-\mathrm{var},[s,t]} + \|\!|R^y|\!\|_{q-\mathrm{var},[s,t]}$ and $\alpha = \log(1+\frac{3}{2C_p})$.

Proof: Notice that the proof on the existence and uniqueness, as well as the Hölder norm estimates, of the solution of (2.4) is already given in [6]. To estimate the norms, we derive similar estimates as in the proof of Theorem 2.3 with note that

$$
y'_s = Cy_s, \quad [Cy]'_s = C^2 y_s, \quad R^{Cy}_{s,t} = CR^y_{s,t}.
$$

Hence the estimate of $\|\!|y,R|\!\|_{p-\mathrm{var},[s,t]}$ in (2.21) is of the form

$$
\begin{aligned}
&\|\!|y,R|\!\|_{p-\mathrm{var},[s,t]}\\
\leq\ & 2\int_s^t (L\,\|\!|y|\!\|_{p-\mathrm{var},[s,u]} + L\|y_s\| + \|f(0)\|)du + \Big(\|C\|\,\|\!|x|\!\|_{p-\mathrm{var},[s,t]} + 2\|C\|^2\,\|\!|\mathbb{X}|\!\|_{q-\mathrm{var},[s,t]}\Big)\|y_s\|\\
&+2C_p\Big\{\|C\|^2\,\|\!|\mathbb{X}|\!\|_{q-\mathrm{var},[s,t]} \vee \|C\|\,\|\!|x|\!\|_{p-\mathrm{var},[s,t]}\Big\}\,\|\!|y,R|\!\|_{p-\mathrm{var},[s,t]}\,.
\end{aligned}
$$

As a result

$$
\|\!|y,R|\!\|_{p-\mathrm{var},[s,t]} \leq \int_s^t 4L\,\|\!|y|\!\|_{p-\mathrm{var},[s,u]}\,du + 4(\|f(0)\| + L\|y_s\|)(t-s) + \frac{3}{2C_p}\|y_s\|
$$

whenever $2C_p\|C\|\,\|\!|\mathbf{x}|\!\|_{p-\mathrm{var},[s,t]} \leq \frac{1}{2}$. Applying the continuous Gronwall lemma 4.1, we obtain

$$
\begin{aligned}
\|\!|y,R|\!\|_{p-\mathrm{var},[s,t]} &\leq 4(\|f(0)\| + L\|y_s\|)(t-s) + \frac{3}{2C_p}\|y_s\|\\
&\quad + \int_s^t 4Le^{4L(t-u)}\Big[4\Big(\|f(0)\| + L\|y_s\|\Big)(u-s) + \frac{3}{2C_p}\|y_s\|\Big]du\\
&\leq \Big(\frac{\|f(0)\|}{L} + (1+\frac{3}{2C_p})\|y_s\|\Big)e^{4L(t-s)} - \|y_s\| \qquad (2.26)
\end{aligned}
$$

whenever $4C_pC_g\,\|\!|\mathbf{x}|\!\|_{p-\mathrm{var},[s,t]} \leq 1$. The rest is the a direct consequence of [9, Theorem 2.1] (see also the proof of Propositions 2.1, 2.2 in the Appendix). □

3 Random attractors

3.1 Generation of rough cocycle and rough flows

In this subsection we would like to present the generation of a random dynamical system from rough differential equations (1.1), which is based mainly on the work in [2] with only a small modification applied for Hölder spaces. Recall that $T_1^2(\mathbb{R}^m) = 1 \oplus \mathbb{R}^m \oplus \mathbb{R}^{m\otimes 2}$ is the set with the tensor product

$$(1, g^1, g^2) \otimes (1, h^1, h^2) = (1, g^1 + h^1, g^1 h^1 + g^2 + h^2), \quad \forall\, \mathbf{g} = (1, g^1, g^2), \mathbf{h} = (1, h^1, h^2) \in T_1^2(\mathbb{R}^m).$$

Then it can be shown that $T_1^2(\mathbb{R}^m), \otimes$ is a topological group with unit element $\mathbf{1} = (1, 0, 0)$.
For $\beta \in (\frac{1}{p}, \nu)$, denote by

$$\mathcal{C}^{0,\beta-\mathrm{Hol}}([a,b], T_1^2(\mathbb{R}^m)) \subset \mathcal{C}^{0,p-\mathrm{var}}([a,b], T_1^2(\mathbb{R}^m))$$

the closure of $\mathcal{C}^\infty([a,b], T_1^2(\mathbb{R}^m))$ in $\mathcal{C}^{\beta-\mathrm{Hol}}([a,b], T_1^2(\mathbb{R}^m))$, and by

$$\mathcal{C}_0^{0,\beta-\mathrm{Hol}}(\mathbb{R}, T_1^2(\mathbb{R}^m)) \subset \mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R}, T_1^2(\mathbb{R}^m))$$

the space of all $x : \mathbb{R} \to \mathbb{R}^m$ such that $x|_I \in \mathcal{C}^{0,\beta-\mathrm{Hol}}(I, T_1^2(\mathbb{R}^m))$ for each compact interval $I \subset \mathbb{R}$ containing 0. Then $\mathcal{C}_0^{0,\beta-\mathrm{Hol}}(\mathbb{R}, T_1^2(\mathbb{R}^m))$ is embeded into $\mathcal{C}_0^{0,p-\mathrm{var}}(\mathbb{R}, T_1^2(\mathbb{R}^m))$, which is equipped with the compact open topology given by the $p-$variation norm, i.e the topology generated by the metric:

$$d_p(\mathbf{x}_1, \mathbf{x}_2) := \sum_{k\geq 1} \frac{1}{2^k} (\|\mathbf{x}_1 - \mathbf{x}_2\|_{p-\mathrm{var},[-k,k]} \wedge 1),$$

where the $p-$var norm is given in (2.9). As a result, they are both separable and thus Polish spaces. Let us consider a stochastic process $\bar{\mathbf{X}}$ defined on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with realizations in $(\mathcal{C}_0^{0,\beta-\mathrm{Hol}}(\mathbb{R}, T_1^2(\mathbb{R}^m)), \mathcal{F})$. Assume further that $\bar{\mathbf{X}}$ has stationary increments. Assign

$$\Omega := \mathcal{C}_0^{0,\beta-\mathrm{Hol}}(\mathbb{R}, T_1^2(\mathbb{R}^m))$$

and equip with the Borel $\sigma-$ algebra $\mathcal{F}$ and let $\mathbb{P}$ be the law of $\bar{\mathbf{X}}$. Denote by θ the *Wiener-type shift*

$$(\theta_t \mathbf{x})_\cdot = \mathbf{x}_t^{-1} \otimes \mathbf{x}_{t+\cdot}, \forall t \in \mathbb{R}, \mathbf{x} \in \mathcal{C}_0^{0,\beta-\mathrm{Hol}}(\mathbb{R}, T_1^2(\mathbb{R}^m)), \tag{3.1}$$

and define the so-called *diagonal process* $\mathbf{X} : \mathbb{R} \times \Omega \to T_1^2(\mathbb{R}^m), \mathbf{X}(t,\omega) = \omega_t$ for all $t \in \mathbb{R}, \omega \in \Omega$. Due to the stationarity of $\bar{\mathbf{X}}$, it can be proved that θ is invariant under $\mathbb{P}$, then forming a continuous (and thus measurable) dynamical system on $(\Omega, \mathcal{F}, \mathbb{P})$ (the proof goes step by step to the one in [2, Theorem 5] with small modification for β-Hölder norm). Moreover, $\mathbf{X}$ forms a $\beta-$ *rough path cocycle*, namely, $\mathbf{X}_\cdot(\omega) \in \mathcal{C}_0^{0,\beta-\mathrm{Hol}}(\mathbb{R}, T_1^2(\mathbb{R}^m))$ for every $\omega \in \Omega$, which satisfies the *cocyle relation*:

$$\mathbf{X}_{t+s}(\omega) = \mathbf{X}_s(\omega) \otimes \mathbf{X}_t(\theta_s \omega), \forall \omega \in \Omega, t, s \in \mathbb{R},$$

in the sense that $\mathbf{X}_{s,s+t} = \mathbf{X}_t(\theta_s\omega)$ with the increment notation $\mathbf{X}_{s,s+t} := \mathbf{X}_s^{-1} \otimes \mathbf{X}_{s+t}$. Under this circumstance, if we assume further that

$$\Big(E \, \|\mathbf{X}_\cdot(\cdot)\|^p_{p-\mathrm{var},[-1,1]}\Big)^{\frac{1}{p}} = \Gamma(p) < \infty$$

then, due to the fact that $\|\mathbf{X}_\cdot(\theta_h\omega)\|_{p-\mathrm{var},[s,t]} = \|\mathbf{X}_\cdot(\omega)\|_{p-\mathrm{var},[s+h,t+h]}$, it follows from Birkhorff ergodic theorem that

$$\Gamma(\mathbf{x}, p) := \limsup_{n\to\infty} \Big(\frac{1}{n} \sum_{k=1}^n \|\theta_{-k}\mathbf{x}\|^p_{p-\mathrm{var},[-1,1]}\Big)^{\frac{1}{p}} = \Gamma(p) \tag{3.2}$$

for almost all realizations $\mathbf{x}_t$ of the form $\mathbf{X}_t(\omega)$. We assume additionally that $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is ergodic.

Remark 3.1 It is important to note that, due to [2, Corollary 9], this construction is possible for $X : \mathbb{R} \to \mathbb{R}^m$ to be a continuous, centered Gaussian process with stationary increments and independent components, satisfying: there exists for any $T > 0$ a constant C_T such that for all $p \geq \frac{1}{\nu}$

$$E\|X_t - X_s\|^p \leq C_T|t-s|^{p\nu}, \quad \forall s,t \in [0,T]. \tag{3.3}$$

By Kolmogorov theorem, for any $\beta \in (\frac{1}{p}, \nu)$ and any interval $[0,T]$ almost all realization of X will be in $C^{0,\beta}([0,T])$. Then X has its covariance function with finite 2-dimensional $\rho-$variation on every square $[s,t]^2 \in \mathbb{R}^2$ for some $\rho \in [1,2)]$, and $\bar{\mathbf{X}}$ is the natural lift of X, in the sense of Friz-Victoir, with sample paths in the space $\mathcal{C}_0^{0,\beta-\mathrm{Hol}}(\mathbb{R}, T_1^2(\mathbb{R}^m))$, for every $p > 2\rho$. For instance, such a stochastic process X, in particular, can be a fractional Brownian motion B^H [19] with Hurst exponent $H \in (\frac{1}{3}, \frac{1}{2})$, i.e. a family of $B^H = \{B_t^H\}_{t\in\mathbb{R}}$ with continuous sample paths and

$$E\|B_t^H - B_s^H\| = |t-s|^{2H}, \forall t,s \in \mathbb{R}.$$

We reformulate a result from [2, Theorem 21] for our situation as follows.

Proposition 3.2 *Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be a measurable metric dynamical system and let $\mathbf{X} : \mathbb{R} \times \Omega \to T_1^2(\mathbb{R}^m)$ be a p- rough cocycle for some $2 \leq p < 3$. Then there exists a unique continuous random dynamical system φ over $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ which solves the rough differential equation*

$$dy_t = [Ay_t + f(y_t)]dt + g(y_t)d\mathbf{X}_t(\omega), t \geq 0. \tag{3.4}$$

3.2 Existence of pullback attractors

Given a random dynamical system φ on $\mathbb{R}^d$, we follow [5], [1, Chapter 9] to present the notion of random pullback attractor. Recall that a set $\hat{M} = \{M(\omega)\}_{\omega\in\Omega}$ a *random set*, if $\omega \mapsto d(x|M(\omega))$ is $\mathcal{F}$-measurable for each $x \in \mathbb{R}^d$, where $d(E|F) = \sup\{\inf\{d(x,y)|y \in F\}|x \in E\}$ for E, F are nonempty subset of $\mathbb{R}^d$ and $d(x|E) = d(\{x\}|E)$. An *universe* $\mathcal{D}$ is a family of random sets which is closed w.r.t. inclusions (i.e. if $\hat{D}_1 \in \mathcal{D}$ and $\hat{D}_2 \subset \hat{D}_1$ then $\hat{D}_2 \in \mathcal{D}$). In our setting, we define the universe $\mathcal{D}$ to be a family of random sets $D(\omega)$ which is *tempered* (see e.g. [1, pp. 164, 386]), namely $D(\omega)$ belongs to the ball $B(0, \rho(\omega))$ for all $\omega \in \Omega$ where the radius $\rho(\omega) > 0$ is a *tempered random varible*, i.e.

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \rho(\theta_t\omega) = 0. \tag{3.5}$$

An invariant random compact set $\mathcal{A} \in \mathcal{D}$ is called a *pullback random attractor* in $\mathcal{D}$, if $\mathcal{A}$ attracts any closed random set $\hat{D} \in \mathcal{D}$ in the pullback sense, i.e.

$$\lim_{t\to\infty} d(\varphi(t, \theta_{-t}\omega)\hat{D}(\theta_{-t}\omega)|\mathcal{A}(\omega)) = 0. \tag{3.6}$$

The existence of a random pullback attractor follows from the existence of a random pullback absorbing set (see [5, Theorem 3]). A random set $\mathcal{B} \in \mathcal{D}$ is called *pullback absorbing* in a universe $\mathcal{D}$ if $\mathcal{B}$ absorbs all sets in $\mathcal{D}$, i.e. for any $\hat{D} \in \mathcal{D}$, there exists a time $t_0 = t_0(\omega, \hat{D})$ such that

$$\varphi(t, \theta_{-t}\omega)\hat{D}(\theta_{-t}\omega) \subset \mathcal{B}(\omega), \text{ for all } t \geq t_0. \tag{3.7}$$

Given a universe $\mathcal{D}$ and a random compact pullback absorbing set $\mathcal{B} \in \mathcal{D}$, there exists a unique random pullback attractor (which is then a weak attractor) in $\mathcal{D}$, given by

$$\mathcal{A}(\omega) = \cap_{s\geq 0}\overline{\cup_{t\geq s}\varphi(t, \theta_{-t}\omega)\mathcal{B}(\theta_{-t}\omega)}. \tag{3.8}$$

We need the following auxiliary results.

Proposition 3.3 *Assume that A has all eigenvalues of negative real parts. Then there exist constant $C_A \geq 1, \lambda_A > 0$ such that*

$$
\begin{aligned}
\|\Phi\|_{\infty,[a,b]} &\leq C_A e^{-\lambda_A a}, &(3.9)\\
\|\Phi\|_{p-\mathrm{var},[a,b]} &\leq \|A\| C_A e^{-\lambda_A a}(b-a), \quad \forall\, 0 \leq a < b, &(3.10)
\end{aligned}
$$

where $\Phi(t) = e^{At}$.

Proof: See the proof in [9, Proposition 3.2]. □

Proposition 3.4 *Given (3.9) and (3.10), the following estimate holds: for any $0 \leq a < b \leq c$*

$$
\begin{aligned}
\Big\| \int_a^b \Phi(c-s) g(y_s) dx_s \Big\| \leq\ & C_A \Big[1 + C_p \|A\|(b-a)\Big] e^{-\lambda_A(c-b)} \Big(C_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[a,b]} + C_g^2 \left\| \mathbf{x} \right\|^2_{p-\mathrm{var},[a,b]} \Big) \\
& + 2C_p C_A e^{-\lambda_A(c-b)} \Big\{ C_g^2 \left\| \mathbf{x} \right\|^2_{p-\mathrm{var},[a,b]} \vee C_g \left\| \mathbf{x} \right\|_{p-\mathrm{var},[a,b]} \Big\} \left\| y, R \right\|_{p-\mathrm{var},[a,b]}. \qquad (3.11)
\end{aligned}
$$

Proof: Since

$$
\Phi(c-t) g(y_t) - \Phi(c-s) g(y_s) = [\Phi(c-t) - \Phi(c-s)] g(y_t) + \Phi(c-s) \Big([g(y)]'_s x_{s,t} + R^{g(y)}_{s,t} \Big)
$$

it follows that

$$
\begin{aligned}
[\Phi(c-\cdot) g(y_\cdot)]'_s &= \Phi(c-s)[g(y)]'_s = \Phi(c-s) Dg(y_s) g(y_s), \\
\Big\| R^{\Phi(c-\cdot)g(y_\cdot)}_{s,t} \Big\| &\leq \|\Phi(c-s) R^{g(y)}_{s,t}\| + \|\Phi(c-t) - \Phi(c-s)\| \|g(y_t)\|,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\left\| [\Phi(c-\cdot) g(y_\cdot)]' \right\|_{p-\mathrm{var},[a,b]} &\leq 2C_g^2 \|\Phi(c-\cdot)\|_{\infty,[a,b]} \left\| y \right\|_{p-\mathrm{var},[a,b]} + C_g^2 \left\| \Phi(c-\cdot) \right\|_{p-\mathrm{var},[a,b]} \\
\left\| R^{\Phi(c-\cdot)g(y_\cdot)} \right\|_{q-\mathrm{var},[a,b]} &\leq \|\Phi(c-b)\| \left\| R^{g(y)} \right\|_{q-\mathrm{var},[a,b]} + C_g \left\| \Phi(c-\cdot) \right\|_{q-\mathrm{var},[a,b]}.
\end{aligned}
$$

Using (4.2) and (3.9), (3.10), we can now estimate

$$
\begin{aligned}
& \Big\| \int_a^b \Phi(c-s) g(y_s) dx_s \Big\| \\
\leq\ & \|\Phi(c-a) g(y_a)\| \|x_{a,b}\| + \|[\Phi(c-\cdot) g(y_\cdot)]'_a\| \|\mathbb{X}_{a,b}\| \\
& + C_p \Big\{ \left\| x \right\|_{p-\mathrm{var},[a,b]} \left\| R^{\Phi(c-\cdot)g(y_\cdot)} \right\|_{q-\mathrm{var},[a,b]} + \left\| \mathbb{X} \right\|_{q-\mathrm{var},[a,b]} \left\| [\Phi(c-\cdot) g(y_\cdot)]' \right\|_{p-\mathrm{var},[a,b]} \Big\} \\
\leq\ & C_A C_g e^{-\lambda_A(c-a)} \left\| x \right\|_{p-\mathrm{var},[a,b]} + C_A C_g^2 e^{-\lambda_A(c-a)} \left\| \mathbb{X} \right\|_{q-\mathrm{var},[a,b]} \\
& + C_p \left\| \mathbb{X} \right\|_{q-\mathrm{var},[a,b]} \Big[2C_A C_g^2 e^{-\lambda_A(c-b)} \left\| y \right\|_{p-\mathrm{var},[a,b]} + C_A C_g^2 \|A\| e^{-\lambda_A(c-b)}(b-a) \Big] \\
& + C_p \left\| x \right\|_{p-\mathrm{var},[a,b]} \Big\{ C_A C_g \|A\| e^{-\lambda_A(c-b)}(b-a) \\
& + C_A e^{-\lambda_A(c-b)} \Big(C_g \left\| R^y \right\|_{q-\mathrm{var},[a,b]} + \frac{1}{2} C_g^2 \left\| x \right\|_{p-\mathrm{var},[a,b]} \left\| y \right\|_{p-\mathrm{var},[a,b]} \Big) \Big\} \\
\leq\ & C_A [1 + C_p \|A\|(b-a)] e^{-\lambda_A(c-b)} \Big(C_g \left\| x \right\|_{p-\mathrm{var},[a,b]} + C_g^2 \left\| \mathbb{X} \right\|_{q-\mathrm{var},[a,b]} \Big) \\
& + C_p C_A e^{-\lambda_A(c-b)} \Big\{ \Big[2C_g^2 \left\| \mathbb{X} \right\|_{q-\mathrm{var},[a,b]} + \frac{1}{2} C_g^2 \left\| x \right\|^2_{p-\mathrm{var},[a,b]} \Big] \vee C_g \left\| x \right\|_{p-\mathrm{var},[a,b]} \Big\} \left\| y, R \right\|_{p-\mathrm{var},[a,b]}.
\end{aligned}
$$

which, together proves (3.11).

□

The following lemma is the crucial technique of this paper.

Lemma 3.5 *Assume that y_t satisfies*

$$y_t = \Phi(t)y_0 + \int_0^t \Phi(t-s)f(y_s)ds + \int_0^t \Phi(t-s)g(y_s)dx_s, \quad \forall t \geq 0. \tag{3.12}$$

Then for any $r > 0$ given and $n \geq 0$,

$$\begin{aligned} \|y_t\|e^{(\lambda_A - L_f)t} &\leq C_A\|y_0\| + \frac{C_A}{\lambda_A - L_f}\|f(0)\|\Big(e^{(\lambda_A-L_f)t} - 1\Big) \\ &\quad + \sum_{k=0}^n e^{\lambda_A r}e^{(\lambda_A - L_f)kr}\Big[\kappa_1(\mathbf{x},\Delta_k^r) + \kappa_2(\mathbf{x},\Delta_k^r)\,\|y,R\|_{p-\mathrm{var},\Delta_k^r}\Big], \quad \forall t \in \Delta_n^r \end{aligned} \tag{3.13}$$

where $\Delta_k^r := [kr,(k+1)r]$, $L_f := C_A C_f$ and

$$\begin{aligned} \kappa_1(\mathbf{x},[a,b]) &:= C_A[1 + C_p\|A\|(b-a)]\Big(C_g\,\|\mathbf{x}\|_{p-\mathrm{var},[a,b]} + C_g^2\,\|\mathbf{x}\|^2_{p-\mathrm{var},[a,b]}\Big) &(3.14)\\ \kappa_2(\mathbf{x},[a,b]) &:= 2C_pC_A\Big\{C_g^2\,\|\mathbf{x}\|^2_{p-\mathrm{var},[a,b]} \vee C_g\,\|\mathbf{x}\|_{p-\mathrm{var},[a,b]}\Big\}. &(3.15) \end{aligned}$$

Proof: First, for any $t \in [nr,(n+1)r)$, it follows from (3.9) and the global Lipschitz continuity of f that

$$\begin{aligned} \|y_t\| &\leq \|\Phi(t)y_0\| + \int_0^t \|\Phi(t-s)f(y_s)\|ds + \Big\|\int_0^t \Phi(t-s)g(y_s)dx_s\Big\| \\ &\leq C_Ae^{-\lambda_A t}\|y_0\| + \int_0^t C_Ae^{-\lambda_A(t-s)}\Big(C_f\|y_s\| + \|f(0)\|\Big)ds + \Big\|\int_0^t \Phi(t-s)g(y_s)dx_s\Big\| \\ &\leq C_Ae^{-\lambda_A t}\|y_0\| + \frac{C_A}{\lambda_A}\|f(0)\|(1 - e^{-\lambda_A t}) + \beta_t + C_AC_f\int_0^t e^{-\lambda_A(t-s)}\|y_s\|ds, \end{aligned}$$

where $\beta_t := \Big\|\int_0^t \Phi(t-s)g(y_s)dx_s\Big\|$. Multiplying both sides with $e^{\lambda_A t}$ yields

$$\|y_t\|e^{\lambda_A t} \leq C_A\|y_0\| + \frac{C_A}{\lambda_A}\|f(0)\|(e^{\lambda_A t} - 1) + \beta_te^{\lambda_A t} + C_AC_f\int_0^t e^{\lambda_A s}\|y_s\|ds.$$

By applying the continuous Gronwall lemma 4.1, we obtain

$$\begin{aligned} \|y_t\|e^{\lambda_A t} &\leq C_A\|y_0\| + \frac{C_A}{\lambda_A}\|f(0)\|(e^{\lambda_A t} - 1) + \beta_te^{\lambda_A t} \\ &\quad + \int_0^t L_fe^{L_f(t-s)}\Big[C_A\|y_0\| + \frac{C_A}{\lambda_A}\|f(0)\|(e^{\lambda_A s} - 1) + \beta_se^{\lambda_A s}\Big]ds. \end{aligned}$$

Multiplying both sides with $e^{-L_f t}$ yields

$$\begin{aligned} \|y_t\|e^{(\lambda_A - L_f)t} &\leq C_A\|y_0\|e^{-L_f t} + \frac{C_A}{\lambda_A}\|f(0)\|\Big(e^{(\lambda_A - L_f)t} - e^{-L_f t}\Big) + \beta_te^{(\lambda_A - L_f)t} \\ &\quad + \int_0^t L_fe^{-L_f s}\Big[C_A\|y_0\| + \frac{C_A}{\lambda_A}\|f(0)\|(e^{\lambda_A s} - 1) + \beta_se^{\lambda_A s}\Big]ds \\ &\leq C_A\|y_0\| + \frac{C_A}{\lambda_A - L_f}\|f(0)\|\Big(e^{(\lambda_A - L_f)t} - 1\Big) + \beta_te^{(\lambda_A - L_f)t} + \int_0^t L_f\beta_se^{(\lambda_A - L_f)s}ds. \end{aligned} \tag{3.16}$$

Next, observe from (3.11) that for all $s \leq t$

$$
\begin{aligned}
\beta_s e^{(\lambda_A - L_f)s} &= e^{(\lambda_A - L_f)s} \Big\| \int_0^s \Phi(s-u) g(y_u) dx_u \Big\| \\
&\leq e^{(\lambda_A - L_f)s} \sum_{k=0}^{\lfloor \frac{s}{r} \rfloor - 1} \Big\| \int_{\Delta_k^r} \Phi(s-u) g(y_u) dx_u \Big\| + \Big\| \int_{r\lfloor s/r \rfloor}^s \Phi(s-u) g(y_u) dx_u \Big\| \\
&\leq e^{(\lambda_A - L_f)s} \sum_{k=0}^{\lfloor \frac{s}{r} \rfloor - 1} e^{-\lambda_A(s-kr-r)} \Big[\kappa_1(\mathbf{x}, \Delta_k^r) + \kappa_2(\mathbf{x}, \Delta_k^r) \|y, R\|_{p-\mathrm{var}, \Delta_k^r}\Big] \\
&\quad + e^{(\lambda_A - L_f)s} \Big[\kappa_1(\mathbf{x}, [r\lfloor \frac{s}{r} \rfloor, s]) + \kappa_2(\mathbf{x}, [r\lfloor \frac{s}{r} \rfloor, s]) \|y, R\|_{p-\mathrm{var}, [r\lfloor \frac{s}{r} \rfloor, s]}\Big] \\
&\leq \sum_{k=0}^{\lfloor \frac{s}{r} \rfloor} e^{(\lambda_A - L_f)s} e^{-\lambda_A(s-kr-r)} \Big[\kappa_1(\mathbf{x}, \Delta_k^r) + \kappa_2(\mathbf{x}, \Delta_k^r) \|y, R\|_{p-\mathrm{var}, \Delta_k^r}\Big] \\
&\leq e^{\lambda_A r} \sum_{k=0}^{\lfloor \frac{s}{r} \rfloor} e^{(\lambda_A - L_f)kr} e^{-L_f(s-kr)} \Big[\kappa_1(\mathbf{x}, \Delta_k^r) + \kappa_2(\mathbf{x}, \Delta_k^r) \|y, R\|_{p-\mathrm{var}, \Delta_k^r}\Big]. \quad (3.17)
\end{aligned}
$$

Replacing (3.17) into (3.16) yields

$$
\begin{aligned}
&\|y_t\| e^{(\lambda_A - L_f)t} \\
\leq\ & C_A \|y_0\| + \frac{C_A}{\lambda_A - L_f} \|f(0)\| \Big(e^{(\lambda_A - L_f)t} - 1\Big) \\
&+ e^{\lambda_A r} \sum_{k=0}^{n} e^{(\lambda_A - L_f)kr} e^{-L_f(t-kr)} \Big[\kappa_1(\mathbf{x}, \Delta_k^r) + \kappa_2(\mathbf{x}, \Delta_k^r) \|y, R\|_{p-\mathrm{var}, \Delta_k^r}\Big] \\
&+ e^{\lambda_A r} \int_0^t \sum_{k=0}^{\lfloor \frac{s}{r} \rfloor} e^{(\lambda_A - L_f)kr} e^{-L_f(s-kr)} \Big[\kappa_1(\mathbf{x}, \Delta_k^r) + \kappa_2(\mathbf{x}, \Delta_k^r) \|y, R\|_{p-\mathrm{var}, \Delta_k^r}\Big] ds \\
\leq\ & C_A \|y_0\| + \frac{C_A}{\lambda_A - L_f} \|f(0)\| \Big(e^{(\lambda_A - L_f)t} - 1\Big) \\
&+ e^{\lambda_A r} \sum_{k=0}^{n} e^{(\lambda_A - L_f)kr} \Big[\kappa_1(\mathbf{x}, \Delta_k^r) + \kappa_2(\mathbf{x}, \Delta_k^r) \|y, R\|_{p-\mathrm{var}, \Delta_k^r}\Big] \Big(e^{-L_f(t-kr)} + \int_{kr}^t L_f e^{-L_f(s-kr)} ds\Big) \\
\leq\ & C_A \|y_0\| + \frac{C_A}{\lambda_A - L_f} \|f(0)\| \Big(e^{(\lambda_A - L_f)t} - 1\Big) \\
&+ \sum_{k=0}^{n} e^{\lambda_A r} e^{(\lambda_A - L_f)kr} \Big[\kappa_1(\mathbf{x}, \Delta_k^r) + \kappa_2(\mathbf{x}, \Delta_k^r) \|y, R\|_{p-\mathrm{var}, \Delta_k^r}\Big], \quad (3.18)
\end{aligned}
$$

where we use the fact that $e^{-L_f(t-kr)} + \int_{kr}^t L_f e^{-L_f(s-kr)} ds = 1$ for all $t \geq kr$. The continuity of y at $t = (n+1)r$ then proves (3.13).

□

We need one more auxiliary proposition.

Proposition 3.6 *Define*

$$
\begin{aligned}
G(\mathbf{x}, [a,b]) &:= e^{\lambda_A + 4L(b-a)} \kappa_2(\mathbf{x}, [a,b]) N_{[a,b]}^{\frac{p-1}{p}}(\mathbf{x}), \quad (3.19) \\
H(\mathbf{x}, [a,b]) &:= e^{\lambda_A r} \Big\{ \frac{C_A \|f(0)\| (e^{\lambda} - 1)}{\lambda} + \kappa_1(\mathbf{x}, [a,b]) \\
&\quad + \kappa_2(\mathbf{x}, [a,b]) \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big) e^{4L(b-a)} N_{[a,b]}^{\frac{2p-1}{p}}(\mathbf{x}) \Big\}, \quad (3.20)
\end{aligned}
$$

and

$$b(\mathbf{x}) := \sum_{k=1}^{\infty} e^{-\lambda k} H(\theta_{-k}\mathbf{x},[-1,1]) \prod_{j=1}^{k-1} \Big[1+G(\theta_{-j}\mathbf{x},[-1,1])\Big] \tag{3.21}$$

(which can be infinity), where $\lambda := \lambda_A - C_A C_f$, κ_1,κ_2 *are given by* (3.14), (3.15). *Assume further that*

$$\lambda > \hat{G} := \frac{1}{2} C_A e^{\lambda_A + 8L} \Big\{ \Big[4C_p C_g \Gamma(p)\Big]^p + \Big[4C_p C_g \Gamma(p)\Big]\Big\}. \tag{3.22}$$

Then $b(\mathbf{x})$ *is finite and tempered a.s., i.e.*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log b(\theta_t \mathbf{x}) = 0. \tag{3.23}$$

Proof: Assign $\Delta_k = [k,k+1]$ and $N_k(\mathbf{x}) := N_{\Delta_k}(\mathbf{x})$. Observe from (2.11) that

$$\begin{aligned} N_k^{\frac{p-1}{p}}(\mathbf{x}) &\leq \Big(1 + [4C_pC_g]^p \, \|\mathbf{x}\|^p_{p-\mathrm{var},\Delta_k}\Big)^{\frac{p-1}{p}} \leq 1 + [4C_pC_g]^{p-1} \, \|\mathbf{x}\|^{p-1}_{p-\mathrm{var},\Delta_k}, \\ N_k^{\frac{2p-1}{p}}(\mathbf{x}) &\leq \Big(1 + [4C_pC_g]^p \, \|\mathbf{x}\|^p_{p-\mathrm{var},\Delta_k}\Big)^{\frac{2p-1}{p}} \leq 2^{\frac{p-1}{p}} \Big(1 + [4C_pC_g]^{2p-1} \, \|\mathbf{x}\|^{2p-1}_{p-\mathrm{var},\Delta_k}\Big), \end{aligned}$$

In addition, it follows from (3.22) that $2C_g\Gamma(p) < 2C_pC_g\Gamma(p) < 1$. As a result, a direct computation shows that

$$G(\mathbf{x},[a,b]) \leq \frac{1}{2} C_A e^{\lambda_A + 4L(b-a)} \Big\{ 4C_pC_g \, \|\mathbf{x}\|_{p-\mathrm{var},[a,b]} + [4C_pC_g]^p \, \|\mathbf{x}\|^p_{p-\mathrm{var},[a,b]} \Big\} \Big\{ C_g \, \|\mathbf{x}\|_{p-\mathrm{var},[a,b]} \vee 1 \Big\}. \tag{3.24}$$

Due to the inequality $\log(1 + \frac{v}{4C_p}(u+u^p)(u\vee 1)) \leq v(u+u^p)$ for all $u \geq 0, v \geq \frac{1}{2}$, (3.24) yields

$$\log\Big(1 + G(\mathbf{x},[-1,1])\Big) \leq \frac{1}{2} C_A e^{\lambda_A + 8L} \Big\{ 4C_pC_g \, \|\mathbf{x}\|_{p-\mathrm{var},[-1,1]} + [4C_pC_g]^p \, \|\mathbf{x}\|^p_{p-\mathrm{var},[-1,1]} \Big\}.$$

It follows that for a.s. all x,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \log \prod_{k=0}^{n-1} \Big[1 + G(\theta_{-k}\mathbf{x},[-1,1])\Big] = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log\Big[1 + G(\theta_k\mathbf{x},[-1,1])\Big] \\ \leq\ & \frac{1}{2} C_A e^{\lambda_A + 8L} \Big\{ \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \|\theta_{-k}\mathbf{x}\|^p_{p-\mathrm{var},[-1,1]} + \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \|\theta_{-k}\mathbf{x}\|_{p-\mathrm{var},[-1,1]} \Big\} \\ \leq\ & \frac{1}{2} C_A e^{\lambda_A + 8L} \Big\{ \Big[4C_pC_g\Gamma(p)\Big]^p + \Big[4C_pC_g\Gamma(p)\Big] \Big\} = \hat{G}. \end{aligned}$$

Similarly, it is easy to show from (3.20) and (3.14), (3.15) that $E|\log H(\mathbf{x},[-1,1])| < \infty$, thus

$$\limsup_{n\to\infty} \frac{\log H(\theta_n\mathbf{x},[-1,1])}{n} = \limsup_{n\to\infty} \frac{\log H(\theta_{-n}\mathbf{x},[-1,1])}{n} = 0.$$

Hence, there exists for each $0 < 2\delta < \lambda - \hat{G}$ an $n_0 = n_0(\delta,x)$ such that for all $n \geq n_0$,

$$e^{(-\delta+\hat{G})n} \leq \prod_{k=0}^{n-1} \Big[1 + G(\theta_{-k}\mathbf{x},[-1,1])\Big], \quad \prod_{k=0}^{n-1} \Big[1 + G(\theta_k\mathbf{x},[-1,1])\Big] \leq e^{(\delta+\hat{G})n}$$

and

$$e^{-\delta n} \leq H(\theta_{-n}\mathbf{x},[-1,1]), \quad H(\theta_n\mathbf{x},[-1,1]) \leq e^{\delta n}.$$

Consequently,

$$
\begin{aligned}
b(\mathbf{x}) &\leq \sum_{k=1}^{n_0-1} e^{-\lambda k} H(\theta_{-k}\mathbf{x},[-1,1]) \prod_{j=1}^{k-1}\Big(1+M_1C_gG(\theta_{-j}\mathbf{x},[-1,1])\Big) + \sum_{k=n_0}^{\infty} e^{-(\lambda-2\delta-\hat{G})k}\\
&\leq \sum_{k=1}^{n_0-1} e^{-\lambda k} H(\theta_{-k}\mathbf{x},[-1,1]) \prod_{j=1}^{k-1}\Big(1+M_1C_gG(\theta_{-j}\mathbf{x},[-1,1])\Big) + \frac{e^{-(\lambda-2\delta-\hat{G})n_0}}{1-e^{-(\lambda-2\delta-\hat{G})}}
\end{aligned}
$$

which is finite. The proof on the temperedness of $b(\mathbf{x})$ is proved similarly to [9, Appendix].

□

We are now able to formulate the first main result of the paper.

Theorem 3.7 *Assume that A has all eigenvalues of negative real parts with λ_A satisfying* (3.9) *and* (3.10)*, and f is globally Lipschitz continuous such that $\lambda_A > C_f C_A$. Assume further that the driving path x satisfies* (3.2)*. Then under the condition*

$$
\lambda_A - C_A C_f > \frac{1}{2} C_A e^{\lambda_A + 8(\|A\|+C_f)} \Big\{ \Big[4C_pC_g\Gamma(p)\Big]^p + \Big[4C_pC_g\Gamma(p)\Big]\Big\}, \tag{3.25}
$$

where $\Gamma(p) = \Big(E\,\|Z\|^p_{p-\mathrm{var},[-1,1]}\Big)^{\frac{1}{p}}$, the random dynamical system φ possesses a pullback attractor $\mathcal{A}(\mathbf{x})$.

Proof: Using the rule of integration by parts for rough integrals, it is easy to prove that y_t satisfies

$$
y_t = \Phi(t)y_0 + \int_0^t \Phi(t-s)f(y_s)ds + \int_0^t \Phi(t-s)g(y_s)dx_s. \tag{3.26}
$$

Then by applying Proposition 3.5 and using the estimate in (2.16)

$$
\|y,R\|_{p-\mathrm{var},\Delta_k} \leq \|y_k\| e^{4L} N_k^{\frac{p-1}{p}}(\mathbf{x}) + \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big) e^{4L} N_k^{\frac{2p-1}{p}}(\mathbf{x}),
$$

where $N_k(x) := N_{\Delta_k}(x)$, we obtain

$$
\begin{aligned}
\|y_n\|e^{\lambda n} &\leq C_A\|y_0\| + (e^{\lambda n}-1)\frac{C_A\|f(0)\|}{\lambda}\\
&\quad + e^{\lambda_A}\sum_{k=0}^{n-1} e^{\lambda k}\Big\{\kappa_1(\mathbf{x},\Delta_k) + \kappa_2(\mathbf{x},\Delta_k)\Big[\|y_k\|e^{4L}N_k^{\frac{p-1}{p}}(\mathbf{x}) + \Big(\frac{\|f(0)\|}{L}+\frac{1}{C_p}\Big)e^{4L}N_k^{\frac{2p-1}{p}}(\mathbf{x})\Big]\Big\}\\
&\leq C_A\|y_0\| + \frac{C_A\|f(0)\|(e^{\lambda}-1)}{\lambda}\sum_{k=0}^{n-1} e^{\lambda k} + \sum_{k=0}^{n-1} e^{\lambda k}\kappa_2(\mathbf{x},\Delta_k)e^{4L}N_k^{\frac{p-1}{p}}(\mathbf{x})\|y_k\|\\
&\quad + e^{\lambda_A r}\sum_{k=0}^{n-1} e^{\lambda k}\Big\{\kappa_1(\mathbf{x},\Delta_k) + \kappa_2(\mathbf{x},\Delta_k)\Big(\frac{\|f(0)\|}{L}+\frac{1}{C_p}\Big)e^{4L}N_k^{\frac{2p-1}{p}}(\mathbf{x})\Big\}\\
&\leq C_A\|y_0\| + \sum_{k=0}^{n-1}\kappa_2(\mathbf{x},\Delta_k)e^{4L}N_k^{\frac{p-1}{p}}(\mathbf{x})e^{\lambda k}\|y_k\| \qquad (3.27)\\
&\quad + e^{\lambda_A r}\sum_{k=0}^{n-1} e^{\lambda k}\Big\{\frac{C_A\|f(0)\|(e^{\lambda}-1)}{\lambda} + \kappa_1(\mathbf{x},\Delta_k) + \kappa_2(\mathbf{x},\Delta_k)\Big(\frac{\|f(0)\|}{L}+\frac{1}{C_p}\Big)e^{4L}N_k^{\frac{2p-1}{p}}(\mathbf{x})\Big\}.
\end{aligned}
$$

By assigning $a := C_A\|y_0\|$, $u_k := \|y_k\|e^{\lambda k},\ k \geq 0$ and using (3.19), (3.20), we obtain

$$
u_n \leq a + \sum_{k=0}^{n-1} G(x,\Delta_k)u_k + \sum_{k=0}^{n-1} e^{\lambda k}H(x,\Delta_k). \tag{3.28}
$$

We are now in the position to apply Lemma 4.2, so that

$$\|y_n(\mathbf{x},y_0)\| \leq C_A \|y_0\| e^{-\lambda n} \prod_{k=0}^{n-1} \Big[1+G(\theta_k x,[0,1])\Big] + \sum_{k=0}^{n-1} e^{-\lambda(n-k)} H(\theta_k x,[0,1]) \prod_{j=k+1}^{n-1} \Big[1+G(\theta_j x,[0,1])\Big]. \tag{3.29}$$

Now using (2.16), it follows that for any $t \in [n,n+1]$

$$\begin{aligned}
\|y_t(\mathbf{x},y_0)\| &\leq \Big[\|y_n(\mathbf{x},y_0)\| + \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big) N_n(\mathbf{x})\Big] e^{4L} \\
&\leq C_A e^{4L} \|y_0\| e^{-\lambda n} \prod_{k=0}^{n-1} \Big[1+G(\theta_k \mathbf{x},[0,1])\Big] \qquad (3.30)\\
&\quad + \sum_{k=0}^{n-1} e^{-\lambda(n-k)} e^{4L} H(\theta_k \mathbf{x},[0,1]) \prod_{j=k+1}^{n-1} \Big[1+G(\theta_j \mathbf{x},[0,1])\Big] + e^{4L}\Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big) N_n(\mathbf{x}).
\end{aligned}$$

Consequently, by assigning x with $\theta_{-t}x$ in (3.30), we obtain

$$\begin{aligned}
&\|y_t(\theta_{-t}\mathbf{x}, y_0(\theta_{-t}x)) \\
\leq\ & C_A e^{4L} \|y_0\| e^{-\lambda n} \prod_{k=0}^{n-1} \Big[1+G(\theta_{k-t} \mathbf{x},[0,1])\Big] \\
&+ \sum_{k=0}^{n-1} e^{-\lambda(n-k)} e^{4L} H(\theta_{k-t} \mathbf{x},[0,1]) \prod_{j=k+1}^{n-1} \Big[1+G(\theta_{j-t} \mathbf{x},[0,1])\Big] + e^{4L}\Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big) N_n(\theta_{-t}\mathbf{x}) \\
\leq\ & C_A e^{4L} \|y_0\| e^{-\lambda n} \prod_{k=0}^{n-1} \Big[1+G(\theta_{k-n} \mathbf{x},[-1,1])\Big] + e^{4L}\Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big) N_{[-1,1]}(\mathbf{x}) \\
&+ \sum_{k=0}^{n-1} e^{-\lambda(n-k)} e^{4L} H(\theta_{k-n} \mathbf{x},[-1,1]) \prod_{j=k+1}^{n-1} \Big[1+G(\theta_{j-n} \mathbf{x},[-1,1])\Big] \qquad (3.31)
\end{aligned}$$

We are now in the position to apply Proposition 3.6 into (3.31) so that for $t \in \Delta_n$ with $0 < \delta < \frac{1}{2}(\lambda - \hat{G})$ and n large enough

$$\begin{aligned}
\|y_t(\theta_{-t}\mathbf{x},y_0)\| &\leq C_A e^{4L} \|y_0(\theta_{-t}x)\| \exp\Big\{-\Big(\lambda - \hat{G} - \delta\Big) n\Big\} + e^{4L}\Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big) N_{[-1,1]}(\mathbf{x}) \\
&\quad + \sum_{k=1}^{n} e^{-\lambda k} e^{4L} H(\theta_{-k}\mathbf{x},[-1,1]) \prod_{j=k+1}^{n-1} \Big[1+G(\theta_{-j}\mathbf{x},[-1,1])\Big] \\
&\leq C_A e^{4L} \|y_0(\theta_{-t}\mathbf{x})\| \exp\Big\{-\Big(\lambda - \hat{G} - \delta\Big) n\Big\} + b(\mathbf{x}) + \Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big) N_{[-1,1]}(\mathbf{x}),
\end{aligned} \tag{3.32}$$

where $b(\mathbf{x})$ is given by (3.21). This implies that, starting from any point $y_0(\theta_{-t}\mathbf{x}) \in D(\theta_{-t}\mathbf{x})$ which is tempered due to (3.5), there exists n large enough such that for $t \in [n,n+1]$

$$\|y_t(\theta_{-t}\mathbf{x},y_0)\| \leq 1 + b(\mathbf{x}) + e^{4L}\Big(\frac{\|f(0)\|}{L} + \frac{1}{C_p}\Big) N_{[-1,1]}(\mathbf{x}) =: \hat{b}(\mathbf{x}). \tag{3.33}$$

Moreover, the temperedness of $\hat{b}(\mathbf{x})$ follows directly from the temperedness (3.23) of $b(\mathbf{x})$ and of $\|\mathbf{x}\|_{p-\mathrm{var},[-1,1]}$. Therefore, there exists a compact absorbing set $\mathcal{B}(\mathbf{x}) = \bar{B}(0,\hat{b}(\mathbf{x}))$ and thus a pullback attractor $\mathcal{A}(\mathbf{x})$ for system (1.1) which is given by (3.8). □

Remark 3.8 (i), Assume that $f(0) = g(0) = 0$ so that $y \equiv 0$ is a solution of (1.1). Then (3.25) in Theorem 3.7 is the exponential stability criterion for the trivial attractor $\mathcal{A}(\mathbf{x}) \equiv 0$.

(ii) It is important to note that the term $e^{\lambda_A + 8(\|A\|+C_f)}$ in (3.25) is the unavoidable effect from the discretization scheme.

(iii), A similar proof of Theorem 3.7 using step size r with $\Delta_k = [kr, (k+1)r]$ then leads to a criterion for the existence of a global random pullback attractor

$$\lambda_A - C_A C_f > \frac{1}{2r} C_A e^{\left[\lambda_A + 8(\|A\|+C_f)\right]r} \Big\{ \Big[4C_p C_g \Gamma(p,r)\Big]^p + \Big[4C_p C_g \Gamma(p,r)\Big] \Big\} \tag{3.34}$$

where $\Gamma(p,r) = \Big(E \, \|Z\|^p_{p-\mathrm{var},[-r,r]}\Big)^{\frac{1}{p}}$ for almost sure all realizations x. As a result, the final criterion can be optimized to

$$\lambda_A - C_A C_f > \inf_{r>0} \frac{1}{2r} C_A e^{\left[\lambda_A + 8(\|A\|+C_f)\right]r} \Big\{ \Big[4C_p C_g \Gamma(p,r)\Big]^p + \Big[4C_p C_g \Gamma(p,r)\Big] \Big\}.$$

In the following, we are going to prove that the diameter of the random attractor can be controlled by parameter C_g. We first introduce a quantity.

Proposition 3.9 *Assume that x satisfies* (3.2)*. Then under criterion* (3.25)*, the following quantity is well defined and finite*

$$\begin{aligned} \xi(\mathbf{x}) \;\; &:= \;\; e^{4L} \sum_{k=1}^{\infty} \Big(\|\!|\theta_{-k}\mathbf{x}|\!\|_{p-\mathrm{var},[0,1]} + \|\!|\theta_{-k}\mathbf{x}|\!\|^2_{p-\mathrm{var},[0,1]} \Big) \times \\ & \quad \times e^{-\lambda(n-k)} \Big[\max\Big\{ \frac{\|f(0)\|}{L}, \frac{1}{C_p} \Big\} N_{[0,1]}(\theta_{-k}\mathbf{x}) + \hat{b}(\theta_{-k}\mathbf{x}) \Big] N_{[0,1]}(\theta_{-k}\mathbf{x})^{\frac{p-1}{p}}. \end{aligned} \tag{3.35}$$

Proof: Observe that the existence of $\Gamma(\mathbf{x}, p)$ implies the temperedness of $N_{[0,1]}(\mathbf{x}), N_{[0,1]}(\mathbf{x})^{\frac{p-1}{p}}$ and $\|\!|\mathbf{x}|\!\|_{p-\mathrm{var},[0,1]} + \|\!|\mathbf{x}|\!\|^2_{p-\mathrm{var},[0,1]}$. The convergence of the series in (3.35) can then be proved similarly to the convergence of $\hat{b}(x)$ in Proposition 3.6. □

Theorem 3.10 *Under the assumptions of Theorem 3.7, the diameter of $\mathcal{A}$ is estimated as*

$$\mathrm{diam}(\mathcal{A}(\mathbf{x})) \leq 2 C_A e^{\lambda_A} (1 + \|A\|) K C_g \xi(\mathbf{x}) \tag{3.36}$$

where $\xi(\mathbf{x})$ is given in (3.35)*.*

Proof: The existence of the pullback attractor $\mathcal{A}$ is followed by Theorem 3.7. Take any two points $a_1, a_2 \in \mathcal{A}(x)$. For a given $n \in \mathbb{N}$, assign $x^* := \theta_{-n}x$ and consider the equation

$$dy_t = [Ay_t + f(y_t)]dt + g(y_t)dx^*_t. \tag{3.37}$$

Due to the invariance of $\mathcal{A}$ under the flow, there exist $b_1, b_2 \in \mathcal{A}(x^*)$ such that $a_i = y_n(x^*, b_i)$. Put $z_t = z_t(x^*) := y_t(x^*, b_1) - y_t(x^*, b_2)$ then $z_n(x^*) = a_1 - a_2$ and we have

$$dz_t = [Az_t + P(t, z_t)]dt + Q(t, z_t)dx^*_t \tag{3.38}$$

where we write in short $y^1_t = y_t(x^*, b_1)$ and

$$\begin{aligned} P(t, z_t) \;\; &= \;\; f(y(t, x^*, b_2)) - f(y(t, x^*, b_1)) = f(y^1_t + z_t) - f(y^1_t), \\ Q(t, z_t) \;\; &= \;\; g(y(t, x^*, b_2)) - g(y(t, x^*, b_1)) = g(y^1_t + z_t) - g(y^1_t). \end{aligned}$$

Observe that

$$\|P(t,z)-P(t,z')\|\leq C_f\|z-z'\|, \|Q(t,z)-Q(t,z')\|\leq C_g\|z-z'\|$$

and $P(t,0)=Q(t,0)\equiv 0$. Consequently,

$$\|P(t,z_t)\|\leq C_f\|z_t\|,\quad \|Q(t,z_t)\|\leq C_g\|z_t\|.$$

Using the rule of integration by parts for rough integrals, it is easy to see that z_t satisfies the equation

$$z_t = z_a + \int_a^t \Phi(t-s)P(s,z_s)ds + \int_a^t \Phi(t-s)Q(s,z_s)dx^*_s.$$

On the other hand, a direct computation shows that

$$\begin{aligned}
[\Phi(c-\cdot)Q(\cdot,z.)]'_s &= \Phi(c-s)[g(y^1)]'_s - \Phi(c-s)[g(y^s)]'_s\\
&= \Phi(c-s)Dg(y^1_s)g(y^1_s) - \Phi(c-s)Dg(y^2_s)g(y^2_s)\\
\left\|R^{\Phi(c-\cdot)Q(\cdot,z.)}_{s,t}\right\| &\leq \|\Phi(c-s)R^Q_{s,t}\| + \|\Phi(c-t)-\Phi(c-s)\|\|Q(t,z_t)\|
\end{aligned}\tag{3.39}$$

which yields

$$\begin{aligned}
\left\|\!\left|[\Phi(c-\cdot)Q(\cdot,z.)]'\right|\!\right\|_{p-\mathrm{var},[a,b]} &\leq 2C_g^2\|\Phi(c-\cdot)\|_{\infty,[a,b]}\Big(\left\|\!\left|y^1\right|\!\right\|_{p-\mathrm{var},[a,b]} + \left\|\!\left|y^2\right|\!\right\|_{p-\mathrm{var},[a,b]}\Big)\\
&\quad +2C_g^2\left\|\!\left|\Phi(c-\cdot)\right|\!\right\|_{p-\mathrm{var},[a,b]}\\
\left\|\!\left|R^{\Phi(c-\cdot)Q(\cdot,z.)}\right|\!\right\|_{q-\mathrm{var},[a,b]} &\leq \|\Phi(c-b)\|\Big(\left\|\!\left|R^{g(y^1)}\right|\!\right\|_{q-\mathrm{var},[a,b]} + \left\|\!\left|R^{g(y^2)}\right|\!\right\|_{q-\mathrm{var},[a,b]}\Big)\\
&\quad +2C_g\left\|\!\left|\Phi(c-\cdot)\right|\!\right\|_{q-\mathrm{var},[a,b]}
\end{aligned}\tag{3.40}$$

As a result, we use similar estimate to Proposition 3.4 to obtain

$$\begin{aligned}
&\left\|\int_a^b \Phi(c-s)Q(s,z_s)dx_s\right\|\\
\leq\ & \|\Phi(c-a)Q(a,z_a)\|\,\left\|\!\left|x\right|\!\right\|_{p-\mathrm{var},[a,b]} + \|[\Phi(c-\cdot)Q(\cdot,z.)]'_a\|\,\left\|\!\left|\mathbb{X}\right|\!\right\|_{q-\mathrm{var},[a,b]}\\
&+C_p\Big\{\left\|\!\left|x\right|\!\right\|_{p-\mathrm{var},[a,b]}\left\|\!\left|R^{\Phi(c-\cdot)Q}\right|\!\right\|_{q-\mathrm{var},[a,b]} + \left\|\!\left|\mathbb{X}\right|\!\right\|_{q-\mathrm{var},[a,b]}\left\|\!\left|[\Phi(c-\cdot)Q]'\right|\!\right\|_{p-\mathrm{var},[a,b]}\Big\}\\
\leq\ & C_Ae^{-\lambda_A(c-a)}\Big(C_g\left\|\!\left|x\right|\!\right\|_{p-\mathrm{var},[a,b]} + 2C_g^2\left\|\!\left|\mathbb{X}\right|\!\right\|_{q-\mathrm{var},[a,b]}\Big)\|z_a\|\\
&+C_AC_p\|A\|(b-a)e^{-\lambda_A(c-b)}\Big(C_g\left\|\!\left|x\right|\!\right\|_{p-\mathrm{var},[a,b]} + C_g^2\left\|\!\left|\mathbb{X}\right|\!\right\|_{q-\mathrm{var},[a,b]}\Big)\\
&+C_pC_Ae^{-\lambda_A(c-b)}\Big\{\Big[2C_g^2\left\|\!\left|\mathbb{X}\right|\!\right\|_{q-\mathrm{var},[a,b]} + \frac{1}{2}C_g^2\left\|\!\left|x\right|\!\right\|^2_{p-\mathrm{var},[a,b]}\Big]\vee C_g\left\|\!\left|x\right|\!\right\|_{p-\mathrm{var},[a,b]}\Big\}\times\\
&\times\Big(\left\|\!\left|y^1,R^{y^1}\right|\!\right\|_{p-\mathrm{var},[a,b]} + \left\|\!\left|y^2,R^{y^2}\right|\!\right\|_{p-\mathrm{var},[a,b]}\Big).
\end{aligned}\tag{3.41}$$

Now, repeating the estimate in the proof of Theorem 3.7 with $\beta^*_t = \|\int_0^t\Phi(t-s)Q(s,z_s)dx^*_s\|$ we obtain

$$e^{\lambda_A t}\|z_t\| \leq C_A\|z_0\| + e^{\lambda_A t}\beta^*_t + L_f\int_0^t \Big(C_A\|z_0\| + e^{\lambda_A s}\beta^*_s\Big)e^{L_f(t-s)}ds$$

and then

$$e^{\lambda t}\|z_t\| \leq C_A\|z_0\| + e^{\lambda t}\beta^*_t + L_f\int_0^t e^{\lambda s}\beta^*_s ds \tag{3.42}$$

Similarly to (3.17) we have

$$\begin{aligned}
\beta^*_t e^{\lambda t} &= e^{\lambda t}\Big\| \int_0^t \Phi(t-s)Q(s,z_s)dx^*_s\Big\| \\
&\leq e^{\lambda t} \sum_{k=0}^{\lfloor t \rfloor} C_p C_A(1+\|A\|)\Big(C_g \left|\!\left|\!\left| \mathbf{x}^* \right|\!\right|\!\right|_{p-\mathrm{var},\Delta_k} + C_g^2 \left|\!\left|\!\left| \mathbf{x}^* \right|\!\right|\!\right|^2_{p-\mathrm{var},\Delta_k}\Big) e^{-\lambda_A(t-k-1)} \times \\
&\quad \times \Big(\|y^1_k\| + \|y^2_k\| + \Big|\!\Big|\!\Big| y^1, R^{y^1} \Big|\!\Big|\!\Big|_{p-\mathrm{var},\Delta_\mathrm{k}} + \Big|\!\Big|\!\Big| y^2, R^{y^2} \Big|\!\Big|\!\Big|_{p-\mathrm{var},\Delta_\mathrm{k}} \Big). \qquad (3.43)
\end{aligned}$$

Therefore the similar estimate to (3.18) in Lemma 3.5 shows that

$$\begin{aligned}
e^{\lambda t}\|z_t\| &\leq C_A\|z_0\| + C_pC_A(1+\|A\|)\sum_{k=0}^{\lfloor t \rfloor} \Big(C_g \left|\!\left|\!\left| \mathbf{x}^* \right|\!\right|\!\right|_{p-\mathrm{var},\Delta_k} + C_g^2 \left|\!\left|\!\left| \mathbf{x}^* \right|\!\right|\!\right|^2_{p-\mathrm{var},\Delta_k}\Big) \times \\
&\quad \times e^{\lambda k}\Big(\|y^1_k\| + \|y^2_k\| + \Big|\!\Big|\!\Big| y^1, R^{y^1} \Big|\!\Big|\!\Big|_{p-\mathrm{var},\Delta_\mathrm{k}} + \Big|\!\Big|\!\Big| y^2, R^{y^2} \Big|\!\Big|\!\Big|_{p-\mathrm{var},\Delta_\mathrm{k}} \Big). \qquad (3.44)
\end{aligned}$$

Since $b_i \in \mathcal{A}(\mathbf{x}^*)$ for $i = 1,2$, it follows from the invariance of $\mathcal{A}$ that $y^i(k,\mathbf{x}^*,b_i) \in \mathcal{A}(\theta_k\mathbf{x}^*)$. Moreover, it follows from (3.8) and (3.33) that

$$\sup_{y\in\mathcal{A}(\mathbf{x})} \|y\| \leq \hat{b}(\mathbf{x}). \qquad (3.45)$$

Indeed, taking $y^* \in \mathcal{A}(\mathbf{x})$ be arbitrary, it follows from (3.8) that there exists a sequence $t_k \to \infty$ such that

$$y^* = \lim_k \varphi(t_k, \theta_{-t_k}\mathbf{x}, y_0(\theta_{-t_k}\mathbf{x}))$$

where $y_0(\theta_{-t_k}\mathbf{x}) \in \mathcal{B}(\theta_{-t_k}\mathbf{x})$. Since $\hat{b}(\mathbf{x})$ is tempered, by choosing t_k large enough so that (3.33) holds, we conclude that (3.45) holds. As a consequence, (3.45) yields $\|y^1(k,\mathbf{x}^*,b_1)\| \leq \hat{b}(\theta_k\mathbf{x}^*)$. Similarly, $\|z_0\| \leq \|b_1\| + \|b_2\| < 2\hat{b}(\mathbf{x}^*)$. On the other hand, due to (2.17) and (3.44) yields

$$\begin{aligned}
\|z_n\| &\leq 2C_A\hat{b}(\mathbf{x}^*)e^{-\lambda n} + 2C_pC_A(1+\|A\|)(C_g\vee C_g^2)\sum_{k=0}^{n-1}\Big(\left|\!\left|\!\left| \mathbf{x}^* \right|\!\right|\!\right|_{p-\mathrm{var},\Delta_k} + \left|\!\left|\!\left| \mathbf{x}^* \right|\!\right|\!\right|^2_{p-\mathrm{var},\Delta_k}\Big)e^{-\lambda(n-k)} \times \\
&\quad \times \Big[\max\Big\{\frac{\|f(0)\|}{L}, \frac{1}{C_p}\Big\} N_{\Delta_k}(\mathbf{x}^*) + \hat{b}(\theta_k\mathbf{x}^*)\Big] e^{4L} N_{\Delta_k}(\mathbf{x}^*)^{\frac{p-1}{p}} \\
&\leq 2C_A\hat{b}(\theta_{-n}\mathbf{x})e^{-\lambda n} + 2C_pC_A(1+\|A\|)(C_g\vee C_g^2)\sum_{k=1}^{n}\Big(\left|\!\left|\!\left| \theta_{-k}\mathbf{x} \right|\!\right|\!\right|_{p-\mathrm{var},[0,1]} + \left|\!\left|\!\left| \theta_{-k}\mathbf{x} \right|\!\right|\!\right|^2_{p-\mathrm{var},[0,1]}\Big) \times \\
&\quad \times e^{-\lambda(n-k)}\Big[\max\Big\{\frac{\|f(0)\|}{L}, \frac{1}{C_p}\Big\} N_{[0,1]}(\theta_{-k}\mathbf{x}) + \hat{b}(\theta_{-k}\mathbf{x})\Big] e^{4L} N_{[0,1]}(\theta_{-k}\mathbf{x})^{\frac{p-1}{p}}. \qquad (3.46)
\end{aligned}$$

Letting n tend to infinity, the first term in the last line of (3.46) tends to zero due to the temperedness of $\hat{b}(\mathbf{x})$. Hence it follows from (3.35) in Proposition 3.9 that

$$\|a_1 - a_2\| \leq 2C_pC_A(1+\|A\|)(C_g\vee C_g^2)\xi(\mathbf{x})$$

which proves (3.36).

□

In the rest of the paper, we are going to prove the result on one-point attractor in case g is of linear form, as proved in [9] for Young equations.

Theorem 3.11 *Assume that $g(y) = Cy$ is a linear map. Then under the condition*

$$\lambda_A - C_A C_f > 2C_A[1+\|A\|]e^{\lambda_A + 4(\|A\|+C_f)}\Big\{\Big[4C_p\|C\|\Gamma(p)\Big] + \Big[4C_p\|C\|\Gamma(p)\Big]^p\Big\}, \qquad (3.47)$$

the attractor is a random point, i.e. $\mathcal{A}(\mathbf{x}) = \{a(\mathbf{x})\}$.

Proof: With the setting in the proof of Theorem 3.10 for $g(y) = Cy$, observe that $Q(t,z_t) = Cz_t$ and

$$\begin{aligned} \left\vert\!\left\vert\!\left\vert [\Phi(c-\cdot)Cz.]' \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} &\leq \|\Phi(c-\cdot)\|_{\infty,[a,b]}\|C\|^2 \left\vert\!\left\vert\!\left\vert z\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} + \left\vert\!\left\vert\!\left\vert \Phi(c-\cdot)\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} \|C\|^2 \left\vert\!\left\vert\!\left\vert z\right\vert\!\right\vert\!\right\vert_{\infty,[a,b]} \\ \left\vert\!\left\vert\!\left\vert R^{\Phi(c-\cdot)Cz.} \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} &\leq \|\Phi(c-\cdot)\|_{\infty,[a,b]}\|C\| \left\vert\!\left\vert\!\left\vert R^z\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} + \left\vert\!\left\vert\!\left\vert \Phi(c-\cdot)\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} \|C\|\|z\|_{\infty,[a,b]}. \end{aligned}$$

As a result, the estimate in (3.41) is of the form

$$\begin{aligned} &\Big\| \int_a^b \Phi(c-s)Q(s,z_s)dx^*_s \Big\| \\ \leq\ & \|\Phi(c-a)Cz_a\| \left\vert\!\left\vert\!\left\vert x^*\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} + \|\Phi(c-a)C^2 z_a\| \left\vert\!\left\vert\!\left\vert \mathbb{X}^*\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} \\ & + C_p\Big\{ \left\vert\!\left\vert\!\left\vert x^*\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} \left\vert\!\left\vert\!\left\vert R^{\Phi(c-\cdot)C}\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} + \left\vert\!\left\vert\!\left\vert \mathbb{X}^*\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} \left\vert\!\left\vert\!\left\vert [\Phi(c-\cdot)C]'\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} \Big\} \\ \leq\ & C_p C_A[1+\|A\|(b-a)]e^{-\lambda_A(c-b)}\Big(\|C\| \left\vert\!\left\vert\!\left\vert \mathbf{x}^*\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} + \|C\|^2 \left\vert\!\left\vert\!\left\vert \mathbf{x}^*\right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},[a,b]} \Big) \times \\ & \times \Big(\|z_a\| + \left\vert\!\left\vert\!\left\vert z, R^z\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} \Big). \end{aligned} \qquad (3.48)$$

Meanwhile, similar estimates to (2.25) in Theorem 2.4, with $P(t,0) = 0$, show that

$$\begin{aligned} \left\vert\!\left\vert\!\left\vert z, R^z\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} + \|z_a\| &\leq \|z_a\| e^{4L(b-a) + \alpha N_{[a,b]}(\mathbf{x}^*)} N^{\frac{p-1}{p}}_{[a,b]}(\mathbf{x}^*), \quad \text{with} \\ N_{[a,b]}(\mathbf{x}^*) &\leq 1 + [4C_p\|C\|]^p \left\vert\!\left\vert\!\left\vert \mathbf{x}^*\right\vert\!\right\vert\!\right\vert^p_{p-\mathrm{var},[a,b]}. \end{aligned}$$

As a result, (3.44) has the form

$$\begin{aligned} e^{\lambda n}\|z_n\| &\leq C_A\|z_0\| + C_pC_A[1+\|A\|]e^{\lambda_A} \sum_{k=0}^{n-1} \Big(\|C\| \left\vert\!\left\vert\!\left\vert \mathbf{x}^*\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},\Delta_k} + \|C\|^2 \left\vert\!\left\vert\!\left\vert \mathbf{x}^*\right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},\Delta_k} \Big) \times \\ & \quad \times e^{4L+\alpha N_{\Delta_k}(\mathbf{x}^*)} N^{\frac{p-1}{p}}_{\Delta_k}(\mathbf{x}^*) e^{\lambda k}\|z_k\| \\ &\leq C_A\|z_0\| + \sum_{k=0}^{n-1} I_{\Delta_k}(\mathbf{x}^*) e^{\lambda k}\|z_k\|, \end{aligned}$$

where

$$I_{[a,b]}(\mathbf{x}) = C_pC_A[1+\|A\|]e^{\lambda_A}\Big(\|C\| \left\vert\!\left\vert\!\left\vert \mathbf{x}\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} + \|C\|^2 \left\vert\!\left\vert\!\left\vert \mathbf{x}\right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},[a,b]} \Big) e^{4L+\alpha N_{[a,b]}(\mathbf{x})} N^{\frac{p-1}{p}}_{[a,b]}(\mathbf{x})$$

is the function of $\mathbf{x}$. Now applying the discrete Gronwall lemma, we obtain

$$e^{\lambda n}\|z_n\| \leq C_A\|z_0\| \prod_{k=0}^{n-1} \Big[1 + I_{[0,1]}(\theta_{k-n}\mathbf{x})\Big]$$

Hence, it follows from Birkhorff's ergodic theorem that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|z_n\| \leq -\lambda + \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \log\Big[1 + I_{[0,a]}(\theta_{-k}\mathbf{x})\Big] \leq -\lambda + E\log\Big[1 + I_{[0,1]}(\mathbf{x})\Big].$$

Given C_p and α, it follows from the estimate of $N_{[0,1]}(\mathbf{x})$ and the inequalities

$$\begin{aligned}\log(1+ue^v) &\leq v+\log(1+u), \quad \forall u,v\geq 0,\\ \log\Big[1+\frac{(2C_p+3)v}{4C_p}(1+u^{p-1})(u+u^2)\Big] &\leq (2-\alpha)v(u+u^p), \quad \forall u\geq 0, v\geq 1,\end{aligned}$$

that

$$\begin{aligned}&\log\Big[1+I_{[0,1]}(\mathbf{x})\Big]\\ \leq\ &\log\Big\{1+C_pC_A[1+\|A\|]e^{\lambda_A}e^{4L+\alpha}N^{\frac{p-1}{p}}_{[a,b]}(\mathbf{x})\Big(\|C\|\,\|\mathbf{x}\|_{p-\mathrm{var},[a,b]}+\|C\|^2\,\|\mathbf{x}\|^2_{p-\mathrm{var},[a,b]}\Big)\Big\}\\ &+\alpha\Big(4C_p\|C\|\Big)^p\,\|\mathbf{x}\|^p_{p-\mathrm{var},[0,1]}\\ \leq\ &\log\Big\{1+\frac{2C_p+3}{4C_p}C_A[1+\|A\|]e^{\lambda_A+4L}\Big[1+(4C_p\|C\|)^{p-1}\,\|\mathbf{x}\|^{p-1}_{p-\mathrm{var},[0,1]}\Big]\times\\ &\times\Big(4C_p\|C\|\,\|\mathbf{x}\|_{p-\mathrm{var},[a,b]}+\Big[4C_p\|C\|\Big]^2\|\mathbf{x}\|^2_{p-\mathrm{var},[a,b]}\Big)\Big\}+\alpha\Big(4C_p\|C\|\Big)^p\,\|\mathbf{x}\|^p_{p-\mathrm{var},[0,1]}\\ \leq\ &2C_A[1+\|A\|]e^{\lambda_A+4L}\Big\{\Big[4C_p\|C\|\,\|\mathbf{x}\|_{p-\mathrm{var},[0,1]}\Big]+\Big[4C_p\|C\|\,\|\mathbf{x}\|_{p-\mathrm{var},[0,1]}\Big]^p\Big\}\end{aligned}$$

Therefore we finally obtain

$$\limsup_{n\to\infty}\frac{1}{n}\log\|z_n\| \quad\leq\quad -\lambda+2C_A[1+\|A\|]e^{\lambda_A+4L}\Big\{\Big[4C_p\|C\|\Gamma(p)\Big]+\Big[4C_p\|C\|\Gamma(p)\Big]^p\Big\}<0$$

under the condition (3.47). This follows that $\lim_{n\to\infty}\|a_1-a_2\|=0$ or $\mathcal{A}$ is an one point set. □

4 Appendix

Lemma 4.1 (Continuous Gronwall Lemma) *Assume that $u_t,\alpha_t,\beta>0$ such that*

$$u_t\leq\alpha_t+\int_a^t\beta u_s ds, \forall t\geq a.$$

Then

$$u_t\leq\alpha_t+\int_a^t\beta e^{\beta(t-s)}\alpha_s ds, \forall t\geq a.$$

Proof: See [**?**, Lemma 6.1, p 89]. □

Lemma 4.2 (Discrete Gronwall Lemma) *Let a be a non negative constant and u_n,α_n,β_n be nonnegative sequences satisfying*

$$u_n\leq a+\sum_{k=0}^{n-1}\alpha_k u_k+\sum_{k=0}^{n-1}\beta_k,\ \ \forall n\geq 1$$

then

$$u_n\leq\max\{a,u_0\}\prod_{k=0}^{n-1}(1+\alpha_k)+\sum_{k=0}^{n-1}\beta_k\prod_{j=k+1}^{n-1}(1+\alpha_j) \tag{4.1}$$

for all $n\geq 1$.

Proof: See [9, Appendix 4.2] □

Proof: [**Proposition 2.1**] The proof is divided into several steps.

Step 1: First we would like to estimate the solution norms of (2.4). To do that, observe that

$$\Big\| \int_s^t g(y_u)dx_u - g(y_s)x_{s,t} - [g(y)]'_s\mathbb{X}_{s,t}\Big\| \leq C_\beta (t-s)^{3\beta}\Big[\left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{\beta,[s,t]} \left\vert\!\left\vert\!\left\vert R^{g(y)} \right\vert\!\right\vert\!\right\vert_{2\beta,[s,t]} + \left\vert\!\left\vert\!\left\vert \mathbb{X} \right\vert\!\right\vert\!\right\vert_{2\beta,[s,t]} \left\vert\!\left\vert\!\left\vert [g(y)]' \right\vert\!\right\vert\!\right\vert_{\beta,[s,t]} \Big]$$

due to (2.7). It then follows that y is controlled by x with $y' = g(y)$. Since

$$\begin{aligned} g(y_t) - g(y_s) &= \int_0^1 Dg(y_s + \eta y_{s,t})y_{s,t}d\eta \\ &= D_g(y_s)y'_s x_{s,t} + \int_0^1 Dg(y_s + \eta y_{s,t})R^y_{s,t}d\eta + \int_0^1 [Dg(y_s + \eta y_{s,t}) - Dg(y_s)]y'_{s,t}x_{s,t}d\eta, \end{aligned}$$

it easy to show that $[g(y)]'_s = Dg(y_s)g(y_s)$, where we use (1.2) to estimate

$$\begin{aligned} \|R^{g(y)}_{s,t}\| &\leq \int_0^1 \|Dg(y_s + \eta y_{s,t})\|\|R^y_{s,t}\|d\eta + \int_0^1 \|Dg(y_s + \eta y_{s,t}) - Dg(y_s)\|\|g(y_s)\|\|x_{s,t}\|d\eta \\ &\leq C_g\|R^y_{s,t}\| + \frac{1}{2}C_g^2\|y_{s,t}\|\|x_{s,t}\|. \end{aligned}$$

Hence, it follows, using p-variation norms and Hölder inequality, that

$$\begin{aligned} \left\vert\!\left\vert\!\left\vert [g(y)]' \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} &\leq 2C_g^2 \left\vert\!\left\vert\!\left\vert y \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]}, \quad \|[g(y)]'\|_{\infty,[s,t]} \leq C_g^2, \\ \left\vert\!\left\vert\!\left\vert R^{g(y)} \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[s,t]} &\leq C_g \left\vert\!\left\vert\!\left\vert R^y \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[s,t]} + \frac{1}{2}C_g^2 \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \left\vert\!\left\vert\!\left\vert y \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]}. \end{aligned} \tag{4.2}$$

As a result, by introducing $\left\vert\!\left\vert\!\left\vert y,R \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} := \left\vert\!\left\vert\!\left\vert y \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} + \left\vert\!\left\vert\!\left\vert R^y \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[s,t]}$, we obtain

$$\begin{aligned} \|y_{s,t}\| &\leq \Big\| \int_s^t g(y_u)dx_u \Big\| \\ &\leq \|g(y_s)\|\|x_{s,t}\| + \|Dg(y_s)g(y_s)\|\|\mathbb{X}_{s,t}\| \\ &\quad + C_p\Big[\left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \left\vert\!\left\vert\!\left\vert R^{g(y)} \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[s,t]} + \left\vert\!\left\vert\!\left\vert \mathbb{X} \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[s,t]} \left\vert\!\left\vert\!\left\vert [g(y)]' \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \Big] \\ &\leq C_g \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} + C_g^2 \left\vert\!\left\vert\!\left\vert \mathbb{X} \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[s,t]} \\ &\quad + C_p\Big\{ C_g \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \left\vert\!\left\vert\!\left\vert R^y \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[s,t]} + \Big[\frac{1}{2} \left\vert\!\left\vert\!\left\vert x \right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},[s,t]} + 2\left\vert\!\left\vert\!\left\vert \mathbb{X} \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[s,t]}\Big] C_g^2 \left\vert\!\left\vert\!\left\vert y \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \Big\} \\ &\leq C_g \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} + C_g^2 \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},[s,t]} + 2C_p\Big\{ C_g \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \vee C_g^2 \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},[s,t]} \Big\} \left\vert\!\left\vert\!\left\vert y,R \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]}, \\ &\leq 2\Big\{ C_g \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \vee C_g^2 \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},[s,t]} \Big\} \Big(1 + C_p \left\vert\!\left\vert\!\left\vert y,R \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \Big), \end{aligned}$$

which yields

$$\left\vert\!\left\vert\!\left\vert y \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \leq 2\Big\{ C_g \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \vee C_g^2 \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},[s,t]} \Big\} \Big(1 + C_p \left\vert\!\left\vert\!\left\vert y,R \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \Big). \tag{4.3}$$

The same estimate for R^y is actually included in the above estimate, hence

$$\left\vert\!\left\vert\!\left\vert R^y \right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[s,t]} \leq 2\Big\{ C_g \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \vee C_g^2 \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},[s,t]} \Big\} \Big(1 + C_p \left\vert\!\left\vert\!\left\vert y,R \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \Big). \tag{4.4}$$

Combining (4.3) and (4.4) gives

$$\left\vert\!\left\vert\!\left\vert y,R \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \leq 4\Big\{ C_g \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \vee C_g^2 \left\vert\!\left\vert\!\left\vert \mathbf{x} \right\vert\!\right\vert\!\right\vert^2_{p-\mathrm{var},[s,t]} \Big\} \Big(1 + C_p \left\vert\!\left\vert\!\left\vert y,R \right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[s,t]} \Big). \tag{4.5}$$

It implies from (4.5) that $\|y,R\|_{p-\mathrm{var},[s,t]} \leq \frac{1}{C_p}$ whenever $\left\{C_g \|\mathbf{x}\|_{p-\mathrm{var},[s,t]} \vee C_g^2 \|\mathbf{x}\|^2_{p-\mathrm{var},[s,t]}\right\} \leq \frac{1}{8C_p} < 1$, which yields

$$\|y,R\|_{p-\mathrm{var},[s,t]} \leq \frac{1}{C_p} \quad \text{whenever} \quad \|\mathbf{x}\|_{p-\mathrm{var},[s,t]} \leq \frac{1}{8C_pC_g}.$$

By constructing the sequence of greedy times $\{\tau_k(\frac{1}{8C_pC_g})\}_{k\in\mathbb{N}}$ on interval $[a,b]$, it follows from induction that

$$\begin{aligned} \|y_{\tau_{k+1}}\| &\leq \|y\|_{\infty,[\tau_k,\tau_{k+1}]} \leq \|y\|_{p-\mathrm{var},[\tau_k,\tau_{k+1}]} \leq \|y_{\tau_k}\| + \|y,R\|_{p-\mathrm{var},[\tau_k,\tau_{k+1}]} \leq \|y_{\tau_k}\| + \frac{1}{C_p} \\ &\leq \ldots \leq \|y_a\| + (k+1)\frac{1}{C_p}, \quad \forall k = 0,\ldots,N_{[a,b]}(\mathbf{x}) - 1. \end{aligned}$$

That means

$$\|y\|_{\infty,[a,b]} \leq \|y_a\| + \frac{1}{C_p} N_{[a,b]}(\mathbf{x}) \tag{4.6}$$

On the other hand, it then follows from inequality of p-variation seminorm in [7] that

$$\|y,R\|_{p-\mathrm{var},[a,b]} \leq N^{\frac{p-1}{p}}_{[a,b]}(\mathbf{x}) \sum_{k=0}^{N_{[a,b]}(\mathbf{x})-1} \|y,R\|_{q-\mathrm{var},[\tau_k,\tau_{k+1}]} \leq \frac{1}{C_p} N^{\frac{2p-1}{p}}_{[a,b]}(\mathbf{x}). \tag{4.7}$$

Step 2: Next, using (4.6) and (4.7) for any two solutions $y_t(\mathbf{x},y_a)$ and $\bar{y}_t(\mathbf{x},\bar{y}_a)$ within the bounded range $\frac{1}{C_p}N^{\frac{2p-1}{p}}_{[a,b]}(\mathbf{x})$, let us consider their difference $z_t = \bar{y}_t - y_t$, which satisfies the integral rough equation

$$z_t = z_a + \int_a^t [g(\bar{y}_s) - g(y_s)]dx_s.$$

As a result, $y'_s = g(y_s), \bar{y}'_s = g(\bar{y}_s)$ and

$$\begin{aligned} & g(\bar{y}_t) - g(y_t) - g(\bar{y}_s) + g(y_s) \\ =\ & \int_0^1 \Big[Dg(\bar{y}_s + \eta\bar{y}_{s,t})\bar{y}_{s,t} - Dg(y_s+\eta y_{s,t})y_{s,t}\Big]d\eta \\ =\ & Dg(\bar{y}_s)g(\bar{y}_s) - Dg(y_s)g(y_s) + \int_0^1 \Big[Dg(\bar{y}_s+\eta\bar{y}_{s,t})R^{\bar{y}}_{s,t} - Dg(y_s+\eta y_{s,t})R^y_{s,t}\Big]d\eta \\ & + \int_0^1 \Big\{\Big[Dg(\bar{y}_s+\eta\bar{y}_{s,t}) - Dg(\bar{y}_s)\Big]g(\bar{y}_s) - \Big[Dg(y_s+\eta y_{s,t}) - Dg(y_s)\Big]g(y_s)\Big\}x_{s,t}d\eta \\ =\ & Dg(\bar{y}_s)g(\bar{y}_s) - Dg(y_s)g(y_s) \\ & + \int_0^1 \Big\{Dg(\bar{y}_s+\eta\bar{y}_{s,t})\Big(R^{\bar{y}}_{s,t} - R^y_{s,t}\Big) + \Big[Dg(\bar{y}_s+\eta\bar{y}_{s,t}) - Dg(y_s+\eta y_{s,t})\Big]R^y_{s,t}\Big\}d\eta \\ & + \int_0^1 \Big[Dg(\bar{y}_s+\eta\bar{y}_{s,t}) - Dg(\bar{y}_s)\Big]\Big[g(\bar{y}_s) - g(y_s)\Big]x_{s,t}d\eta \\ & + \int_0^1 \Big[Dg(\bar{y}_s+\eta\bar{y}_{s,t}) - Dg(\bar{y}_s) - Dg(y_s+\eta y_{s,t}) + Dg(y_s)\Big]g(y_s)x_{s,t}d\eta \\ =\ & Dg(\bar{y}_s)g(\bar{y}_s) - Dg(y_s)g(y_s) \\ & + \int_0^1 \Big\{Dg(\bar{y}_s+\eta\bar{y}_{s,t})R^{\bar{y}-y}_{s,t} + \Big[Dg(\bar{y}_s+\eta\bar{y}_{s,t}) - Dg(y_s+\eta y_{s,t})\Big]R^y_{s,t}\Big\}d\eta \\ & + \int_0^1 \Big[Dg(\bar{y}_s+\eta\bar{y}_{s,t}) - Dg(\bar{y}_s)\Big]\Big[g(\bar{y}_s) - g(y_s)\Big]x_{s,t}d\eta \end{aligned}$$

$$+\Big(\int_0^1\int_0^1 D^2 g(\bar{y}_s+\mu\eta\bar{y}_{s,t})\eta(\bar{y}_{s,t}-y_{s,t})d\mu d\eta\Big)g(y_s)x_{s,t}$$
$$+\Big(\int_0^1\int_0^1 \Big[D^2 g(\bar{y}_s+\mu\eta\bar{y}_{s,t})-D^2 g(y_s+\mu\eta y_{s,t})\eta y_{s,t}\Big]d\mu d\eta\Big)g(y_s)x_{s,t}.$$

This proves $[g(\bar{y})-g(y)]'_s = Dg(\bar{y}_s)g(\bar{y}_s)-Dg(y_s)g(y_s) = Q(\bar{y}_s)-Q(y_s)$ with $\|Q(\bar{y}_s)-Q(y_s)\|\leq 2C_g^2\|z_s\|$ and

$$\begin{aligned}\left\|\!\left|Q(\bar{y})-Q(y)\right|\!\right\|_{p-\mathrm{var},[s,t]} &\leq C_Q\Big(\left\|\!\left|z\right|\!\right\|_{p-\mathrm{var},[s,t]}+\|z\|_{\infty,[s,t]}\left\|\!\left|y\right|\!\right\|_{p-\mathrm{var},[s,t]}\Big)\\ &\leq 2C_g^2\Big(\left\|\!\left|z\right|\!\right\|_{p-\mathrm{var},[s,t]}+\|z\|_{\infty,[s,t]}\left\|\!\left|y\right|\!\right\|_{p-\mathrm{var},[s,t]}\Big);\end{aligned}$$

and moreover

$$\begin{aligned}\Big\|\!\Big|R^{g(\bar{y})-g(y)}\Big|\!\Big\|_{q-\mathrm{var},[s,t]} &\leq C_g\left\|\!\left|R^z\right|\!\right\|_{q-\mathrm{var},[s,t]}+C_g\|z\|_{\infty,[s,t]}\left\|\!\left|R^y\right|\!\right\|_{q-\mathrm{var},[s,t]}\\ &\quad+\frac{1}{2}C_g^2\left\|\!\left|x\right|\!\right\|_{p-\mathrm{var},[s,t]}\Big[\left\|\!\left|z\right|\!\right\|_{p-\mathrm{var},[s,t]}+\|z\|_\infty\Big(\left\|\!\left|\bar{y}\right|\!\right\|_{p-\mathrm{var},[s,t]}+\left\|\!\left|y\right|\!\right\|_{p-\mathrm{var},[s,t]}\Big)\Big].\end{aligned}$$

Using the fact that

$$\begin{aligned}\|z_{s,t}\| &\leq \Big\|\int_s^t[g(\bar{y}_u)-g(y_u)]dx_u\Big\|\\ &\leq C_g\|z_s\|\left\|\!\left|x\right|\!\right\|_{p-\mathrm{var},[s,t]}+2C_g^2\|z_s\|\left\|\!\left|\mathbb{X}\right|\!\right\|_{q-\mathrm{var},[s,t]}\\ &\quad+C_p\Big\{\left\|\!\left|x\right|\!\right\|_{p-\mathrm{var},[s,t]}\Big\|\!\Big|R^{g(\bar{y})-g(y)}\Big|\!\Big\|_{q-\mathrm{var},[s,t]}+\left\|\!\left|\mathbb{X}\right|\!\right\|_{q-\mathrm{var},[s,t]}\left\|\!\left|[g(\bar{y})-g(y)]'\right|\!\right\|_{p-\mathrm{var},[s,t]}\Big\},\end{aligned}$$

we can now estimate

$$\begin{aligned}\left\|\!\left|z\right|\!\right\|_{p-\mathrm{var},[s,t]} &\leq 2\Big\{C_g\left\|\!\left|\mathbf{x}\right|\!\right\|_{p-\mathrm{var},[s,t]}\vee C_g^2\left\|\!\left|\mathbf{x}\right|\!\right\|^2_{p-\mathrm{var},[s,t]}\Big\}\times\\ &\quad\times\Big\{\|z\|_{\infty,[s,t]}\Big[1+C_p(\left\|\!\left|\bar{y},R\right|\!\right\|_{p-\mathrm{var},[s,t]}+\left\|\!\left|y,R\right|\!\right\|_{p-\mathrm{var},[s,t]})\Big]+C_p\left\|\!\left|z,R\right|\!\right\|_{p-\mathrm{var},[s,t]}\Big\}\\ &\leq 2C_p\Big\{C_g\left\|\!\left|\mathbf{x}\right|\!\right\|_{p-\mathrm{var},[s,t]}\vee C_g^2\left\|\!\left|\mathbf{x}\right|\!\right\|^2_{p-\mathrm{var},[s,t]}\Big\}\times\\ &\quad\times\Big(1+\left\|\!\left|\bar{y},R\right|\!\right\|_{p-\mathrm{var},[s,t]}+\left\|\!\left|y,R\right|\!\right\|_{p-\mathrm{var},[s,t]}\Big)\Big(\|z_s\|+\left\|\!\left|z,R\right|\!\right\|_{p-\mathrm{var},[s,t]}\Big).\end{aligned}$$

The similar estimate for $\left\|\!\left|R^z\right|\!\right\|_{q-\mathrm{var},[s,t]}$ is already included in the above computation. Therefore by combining with (4.7), we obtain

$$\begin{aligned}&\left\|\!\left|z,R\right|\!\right\|_{p-\mathrm{var},[s,t]}\\ \leq\;& 4C_p\Big\{C_g\left\|\!\left|\mathbf{x}\right|\!\right\|_{p-\mathrm{var},[s,t]}\vee C_g^2\left\|\!\left|\mathbf{x}\right|\!\right\|^2_{p-\mathrm{var},[s,t]}\Big\}\times\\ &\times\Big(1+\left\|\!\left|\bar{y},R\right|\!\right\|_{p-\mathrm{var},[s,t]}+\left\|\!\left|y,R\right|\!\right\|_{p-\mathrm{var},[s,t]}\Big)\Big(\|z_s\|+\left\|\!\left|z,R\right|\!\right\|_{p-\mathrm{var},[s,t]}\Big)\\ \leq\;& 4C_p\Big\{C_g\left\|\!\left|\mathbf{x}\right|\!\right\|_{p-\mathrm{var},[s,t]}\vee C_g^2\left\|\!\left|\mathbf{x}\right|\!\right\|^2_{p-\mathrm{var},[s,t]}\Big\}\Big(1+\frac{2}{C_p}N^{\frac{2p-1}{p}}_{\frac{1}{8C_pC_g},[a,b]}(\mathbf{x})\Big)\Big(\|z_s\|+\left\|\!\left|z,R\right|\!\right\|_{p-\mathrm{var},[a,b]}\Big),\end{aligned}$$

which follows that

$$\left\|\!\left|z,R\right|\!\right\|_{p-\mathrm{var},[s,t]}\leq\|z_s\|\quad\text{whenever}\quad 8C_pC_g\Big(1+\frac{2}{C_p}N^{\frac{2p-1}{p}}_{\frac{1}{8C_pC_g},[a,b]}(\mathbf{x})\Big)\left\|\!\left|\mathbf{x}\right|\!\right\|_{p-\mathrm{var},[s,t]}\leq 1.$$

Therefore, (2.13) is followed directly from the usage of greedy times (2.14), which is similar to (4.6) and (4.7). $\square$

Proof: [**Proposition 2.2**] The proof is divided into several steps.

Step 1: First, for a fixed solution $y_t(\mathbf{x}, y_a)$ on a given interval $[a,b]$, we need to prove the existence and uniqueness of the solution of the linearized rough differential equation (2.15), which has the time dependent coefficient $\Sigma_t := Dg(y_t)$. To do that, we simply follow Gubinelli's method by considering the Ito-Lyons map $H_t = \xi_a + \int_a^t \Sigma_s \xi_s dx_s$ on the set $\mathcal{D}_x^{2\beta}([a,b], \xi_a, \Sigma_a\xi_a)$ of controlled paths ξ_t such that ξ_a is fixed, $\xi'_a = \Sigma_a \xi_a$. Note that $\Sigma_t = Dg(y_t)$ is also controlled by x with

$$\Sigma_{s,t} = \int_0^1 D^2 g(y_s + \eta y_{s,t})(g(y_s)x_{s,t} + R^y_{s,t}) d\eta,$$

thus

$$\Sigma'_s = D^2_g(y_s)g(y_s), \quad \|R^\Sigma_{s,t}\| \leq C_g \mathbb{R}^y_{s,t}\| + \frac{1}{2}C_g^2 \|y_{s,t}\|\|x_{s,t}\|.$$

As a result $\Sigma_t\xi_t$ is also controlled by x with $[\Sigma.\xi.]'_s = \Sigma'_s\xi_s + \Sigma_s\xi'_s$ and

$$\|R^{\Sigma.\xi.}_{s,t}\| \leq \|\Sigma_{s,t}\|\|\xi_{s,t}\| + \|\xi_s\|\|R^\Sigma_{s,t}\| + \|\Sigma_s\|\|R^\xi_{s,t}\|.$$

It then enable to estimate

$$\begin{aligned}
& \|H_{s,t} - \Sigma_s\xi_s x_{s,t} + [\Sigma'_s\xi_s + \Sigma_s\xi'_s]\mathbb{X}_{s,t}\| \\
= \; & \|\int_s^t \Sigma_u\xi_u dx_u - \Sigma_s\xi_s x_{s,t} + [\Sigma'_s\xi_s + \Sigma_s\xi'_s]\mathbb{X}_{s,t}\| \\
& + C_\beta (t-s)^{3\beta}\Big(\left\|\!\left| x \right|\!\right\|_{\beta,[s,t]} \left\|\!\left| R^{\Sigma.\xi.} \right|\!\right\|_{2\beta,[s,t]} + \left\|\!\left| \mathbb{X} \right|\!\right\|_{2\beta,[s,t]} \left\|\!\left| [\Sigma.\xi.]' \right|\!\right\|_{\beta,[s,t]} \Big)
\end{aligned}$$

where

$$\begin{aligned}
\left\|\!\left| [\Sigma.\xi.]' \right|\!\right\|_\beta &\leq \|\Sigma'\|_\infty \left\|\!\left| \xi \right|\!\right\|_\beta + \left\|\!\left| \Sigma' \right|\!\right\|_\beta \|\xi\|_\infty + \|\Sigma\|_\infty \left\|\!\left| \xi' \right|\!\right\|_\beta + \left\|\!\left| \Sigma \right|\!\right\|_\beta \|\xi'\|_\infty, \\
\left\|\!\left| R^{\Sigma.\xi.} \right|\!\right\|_{2\beta} &\leq \left\|\!\left| \Sigma \right|\!\right\|_\beta \left\|\!\left| \xi \right|\!\right\|_\beta + \left\|\!\left| R^\Sigma \right|\!\right\|_{2\beta} \|\xi\|_\infty + \|\Sigma\|_\infty \left\|\!\left| R^\xi \right|\!\right\|_{2\beta} \quad \text{with} \\
\|\xi'\| &\leq \|\xi'_a\| + (t-a)^\beta \left\|\!\left| \xi' \right|\!\right\|_\beta \leq \|\Sigma\|_\infty \|\xi_a\| + (t-a)^\beta \left\|\!\left| \xi, \xi' \right|\!\right\|_{2\beta}, \\
\left\|\!\left| \xi \right|\!\right\|_\beta &\leq \|\xi'\|_\infty \left\|\!\left| x \right|\!\right\|_\beta + (t-a)^\beta \left\|\!\left| R^\xi \right|\!\right\|_{2\beta} \leq \|\Sigma\|_\infty \left\|\!\left| x \right|\!\right\|_\beta \|\xi_a\| + \Big(\left\|\!\left| x \right|\!\right\|_\beta \vee 1\Big)(t-a)^\beta \left\|\!\left| \xi, \xi' \right|\!\right\|_{2\beta}, \\
\|\xi\|_\infty &\leq \|\xi_a\| + (t-a)^\beta \left\|\!\left| \xi \right|\!\right\|_\beta \leq \|\xi_a\| \Big(1 + \|\Sigma\|_\infty (t-a)^\beta \left\|\!\left| x \right|\!\right\|_\beta\Big) + (t-a)^{2\beta} \Big(\left\|\!\left| x \right|\!\right\|_\beta \vee 1\Big) \left\|\!\left| \xi, \xi' \right|\!\right\|_{2\beta}.
\end{aligned}$$

A direct computation then shows that

$$\begin{aligned}
\left\|\!\left| [\Sigma.\xi.]' \right|\!\right\|_\beta &\leq \Big\{ \|\Sigma\|_\infty + (t-a)^\beta \left\|\!\left| \Sigma \right|\!\right\|_\beta + \|\Sigma'\|_\infty (t-a)^\beta (\left\|\!\left| x \right|\!\right\|_\beta \vee 1) + \left\|\!\left| \Sigma' \right|\!\right\|_\beta (t-a)^{2\beta} (\left\|\!\left| x \right|\!\right\|_\beta \vee 1)\Big\} \left\|\!\left| \xi, \xi' \right|\!\right\|_{2\beta} \\
&\quad + \Big\{ \left\|\!\left| \Sigma \right|\!\right\|_\beta \|\Sigma\|_\infty + \|\Sigma'\|_\infty \|\Sigma\|_\infty \left\|\!\left| x \right|\!\right\|_\beta + \left\|\!\left| \Sigma' \right|\!\right\|_\beta \Big(1 + \|\Sigma\|_\infty (t-a)^\beta \left\|\!\left| x \right|\!\right\|_\beta\Big)\Big\} \|\xi_a\| \\
\left\|\!\left| R^{\Sigma.\xi.} \right|\!\right\|_{2\beta} &\leq \Big\{ \|\Sigma\|_\infty + \left\|\!\left| \Sigma \right|\!\right\|_\beta (t-a)^\beta (\left\|\!\left| x \right|\!\right\|_\beta \vee 1) + \left\|\!\left| R^\Sigma \right|\!\right\|_{2\beta} (t-a)^{2\beta} (\left\|\!\left| x \right|\!\right\|_\beta \vee 1)\Big\} \left\|\!\left| \xi, \xi' \right|\!\right\|_{2\beta} \\
&\quad + \Big\{ \left\|\!\left| \Sigma \right|\!\right\|_\beta \|\Sigma\|_\infty \left\|\!\left| x \right|\!\right\|_\beta + \left\|\!\left| R^\Sigma \right|\!\right\|_{2\beta} \Big(1 + \|\Sigma\|_\infty (t-a)^\beta \left\|\!\left| x \right|\!\right\|_\beta\Big)\Big\} \|\xi_a\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\left\|\!\left| R^H \right|\!\right\|_{2\beta} &\leq \Big(\|\Sigma'\|_\infty \|\xi\|_\infty + \|\Sigma\|_\infty \|\xi'\|_\infty\Big) \left\|\!\left| \mathbb{X} \right|\!\right\|_{2\beta} + C_\beta (t-a)^\beta \Big(\left\|\!\left| x \right|\!\right\|_\beta \left\|\!\left| R^{\Sigma.\xi.} \right|\!\right\|_{2\beta} + \left\|\!\left| \mathbb{X} \right|\!\right\|_{2\beta} \left\|\!\left| [\Sigma.\xi.]' \right|\!\right\|_\beta \Big) \\
\left\|\!\left| H' \right|\!\right\|_\beta &\leq \|\Sigma\|_\infty \left\|\!\left| \xi \right|\!\right\|_\beta + \left\|\!\left| \Sigma \right|\!\right\|_\beta \|\xi\|_\infty \\
&\leq \Big\{ \|\Sigma\|_\infty (t-a)^\beta (\left\|\!\left| x \right|\!\right\|_\beta \vee 1) + \left\|\!\left| \Sigma \right|\!\right\|_\beta (t-a)^{2\beta} (\left\|\!\left| x \right|\!\right\|_\beta \vee 1)\Big\} \left\|\!\left| \xi, \xi' \right|\!\right\|_{2\beta} \\
&\quad + \Big\{ \|\Sigma\|^2_\infty \left\|\!\left| x \right|\!\right\|_\beta + \left\|\!\left| \Sigma \right|\!\right\|_\beta \Big[1 + \|\Sigma\|_\infty (t-a)^\beta \left\|\!\left| x \right|\!\right\|_\beta\Big]\Big\} \|\xi_a\|.
\end{aligned}$$

Combining everything together, we have just showed that there exists constants

$$M_1 = M_1(\Sigma, [a,b], x, \mathbb{X}), M_2 = M_2(\Sigma, [a,b], x, \mathbb{X})$$

such that

$$\left\|\!\left| H' \right|\!\right\|_{\beta} + \left\|\!\left| R^H \right|\!\right\|_{2\beta} \leq M_2 \|\xi_a\| + M_1 \Big[(t-a)^\beta + \|\!| x |\!\|_{\beta} + \|\!| \mathbb{X} |\!\|_{2\beta} \Big] \left\|\!\left| \xi, \xi' \right|\!\right\|_{2\beta}.$$

This implies that on every interval $[t_k, t_{k+1}]$ of the greedy times

$$t_0 = a, t_{k+1} = \inf\{t > t_k : M_1 \Big[(t-t_k)^\beta + \|\!| x |\!\|_{\beta,[t_k,t]} + \|\!| \mathbb{X} |\!\|_{2\beta,[t_k,t]} = \frac{1}{2} \} \wedge b,$$

the Ito-Lyons map is a contraction from the set $\Big\{ \mathcal{D}^{2\beta}_x([a,b], \xi_a, \Sigma_a \xi_a) : \|\!| \xi, \xi' |\!\|_{2\beta,[t_k,t_{k+1}]} \leq 2M_2 \|\xi_{t_k}\| \Big\}$ into itself, hence there exists a unique solution of (2.15) on every interval $[t_k, t_{k+1}]$. The concatenation of solutions on intervals $[t_k, t_{k+1}]$ then proves the existence and uniqueness of the solution of (2.15) on $[a,b]$.

Step 2: Denote by $\Phi(t, x, z_a)$ the solution matrix of the linearized system (2.15), then $\xi = \Phi(t, x, z_a)(\bar{z}_a - z_a)$ is the solution of (2.15) given initial point $\xi_a = \bar{z}_a - z_a$. Assign $r_t := \bar{z}_t - z_t - \xi_t$, then $r_a = 0$ and

$$\begin{aligned} r_t &= \int_a^t \Big[\int_0^1 D_z g(z_s + \eta(\bar{z}_s - z_s)) - D_z g(z_s) \Big] (\bar{z}_s - z_s) d\eta dx_s + \int_a^t D_z g(z_s) r_s dx_s, \\ &= e_{a,t} + \int_a^t D_z g(z_s) r_s dx_s, \quad \forall t \in [a,b], \end{aligned} \tag{4.8}$$

where

$$e_{a,t} = \int_a^t \int_0^1 \Big[D_z g(z_s + \eta(\bar{z}_s - z_s)) - D_z g(z_s) \Big] (\bar{z}_s - z_s) d\eta dx_s$$

and e is also controlled by x with $e_a = 0$. We are going to estimate $\|r\|_{\infty,[a,b]}$ and $\|\!| r, R |\!\|_{p-\mathrm{var},[a,b]}$ through $\|e'\|_{\infty,[a,b]}, \|\!| e' |\!\|_{p-\mathrm{var},[a,b]}, \|\!| R^e |\!\|_{q-\mathrm{var},[a,b]}$. First observe that

$$r_{s,t} = e_{s,t} + \int_s^t Dg(y_u) r_u dx_u = e' x_{s,t} + R^e_{s,t} + \int_s^t Dg(y_u) r_u dx_u,$$

which yields $r'_s = e'_s + Dg(y_s) r_s$ and

$$\|R^r_{s,t}\| \leq \|R^e_{s,t}\| + \|[Dg(y)r]'_s\| \|\mathbb{X}_{s,t}\| + C_p \Big(\|\!| x |\!\|_p \left\|\!\left| R^{Dg(y)r} \right|\!\right\|_q + \|\!| \mathbb{X} |\!\|_q \left\|\!\left| [Dg(y)r]' \right|\!\right\|_p \Big). \tag{4.9}$$

A direct computation shows that

$$\begin{aligned} &\Big\| Dg(y_t) r_t - Dg(y_s) r_s - \Big[D^2_g(y_s) g(y_s) r_s + Dg(y_s) r'_s \Big] x_{s,t} \Big\| \\ \leq \quad & \Big\| \int_0^1 D^2 g(y_s + \eta y_{s,t}) R^y_{s,t} r_s d\eta \Big\| + \frac{1}{2} C_g^2 \|y_{s,t}\| \|r_s\| \|x_{s,t}\| + \|Dg(y_s) R^y_{s,t}\| + C_g \|y_{s,t}\| \|r_{s,t}\|, \end{aligned}$$

which yields $[Dg(y)r]'_s = D^2 g(y_s)[g(y_s), r_s] + Dg(y_s) r'_s$ and

$$\begin{aligned} &\|R^{D(y)r}_{s,t}\| \leq C_g \|R^y_{s,t}\| \|r\|_\infty + \frac{1}{2} C_g^2 \|r\|_\infty \|y_{s,t}\| \|x_{s,t}\| + C_g \|R^r_{s,t}\| + C_g \|y_{s,t}\| \|r_{s,t}\| \\ \Rightarrow \quad & \left\|\!\left| R^{Dg(y)r} \right|\!\right\|_q \leq C_g \|r\|_\infty \|\!| R^y |\!\|_q + \frac{1}{2} C_g^2 \|r\|_\infty \|\!| y |\!\|_p \|\!| x |\!\|_p + C_g \|\!| R^r |\!\|_q + C_g \|\!| y |\!\|_p \|\!| r |\!\|_p. \end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
\|[Dg(y)r]'\|_\infty &\leq C_g^2\|r\|_\infty + C_g\|r'\|_\infty \\
\left|\!\left|\!\left|[Dg(y)r]'\right|\!\right|\!\right|_p &\leq 2C_g^2\|r\|_\infty \left|\!\left|\!\left|y\right|\!\right|\!\right|_p + C_g^2 \left|\!\left|\!\left|r\right|\!\right|\!\right|_p + C_g \left|\!\left|\!\left|r'\right|\!\right|\!\right|_p + C_g\|r'\|_\infty \left|\!\left|\!\left|y\right|\!\right|\!\right|_p .
\end{aligned}$$

Combining all the above estimates into (4.9), we obtain

$$\begin{aligned}
&\left|\!\left|\!\left|R^r\right|\!\right|\!\right|_q \leq C_pC_g \left|\!\left|\!\left|x\right|\!\right|\!\right|_p \left|\!\left|\!\left|R^r\right|\!\right|\!\right|_q \\
&+ \left|\!\left|\!\left|r\right|\!\right|\!\right|_p \Big\{2C_g^2 \left|\!\left|\!\left|\mathbb{X}\right|\!\right|\!\right|_q + C_p \left|\!\left|\!\left|x\right|\!\right|\!\right|_p \Big[C_g \left|\!\left|\!\left|R^y\right|\!\right|\!\right|_q + \frac{1}{2}C_g^2 \left|\!\left|\!\left|x\right|\!\right|\!\right|_p \left|\!\left|\!\left|y\right|\!\right|\!\right|_p + C_g \left|\!\left|\!\left|y\right|\!\right|\!\right|_p\Big] + C_g \left|\!\left|\!\left|\mathbb{X}\right|\!\right|\!\right|_q (4C_g^2 \left|\!\left|\!\left|y\right|\!\right|\!\right|_p + 2C_g^2)\Big\} \\
&+\|r_s\|\Big\{2C_g^2 \left|\!\left|\!\left|\mathbb{X}\right|\!\right|\!\right|_q + C_p \left|\!\left|\!\left|x\right|\!\right|\!\right|_p \Big[C_g \left|\!\left|\!\left|R^y\right|\!\right|\!\right|_q + \frac{1}{2}C_g^2 \left|\!\left|\!\left|x\right|\!\right|\!\right|_p \left|\!\left|\!\left|y\right|\!\right|\!\right|_p\Big] + 4C_pC_g^2 \left|\!\left|\!\left|\mathbb{X}\right|\!\right|\!\right|_q \Big\} \\
&+ \left|\!\left|\!\left|R^e\right|\!\right|\!\right|_q + C_g \left|\!\left|\!\left|\mathbb{X}\right|\!\right|\!\right|_q \|e'\|_\infty + C_p \left|\!\left|\!\left|\mathbb{X}\right|\!\right|\!\right|_q (C_g \left|\!\left|\!\left|e'\right|\!\right|\!\right|_p + C_g \left|\!\left|\!\left|y\right|\!\right|\!\right|_p \|e'\|_\infty) =: \bar{R},
\end{aligned}$$

and similarly

$$\left|\!\left|\!\left|r\right|\!\right|\!\right|_p \leq \|e'\|_\infty \left|\!\left|\!\left|x\right|\!\right|\!\right|_p + C_g\|r\|_\infty \left|\!\left|\!\left|x\right|\!\right|\!\right|_p + \bar{R} \leq \|e'\|_\infty \left|\!\left|\!\left|x\right|\!\right|\!\right|_p + C_g \left|\!\left|\!\left|x\right|\!\right|\!\right|_p (\|r_s\| + \left|\!\left|\!\left|r\right|\!\right|\!\right|_p) + \bar{R}.$$

Therefore, taking into account (4.7) we have just proved that there exists a constant $M = M(p, [a,b], \left|\!\left|\!\left|\mathbf{x}\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]}) > 1$ such that

$$\begin{aligned}
\left|\!\left|\!\left|r,R\right|\!\right|\!\right|_{p-\mathrm{var},[s,t]} &\leq M\Big(C_g \left|\!\left|\!\left|x\right|\!\right|\!\right|_{p-\mathrm{var},[s,t]} + C_g^2 \left|\!\left|\!\left|\mathbb{X}\right|\!\right|\!\right|_{q-\mathrm{var},[s,t]}\Big)\Big(\|r_s\| + \left|\!\left|\!\left|r,R\right|\!\right|\!\right|_{p-\mathrm{var},[s,t]}\Big) \\
&\quad +M\Big(\|e'\|_{\infty,[a,b]} + \left|\!\left|\!\left|e'\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]} + \left|\!\left|\!\left|R^e\right|\!\right|\!\right|_{q-\mathrm{var},[a,b]}\Big) \\
&\leq 2M\Big(C_g \left|\!\left|\!\left|\mathbf{x}\right|\!\right|\!\right|_{p-\mathrm{var},[s,t]} \vee C_g^2 \left|\!\left|\!\left|\mathbf{x}\right|\!\right|\!\right|^2_{q-\mathrm{var},[s,t]}\Big)\Big(\|r_s\| + \left|\!\left|\!\left|r,R\right|\!\right|\!\right|_{p-\mathrm{var},[s,t]}\Big) \\
&\quad +M\Big(\|e'\|_{\infty,[a,b]} + \left|\!\left|\!\left|e'\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]} + \left|\!\left|\!\left|R^e\right|\!\right|\!\right|_{q-\mathrm{var},[a,b]}\Big),
\end{aligned}$$

which implies

$$\left|\!\left|\!\left|r,R\right|\!\right|\!\right|_{p-\mathrm{var},[s,t]} \leq \|r_s\| + 2M\Big(\|e'\|_{\infty,[a,b]} + \left|\!\left|\!\left|e'\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]} + \left|\!\left|\!\left|R^e\right|\!\right|\!\right|_{q-\mathrm{var},[a,b]}\Big)$$

whenever $2MC_g \left|\!\left|\!\left|\mathbf{x}\right|\!\right|\!\right|_{p-\mathrm{var},[s,t]} \leq \frac{1}{2}$. Similar estimates to (4.6) and (4.7), using the sequence of greedy times $\{\tau_k(\frac{1}{4MC_g})\}_{k\in\mathbb{N}}$, lead to

$$\begin{aligned}
&\|r\|_{\infty,[a,b]} \vee \left|\!\left|\!\left|r,R\right|\!\right|\!\right|_{p-\mathrm{var},[s,t]} \\
\leq\ & N_{\frac{1}{4MC_g},[a,b]}(\mathbf{x})^{\frac{p-1}{p}}\Big\{\|r_a\| + 2M\Big(\|e'\|_{\infty,[a,b]} + \left|\!\left|\!\left|e'\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]} + \left|\!\left|\!\left|R^e\right|\!\right|\!\right|_{q-\mathrm{var},[a,b]}\Big)\Big\}e^{(\log 2)N_{\frac{1}{4MC_g},[a,b]}(\mathbf{x})} \\
\leq\ & 2M\Big(\|e'\|_{\infty,[a,b]} + \left|\!\left|\!\left|e'\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]} + \left|\!\left|\!\left|R^e\right|\!\right|\!\right|_{q-\mathrm{var},[a,b]}\Big)e^{(1+\log 2)N_{\frac{1}{4MC_g},[a,b]}(\mathbf{x})}, \qquad (4.10)
\end{aligned}$$

where we use the fact that $r_a = 0$.

Step 3: From (4.8), it follows that

$$e'_s = \int_0^1 [Dg(y_s+\eta z_s) - Dg(y_s)]z_s d\eta = \int_0^1\int_0^1 D^2g\Big((1-\mu\eta)y_s + \mu\eta\bar{y}_s\Big)[z_s,z_s]\eta d\mu d\eta = \bar{e}_s,$$

is controlled by x. As a result $\|e'_s\| \leq \frac{1}{2}C_g\|z_s\|^2$ thus

$$\begin{aligned}
\|e'\|_{\infty,[a,b]} &\leq \frac{1}{2}C_g\|z\|^2_{\infty,[a,b]}, \qquad (4.11)\\
\left|\!\left|\!\left|e'\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]} &\leq C_g\Big(\left|\!\left|\!\left|y\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]} \vee \left|\!\left|\!\left|\bar{y}\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]}\Big)\|z\|^2_{\infty,[a,b]} + 2C_g\|z\|_{\infty,[a,b]} \left|\!\left|\!\left|z\right|\!\right|\!\right|_{p-\mathrm{var},[a,b]} .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|R^e_{s,t}\| &\leq \|\bar{e}'\|_\infty \|\mathbb{X}_{s,t}\| + C_p\Big(\left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} \left\vert\!\left\vert\!\left\vert R^{\bar{e}}\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} + \left\vert\!\left\vert\!\left\vert \mathbb{X}\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} \left\vert\!\left\vert\!\left\vert \bar{e}'\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} \Big) \\
\text{thus } \left\vert\!\left\vert\!\left\vert R^e\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} &\leq \|\bar{e}'\|_{\infty,[a,b]} \left\vert\!\left\vert\!\left\vert \mathbb{X}\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} \\
&\quad + C_p\Big(\left\vert\!\left\vert\!\left\vert x\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} \left\vert\!\left\vert\!\left\vert R^{\bar{e}}\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} + \left\vert\!\left\vert\!\left\vert \mathbb{X}\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} \left\vert\!\left\vert\!\left\vert \bar{e}'\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} \Big), \qquad (4.12)
\end{aligned}$$

where a direct computation shows that

$$\begin{aligned}
\bar{e}'_s &= \int_0^1 \int_0^1 \Big\{ D^3 g\Big((1-\mu\eta)y_s + \mu\eta\bar{y}_s\Big)[(1-\mu\eta)y'_s + \mu\eta\bar{y}'_s, z_s, z_s] \\
&\quad + 2D^2 g\Big((1-\mu\eta)y_s + \mu\eta\bar{y}_s\Big)[z'_s, z_s]\Big\}\eta d\mu d\eta, \\
R^{\bar{e}}_{s,t} &= \int_0^1 \int_0^1 \Big\{ R^{D^2 g(\bar{y},y)}_{s,t}[z_t, z_t] + D^2 g(\bar{y}, y)\Big([z_s, R^z_{s,t}] + [z'_s x_{s,t}, z'_s x_{s,t} + R^z_{s,t}] + [R^z_{s,t}, z_t]\Big) \\
&\quad + [D^2 g(\bar{y}, y)]'_s x_{s,t}\Big([z_s + z_t, z'_s + R^z_{s,t}]\Big)\Big\}\eta d\mu d\eta.
\end{aligned}$$

We therefore can show that there exists a generic constant α such that

$$\begin{aligned}
&\|\bar{e}'\|_{\infty,[a,b]} \vee \left\vert\!\left\vert\!\left\vert \bar{e}'\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} \vee \left\vert\!\left\vert\!\left\vert R^{\bar{e}}\right\vert\!\right\vert\!\right\vert_{q-\mathrm{var},[a,b]} \\
\leq\ & \alpha\Big(\|z\|_{\infty,[a,b]} + \left\vert\!\left\vert\!\left\vert z, R\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} + \|z'\|_{\infty,[a,b]} + \left\vert\!\left\vert\!\left\vert z'\right\vert\!\right\vert\!\right\vert_{p-\mathrm{var},[a,b]} \Big)^2. \qquad (4.13)
\end{aligned}$$

By replacing (4.11), (4.12), (4.13) into (4.10), and using (2.13), we derive that there exists a generic constant such that

$$\|\bar{y}_\cdot(\mathbf{x}, \bar{y}_a) - y_\cdot(\mathbf{x}, y_a) - \xi_\cdot(\mathbf{x}, \bar{y}_a - y_a)\|_{\infty,[a,b]} \leq \alpha \|\bar{y}_a - y_a\|^2.$$

This, combined with the linearity of ξ w.r.t. $\bar{y}_a - y_a$, shows the differentiability of $y_t(\mathbf{x}, y_a)$ w.r.t. y_a.

□

Acknowledgments

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