A Bohr-Nikol'skii inequality for weighted Lebesgue spaces

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Dedicated to Professor Le Tuan Hoa on the occasion of his 60th-birthday

Abstract

In this paper, we give a new inequality for weighted Lebesgue spaces called Bohr-Nikol'skii inequality, which combines the inequality of Bohr-Favard and the Nikol'skii idea of inequality for functions in different metrics.

Key words: L^p- spaces, Bohr-Favard inequality, Nikol'skii inequality
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1. Introduction

Let $m \geq 1$, $f \in C^m(\mathbb{R})$,$D^j f \in L^\infty(\mathbb{R})$, $j = 0, 1, ..., m$, $\sigma > 0$ and supp$\hat{f} \cap (-\sigma, \sigma) = \emptyset$, where $\hat{f}$ is the Fourier transform of f. Then it is known the following Bohr-Favard inequality (see [5, 6]):

$$\|f\|_\infty \leq \sigma^{-m} K_m \|D^m f\|_\infty,$$

where the Favard constants K_m are sharp in the sense that these cannot be replaced by smaller ones and defined by

$$K_m = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^{j(m+1)}}{(2j+1)^{m+1}} , \quad m \in \mathbb{Z}_+.$$

The Favard constants satisfy the following properties

$$1 = K_0 < K_2 = \frac{\pi^2}{8} < K_4 < ... < \frac{4}{\pi} < ... < K_3 = \frac{\pi^3}{24} < K_1 = \frac{\pi}{2}.$$

This inequality was extended to L^p-norm in [1]: Let $1 \leq p \leq \infty, m \geq 1$, $\sigma > 0$, $f \in L^p(\mathbb{R})$, $D^m f \in L^p(\mathbb{R})$ and supp$\hat{f} \cap (-\sigma, \sigma) = \emptyset$, where $D^m f$ is the m^{th}-generalized derivative of f. Then

$$\|f\|_p \leq \sigma^{-m} K_m \|D^m f\|_p,$$

where K_m are the Favard constants defined above.

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The Bohr-Favard inequality was studied also in [7, 9, 4]. The main purpose of this paper is to derive a new Bohr-Nikol'skii inequality for weighted Lebesgue spaces, which combines the inequality of Bohr-Favard and the Nikol'skii idea of inequality for functions in different metrics (see [12, 13]). Note that the Nikol'skii inequality was studied in [9 - 14] and the Bohr-Nikol'skii inequality for Lebesgue spaces was studied in [3].

2. Main results

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions and by $\mathcal{S}^{'}(\mathbb{R})$ the dual space of $\mathcal{S}(\mathbb{R})$, the space of tempered distributions on $\mathbb{R}$. If $f \in \mathcal{S}^{'}(\mathbb{R})$ then the support of f, denoted supp$f$, is the set of points in $\mathbb{R}$ having no open neighborhood to which the restriction of f is 0. The Fourier transform $\mathcal{F}$ of a tempered generalized function f can be defined via the formula

$$\big\langle \mathcal{F}f, \varphi \big\rangle = \big\langle f, \mathcal{F}\varphi \big\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}),$$

and the m^{th}-generalized derivative of f, denote by $D^m f$, can be defined as follows

$$\big\langle D^m f, \varphi \big\rangle = (-1)^m \big\langle f, D^m \varphi \big\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

Let $1 \le p < \infty, s \in \mathbb{R}$. The weighted Lebesgue space $L^p_s := L^p_s(\mathbb{R})$ consists of all measurable functions such that

$$\|f\|_{L^p_s} = \Big(\int_{\mathbb{R}} |f(x)|^p |x|^{ps} dx \Big)^{1/p} < \infty.$$

Note that $L^p_s(\mathbb{R})$ is a Banach space and $L^p_s(\mathbb{R})$ becomes the usual $L^p(\mathbb{R})$ space if $s = 0$.
We recall the following result in [8] as known an extension of Young's Inequality for the weighted Lebesgue spaces.

Lemma 1. *Let* $1 < u, p, r < \infty, 1/p \le 1/u + 1/r,\ 1/p = 1/u + 1/r + v + q + \gamma - 1,$ $v < 1 - 1/u, q < 1/p, \gamma < 1 - 1/r, \gamma + q \ge 0, \gamma + v \ge 0, q + v \ge 0$ *and let* $f \in L^u_v(\mathbb{R}), g \in L^r_\gamma(\mathbb{R})$. *Then* $f * g \in L^p_{-q}(\mathbb{R})$ *and there exists a constant* C *independent of* f, g *such that*

$$\|f * g\|_{L^p_{-q}} \le C \|f\|_{L^u_v} \|g\|_{L^r_\gamma} \ ,$$

where

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y) g(y) dy.$$

Now, we state our main theorem.

Theorem 2. *Let* $1 < u, p < \infty, 0 < q + 1/p < v + 1/u < 1,\ v - q \ge 0,\ m \ge 3,\ \sigma > 0,$ *and* $f \in \mathcal{S}^{'}(\mathbb{R})$ *such that it's* m^{th}*-generalized derivative* $D^m f \in L^u_v(\mathbb{R})$ *and* $\mathrm{supp}\hat{f} \cap (-\sigma, \sigma) = \emptyset$. *Then* $f \in L^p_q(\mathbb{R})$ *and there exists a constant* $C > 0$ *independent of* f, m, σ *such that*

$$\|D^m f\|_{L^u_v} \ge C m^\lambda \sigma^{m - \lambda} \|f\|_{L^p_q}, \tag{1}$$

where

$$\lambda = v + \frac{1}{u} - q - \frac{1}{p} > 0.$$

PROOF. Let us first prove (1) for the case $\sigma = 1$. To do that, we denote $K := (-\infty, -1] \cup [1, +\infty)$, $K_\epsilon := (-\infty, -(1+\epsilon)] \cup [1+\epsilon, +\infty)$ for each $\epsilon > 0$, and

$$\Upsilon(y) = \begin{cases} C_1 e^{1/(y^2-1)} & \text{if } |y| < 1, \\ 0 & \text{if } |y| \geq 1, \end{cases}$$

where C_1 is chosen such that $\int_{\mathbb{R}} \Upsilon(y) dy = 1$. We define the sequence of functions $(\phi_m(y))_{m\in\mathbb{N}}$ via the formula

$$\phi_m(y) = (1_{K_{3/(4m)}} * \Upsilon_{1/(4m)})(y),$$

where

$$\Upsilon_{1/(4m)}(y) = 4m\Upsilon(4my).$$

Then $\Upsilon_{1/(4m)}(y) = 0$ for all $y \notin [-1/(4m), 1/(4m)]$, $\int_{\mathbb{R}} \Upsilon_{1/(4m)}(y)dy = 1$. Hence, for all $m \in \mathbb{N}$ we have $\phi_m(y) \in C^\infty(\mathbb{R})$ and

$$\phi_m(y) = 1 \quad \forall y \in K_{1/(2m)}, \phi_m(y) = 0 \quad \forall y \notin K_{1/m}. \tag{2}$$

So, it follows from supp$\widehat{D^m f} \subset K$ that $\phi_m(y)\widehat{D^m f} = \widehat{D^m f}$. Therefore, since

$$\widehat{D^m f} = (-iy)^m \hat{f},$$

we get

$$\phi_m(y)\widehat{D^m f} = (-iy)^m \hat{f},$$

and then

$$\widehat{D^m f}\phi_m(y)/(-iy)^m = \hat{f}.$$

Hence

$$f = (2\pi)^{-1/2}(D^m f) * \mathcal{F}^{-1}(\phi_m(y)/(-iy)^m). \tag{3}$$

We consider two numbers r, γ satisfies $1 < r < \infty$, $q + \frac{1}{p} - v - \frac{1}{u} = \frac{1}{r} + \gamma - 1$, $\gamma + v \geq 0$, $\gamma - q \geq 0$, $v - q + \gamma \leq 1$. From the hypothesis, we have $\frac{1}{p} \leq \frac{1}{u} + \frac{1}{r}$, $\gamma < 1 - \frac{1}{r}$, $v < 1 - 1/u$ and $-q < 1/p$. Therefore, due to (3) and Lemma 1, we obtain the following estimate

$$\begin{aligned} \|f\|_{L_q^p} &\leq (2\pi)^{-1/2}\|D^m f\|_{L_v^u}\|\mathcal{F}^{-1}(\phi_m(y)/y^m)\|_{L_\gamma^r} \\ &= (2\pi)^{-1/2}\|D^m f\|_{L_v^u}\|\mathcal{F}(\phi_m(y)/y^m)\|_{L_\gamma^r}. \end{aligned} \tag{4}$$

We define

$$k_m := 1 + \frac{1}{m}, \vartheta_m(y) = \phi_m(k_m y), \Phi_m(y) = \phi_m(y) - \vartheta_m(y).$$

Then

$$(\mathcal{F}(\vartheta_m(y)/y^m))(x) = (k_m)^m(\mathcal{F}(\phi_m(k_m y)/(k_m y)^m)(x) = (k_m)^{m-1}(\mathcal{F}(\phi_m(y)/y^m))(x/k_m).$$

So,

$$\left\|\mathcal{F}(\vartheta_m(y)/y^m)\right\|_{L_\gamma^r} = (k_m)^{m-1+\gamma+\frac{1}{r}}\left\|\mathcal{F}(\phi_m(y)/y^m)\right\|_{L_\gamma^r}.$$

Hence, it follows from $(k_m)^{m-1+\gamma+\frac{1}{r}} \geq (k_m)^{m-1} = (1+\frac{1}{m})^{m-1} \geq \frac{3}{2}$ that

$$\Big\|\mathcal{F}(\vartheta_m(y)/y^m)\Big\|_{L^r_\gamma} \geq \frac{3}{2}\Big\|\mathcal{F}(\phi_m(y)/y^m)\Big\|_{L^r_\gamma}.$$

Therefore, since $\Phi_m(y) = \phi_m(y) - \vartheta_m(y)$ we get

$$\begin{aligned}\Big\|\mathcal{F}(\Phi_m(y)/y^m)\Big\|_{L^r_\gamma} &\geq \Big\|\mathcal{F}(\vartheta_m(y)/y^m)\Big\|_{L^r_\gamma} - \Big\|\mathcal{F}(\phi_m(y)/y^m)\Big\|_{L^r_\gamma} \\ &\geq \frac{1}{2}\Big\|\mathcal{F}(\phi_m(y)/y^m)\Big\|_{L^r_\gamma}. \end{aligned} \tag{5}$$

From (4)-(5) we obtain

$$\|f\|_{L^p_q} \leq 2(2\pi)^{-1/2}\|D^m f\|_{L^u_v}\|\mathcal{F}(\Phi_m(y)/y^m)\|_{L^r_\gamma}. \tag{6}$$

Now, we will estimate $\|\mathcal{F}(\Phi_m(y)/y^m)\|_{L^r_\gamma}$. To do that, we put $C_2 = \max\{\|\Upsilon^{(j)}\|_{L^1}, \quad j \leq 3\}$. By $\Upsilon_{1/(4m)}(x) = 4m\Upsilon(4mx)$, we obtain $\Upsilon^{(j)}_{1/(4m)}(x) = (4m)^{j+1}\Upsilon^{(j)}(4mx)$ and then

$$\|\Upsilon^{(j)}_{1/(4m)}\|_{L^1} = (4m)^j\|\Upsilon^{(j)}\|_{L^1} \leq C_2(4m)^j, \quad \forall j \leq 3.$$

Therefore,

$$\Big\|\phi^{(j)}_m\Big\|_{L^\infty} = \Big\|(1_{K_{3/(4m)}} * \Upsilon^{(j)}_{1/(4m)})\Big\|_{L^\infty} \leq \Big\|\Upsilon^{(j)}_{1/(4m)}\Big\|_{L^1} \leq (4m)^j C_2, \quad \forall j \leq 3. \tag{7}$$

Note that $\phi_m(y) = 1 \quad \forall y \in (-\infty, -1+(1/2m)] \cup [1-(1/2m), +\infty)$ and $\phi_m(y) = 0 \quad \forall y \in [-1+(1/m), 1-(1/m)]$.

So, if $|y| < 1-(3/m)$, then $|y| < |k_m y| < 1-(1/m)$ and then $\phi_m(y) = \phi_m(k_m y) = 0$, which implies $\Phi_m(y) = 0$. Further, if $|y| > 1$ then $|k_m y| > |y| > 1$ and then $\phi_m(y) = \phi_m(k_m y) = 1$, which implies $\Phi_m(y) = 0$. From these we have

$$\text{supp}\Phi_m \subset [1-(3/m), 1] \cup [-1, (3/m)-1]. \tag{8}$$

Now, for $y \in [1-(3/m), 1] \cup [-1, (3/m)-1]$ we get

$$\Big|y - k_m y\Big| = \Big|\frac{y}{m}\Big| \leq \frac{1}{m}. \tag{9}$$

From (7) and (9) we have the following estimate for $y \in [1-(3/m), 1] \cup [-1, (3/m)-1]$

$$\begin{aligned}\Big|\Phi_m(y)\Big| &= \Big|\phi_m(y) - \vartheta_m(y)\Big| = \Big|\phi_m(y) - \phi(k_m y)\Big| \\ &\leq \Big\|y - k_m y\Big|.\Big\|\phi'_m\Big\|_{L^\infty} \leq \frac{1}{m}4mC_2 = 4C_2, \end{aligned} \tag{10}$$

and

$$\begin{aligned}\Big|\Phi'_m(y)\Big| &= \Big|\phi'_m(y) - \vartheta'_m(y)\Big| = \Big|\phi'_m(y) - \phi'_m(k_m y)\Big| \\ &= \Big|\phi'_m(y) - k_m\phi'_m(k_m y)\Big| \leq \Big|\phi'_m(y) - \phi'_m(k_m y)\Big| + \Big|(1-k_m)\phi'_m(k_m y)\Big| \\ &\leq \Big|y - k_m y\Big|.\Big\|\phi''_m\Big\|_{L^\infty} + \Big|1 - k_m\Big|.\Big\|\phi'_m\Big\|_{L^\infty}\end{aligned}$$

$$\leq \frac{1}{m}(4m)^2C_2 + \Big|1-k_m\Big|4mC_2$$
$$\leq 20mC_2. \tag{11}$$

Put $\Psi_m(x) = (\mathcal{F}(\Phi_m(y)/y^m))(x)$. Then

$$\Psi_m(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} e^{ixy}\Phi_m(y)/y^m dy.$$

Therefore, due to (8), we have

$$\sup_{x\in\mathbb{R}}\Big|\Psi_m(x)\Big| \leq \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}}\Big|\Phi_m(y)/y^m\Big|dy = \frac{1}{\sqrt{2\pi}}\int\limits_{1-\frac{3}{m}\leq|y|\leq 1}\Big|\Phi_m(y)/y^m\Big|dy$$

and it follows from (7) that

$$\sup_{x\in\mathbb{R}}\Big|\Psi_m(x)\Big| \leq \frac{6}{m\sqrt{2\pi}}\sup_{y\in\mathbb{R}}\Big|\Phi_m(y)\Big|(1-\frac{3}{m})^{-m} \leq \frac{24e^4C_2}{m\sqrt{2\pi}}. \tag{12}$$

We also obtain

$$\sup_{x\in\mathbb{R}}\Big|x\Psi_m(x)\Big| = \frac{1}{\sqrt{2\pi}}\sup_{x\in\mathbb{R}}\Big|\int\limits_{\mathbb{R}} e^{ixy}(\frac{m\Phi_m(y)}{y^{m+1}} - \frac{\Phi_m'(y)}{y^m})dy\Big|$$
$$\leq \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}}\Big|\frac{m\Phi_m(y)}{y^{m+1}} - \frac{\Phi_m'(y)}{y^m}\Big|dy.$$

Therefore, due to (7)-(8), we have

$$\sup_{x\in\mathbb{R}}\Big|x\Psi_m(x)\Big| \leq \frac{1}{\sqrt{2\pi}}\int\limits_{1-\frac{3}{m}\leq|y|\leq 1}\Big|\frac{m\Phi_m(y)}{y^{m+1}} - \frac{\Phi_m'(y)}{y^m}\Big|dy$$
$$\leq \frac{6}{m\sqrt{2\pi}}\sup_{1-\frac{3}{m}\leq|y|\leq 1}\Big|\frac{m\Phi_m(y)}{y^{m+1}} - \frac{\Phi_m'(y)}{y^m}\Big|$$
$$\leq \frac{6}{m\sqrt{2\pi}}\Big[\sup_{y\in\mathbb{R}}\Big|\Phi_m(y)\Big|m(1-\frac{3}{m})^{-m-1} + \sup_{y\in\mathbb{R}}\Big|\Phi_m'(y)\Big|(1-\frac{3}{m})^{-m}\Big]$$
$$\leq \frac{6}{m\sqrt{2\pi}}\Big[4C_2me^4 + 20C_2me^4\Big] = \frac{144e^4C_2}{\sqrt{2\pi}}. \tag{13}$$

We see that

$$\big\|\Psi_m\big\|^r_{L^r_\gamma} = \int\limits_{|x|\leq m}\big|x^\gamma\Psi_m(x)\big|^r dx + \int\limits_{|x|\geq m}\big|x^\gamma\Psi_m(x)\big|^r dx$$
$$\leq \sup_{x\in\mathbb{R}}\big|\Psi_m(x)\big|^r\int\limits_{|x|\leq m} x^{\gamma r}dx + \sup_{x\in\mathbb{R}}\big|x\Psi_m(x)\big|^r\int\limits_{|x|\geq m}\Big|\frac{1}{x^{r-\gamma r}}\Big|dx.$$

Due to $r-\gamma r > 1$, we have $\int\limits_{|x|\geq m}\Big|\frac{1}{x^{r-\gamma r}}\Big|dx < \infty$, and then

$$\big\|\Psi_m\big\|^r_{L^r_\gamma} \leq 2m^{\gamma r+1}\sup_{x\in\mathbb{R}}\big|\Psi_m(x)\big|^r + \frac{2m^{\gamma r+1-r}}{r-\gamma r-1}\sup_{x\in\mathbb{R}}\big|x\Psi_m(x)\big|^r. \tag{14}$$

From (12)-(14), we obtain

$$\Big\|\Psi_m\Big\|_{L^r_\gamma}^r \le 2m^{\gamma r+1}\Big(\frac{24e^4C_2}{m\sqrt{2\pi}}\Big)^r + \frac{2m^{\gamma r+1-r}}{r-\gamma r-1}\Big(\frac{144e^4C_2}{\sqrt{2\pi}}\Big)^r = 2m^{-r+\gamma r+1}\Big(\frac{e^4C_2}{\sqrt{2\pi}}\Big)^r(\frac{144^r}{r-\gamma r-1}+96^r),$$

and then

$$\Big\|\Psi_m\Big\|_{L^r_\gamma} \le \frac{e^4C_2}{\sqrt{2\pi}}(\frac{144^r2}{r-\gamma r-1}+96^r2)^{\frac{1}{r}}m^{-1+\gamma+\frac{1}{r}} = m^{-1+\gamma+\frac{1}{r}}/C_3, \tag{15}$$

where $C_3 = \sqrt{2\pi}/e^4C_2(\frac{144^r2}{r-\gamma r-1}+96^r2)^{\frac{1}{r}}$. From (6) and (15) we have

$$\|D^m f\|_{L^u_v} \ge Cm^{1-\gamma-\frac{1}{r}}\|f\|_{L^p_q}.$$

So, (1) has been proved for the case $\sigma = 1$.

Next, we prove (1) for any $\sigma > 0$. To do that, we define a function f_σ as follows

$$f_\sigma(x) = f(x/\sigma), \quad x \in \mathbb{R}.$$

Clearly, $(D^m f_\sigma)(x) = \sigma^{-m}(D^m f)(x/\sigma)$. Hence,

$$\|f_\sigma\|_{L^p_q} = \sigma^{q+\frac{1}{p}}\|f\|_{L^p_q}, \|D^m f_\sigma\|_{L^u_v} = \sigma^{-m+v+\frac{1}{u}}\|D^m f\|_{L^u_v}. \tag{16}$$

From $\mathrm{supp}\hat{f}\cap(-\sigma,\sigma)=\emptyset$ we deduce $\mathrm{supp}\widehat{f_\sigma}\cap(-1,1)=\emptyset$. Therefore,

$$\|D^m f_\sigma\|_{L^u_v} \ge Cm^\lambda\|f_\sigma\|_{L^p_q},$$

where $\lambda = v+\frac{1}{u}-q-\frac{1}{p}$. Hence, it follows from (16) that

$$\sigma^{-m+v+\frac{1}{u}}\|D^m f\|_{L^u_v} \ge Cm^\lambda\sigma^{q+\frac{1}{p}}\|f\|_{L^p_q}.$$

So,

$$\begin{aligned}\|D^m f\|_{L^u_v} &\ge Cm^\lambda\sigma^{m+q+\frac{1}{p}-v-\frac{1}{u}}\|f\|_{L^p_q}\\ &= Cm^\lambda\sigma^{m-\lambda}\|f\|_{L^p_q}.\end{aligned}$$

The proof is complete.

For $\sigma > 0$ we denote

$$L^u_{v,\sigma} = \{f \in L^u_v(\mathbb{R}) : \mathrm{supp}\hat{f}\cap(-\sigma,\sigma)=\emptyset\}.$$

The norm of the derivative operator D^m is given by

$$\|D^m\|_{L^u_{v,\sigma}\to L^p_q} = \sup_{\|f\|_{L^u_{v,\sigma}\le 1}}\|D^m f\|_{L^p_q}.$$

From Theorem 2, we have the following corollary about the norm of derivative operators.

Corollary 1. Let $1 < u, p < \infty, 0 < q+1/p < v+1/u < 1$, $v-q \ge 0$, $m \ge 3$, $\sigma > 0$. Then there exists a constant $C > 0$ independent of m, σ such that

$$\|D^m\|_{L^u_{v,\sigma}\to L^p_q} \ge Cm^\lambda\sigma^{m-\lambda},$$

where

$$\lambda = v+\frac{1}{u}-q-\frac{1}{p} > 0.$$

If $p=u$,using Theorem 2, we have the following result.

Corollary 2. Let $1<p<\infty, -1/p<q<v<1-1/p$, $m\geq 3$, $\sigma>0$, and $f\in \mathcal{S}^{'}(\mathbb{R})$ such that it's m^{th}-generalized derivative $D^m f\in L^p_v(\mathbb{R})$ and $\text{supp}\hat{f}\cap(-\sigma,\sigma)=\emptyset$. Then $f\in L^p_q(\mathbb{R})$ and there exists a constant $C>0$ independent of f, m, σ such that

$$\|D^m f\|_{L^p_v}\geq Cm^{\lambda}\sigma^{m-\lambda}\|f\|_{L^p_q},$$

where

$$\lambda=v-q>0.$$

If $q=v$, we have the following result from Theorem 2.

Corollary 3. Let $1<u<p<\infty, -1/p<q<1-1/u$, $m\geq 3$, $\sigma>0$, and $f\in \mathcal{S}^{'}(\mathbb{R})$ such that it's m^{th}-generalized derivative $D^m f\in L^u_q(\mathbb{R})$ and $\text{supp}\hat{f}\cap(-\sigma,\sigma)=\emptyset$. Then there exists a constant $C>0$ independent of f, m, σ such that

$$\|D^m f\|_{L^u_q}\geq Cm^{\lambda}\sigma^{m-\lambda}\|f\|_{L^p_q},$$

where

$$\lambda=\frac{1}{u}-\frac{1}{p}>0.$$

Using Theorem 2 in the case $q=0$, we have the following:

Corollary 4. Let $1<u,p<\infty, 1/p<v+1/u<1$, $v\geq 0$, $m\geq 3$, $\sigma>0$, $f\in \mathcal{S}^{'}(\mathbb{R})$ such that it's m^{th}-generalized derivative $D^m f\in L^u_v(\mathbb{R})$ and $\text{supp}\hat{f}\cap(-\sigma,\sigma)=\emptyset$, $f\not\equiv 0$. Then there exists a constant $C>0$ independent of f, m, σ such that

$$\|D^m f\|_{L^u_v}\geq Cm^{\lambda}\sigma^{m-\lambda}\|f\|_{L^p}, \quad (\lambda=v+\frac{1}{u}-\frac{1}{p}).$$

In particular,

$$\lim_{m\to\infty}\|D^m f\|_{L^u_v}/\sigma^m=\infty, \liminf_{m\to\infty}\|D^m f\|^{1/m}_{L^u_v}\geq\sigma.$$

Further, if $v=0$, we have

Corollary 5. Let $1<u,p<\infty, 0<q+1/p<1/u$, $q\leq 0$, $m\geq 3$, $\sigma>0$, $f\in \mathcal{S}^{'}(\mathbb{R})$ such that it's m^{th}-generalized derivative $D^m f\in L^u(\mathbb{R})$ and $\text{supp}\hat{f}\cap(-\sigma,\sigma)=\emptyset$, $f\not\equiv 0$. Then there exists a constant $C>0$ independent of f, m, σ such that

$$\|D^m f\|_{L^u}\geq Cm^{\lambda}\sigma^{m-\lambda}\|f\|_{L^p_q},$$

where

$$\lambda=\frac{1}{u}-q-\frac{1}{p}>0.$$

In particular,

$$\lim_{m\to\infty}\|D^m f\|_{L^u}/\sigma^m=\infty, \liminf_{m\to\infty}\|D^m f\|^{1/m}_{L^u}\geq\sigma.$$

Moreover, if $v=q=0$ then we have the following result from Theorem 2.

Corollary 6. Let $1 < u < p < \infty, m \geq 3$, $\sigma > 0$, $f \in \mathcal{S}^{'}(\mathbb{R})$ such that it's m^{th}-generalized derivative $D^m f \in L^u(\mathbb{R})$ and $\text{supp}\hat{f} \cap (-\sigma, \sigma) = \emptyset$. Then $f \in L^p(\mathbb{R})$ and there exists a constant $C > 0$ independent of f, m, σ such that

$$\|D^m f\|_{L^u} \geq C m^\lambda \sigma^{m-\lambda} \|f\|_{L^p},$$

where

$$\lambda = \frac{1}{u} - \frac{1}{p} > 0.$$

Remark 1. For comparison, using Bohr-Favard inequality for $L^u(\mathbb{R})$, we get $K_m \|D^m f\|_{L^u} \geq \sigma^m \|f\|_{L^u}$ and then the sequence $\{\|D^m f\|_{L^u}/\sigma^m\}_{m=1}^{\infty}$ is separated with the origin, while by Corollary 5 we have the following stronger result: $\lim_{m\to\infty} \|D^m f\|_{L^u}/(m^a \sigma^m) = \infty$ for all $0 < a < \frac{1}{u} - q - \frac{1}{p}$ and for all $f \in L^p_q(\mathbb{R})$, and then the sequence $\{\|D^m f\|_{L^u}/\sigma^m\}_{m=1}^{\infty}$ converges to ∞.

Using Theorem 2 and the Bohr-Favard inequality, we can prove the following result.

Corollary 7. Let $1 < u < p < \infty, \sigma > 0$. Denote

$$N_{\sigma,u} := \{f \in \mathcal{S}^{'}(\mathbb{R}) : \quad \text{supp}\hat{f} \cap (-\sigma, \sigma) = \emptyset, D^m f \in L^u(\mathbb{R}) \text{ for all } m = 0, 1, 2, ...\}$$

and

$$\gamma_m := \inf_{f \in N_{\sigma,u}} \frac{\|D^m f\|_{L^u}}{\sigma^m \|f\|_{L^p}}.$$

Then $\gamma_m \leq \frac{\pi}{2} \gamma_{m+1}$ and

$$\lim_{m\to\infty} \gamma_m = \infty.$$

Let $1 \leq p < \infty$ and $s \in \mathbb{R}$. The weighted Lebesgue space $L^p_s := L^p_s(\mathbb{R}^n)$ consists of all measurable functions such that

$$\|f\|_{L^p_s} = \Big(\int_{\mathbb{R}^n} |f(\mathbf{x})|^p \prod_{j=1}^{n} |x_j|^{ps} d\mathbf{x} \Big)^{1/p} < \infty,$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Consecutively applying Theorem 2 to each variable, we get the following result for the n-dimensional case.

Theorem 3. *Let* $1 < u, p < \infty, 0 < q + 1/p < v + 1/u < 1$, $v - q \geq 0$, $m \geq 3$, $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n_+$, *and* $f \in \mathcal{S}^{'}(\mathbb{R}^n)$ *such that it's* α^{th}*-generalized derivative* $D^\alpha f \in L^u_v(\mathbb{R}^n)$ *and* $\text{supp}\hat{f} \cap (-\sigma_1, \sigma_1) \times \cdots \times (-\sigma_n, \sigma_n) = \emptyset$. *Then* $f \in L^p_q(\mathbb{R}^n)$ *and there exists a constant* $C > 0$ *independent of* f, α, σ *such that*

$$\|D^\alpha f\|_{L^u_v} \geq C \|f\|_{L^p_q} \prod_{j=1, \alpha_j \neq 0}^{n} {\alpha_j}^\lambda \sigma_j^{m-\lambda}, \tag{17}$$

where

$$\lambda = v + \frac{1}{u} - q - \frac{1}{p} > 0.$$

In the following theorem, we give a result for the sequence of L^p_q-norm of primitives of a function (see the notion of primitives of functions in [2, 16]).

Theorem 4. *Let $1 < u,p < \infty, 0 < q+1/p < v+1/u < 1,\ v-q \geq 0,\ f \in L^u_v(\mathbb{R})$, $\sigma = \inf\{|\xi| : \xi \in \mathrm{supp}\hat{f}\} > 0$, and $\{I^m f\}_{m=0}^{\infty} \subset L^p_q(\mathbb{R})$, where $I^0 f = f$, $I^m f$ is a primitive of $I^{m-1}f$, $m = 1,2,\ldots$. Then for $0 < a < \lambda = v + \frac{1}{u} - q - \frac{1}{p}$ we have the following limit*

$$\lim_{m\to\infty} m^a \sigma^m \|I^m f\|_{L^p_q} = 0$$

and

$$\lim_{m\to\infty} \|I^m f\|^{1/m}_{L^p_q} = 1/\sigma. \tag{18}$$

Proof. Similar to the proof in [2] we have

$$\mathrm{supp}\widehat{I^m f} = \mathrm{supp}\hat{f} \quad \forall m \in \mathbb{N}.$$

Therefore, $\mathrm{supp}\widehat{I^m f} \cap (-\sigma, \sigma) = \emptyset$ and then it follows from Theorem 2 that

$$\|f\|_{L^u_v} = \|D^m(I^m f)\|_{L^u_v} \geq C m^\lambda \sigma^{m-\lambda} \|I^m f\|_{L^p_q}.$$

Hence,

$$\lim_{m\to\infty} m^a \sigma^m \|I^m f\|_{L^p_q} = 0$$

for all $0 < a < \lambda$. Consequently,

$$\limsup_{m\to\infty} \|I^m f\|^{1/m}_{L^p_q} \leq 1/\sigma.$$

To complete the proof, we have to obtain

$$\liminf_{m\to\infty} \|I^m f\|^{1/m}_{L^p_q} \geq 1/\sigma. \tag{19}$$

To do that, we consider $0 < \epsilon < \sigma$. Without loss of generality we may assume that $\sigma \in \mathrm{supp}\hat{f}$. Then there exists a function $\varphi \in C_0^\infty(\mathbb{R}), \mathrm{supp}\mathcal{F}^{-1}\varphi \subset [\sigma-\epsilon, \sigma+\epsilon]$ such that $\langle f, \varphi\rangle \neq 0$. Hence,

$$0 < |\langle f, \varphi\rangle| = |\langle I^m f, D^m\varphi\rangle| = \int_{\mathbb{R}} I^m f(x) D^m\varphi(x)dx \leq \int_{\mathbb{R}} |x^q I^m f(x)||x^{-q} D^m\varphi(x)|dx.$$

Using Hölder inequality, we have

$$0 < |\langle f, \varphi\rangle| \leq \Big(\int_{\mathbb{R}} |x^q I^m f(x)|^p dx\Big)^{1/p} \Big(\int_{\mathbb{R}} |x^{-q} D^m\varphi(x)|^{p'} dx\Big)^{1/p'} = \|I^m f\|_{L^p_q} \|D^m\varphi\|_{L^{p'}_{-q}},$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

So,

$$\liminf_{m\to\infty} \|I^m f\|^{1/m}_{L^p_q} \geq 1/\limsup_{m\to\infty} \|D^m\varphi\|^{1/m}_{L^{p'}_{-q}}. \tag{20}$$

Note that

$$\begin{aligned}\sup_{x\in\mathbb{R}} (1+x^2)|D^m\varphi(x)| &\leq \int_{[\sigma-\epsilon,\sigma+\epsilon]} (|x^m(\mathcal{F}^{-1}\varphi)(x)| + |(x^m(\mathcal{F}^{-1}\varphi)(x))''|)dx \\ &\leq Cm^2(\sigma+\epsilon)^m,\end{aligned}$$

where C is independent of m, and so,

$$\limsup_{m\to\infty} \|D^m\varphi\|_{L^{p'}_{-q}}^{1/m} \le \sigma + \epsilon.$$

Then it follows from (20) that

$$\liminf_{m\to\infty} \|I^m f\|_{L^p_q}^{1/m} \ge 1/(\sigma + \epsilon).$$

Letting $\epsilon \to 0$, we confirm (19). The proof is complete.

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