ON THE PROBABILITY DISTRIBUTION OF THE PRODUCT OF POWERS OF ELEMENTS IN COMPACT LIE GROUPS

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Abstract. In this paper, we study the probability distribution of the word map $w(x_1, x_2, \cdots, x_k) = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ in a compact Lie group. We show that the probability distribution can be represented as an infinite series. Moreover, in the case of the Lie group SU(2), our computations give a nice convergent series for the probability distribution.

1. Introduction

The literature on the study of probabilistic questions in finite group is huge. The starting point is the work of P. Erdös and P. Turán [3, 4]. Since then several papers are devoted to the study of the number of solutions of equations and the probability distribution of word maps in finite groups. In the general setting, the problem can be formulated as follows. Let F_k be the free group of rank k , and let $w \in F_k$ be a word. For a group G, the word w induces a word map $w_G : G^k \to G$. The problem is to study the probability distribution $P_{w,G}$ on G defined by $P_{w,G}(X) = \frac{|w_G^{-1}(X)|}{|G|^k}$, for $X \subset G$, where we use the counting measure on the finite group G.

The probability distribution of the commutator word has been studied in [6, 15] for finite groups. The problem for other words on finite groups has been investigated in [2, 10, 13, 14].

Other direction is to study the same question for compact group by using the Haar measure, see [5, 7, 8]. In these papers, the authors study the probability distribution of the commutator word on compact groups by using the product Haar measure to define $|w_G^{-1}(X)|$.

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In this paper, following M. Mulase and M. Penkava [12], we define $|w_G^{-1}(g)|, g \in G$, by the volume distribution:

$$|w_G^{-1}(g)| = \int_{G^k} \delta(w_G(x_1, x_2, \cdots, x_k)g^{-1})dx_1 \cdots dx_k.$$

Here, G is a compact Lie group, δ is the Dirac delta distribution on G and $dx_1 \cdots dx_k$ is the product Haar measure on G^k. Recall that the Dirac delta distribution has the following expansion:

$$\delta(x) = \frac{1}{|G|} \sum_{\lambda \in \widehat{G}} d_\lambda \chi_\lambda,$$

where $\widehat{G}$ is the set of all the irreducible representations of G. Also d_λ and χ_λ are respectively the dimension and character of the representation λ. Therefore the probability distribution $P_{w,G}$ is given by:

$$P_{w,G}(g) = \frac{1}{|G|^{k+1}} \sum_{\lambda \in \widehat{G}} \Big(\int_{G^k} d_\lambda \chi_\lambda (w(x_1, x_2, \cdots, x_k)g^{-1}) dx_1 dx_2 \cdots dx_k \Big), \tag{1.1}$$

provided that the sum on right hand side converges. The readers are referred to [12] for the details on volume distribution.

The purpose of this paper is to give a formula for $P_{w,G}$ in the case $w = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$, where $(n_1, n_2, \cdots, n_k)$ is a $k-$tuples of integers. Moreover, We will show that in the case $G = \mathrm{SU}(2)$, the sum in the right hand side of (1.1) converges and $P_{w,G}$ has a nice form. In the next section we will state and prove the main results.

2. Main results

We first note that it is enough to consider the case where all the exponents $n_i's$ are positive since changing x_i to x_i^{-1} does not affect $P_{w,G}$. Our first result is the following

Theorem 2.1. *Let G be a compact Lie group and $w = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$. The probability distribution of w on G is given by*

$$P_{w,G}(g) = \frac{1}{|G|^{k+1}} \sum_{\lambda \in \widehat{G}} \frac{\chi_\lambda(g^{-1})}{d_\lambda^{k-1}} \int_G \chi_\lambda(x_1^{n_1}) dx_1 \int_G \chi_\lambda(x_2^{n_2}) dx_2 \cdots \int_G \chi_\lambda(x_k^{n_k}) dx_k,$$

provided that the sum on the right hand side converges.

Proof. To prove the formula we need the following lemma.

Lemma 2.2. *The following equality holds*

$$\int_G \chi_\lambda(x^n u)dx = \frac{1}{d_\lambda}\int_G \chi_\lambda(x^n)dx\chi_\lambda(u)$$

for any $u \in G$ *and* $\lambda \in \widehat{G}$.

Proof. We consider the action $\Phi : L^2(G) \to L^2(G)$ defined by $\Phi(f)(u) := \int_G f(x^n u)dx$. We can see that Φ commutes with the action of G on $L^2(G)$ by left multiplication. Indeed,

$$\Phi(g \cdot f)(u) = \int_G (g \cdot f)(x^n u)dx = \int_G f(g^{-1}x^n u)dx = \int_G f(g^{-1}(gx^n g^{-1})^n u)dx$$

$$= \int_G f(x^n g^{-1}u)dx = \Phi(f)(g^{-1}u) = g \cdot \Phi(f)(u).$$

So Φ acts on $L^2(G)$ by multiplication by a constant C. In particular, $\Phi(\chi_\lambda) = C\chi_\lambda$. Therefore, we get

$$\Phi(\chi_\lambda)(1) = \int_G \chi_\lambda(x^n)dx = C\chi_\lambda(1) = Cd_\lambda.$$

So $C = \frac{1}{d_\lambda}\int_G \chi_\lambda(x^n)dx$ and we get the desire equality

$$\int_G \chi_\lambda(x^n u)dx = \Phi(\chi_\lambda)(u) = C\chi_\lambda(u) = \frac{1}{d_\lambda}\int_G \chi_\lambda(x^n)\chi_\lambda(u).$$

□

Now using (1.1) we have

$$P_{w,G}(g) = \frac{1}{|G|^{k+1}}\sum_{\lambda\in\widehat{G}}\Big(\int_{G^k} d_\lambda\chi_\lambda(x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}g^{-1})dx_1dx_2\cdots dx_k\Big).$$

By using Lemma 2.2 and taking integration over one variable at a time, we obtain the desired formula in the theorem.

□

Note that for a compact Lie group

$$\int_G \chi_\lambda(x)dx = \begin{cases} |G| & \text{if } \lambda \text{ is the trivial representation,} \\ 0 & \text{otherwise.} \end{cases}.$$

So we immediately deduce the following corollary.

Corollary 2.3. *Let G be a compact Lie group. Suppose that $w = x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ is a word where at least one of the exponents $n_i's$ is equal to 1 then $P_{w,G}$ is the uniform distribution, that is $P_{w,G}(g) = \frac{1}{|G|}$ for all g.*

We note that the integral $\int_G \chi_\lambda(x^n)dx$ is a generalization of the Frobenius-Schur indicator on finite groups and is hard to compute in general. Therefore in general, it is hard to check if the right hand side of the formula in Theorem 2.1 converges or not. However, for the case of the group SU(2), we get an explicit formula which converges.

The case when the word $w = x_1^2x_2^2\cdots x_k^2$ has been covered in [12]. For higher exponents cases, we get the following result.

Theorem 2.4. *Let $G = \mathrm{SU}(2)$ with the normalized Haar measure. Suppose that $w = x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ is a word where all the $n_i's$ are bigger than or equal to 2 and at least one of the $n_i's$ is bigger than 2. Then for $k \geq 4$ the probability distribution of w is given by*

$$P_{w,G}(g) = \sum_{j=0}^{\infty} \frac{\chi_{2j}(g^{-1})}{(2j+1)^{k-1}},$$

where χ_{2j} is the $2j$-th irreducible character of SU(2).

Proof. First, we recall some standard facts about representation of the group SU(2) in [1]. Note that $\widehat{G}$ can be identified with the set of non-negative integers $\mathbb{Z}_{\geq}$. The j-th irreducible representation has the dimension $d_j = j+1$ and the character

$$\chi_j(e(\theta)) = \frac{\sin(j+1)\theta}{\sin\theta}, \text{ where } e(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \ j \in \mathbb{Z}_{\geq}. \tag{2.1}$$

To prove the theorem we need to compute $\int_G \chi_j(x^n)dx$. For $n = 2$, it is the Frobenius-Schur indicator $\int_G \chi_j(x^2)dx = (-1)^j$. For $n \geq 3$, we have the following result.

Lemma 2.5. *Let $G = \mathrm{SU}(2)$ with the normalized Haar measure, then the following equality holds for all $j \in \mathbb{Z}_{\geq}$ and $n \geq 3$*

$$\int_G \chi_j(x^n)dx = \begin{cases} 0 & \text{if } j \text{ is odd,} \\ 1 & \text{if } j \text{ is even.} \end{cases}$$

Proof. We recall that the Chebyshev polynomials of the first kind and the second kind are defined respectively by

$$T_j(\cos\theta) = \cos(j\theta) \quad \text{and} \quad U_j(\cos\theta) = \frac{\sin((j+1)\theta)}{\sin\theta}, \quad j \in \mathbb{Z}_{\geq}.$$

To compute integral of class functions on Lie group we use the following result ([1] page 86)

$$\int_{\mathrm{SU}(2)} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(e(\theta))sin^2\theta d\theta.$$

So, we get

$$\int_G \chi_j(x^n)dx = \frac{2}{\pi}\int_0^{\pi} \frac{\sin((j+1)n\theta)}{\sin(n\theta)}sin^2\theta d\theta = \frac{2}{\pi}\int_0^{\pi} U_j(T_n(\cos\theta))sin^2\theta d\theta.$$

We need the following identities of Chebyshev polynomials from Exercises 3, section 2.5 of [11]

$$\frac{1}{2}U_{2m+1} = T_1 + T_2 + \cdots + T_{2m+1} \quad \text{and} \quad \frac{1}{2}U_{2m} = \frac{1}{2}T_0 + T_2 + \cdots + T_{2m}.$$

These identities together with the fact that $T_n \circ T_m = T_{mn}$ allow us to write

$$U_{2m+1}(T_n(\cos\theta)) = 2T_n(\cos\theta) + 2T_{2n}(\cos\theta) + \cdots + 2T_{(2m+1)n}(\cos\theta), \quad \text{and}$$

$$U_{2m}(T_n(\cos\theta)) = T_0(\cos\theta) + 2T_{2n}(\cos\theta) + \cdots + 2T_{2mn}(\cos\theta).$$

Therefore, we get

$$U_{2m+1}(T_n(\cos\theta)) = 2\cos(n\theta) + 2\cos(2n\theta) + \cdots + 2\cos((2m+1)n\theta) \text{ and} \tag{2.2}$$

$$U_{2m}(T_n(\cos\theta)) = 1 + 2\cos(2n\theta) + \cdots + 2\cos(2mn\theta) \tag{2.3}$$

Now the formula 7 and 12, section 3.631, page 397 of [9] imply that

$$\int_0^{\pi} cos(m\theta)sin^2(\theta)d\theta = 0 \quad \text{if} \quad m \geq 3. \tag{2.4}$$

So in the case j is odd, it follows from (2.2) and (2.4) that the integrals of all the terms vanish, we get $\int_G \chi_j(x^n)dx = 0$. If j is even, then it also follows from (2.3) and (2.4) that $\int_G \chi_j(x^n)dx = \frac{2}{\pi}\int_0^{\pi} sin^2\theta d\theta = 1$ as required. □

Combining Theorem 2.1 and Lemma 2.5 we arrive at the formula

$$P_{w,G}(g) = \sum_{j=0}^{\infty} \frac{\chi_{2j}(g^{-1})}{(2j+1)^{k-1}}.$$

As noted above, the value of the character $\chi_{2j}(g^{-1})$ is of the form

$$U_{2j}(\cos\theta) = T_0 + 2T_2(\cos\theta) + \cdots + 2T_{2j}(\cos\theta).$$

So we deduce that $|\chi_{2j}(g^{-1})| \leq 2j+1$ for all j. Therefore, the series $\sum_{j=0}^{\infty} \frac{\chi_{2j}(g^{-1})}{(2j+1)^{k-1}}$ converges for $k \geq 4$ and the theorem is proved.

□

It is interesting that for the group SU(2) the probability distribution $P_{w,G}$ does not depend on the exponents $n_i's$ appearing in w. It is natural to ask the following question:

It is true that on the Lie group SU(n), for a fixed n, the probability distribution $P_{w,G}$ with $w = x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ is finite when k is big enough and it does not depend on the $n_i's$ when $n_i's$ are big enough?

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