JACOBIAN CONJECTURE AS A PROBLEM ON INTEGRAL POINTS ON AFFINE CURVES

NGUYEN VAN CHAU

ABSTRACT. It is shown that the Jacobian conjecture over algebraic number fields may be considered as an existence problem of integral points on affine curves. More specially, if the Jacobian conjecture over $\mathbb{C}$ is false, then for some $n \gg 1$ there exists a counterexample $F \in \mathbb{Z}[X]^n$ of the form $F_i(X) = X_i + (a_{i1}X_1 + \cdots + a_{in}X_n)^3$, $a_{ij} \in \mathbb{Z}$, $i, j = \overline{1,n}$, such that the affine curve $F_1(X) = F_2(X) = \cdots = F_n(X)$ has no non-zero integer points.

1. INTRODUCTION

Let k be a field of characteristic zero and $k[X]$ the ring of polynomials of the variable $X := (X_1, X_2, \ldots, X_n)$, $n > 1$. In tradition, polynomial maps $F = (F_1, F_2, \ldots, F_n) \in k[X]^n$ with $JF := \det DF \equiv 1$ will be called *Keller maps*. The n-dimensional Jacobian conjecture over k ($JC(k,n)$) , which was posed firstly in 1939 by Ott-Heinrich Keller [7] and still open even for $n = 2$, asserts that *every Keller map $F \in k[X]^n$ is invertible, and hence, has an inverse in $k[X]^n$*. We refer the readers to [1, 4] for nice surveys on this conjecture and related topics.

It is known that the Jacobian conjecture is true if it is true for the field $\overline{\mathbb{Q}}$ of all algebraic numbers and that $JC(K,n)$ for a number field K can be reduced to consider the existence of solutions in O_K^n of the Diophantine equation $F(X) = 0$ for Keller maps $F \in O_K^n[X]^n$, where O_K is the ring of integers of K (see in [1, 9]). These facts show another aspect of the Jacobian conjecture from view points of the Diophantine geometry and would be useful in attempting to understand the nature of this conjecture. This article is to present the following results that reduces the Jacobian conjecture to an existence problem of integral points on affine curves.

Theorem A. *Let K be an algebraic number fields and O_K the ring of integers in K. For every $n > 1$ the conjecture $JC(K,n)$ is equivalent to the following statement:*

$DJC(K,n)$: For every Keller map $F \in O_K[X]^n$ with $F(0) = 0$ the affine curve

$$F_1(X) = F_2(X) = \cdots = F_n(X). \qquad (\mathscr{C}_F)$$

always has non-zero points in O_K^n.

Theorem B. *If the Jacobian conjecture over $\mathbb{C}$ is not true, then for some $n \gg 1$ there exists a Keller map $F \in \mathbb{Z}[X]^n$ of the linear cubic form over $\mathbb{Z}$, $F_i(X) = X_i + < a_i, X >^3$, $a_i \in \mathbb{Z}^n$, $i = \overline{1,n}$, such that the affine curve $\mathscr{C}_F$ has no non-zero points in $\mathbb{Z}^n$.*

Theorem B leads to a little surprise consequence that the Jacobian conjecture over $\mathbb{C}$ for all $n > 1$ can be reduced to the question whether for every Keller map $F \in \mathbb{Z}[X]^n$ of the

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linear cubic form over $\mathbb{Z}$ the 6-degree diophantine equation $F_1(X)^2 + \cdots + F_{n-1}(X)^2 = 0$ always has non-zero integer solutions (Cor. 1, Sec. 5).

Our approach here is based on the celebrated Siegel's theorem on the integral points on affine curves and the reduction theorems for the Jacobian conjecture, due to Bass, Connell and Wright [1], Yagzhev [13], Drużkowski [3] and Connell and Van den Dries [2]. Theorem A and Theorem B will be proved in the sections 3 and 4. The essential key in the proofs is Main Lemma in Section 2, which shows that if K is an algebraic number field and if $F \in K[X]^n$ is a non-invertible Keller map, then for general lines l in K^n the inverse images $F^{-1}(l)$ are irreducible affine curves having at most finitely many points in O_K^n.

2. Lemma on inverse images of generic lines

In this section we consider the possible behavior of non-invertible Keller maps, if exist. We will try to estimate the topological type and the number of integral points of the inverse images of generic lines by such Keller maps.

In sequences, let us denote $l(u,v) := \{u + tv : t \in \mathbb{C}\}$ - the complex line of direction $0 \neq v \in \mathbb{C}^n$ passing through a point $u \in \mathbb{C}^n$. By the *bifurcation value set*, denoted by E_f, of a dominant polynomial mapping $f : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ we mean the smallest algebraic subset of $\mathbb{C}^m$ such that the restriction $f : \mathbb{C}^n \setminus f^{-1}(E_f) \longrightarrow \mathbb{C}^m \setminus E_f$ defines a locally trivial smooth fibration.

Suppose $F \in \mathbb{C}[X]^n$ is a Keller map, but not invertible. In this situation, the field extension $\mathbb{C}(F) \subset \mathbb{C}(X)$ is algebraic and has degree $d_F := [\mathbb{C}(X) : \mathbb{C}(F)] > 1$. The bifurcation value set E_F is just the set of all $a \in \mathbb{C}^n$ such that $\#F^{-1}(a) < d_F$, and the restriction $F : \mathbb{C}^n \setminus F^{-1}(E_F) \longrightarrow \mathbb{C}^n \setminus E_F$ gives a unbranched smooth covering of d_F sheets. Since the extension $\mathbb{C}(F) \subset \mathbb{C}(X)$ is algebraic, there exists irreducible polynomials $h_i \in \mathbb{C}[X][T]$, $i = \overline{1,n}$, such that $h_i(F(X), X_i) = 0$. Following [6, 4], the bifurcation value set E_F then is the hypersurface defined by the equation $a_1(X)\ldots a_n(X) = 0$, where $a_i(X)$ are coefficients of $T^{\deg_T h_i}$ in h_i. Let $H_F(X)$ be the product of all distinct irreducible polynomial factors of $a_i(X)$s. Then $E_F = \{a \in \mathbb{C}^n : H_F(a) = 0\}$ and, in particular, $H_F \in \overline{\mathbb{Q}}[X]$ if $F \in \overline{\mathbb{Q}}[X]^n$. Let us denote by K_F the cone of tangents at infinity of E_F, i.e. $K_F = \{v \in \mathbb{C}^n : h_F(v) = 0\}$, where h_F is the leading homogeneous of H_F. The following lemma is an essential key in the proofs of Theorem A and Theorem B.

Main Lemma *Suppose $F \in \mathbb{C}[X]^n$ is a Keller map, but not invertible. Then, there exists a non-constant polynomial $\sigma_F \in \mathbb{C}[U,V]$, $U = (U_1 \ldots, U_n)$ and $V = (V_1, \ldots, V_n)$, such that*

a) *$\sigma_F(\bar{u}, V) \not\equiv 0$ and $\sigma_F(U, \bar{v}) \not\equiv 0$ for $\bar{u} \notin E_F$ and $\bar{v} \notin K_F$, and*
b) *the inverse images $F^{-1}(l(u,v))$ with $u, v \in \mathbb{C}^n$ and $\sigma_F(u,v) \neq 0$ are irreducible affine curves of same a genus g_F and a number $n_F > 2$ of irreducible branches at infinity .*

Moreover, in the case $F \in K[X]^n$ for an algebraic number field K,

c) *$H_F, \sigma_F \in \overline{\mathbb{Q}}[X]$ and*
d) *for every $u, v \in K^n$ with $\sigma_F(u,v) \neq 0$ the inverse image $F^{-1}(l(u,v))$ has at most finitely many integral points in K^n.*

Proof. Let $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be as in the statement. We will try to construct the desired polynomial σ_F and prove the conclusions (a-c). Conclusion (d) then is an immediate consequence of (a), (b) and Siegel's theorem, which asserts that any irreducible affine curve over a number field K with positive genus or with more than two irreducible branches at infinity has at most finitely many K-integral points.

In view of the results on the topological equisingularity, due to Verdier [12] and Varchenko [11], there exists a proper algebraic subset $\Sigma \subset \mathbb{C}^n \times \mathbb{C}^n$ such that the inverse images $F^{-1}(l)$, $l := l(u,v)$, $(u,v) \in (\mathbb{C}^n \times \mathbb{C}^n) \setminus \Sigma$, are diffeomorphic to same a connected Riemann surface. The polynomial σ_F then may be taken to be a polynomial that defines an algebraic hypersurface containing Σ. Below, we will determine σ_F by a constructive way that enables us to handle other properties in (a-c). To do that we need to use the following fact due to Jelonek [5, 6]:

Theorem (*) *Let $f : V \longrightarrow W$ be a dominant generically-finite polynomial mapping of irreducible affine varieties $V \subset \mathbb{C}^n$ and $W \subset \mathbb{C}^m$. Let S_f be the so-called non-proper value set of f, $S_f := \{a \in W : a = \lim_k f(x_k), V \ni x_k \mapsto \infty\}$. Then, S_f is empty or an algebraic hypersurface in W. Moreover, if the coefficients in the polynomials defining V,W and f are algebraic numbers, then all coefficients in the polynomial defining S_f are also algebraic numbers.* (see Theorem 2.3, [6] and it's proof.)

Now, consider the d_F-sheeted unbranched covering

$$F : \mathbb{C}^n \setminus F^{-1}(E_F) \longrightarrow \mathbb{C}^n \setminus E_F. \tag{1}$$

Since F is locally diffeomorphic, the bifurcation value set E_F coincides with the non-proper value set S_F of F. By Theorem (*) we have

i) E_F is an affine hypersurface in $\mathbb{C}^n$ defined by a reduced polynomial $H_F \in \mathbb{C}[X]$. If $F \in \overline{\mathbb{Q}}[X]^n$, then $H_F \in \overline{\mathbb{Q}}[X]$.

Let $W \subset E_F$ be an irreducible component of E_F, given by $h_W(X) = 0$ for an irreducible factor $h_W \in \mathbb{C}[X]$ of H_F. Let V be an irreducible component of the inverse image $F^{-1}(W)$ and $f_{VW} := F_V : V \longrightarrow W$. Denote by $\Sigma(W)$ the set of all singular points of W and by E_{VW} the non-proper value set of f_{VW}. Again by Theorem (*)

ii) E_{VW} is either empty set or an algebraic variety of dimension $n-2$, given by the equations $h_W(X) = 0$ and $g_{VW}(X) = 0$ for a $g_{VW} \in \mathbb{C}[X]$;
iii) the restriction $f_{VW} : V \setminus f_{VW}^{-1}(\Sigma(W) \cup E_{VW}) \longrightarrow W \setminus (\Sigma(W) \cup E_{VW})$ gives a unramified covering;
iv) If $F \in \overline{\mathbb{Q}}[X]^n$, then $h_W, g_{VW} \in \overline{\mathbb{Q}}[X]$.

Let $B_F := \bigcup E_{VW}$, where W and V run through over the irreducible components of E_F and the irreducible components of $F^{-1}(W)$, respectively. Let E_F^1 denote the bifurcation value set of the restriction $F : F^{-1}(E_F) \longrightarrow E_F$. By definitions E_F^1 is a proper algebraic subset of E_F, $E_F \setminus E_F^1$ is smooth and the restriction

$$F : F^{-1}(E_F \setminus E_F^1) \longrightarrow E_F \setminus E_F^1 \tag{2}$$

gives a unramified covering. By (ii-iv) we can see

v) E_F^1 is the union of B_F and the set of all singular points of E_F.

Let us denote by Σ_F the complement of the set of all $(u,v) \in \mathbb{C}^n \times \mathbb{C}^n$ satisfying the following conditions:

a1) $l(u,v)$ intersects transversally E_F,
a2) $v \notin K_F$ and
a3) $l(u,v) \cap E_F^1 = \emptyset$.

By (v) we can easy verify that (u,v) satisfies the conditions (a1-a3) if and only if it satisfies

b1) $l(u,v)$ intersects E_F at $\deg H$ different points, and
b2) $l(u,v) \cap B_F = \emptyset$.

Now, we define

$$\begin{aligned} D(U,V) &:= Disc_t(H(U+tV)) \\ R(U,V) &:= \prod_W \prod_{VW} Res_t(h_W(U+tV), g_{VW}(U+tV)) \\ \sigma_F(U,V) &:= D(U,V)R(U,V). \end{aligned}$$

Assertion 1. *We have*

c1) $\Sigma_F = \{(u,v) \in \mathbb{C}^n \times \mathbb{C}^n : \sigma_F(u,v) = 0\}$;
c2) $\sigma_F(\bar{u},V) \not\equiv 0$ *and* $\sigma_F(U,\bar{v}) \not\equiv 0$ *for* $\bar{u} \in \mathbb{C}^n \setminus E_F$ *and* $0 \neq \bar{v} \in \mathbb{C}^n \setminus K_F$*;*
c3) $\sigma_F \in \overline{\mathbb{Q}}[U,V]$ *if* $F \in \overline{\mathbb{Q}}[X]^n$.

Proof. Observe that the conditions (b1) and (b2) can be expressed as $D(u,v) \neq 0$ and $R(u,v) \neq 0$, respectively, that implies (c1). (c3) follows from the constructing of σ_F and (iv).

We now prove (c2).

Given $\bar{u} \in \mathbb{C}^n \setminus E_F$. For each irreducible component W of E_F let $W_{\bar{u}} := \{x \in W : \varphi(x) = 0\}$ and $V_{\bar{u}} := \bigcup_W W_{\bar{u}}$, where $\varphi(X) := Dh_W(X)(X - \bar{u})$. We define the cone

$$K_{\bar{u}} := K_F \cup \{tv : v \in V_{\bar{u}} \cup E_F^1, t \in \mathbb{C}\}.$$

Then, by the conditions (a1-a3) $\sigma_F(\bar{u},v) \neq 0$ if and only if $v \in \mathbb{C}^n \setminus K_{\bar{u}}$. Obviously, $\deg \varphi(X) = \deg h_W(X)$ and $\varphi(\bar{u}) = 0$. Since h_W is irreducible and $\bar{u} \notin E_F$, it follows that $W_{\bar{u}}$ is a subset of W and of pure dimension less than $n-1$. Therefore, $V_{\bar{u}}$ is an algebraic subset of W and has a pure dimension less than $n-1$. Note that $\dim E_F^1 < n-1$ and $\dim K_F < n$. This follows that $K_{\bar{u}}$ is a closed set of pure dimension less than n. Thus, $\mathbb{C}^n \setminus K_{\bar{u}}$ is open dense in $\mathbb{C}^n$, and hence, $\sigma_F(\bar{u},V) \not\equiv 0$.

Given $\bar{v} \notin K_F$. For each irreducible component W of E_F let $W_{\bar{v}} := \{x \in W : Dh_W(x)\bar{v} = 0\}$ and $U_{\bar{v}} := \bigcup_W W_{\bar{v}}$. We define $S_{\bar{v}} := \{u \in \mathbb{C}^n : u = x + t\bar{v}, t \in \mathbb{C}, x \in U_{\bar{v}} \cup E_F^1\}$. Since $\bar{v} \notin K_F$, by the conditions (a1-a3) we can verify that $\sigma_F(u,\bar{v}) \neq 0$ if and only if $u \notin S_{\bar{v}}$. Furthermore, $Dh_W(X)\bar{v} \not\equiv 0$. Otherwise, $\bar{v}$ belongs to the tangent space T_xW for all smooth point x in W that is impossible. Obviously, $\deg Dh_W(X)\bar{v} \leq \deg h_W(X) - 1$. So, $W_{\bar{v}}$ is a proper algebraic subset of W and has a pure dimension less than $n-1$. Hence, $U_{\bar{v}}$ is of pure dimension less than $n-1$. Again we can see that $S_{\bar{v}}$ is of dimension less than n that ensures $\sigma_F(U,\bar{v}) \not\equiv 0$.

□

Assertion 2. *The family* $\mathscr{F} := \{F^{-1}(l(u,v)) : \sigma_F(u,v) \neq 0\}$ *consists of irreducible affine curves of same topological type.*

Proof. It suffices to show

d1) $\mathscr{F}$ contains an irreducible affine curve, and
d2) The affine curves in $\mathscr{F}$ are homeomorphic.

We first prove (d1). Fix $0 \neq v \in \mathbb{C}^n \setminus K_F$. Let E denote the vector space orthogonal to v, $E = \{u \in \mathbb{C}^n : <u,v> = 0\}$. We will show that there is an open dense algebraic subset E' of E such that for every $z \in E'$ the curve $F^{-1}(l(u,v))$ is irreducible and $\sigma_F(u,v) \neq 0$. Let $\pi : \mathbb{C}^n \longrightarrow E \cong \mathbb{C}^{n-1}$ the projection $u \mapsto u - \frac{<u,v>}{<v,v>}v \in E$ and $\varphi := \pi \circ F : \mathbb{C}^n \longrightarrow E$. We have that $F^{-1}(l(u,v)) = \varphi^{-1}(u)$ for $u \in E$. Observe that φ is a dominant morphism. Moreover, the restriction $\pi : E_F \longrightarrow E$ is proper, since $v \notin K_F$. Then, by Theorem 2 in [8] there exists an open dense algebraic subset A of E such that for every $u \in A$ the fiber $\varphi^{-1}(u)$ is irreducible. On the other hand, since $v \notin K_F$, by Assertion 1 (c2) the set $B := \{u \in E : \sigma_F(u,\bar{v}) \neq 0\}$ is open dense in E. So, $E' := A \cap B$ provides the desired subset.

To prove (d2) it suffices to show that for two arbitrary $z^i \in \mathbb{C}^n \times \mathbb{C}^n$ with $\sigma_F(z^i) \neq 0$, $i = 0;1$, the curves $F^{-1}(l(z^i))$ are homeomorphic. Given such points z^i. Since the polynomial σ_F is not constant, we can take a smooth path $\gamma : [0,1] \longrightarrow \mathbb{C}^n \times \mathbb{C}^n$ such that $\gamma(0) = z^0$, $\gamma(1) = z^1$ and $\sigma_F(\gamma(t)) \neq 0$ for $t \in [0,1]$. Let $l_t := l(\gamma(t))$, $t \in [0,1]$. Note that $E_F \setminus E_F^1$ is a smooth open algebraic subset of the hypersurface E_F and by(1-2) the restriction

$$F : (\mathbb{C}^n \setminus F^{-1}(E_F), F^{-1}(E_F \setminus E_F^1)) \longrightarrow (\mathbb{C}^n \setminus E_F, E_F \setminus E_F^1) \tag{3}$$

defines a unramified covering. Furthermore, by the construction of σ_F the lines l_t intersect transversally E_F, do not pass through E_F^1 as well as tangent to E_F at infinity. Then by a standard way we can construct a continue family of homeomorphisms

$$\Gamma_t : (l_0, l_0 \setminus E_F, l_0 \cap E_F) \longrightarrow (l_t, l_t \setminus E_F, l_t \cap E_F),\ t \in [0,1].$$

The lift of Γ_t by the unramified covering (3) then induces homeomorphisms

$$\Phi_t : (F^{-1}(l_0), F^{-1}(l_0 \setminus E_F), F^{-1}(l_0 \cap E_F)) \longrightarrow (F^{-1}(l_t), F^{-1}(l_t \setminus E_F), F^{-1}(l_t \cap E_F)).$$

In particular, Φ_1 gives a homeomorphism of $F^{-1}(l_0)$ and $F^{-1}(l_1)$. □

Let (g_F, n_F) denote the topological type of curves in $\mathscr{F}$, where g_F and n_F are genus and the number of irreducible branches at infinity, respectively.

Assertion 3. $n_F > 2$.

Proof. We first show that there exists $(u,v) \in \mathbb{C}^n \times \mathbb{C}^n$ such that $\sigma_F(u,v) \neq 0$ and the line $l := l(u,v)$ is contained in the image $F(\mathbb{C}^n)$. To see it, fix $u \in \mathbb{C}^n \setminus E_F$ and let $V := \{v \in \mathbb{C}^n : v \neq 0, \sigma(u,v) \neq 0\}$. By Lemma 4 V is open dense in $\mathbb{C}^n$ and $\sigma_F(u,v) \neq 0$ for all $v \in V$. Let $S := \mathbb{C}^n \setminus F(\mathbb{C}^n)$ and $K_u := \{v \in \mathbb{C}^n : v = t(s - \bar{u}), s \in S, t \in \mathbb{C}\}$. Then $l(u,v)$ intersect S if and only if $v \in K_u$. So it suffices to show $V \setminus K_u \neq \emptyset$. Observe, S is a closed proper algebraic subset of $\mathbb{C}^n$, since F is locally diffeomorphic on $\mathbb{C}^n$. Furthermore, $\dim S < n-1$. Otherwise, S would contains a hypersurface $h(X) = 0$ for a non-constant polynomial $h \in \mathbb{C}[X]$. This would imply that $h \circ F(X) \equiv c \neq 0$, and consequently, $Dh(F(X)).DF(X) \equiv 0$ that is impossible. So, the cone K_v is of pure dimension $< n$. Hence $V \setminus K_u$ is open dense in $\mathbb{C}^n$.

Now, we will prove $n_F > 2$. By the above observation we can take a line $l = l(u,v)$ such that $\sigma_F(u,v) \neq 0$ and $l \subset F(\mathbb{C}^n)$. Let $C := F^{-1}(l)$ and $\hat{C}$ be a smooth compactification of C. By Assertion 2 $\hat{C}$ is a connected Riemann surface of genus g_F and $\hat{C} \setminus C$ consists of n_F distinct points Regarding l as the line $\mathbb{C}$ in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, we can extend the restriction of F on C to a regular morphism $f : \hat{C} \longrightarrow \mathbb{P}^1$ that gives a ramified covering of d_F-sheets over $\mathbb{P}^1$. The morphism f may be ramified only at points in $\hat{C} \setminus C$, since $JF \equiv 1$. Moreover, by the choice of l we have $\mathbb{P}^1 \setminus f(C) = \{\infty\}$. Now, applying Hurwitz Relation to f we have

$$2 - 2g_F = 2d_F - \sum_{a \in \hat{C} \setminus C} (\deg_a f - 1) \tag{4}$$

, where $\deg_a f$ is the local degree of f at a. Note that $\deg_a f \leq d_F$ and $\deg_a f = d_F$ only if $f^{-1}(\{f(a)\}) = \{a\}$. From (4) it follows that $g_F = 0$ and $d_F = 1$ for when $n_F = 1$ and that $g_F = 0$ and $\mathbb{P}^1 \setminus f(C)$ consists of two distinct points for when $n_F = 2$. So, both of the cases $n_F = 1;2$ are impossible. □

All (a-c) now follow from the assertions 1-3. □

3. Proof of Theorem A

In sequences, for any number field K we denote by $\mathbb{S}^n_K$ the set of all primitive vectors in O^n_K, $\mathbb{S}^n_K := \{v = (v_1,\dots,v_n) \in O^n_K : \gcd\{v_i\} = 1\}$. We will prove the following variant of Theorem A.

Theorem A' *Let K be a number field K. The conjecture $JC(K,n)$, $n>1$, is equivalent to the statement:*

$DJC(K,n,m)$: Let $m \in \mathbb{N}$, $n > m \geq 0$. For every Keller map $F \in O_K[X]$ with $F(0)=0$ the affine curve

$$\begin{cases} F_1(X) = \cdots = F_m(X) = 0 \\ F_{m+1}(X) = \cdots = F_n(X) \end{cases} \qquad \mathscr{C}_F(m)$$

has non-zero points in O^n_K.

Proof. We need to use the following elementary fact.

Claim 1. *Let K be a number field K. For $v,w \in \mathbb{S}^n_K$ there is a matrix $A \in SL(n,O_K)$ such that $Av = w$.*

Proof. It suffices to show that for each $v \in \mathbb{S}^n_K$ there is $A \in SL(n,O_K)$ such that $Ae_1 = v$, where $e_1 := (1,0,0,...,0)^T$. We do prove that by induction on n. The cases $n = 1,2$ are obvious. Let $n \geq 3$ and let $v = (v_1,v_2,\dots,v_n)^T \in S^n_K$ be given. Without a loss of generality we may assume that $v_1 \neq 0$. Let $\bar{v} := r^{-1}(v_2,...,v_n)^T$ with $r := \gcd(v_2,...,v_n)$. Note that $\gcd(r,v_1) = 1$. By induction assumption there is a $(n-1)\times(n-1)$ matrix $\bar{A} := [\bar{v}\ \ B] \in SL(n-1,O_K)$, for which $\bar{A}\bar{e}_1 = \bar{v}$, where $\bar{e}_1 = (1,0,\dots,0)^T \in \mathbb{S}^{n-1}_K$. Let

$$A(\alpha,\beta) := \begin{bmatrix} v_1 & O & \beta \\ r\bar{v} & B & \alpha\bar{v} \end{bmatrix},$$

which is a $n\times n$-matrix with entries in O_K and parameters $\alpha,\beta \in O_K$. Obviously, $A(\alpha,\beta)e_1 = v$ and $\det A(\alpha,\beta) = \alpha v_1 + (-1)^n\beta r$. Since $\gcd(r,v_1) = 1$, we can choose $0 \neq \alpha_0,\beta_0 \in O_K$ such that $\det A(\alpha_0,\beta_0) = 1$, i.e. $A(\alpha_0,\beta_0) \in SL(n,O_K)$. This concludes the proof. □

To prove the theorem it suffices to verify that if $JC(K,n)$ is false, then $DJC(K,n,m)$ has a counterexample. Assume that $G \in K[X]^n$ is a non-invertible Keller map $G \in K[X]^n$. We will show that there is a non-invertible Keller map $H \in O_K[X]^n$ with $H(0) = 0$ and a line $l = l(0,v)$, $v \in \mathbb{S}^n_K$, such that $H^{-1}(l) \cap O^n_K = \{0\}$. Then, by the above claim we may choose a matrix $A \in SL(n,O_K)$ such that $Av = w$, where $w_i = 0$ for $i = \overline{1,m}$ and $w_i = 1$ for $i = \overline{m+1,n}$. The map $F(X) := A \circ H \circ A^{-1}(X)$ then provides a counterexample to $DJC(K,n,m)$.

Now, we determine such desired H and v. By changing coordinates, if necessary, we may assume that $G \in O_K[X]^n$, $G(0) = 0$ and 0 is not a bifurcation value of G. Note that the algebraic closure of $\mathbb{S}^n_K$ is the whole space $\mathbb{C}^n$. Since $0 \notin E_G$, by Main Lemma we may take a line $l := l(0,v)$ with $v \in \mathbb{S}^n_K$ and $\sigma_F(0,v) \neq 0$ such that $S := G^{-1}(l) \cap O^n_K$ is a finite set. If $S = \{0\}$, we put $H(X) := G(X)$. If $S \neq \{0\}$, we may choose a number $0 \neq r \in O_K$ such that $S \cap rO^n_K = \{0\}$, and then put $H(X) := \frac{1}{r}G(rX) = G^{(1)}(X) + rG^{(2)}(X) + \cdots + r^{k-1}G^{(k)}(X)$, where $G^{(i)}$ are i-degree homogeneous components of G. Observe, $H \in O_K[X]^n$ and is a Keller map. . Moreover, if $x \in H^{-1}(l) \cap O^n_K$, then $G(rx) = rH(x) \in l$, i.e. $rx \in S \cap O^n_K$. Therefore, $x = 0$ by the choice of r. Hence, $H^{-1}(l) \cap O^n_K = \{0\}$. □

4. PROOF OF THEOREM B

First, we recall the following elementary properties of the bifurcation value sets, which will be used later.

Proposition 1. *Let $F \in \mathbb{C}[X]^n$ with $F(0)=0$.*

i) $E_{P\circ F} = P(E_F)$ *for polynomial automorphisms P of $\mathbb{C}^n$.*
ii) $E_{(^rF)} = \frac{1}{r}E_F$ *for* $(^rF)(X) := \frac{1}{r}F(rX)$, $0 \neq r \in \mathbb{C}$;
iii) $E_{\hat{F}} = E_F \times \mathbb{C}^m$ *for* $\hat{F}(X,Y) := (F(X),Y) \in \mathbb{C}[X,Y]^{n+m}$.

The proof is elementary and left to the readers.

After Bass, Connell and Wright [1], Yagzhev [13], Drużkowski [3] and Connell and Van den Dries [2], it is well known that if the Jacobian conjecture over $\mathbb{C}$ is false, then for some $n \gg 1$ there exists counterexample of the cubic homogeneous form $F_i(X) = X_i + H_i(X)$ with 3-degree homogeneous polynomials $H_i \in R[X]$, or even of the cubic linear form $F_i(X) = X_i + <a_i, X>^3$, $a_i \in R^n$, where $R = \mathbb{R};\mathbb{C}$ and even $R = \mathbb{Z}$, as announced personally by Drużkowski. To prove Theorem B we need the following version of the above reductions.

Theorem C *If the Jacobian conjecture over $\mathbb{C}$ is false, then it has a counterexample $F \in \mathbb{Z}[X]^n$ of the cubic linear form $F_i(X) = X_i + <a_i, X>^3$, $a_i \in \mathbb{Z}^n$, such that $0 \notin E_F$.*

Proof. Assume the Jacobian conjecture over $\mathbb{C}$ is false. Following [2], there exists a non-invertible Keller map $H \in \mathbb{Z}[Z]^l$ of degree 3 for some $l \gg 1$. For such a given H, we may take a point $a \in \mathbb{Z}^n$ such that $H(a)$ is not a bifurcation value of H. Replacing H by $H(Z-a) - H(a)$ we obtain a non-invertible Keller map $H \in \mathbb{Z}[Z]^l$ with $\deg H = 3$, $H(0) = 0$ and $0 \notin E_H$. Then, by Drużkowski's reduction procedure in [3] for some $m \gg 1$ we may take a new variable $X = (Z, Y_1, Y_2, ..., Y_m)$, define a new map $\hat{H}(X) := (H(Z), Y_1, \dots, Y_m)$ and determine polynomial isomorphisms P and Q of $\mathbb{Q}^{l+m}$ with $P(0) = 0$ and $Q(0) = 0$ such that the map $G(X) := P \circ \hat{H} \circ Q(X)$ is a non-invertible Keller map of the form $G_i(X) = X_i + <b_i, X>^3$, $b_i \in \mathbb{Q}^n$. By Proposition 1 we have that $E_{\hat{H}} = E_H \times \mathbb{C}^m$ and $E_G = P(E_{\hat{H}})$. Therefore, $0 \notin E_G$, since $P(0) = Q(0) = 0$ and $0 \notin E_H$.

Now, we can choose an integer $r \neq 0$ such that all $rb_i \in \mathbb{Z}^n$. Define $F(X) := \frac{1}{r^3}G(r^3X)$. We have $F_i(X) = X_i + r^{-6} <b_i, r^3X>^3 = X_i + <rb_i, X>^3$, $i = \overline{1,n}$. Moreover $0 \notin E_F$, since $0 \notin E_G$ and $E_F = \frac{1}{r^3}E_G$ by Proposition 1. So F provides a desired counterexample. □

Proof of Theorem B Assume that the Jacobian conjecture is false. First, by Theorem C there exists a non-invertible Keller map $F \in \mathbb{Z}[X]^n$ of the cubic linear form over $\mathbb{Z}$, $F_i(X) = X_i + <c_i, X>^3$, $c_i \in \mathbb{Z}^n$, such that $0 \notin E_F$.

Let $\sigma_F(U,V) \in \overline{\mathbb{Q}}[U,V]$ be as in Main Lemma. Since $0 \notin E_F$, by Main Lemma (a) $\sigma_F(0,V) \not\equiv 0$. Note that the algebraic closure of $\{(w_1^3, \dots, w_n^3) : (w_1, \dots, w_n) \in \mathbb{S}^n_{\mathbb{Z}}\}$ is the whole $\mathbb{C}^n$. So we can choose a vectors $w \in \mathbb{S}^n_Z$ with all $w_i \neq 0$ such that $\sigma_F(0,v) \neq 0$ for $v := (w_1^3, \dots, w_n^3)$. Then by Main Lemma (b) the curve $F^{-1}(l(0,v))$ has at most finitely many points in $\mathbb{Z}^n$. Therefore, there is $0 \neq r \in \mathbb{Z}$ such that the curve $F^{-1}(l(0,v))$ has no non-zero points in $r\mathbb{Z}^n$. Let $\hat{F}(X) := \frac{1}{r^3}F(r^3X)$. Obviously, $\hat{F}$ is a Keller map of the linear cubic form, $\hat{F}_i(X) = X_i + <b_i, X>^3$, where $b_i := rc_i \in \mathbb{Z}^n$, and the curve $\hat{F}^{-1}(l(0,v))$ has no non-zero integer points.

Now, let $T : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the linear transform of $\mathbb{C}^n$ defined by $T_i(X) = v_iX_i$ and

$$\delta := \prod_{i=1}^n v_i = \prod_{i=1}^n w_i^3.$$

We define a new Keller map $G(X) := \frac{1}{\delta} T^{-1} \circ \hat{F} \circ T(\delta X)$. For this map, the curve $\mathscr{C}_G$ has no non-zero integer point. Indeed, if $x \in \mathscr{C}_G \cap \mathbb{Z}^n$, $G(x) = t(1,\ldots,1)$, then $\frac{1}{\delta}\hat{F} \circ T(\delta x) = T \circ G(x) = tv$, and hence, $T(\delta x) \in \hat{F}^{-1}(l(0,v)) \cap \mathbb{Z}^n$. So, $x = 0$ by the choice of $\hat{F}$. On the other hand, we have

$$\begin{aligned} G_i(X) &= X_i + \frac{1}{\delta}\frac{1}{v_i} < b_i, T(\delta X) >^3 \\ &= X_i + \frac{\delta^2}{v_i} < T(b_i), X) >^3 \\ &= X_i + < a_i, X >^3 \end{aligned}$$

, where $a_i := (w_i^{-1}\prod_{j=1}^n w_j^2)T(b_i) \in \mathbb{Z}^n$. Thus, G provides a desired Keller map. □

Corollary 2. *To prove the Jacobian conjecture over $\mathbb{C}$ for all $n > 1$ it suffices to verify that there does not exist Keller maps $F \in \mathbb{Z}[X]^n$ of the linear cubic form over $\mathbb{Z}$, $F_i(X) = X_i +, a_i, X >^3$, $a_i \in \mathbb{Z}^n$, such that the equation $F_1(X)^2 + \cdots + F_{n-1}(X)^2 = 0$ has no non-zero integer solutions.*

Proof. It suffices to show that if the Jacobian conjecture over $\mathbb{C}$ is false, then there exists $n >> 1$ and a Keller map $F \in \mathbb{Z}[X]^n$ of the linear cubic form over $\mathbb{Z}$ such that the equation system $F_1(X) = \cdots = F_{n-1}(X) = 0$ has only trivial integer solution. Assume that we have a counterexample to the Jacobian conjecture over $\mathbb{C}$. In view of Theorem B for some $n \gg 1$ we can construct a non-invertible Keller map $F \in \mathbb{Z}[X]^n$ of the cubic linear form over $\mathbb{Z}$, $F_i(X) = X_i + < a_i, X >^3$, $a_i \in \mathbb{Z}^n$, such that the curve $F_1(X) = \cdots = F_n(X)$ has no non-zero integer points. Let $G \in \mathbb{Z}[X, X_{n+1}]^{n+1}$ given by

$$G_i(X, X_{n+1}) := \begin{cases} F_i(X_1 + X_{n+1}, \ldots, X_n + X_{n+1}) - X_{n+1} & i = \overline{1,n} \\ X_{n+1} & i = n+1. \end{cases}.$$

Obviously, G is a Keller map and the system $G_1(X, X_{n+1}) = \cdots = G_n(X, X_{n+1}) = 0$ also has no non-zero integer solutions. Moreover, for each $i = \overline{1,n}$

$$\begin{aligned} G_i(X, X_{n+1}) &= X_i + X_{n+1} + (\sum_{j=1}^n a_{ij}(X_j + X_{n+1}))^3 - X_{n+1} \\ &= X_i + (\sum_{j=1}^n a_{ij}X_j + (\sum_{j=1}^n a_{ij})X_{n+1})^3 \end{aligned}.$$

Thus, G provides a desired Keller map of linear cubic form over $\mathbb{Z}$. □

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INSTITUTE OF MATHEMATICS, VIETNAM ACADEMY OF SCIENCE AND TECHNOLOGY, 18 HOANG QUOC VIET, 10307 HANOI, VIETNAM.
E-mail address: nvchau@math.ac.vn