POLYNOMIAL OPTIMIZATION ON SOME UNBOUNDED CLOSED SEMI-ALGEBRAIC SETS

TRANG T. DU AND TOAN M. HO

Abstract. We study the polynomial optimization problem on an unbounded semialgebraic set determined by a finite system of polynomial inequalities $f_i(X) \leq r_i; i = 1, 2, \ldots, m$. The method is to make a change of variables to obtain the compact case. Precisely, we characterize the algebra of polynomials bounded on the semialgebraic to be the algebra of polynomials whose supports contained in a convex cone. When the cone is unimodular, by building the monomials mapping $\Phi(X) = X^A$, we can transform the problem on unbounded sets into the one on bounded sets which have been solved by the Lasserre's hierarchy of semidefinite program.

1. Introduction

The Polynomial Optimization Problems (POP) on compact semialgebraic sets have been solved thanks to Lasserre's hierarchy of semidefinite programs ([3]) and Positivstellensatze Theorems ([9], [8]). We can find details the application of 'sums of squares and moment problem' to POP in [4], [7] and the references therein.

For the case when the feasible set is a non-compact semialgebraic set, the Positivstellensatze Theorems mentioned above do not hold anymore. Moreover, there is a partial solution in [2], where the associated quadratic module, that is generated in terms of both the objective function and the constraints, is Archimedean (hence, in particular, the intersection of the sub-level set of the objective function and the constraints is compact).

In this paper, we consider the polynomial optimization problems on a class of noncompact semi-algebraic feasible sets. Moreover, we restrict our attention to finding the optimal value of polynomials which are bounded on the basic semi-algebraic feasible set. In general, checking a polynomial which is bounded on a given semialgebraic feasible set and finding its optimal values on that feasible set are NP hard unless the semialgebraic feasible set is

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a compact set, a two dimensional tentacle [6], a nondegenerate set [5], etc.. However, under some conditions so that the semialgebraic feasible set is 'narrow at infinity', the above problems can be solved by semidefinite programming.

For $F = \{f_1, \ldots, f_m\}$ a family of polynomials in $\mathbb{R}[X]$ and positive numbers $r_1, r_2, \ldots, r_m$. An interesting semi-algebraic set corresponding to F and $r = (r_1, \ldots, r_m)$ under consideration is the set

$$[F \leq r] := \{X \in \mathbb{R}^n \mid f_1(X) \leq r_1; f_2(X) \leq r_2, ..., f_m(X) \leq r_m\}.$$

(The system of inequalities can also be written as $F \leq r$ for short). We also mention that *throughout the paper, $f_i(0)$ is always assumed to be zero for every i.* The support of a polynomial $f = \sum_\alpha f_\alpha X^\alpha$ is the finite set of the exponents α such that f_α is nonzero. The support of the family F is the union of the supports of every member in F. The cone $C(F)$ is the convex cone generated by the support of F. In this work, we study the optimization problem on $[F \leq r]$ with the assumption that the system F is W-asymptotic (see Definition 2.1), where $W \subset \mathbb{N}^n$ generates the unimodular cone $C(F)$. The cone $C(F)$ is unimodular if there exists a set of finite generators $A \subset \mathbb{N}^n$ such that $det(A) = 1$. Then after making the change of variable $U = X^A$, we obtain a compact semi-algebraic set $[\widetilde{F} \leq r]$, where $F(X) = \widetilde{F}(X^A)$. Then, the infimum of a polynomial f on $[F \leq r]$ is the same as the infimum on $[\widetilde{F} \leq r]$ of $\widetilde{f}$, where $\widetilde{f}(X^A) = f(X)$, provided a density assumption (in general, the infimum of $\widetilde{f}$ on $[\widetilde{F} \leq r]$ is a below bound of the infimum of a polynomial f on $[F \leq r]$). Furthermore, calculating the infimum of f mentioned above is only applicable for the polynomial f which is bounded on $[F \leq r]$. We also characterize when the algebra $B[F \leq r]$ of polynomials which are bounded on $[F \leq r]$ is equal to the algebra of the unimodular cone $C(F)$. Checking a polynomial which is bounded on an unbounded semialgebraic set is hard in general, while checking a polynomial which belongs to the algebra of a finitely generated convex cone can be reduced to the problem of checking finitely many points (in the support of that polynomial) to belong to the cone.

The paper is organized as follows: Section 2 contains some results about the algebra (denoted by $B[F \leq r]$) of polynomials which are bounded on $[F \leq r]$. We give a sufficient and necessary conditions when $B[F \leq r]$ is the algebra of a convex cone. When F is W-asymptotic and W generates the cone $C(F)$, the algebra $B[F \leq r]$ is equal to the algebra of the cone $C(F)$. Section 3 presents the study of POP on $[F \leq r]$ when F is F is W-asymptotic, W generates the cone $C(F)$ and $C(F)$ is unimodular.

2. Algebra of polynomials bounded on a semialgebraic set

Let $f \in \mathbb{R}[X]$ be given by $f(X) = \sum f_\alpha X^\alpha$. The support of f, denoted by $supp(f)$ defined by

$$supp(f) = \{\alpha \mid f_\alpha \neq 0\}.$$

Let $\mathcal{A} \subset \mathbb{R}[X]$ be a sub-algebra. The support of $\mathcal{A}$, denoted by $supp(\mathcal{A})$, is a union of the supports of all polynomials in $\mathcal{A}$. Hence $supp(\mathcal{A})$ is a subset of $\mathbb{N}^n$. If C is a convex cone in the first orthant, then the set of all polynomials whose supports lie in C is a sub-algebra of $\mathbb{R}[X]$ and denoted by $\mathcal{A}(C)$. Then the support of $\mathcal{A}(C)$ is equal to C. Given a semi-algebraic set K, denoted by $B(K)$ the set of all polynomials in $\mathbb{R}[X]$ which are bounded on K. Then $B(K)$ is a real sub-algebra of $\mathbb{R}[X]$ and is sometime called the bounded algebra for short. In this section, we will give some class of K for which $B(K)$ is equal to $\mathcal{A}(C)$ for some convex cone C. To check a polynomial belonging to $B(K)$ is difficult in general. However, checking a polynomial which belongs to $\mathcal{A}(C)$ reduces to check the vectors (in the support of the polynomial) belonging to the cone C (the finitely generated convex cone). Hence, the later is just a linear programming problem.

Let $\mathcal{Z} = \{z_1, z_2, \ldots, z_m\}$ be a finite subset of $\mathbb{R}[X]$. Set

$$\mathbb{R}[\mathcal{Z}] = \{f(z_1, z_2, \ldots, z_m) \mid f \in \mathbb{R}[X_1, X_2, \ldots, X_m]\}$$

Let $\mathcal{A}$ be a sub-algebraic of $\mathbb{R}[X]$. $\mathcal{Z}$ *generates* $\mathcal{A}$ if $\mathcal{A} = \mathbb{R}[\mathcal{Z}]$. If we assume further that $\mathcal{Z}$ is a set of monomials, $\mathcal{A}$ is *momomial gennerated.*
Let V be a finite subset of $\mathbb{N}^n$, $\mathbb{R}[X^v \mid v \in V] := \mathbb{R}[\mathcal{Z}]$, where $\mathcal{Z} = \{X^v \mid v \in V\}$.

2.1. **Algebra with a convex unimodular cone support.** *A convex cone, in this paper, is always assumed to be a finitely generated convex cone in the first orthant.* A convex cone C generated by $\alpha^1, \ldots, \alpha^m$ in $\mathbb{N}^n$ is the cone:

$$C := \{\lambda_1\alpha^1 + \cdots + \lambda_m\alpha^m \mid \lambda_1 \geq 0, \ldots, \lambda_m \geq 0\}.$$

Let d_i be great common divisor of the coordinates of α^i for $i = 1, \ldots, n$. Then $d_i^{-1}\alpha^i$ belongs to $\mathbb{N}^n$ and the cone generated by $d_1^{-1}\alpha^1, \ldots, d_n^{-1}\alpha^n$ is equal to C. Thus, from now on, *we can always assume that the great common divisor of the coordinates of each α^i is one for every* $i = 1, \ldots, m$. A convex cone $C \subset \overline{\mathbb{R}^n_+}$ is said to be *unimodular* if there exists a generator set of n-vectors $\alpha^1, \ldots, \alpha^n \in \mathbb{N}^n$ such that $\det[\alpha^1\ \alpha^2 \cdots \alpha^n] = 1$.

Since C is a convex cone, $\mathcal{A}(C)$ is a subalgebra of $\mathbb{R}[X]$. It is clear that

$$Supp(\mathcal{A}(C)) = C \cap \mathbb{N}^n$$

is a sub-semigroup of $\mathbb{N}^n$. In addition, we have

Proposition 2.1. *Let C be a unimodular convex cone generated by n vectors $\alpha^1, \ldots, \alpha^n$ in $\mathbb{N}^n$ with $\det[\alpha^1\ \alpha^2 \cdots \alpha^n] = 1$. Then*

$$\mathcal{A}(C) = \mathbb{R}[X^{\alpha^1}, X^{\alpha^2}, \ldots, X^{\alpha^n}].$$

Proof. For any $\alpha \in C \cap \mathbb{N}^n$, by the definition of C, there exist $l_i \geq 0, i = 1, 2, ..., n$ such that

$$\alpha = \sum_{i=1}^{n} l_i \alpha^i.$$

Furthermore, $[l_i, l_2, \ldots, l_n]^{\mathrm{T}}$ is a solution of the linear system $Al = \alpha$, where $A = [\alpha^1\ \alpha^2\ \cdots \alpha^n]$. Hence, $[l_i, l_2, \ldots, l_n]^{\mathrm{T}} = A^{-1}\alpha$. Since $\det(A) = 1, A^{-1}$ is a matrix with integer coefficients. So $[l_i, l_2, \ldots, l_n]^{\mathrm{T}}$ belongs to $\mathbb{N}^n$. Hence,

$$X^\alpha = \prod_{i=1}^{n} \left(X^{\alpha^i}\right)^{l_i} \in \mathbb{R}[X^{\alpha^1}, X^{\alpha^2}, \ldots, X^{\alpha^n}].$$

Therefore,

$$\mathcal{A}(C) = \mathbb{R}[X^{\alpha^1}, X^{\alpha^2}, \ldots, X^{\alpha^n}].$$

□

2.2. **Bounded algebras with convex cone supports.** *Logarithmic polyhedra* are the interesting examples of semi-algebraic sets whose bounded algebras are the same as the algebras of the cones generated by the corresponding Newton polyhedra. Precisely,

Lemma 2.1. [5, Proposition 3.3] *Let V be a finite set of nonzero vectors with positive even coordinates. Set*

$$L(V, r) := \{X \in \mathbb{R}^n \mid X^v \leq r_v,\ \forall v \in V\},$$

where $r = (r_v)_{v \in V}$ is a vector of positive numbers. Then the algebra of all polynomials in $\mathbb{R}[X]$ which are bounded on $L(V, r)$ is independent on the choice of $r = (r_v)$ and equal to $\mathcal{A}(C(V))$, where $C(V)$ is the cone generated by V.

Next, we will introduce a large class of basic semi-algebraic sets with convex cone support bounded algebras.

Let $F = \{f_1, \ldots, f_m\} \subset \mathbb{R}[X]$ be represented by

$$f_i = \sum_{\alpha^i} f_{\alpha^i} X^{\alpha^i},\ \alpha^i \in \mathbb{N}^n, f_{\alpha^i} \in \mathbb{R}; i = 1, 2, \ldots, m.$$

Denoted by *supp*(F) the set of all exponents $\alpha^i \in \mathbb{N}^n$ where f_{α^i} is nonzero, for all i=1,2, ..., m. We will use $\Gamma(F)$ (or Γ without confusing) to denote the convex polyhedron generated by *supp*(F) and call it the Newton polyhedron of F. Let $V(F)$ and $C(V(F))$ (or $C(F)$ for short) be the set of all vertices of and the convex cone generated by the polyhedron $\Gamma(F)$, respectively. Then every vertex in $V(F)$ is *non-zero* since $f_i(0)$ is assumed to be 0.

For any vector $\alpha \in \Gamma(F) \cap \mathbb{N}^n$, there exist nonnegative numbers $\lambda_v, v \in V(F)$ such that $\sum_{v \in V(F)} \lambda_v = 1$ and $\alpha = \sum_{v \in V(F)} \lambda_v v$. Hence, by Young's inequality (or [1, Chapetr 1, Lemma 2.1]), there is a positive constant M such that

$$|X^\alpha| \leq M\Xi_{V(F)}(|X|),$$

where $\Xi_V(|X|) = \sum\limits_{v \in V} |X^v|$ for a finite set $V \subset \mathbb{N}^n$. Therefore, there exists a constant C such that

$$\max_{1 \leq i \leq m} f_i(X) \leq C\Xi_{V(F)}(|X|),\ \forall X \in \mathbb{R}^n. \tag{1}$$

Definition 2.1. *Let $F \subset \mathbb{R}[X]$ be a finite family as above, $W \subset \mathbb{N}^n$ be a finite subset of nonzero vectors. F is said to be W-asymptotic if there exists positive numbers c, R such that*

$$c|X^\alpha| \leq \max_{1 \leq i \leq m} f_i(X),\ \forall \alpha \in W,\ \forall X \in \mathbb{R}^n,\ ||X|| \geq R. \tag{2}$$

The inequalities (2) are equivalent to

$$c\Xi_W(|X|) \leq \max_{1 \leq i \leq m} f_i(X),\ \forall X \in \mathbb{R}^n,\ ||X|| \geq R. \tag{3}$$

Note that c in the equality (2) and (3) are not necessarily the same.

Proposition 2.2. *Given a finite family $F = \{f_1, f_2, \ldots, f_m\} \subset \mathbb{R}[X]$ with Newton polyhedron $\Gamma(F)$; W is a finite subset of $\Gamma(F)$; $r = (r_1, r_2, \ldots, r_m)$ is a sequence of positive numbers. If F is W-asymptotic then*

$$\mathcal{A}(C(W)) \subset B([F \leq r]) \subset \mathcal{A}(C(F)).$$

In particular, $\mathcal{A}(C(W)) = B([F \leq r])$ provided the convex cone generated by W is equal to $C(F)$.

Thanks to this proposition, from now on, we always assume that the convex cone generated by W *is equal to* $C(F)$, i.e., W *generates* the cone $C(F)$ and so $B([F \leq r]) = \mathcal{A}(C(F))$.

Proof. Put $r^* := \max\{r_1, r_2, ..., r_m\}$. We have

$$[F \leq r] \subset [f_1 \leq r^*, f_2 \leq r^*, \ldots, f_m \leq r^*] = [\max_{1 \leq i \leq m} f_i \leq r^*].$$

Since F is W-asymptotic,

$$\left\{X \in \mathbb{R}^n | \max_{1 \leq i \leq m} f_i(X) \leq r^*; ||X|| \geq R\right\} \subset \left\{X \in \mathbb{R}^n | \ |X^\alpha| \leq \frac{r^*}{c}, \forall \alpha \in W; ||X|| \geq R\right\}.$$

We note that

$$B\left(\left\{X \in \mathbb{R}^n | \max_{1 \leq i \leq m} f_i(X) \leq r^*; ||X|| \geq R\right\}\right) = B\left([\max_{1 \leq i \leq m} f_i \leq r_*]\right),$$

$$B\left(\left\{X \in \mathbb{R}^n | \ |X^\alpha| \leq \frac{r^*}{c}, \forall \alpha \in W; ||X|| \geq R\right\}\right) = B\left(\left\{X \in \mathbb{R}^n | \ |X^\alpha| \leq \frac{r^*}{c}, \forall \alpha \in W\right\}\right).$$

By Lemma 2.1, $B(\{X \in \mathbb{R}^n \mid |X^\alpha| \leq \frac{r^*}{c}, \forall \alpha \in W\}) = \mathcal{A}(C(W))$. So

$$\mathcal{A}(C(W)) \subset B([\max_{1 \leq i \leq m} f_i \leq r_*]) \subset B([F \leq r]).$$

On the other hand, put $r_0 := \min\{r_1, r_2, ..., r_m\}$, since the equality (1), we have

$$[\Xi_{V(F)}(|X|) \leq r_0] \subset [\max_{1 \leq i \leq m} f_i \leq r_0] \subset [F \leq r].$$

Using the Lemma (2.1), it is straightforward to show that $B([\Xi_{V(F)}(|X|) \leq r_0]) = A(C(F))$. Therefore, we get $B([F \leq r]) \subset (B([\Xi_{V(F)}(|X|) \leq r_0]) = A(C(F)))$. Note that, this uper bound of $B([F \leq r])$ is independent of the difinition W- asymptotic of F.

□

Example 1. (i) If F is nondegenerate in the sense of [5, Definition 1] then F is V-asymptotic, where V is the set of vertices of the Newton polyhedron $\Gamma(F)$.

(ii) Let $W \subset \mathbb{N}^n$ be an independent finite set of nonzero even vectors. Define

$$f(X) := \sum_{\alpha \in W} X^\alpha + g(X),$$

where $g(X)$ is a nonnegative polynomial. Then $f(X)$ is W-asymptotic.

(iii) Let $h(x, y) = x^2 + y^2 + (x - y)^4$. By (ii), h is $\{(1, 0), (0, 1)\}$-asymptotic. However, we can check by definition that h is not nondegenerate in the sense of [5, Definition 1]. More general, suppose that a polynomial mapping H is proper, there exists a positive number γ such that H is $\{(\gamma, 0, \ldots, 0); \ldots; (0, \ldots, 0, \gamma)\}$-asymptotic (since the Łojasiewicz number of H at infinity is positive). However, $H(x)$ need not be nondegenerate in the sense of [5, Definition 1].

3. POLYNOMIAL OPTIMIZATION

The polynomial optimazation problem on *compact* semi-algebraic sets is solved by the Lasserre's hierarchy SPD (see [3] and see [8], [9]). In this paper, we consider the polynomial optimization on unbounded semi-algebraic sets. The method here is that by making the change of variable, we can transform the problem on unbounded sets into the one on bounded sets and then we can apply the Lasserre's hierarchy SPD.

Let A be a square matrix whose coefficients are nonnegative integers. Set

$$\Phi(\text{ or } \Phi_A) : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad X \longmapsto U = X^A = (X^{\alpha^1}, \dots, X^{\alpha^n}),$$

where α^i is the i^{th}-column of A. For any polynomial $\widetilde{f}(U)$ in $\mathbb{R}[U]$, let $f(X) = \widetilde{f}(X^A)$. Then $f(X)$ belongs to $\mathbb{R}[X^A]$, the algebra of the convex cone generated by the columns of A. Conversely, for $f(X) \in \mathbb{R}[X]$, then $f(X)$ belongs to $\mathbb{R}[X^A]$ if there exists a polynomial $\widetilde{f}(U) \in \mathbb{R}[U]$ such that $f(X) = \widetilde{f}(X^A)$ (or, we can also write $f(X) = \widetilde{f}(\Phi(X))$).

Let $F = (f_1, \dots, f_m)$ be a family of polynomials in $\mathbb{R}[X]$ and $r = (r_1, \dots, r_m) \in \mathbb{R}^n_+$. In this section, the cone $C(F)$ is assumed to be unimodular, that is $C(F)$ is generated by n-vectors $\alpha^1, \dots, \alpha^n \in \mathbb{N}^n$ with $\det[\alpha^1\ \alpha^2 \cdots \alpha^n] = 1$. Assume that F is W-asymptotic where $C(W) = C(F)$, then by Proposition 2.2, $B([F \leq r])$ is equal to $A(C(F)) = \mathbb{R}[X^A]$, where $A = [\alpha^1\ \alpha^2\ ...\alpha^n]$. Hence,

$$f \in B([F \leq r]) \Longleftrightarrow \exists \widetilde{f} \in \mathbb{R}[U] \ : \ f(X) = \widetilde{f}(\Phi(X)).$$

For $i = 1, 2, ..., m$ by $f_i \in \mathcal{A}(C(F))$, there exist $\widetilde{f_i} \in \mathbb{R}[U]$ such that $f_i(X) = \widetilde{f_i}(X^A)$. Put

$$\widetilde{K} := \left\{ U \in \mathbb{R}^n \mid \widetilde{f_1}(U) \leq r_1, \widetilde{f_2}(U) \leq r_2, ..., \widetilde{f_m}(U) \leq r_m \right\}.$$

Then $\widetilde{K}$ is a basic semi-algebraic set.

Theorem 3.1. *Let $K = [F \leq r]$ be a basic closed semi-algebraic set as above. Assume that F is W- asymptotic, $C(W) = C(F)$ is unimodular and $\widetilde{K} = \overline{\widetilde{K} \cap (\mathbb{R}^*)^n}$. Then $\widetilde{K}$ is compact and, for every polynomial f bounded on K,*

$$\inf_K f = \inf_{\widetilde{K}} \widetilde{f}.$$

We would like to mention that f is bounded on K if and only if $supp(f)$ lies in the convex cone $C(W)$ generated by W by Proposition 2.2. In order to prove the theorem above, we need some more results as follows.

When $C(W) = C(F)$ is unimodular, there are $\alpha^1, \ldots, \alpha^n \in \mathbb{N}^n$ with $\det[\alpha^1\ \alpha^2 \cdots \alpha^n] = 1$ such that $\alpha^1,\ \alpha^2, \ldots, \alpha^n$ gennerate the cone $C(W)$. Consider the monomial mapping $\Phi(X) = X^A$, where $A = [\alpha^1\ \alpha^2 \cdots \alpha^n]$. We have

Remark 3.1. *Let $K = [F \leq r]$ be a basic closed semi-algebraic set as above. Assume that F is W- asymptotic, $C(W) = C(F)$ is unimodular. Then the following statements hold true.*

(i) $\overline{\Phi(K)} \subset \widetilde{K}$.

(ii) *The restriction $\Phi : K \cap (\mathbb{R}^*)^n \longrightarrow \widetilde{K} \cap (\mathbb{R}^*)^n$ is one-to-one and onto.*

(iii) *For any polynomial $f \in \mathbb{R}[X]$ which is bounded on K, we have $f \in \mathbb{R}[U]$ such that $f(X) = \widetilde{f}(X^A)$ and*

$$\inf_K f = \inf_{\overline{\Phi(K)}} \widetilde{f} \geq \inf_{\widetilde{K}} \widetilde{f}.$$

Proof. (i) Let $U = \Phi(X)$, for some $X \in K$. Then $\widetilde{f_i}(U) = f_i(X) \leq r_i$ for $i = 1, 2, \ldots, m$. Thus $\Phi(K) \subset \widetilde{K}$. This implies (i).

(ii) Let $U \in \widetilde{K} \cap (\mathbb{R}^*)^n$. Then $X := U^{A^{-1}}$ belongs to $(\mathbb{R}^*)^n$ and $f_i(X) = \widetilde{f_i}(U)$ for every i. Hence, $X \in K$.

(iii) Since f is bounded on K, $f \in \mathcal{A}(C(W))$ by Proposition 2.2. Thus, there exists a polynomial $\widetilde{f}$ such that $f(X) = \widetilde{f}(X^A)$. Let $(y_n)_{n=1}^{\infty}$ be a sequence in $\Phi(K)$. Then there exists a sequence $(x_n)_{n=1}^{\infty}$ in K such that $y_n = \Phi(x_n)$. If, assume further that $\widetilde{f}(y_n)$ converges to l as $n \to \infty$, so does $f(x_n) = \widetilde{f}(\Phi(x_n))$. Hence,

$$\inf_K f = \inf_{\Phi(K)} \widetilde{f} = \inf_{\overline{\Phi(K)}} \widetilde{f}.$$

□

Lemma 3.1. *Let's adopt the assumption of Theorem 3.1. Then $\widetilde{K}$ is compact.*

Proof. By Proposition 2.2, we have $B(K) = \mathcal{A}(C(W)) = \mathbb{R}[X^{\alpha^1}, \ldots, X^{\alpha^n}]$. For all $i = 1, 2, ..., n$ and $X \in K$, $X^{\alpha^i} \in B(K)$ since $\alpha^i \in C(F)$. Hence, there exists $M_i > 0$ such that $|X^{\alpha^i}| < M_i$ for every i. Let $U = (U_1, U_2, \ldots, U_n) \in \widetilde{K}$. Consider two cases:

- Case 1: Assume $U \in (\mathbb{R}^*)^n$. Put $X = U^{A^{-1}}$ where $A = [\alpha^1 \cdots \alpha^n]$. Then $X \in K$. Indeed, $f_i(X) = \widetilde{f_i}(X^A) = \widetilde{f_i}(U) \leq r_i$, since $U = X^A$. Therefore, $|U_i| = |X^{\alpha^i}| \leq M_i$ for all $i = 1, 2, ..., n$.

- Case 2: Assume $U \notin (\mathbb{R}^*)^n$. From $\widetilde{K} = \overline{\widetilde{K} \cap (\mathbb{R}^*)^n}$, there exists a sequence $\{U^m\} \subset \widetilde{K} \cap (\mathbb{R}^*)^n$ such that $U^m \longrightarrow U$ as $m \to \infty$. By Case 1, $|U_i^m| \leq M_i$. Let m tend to infinity, we get $|U_i| \leq M_i$ for all $i = 1, 2, ..., n$.

Finally, we have $\widetilde{K} \subset \{|U_i| \leq M_i\}$, so $\widetilde{K}$ is compact. $\square$

Proof of Theorem 3.1. By Lemma 3.1, $\widetilde{K}$ is compact. By Remark 3.1, $\Phi(K \cap (\mathbb{R}^*)^n) = \widetilde{K} \cap (\mathbb{R}^*)^n$ and, combining with the hypothesis, we obtain

$$\widetilde{K} \cap (\mathbb{R}^*)^n \subset \Phi(K) \subset \widetilde{K} = \overline{\widetilde{K} \cap (\mathbb{R}^*)^n}.$$

So,

$$\widetilde{K} = \overline{\Phi(K)} = [\widetilde{f}_1 \leq r_1, \ldots, \widetilde{f}_m \leq r_m].$$

According to Remark 3.1, we get

$$\inf_K f = \inf_{\widetilde{K}} \widetilde{f}.$$

$\square$

Note that, according the proof above, we always have

$$\inf_{\widetilde{K}} \widetilde{f} \leq \inf_K f$$

without hypothesis $\widetilde{K} = \overline{\widetilde{K} \cap (\mathbb{R}^*)^n}$.

Corollary 3.1. *Let $L = L(V, r)$ be as in Lemma 2.1. Suppose that the cone $C(V)$ is unimodular and is generated by the column vectors of A with $det(A) = 1$. Let $\widetilde{V} = \{A^{-1}v \mid v \in V\}$. Then $\widetilde{L} = L(\widetilde{V}, r)$ is a compact basic semi-algebraic set and, for any polynomial f with $supp(f) \subset C(V)$,*

$$\inf_L f = \inf_{\widetilde{L}} \widetilde{f}.$$

Proof. It is straightforward to show that $\widetilde{L} = \overline{\widetilde{L} \cap (\mathbb{R}^*)^n}$. $\square$

Corollary 3.2. *Let $K = [F \leq r]$ be a basic closed semi-algebraic set as above. Suppose that F is W- asymptotic, the cone $C(W) = C(F)$ is unimodular and 0 is not a local minimal value of $\max_{1 \leq i \leq m} \widetilde{f}_i - r_i$. Then $\widetilde{K} = [\widetilde{F} \leq r]$ is compact and*

$$\inf_K f = \inf_{\widetilde{K}} \widetilde{f}.$$

To prove this corollary, we need some lemmas.

Lemma 3.2. *Let's adopt the notations above. Suppose that $C(F) = C(W)$ is unimodular. If $U^* \in \widetilde{K}$ and $\Lambda(U^*) < 0$ then U^* belongs to the closure (in the usual topology) of $\widetilde{K} \cap (\mathbb{R}^*)^n$, where*

$$\Lambda(U) := \max_{1 \leq i \leq m} (\widetilde{f}_i(U) - r_i), \quad U \in \mathbb{R}^n.$$

Proof. We have $\widetilde{K} = \left\{U \in \mathbb{R}^n \mid \widetilde{f}_i(U) \leq r_i \ \forall 1 \leq i \leq m\right\} = \{U \in \mathbb{R}^n \mid \Lambda(U) \leq 0\}$. Since Λ is continuous, $\Lambda^{-1}(-\infty, 0)$ is an open neighborhood of U^*. There exists an open ball $S(U^*, \eta)$ centered at U^* with radius $\eta > 0$ such that $\Lambda(S(U^*, \eta)) \subset (-\infty, 0)$. So, for every $0 < \varepsilon \leq \eta$, there exists $U_\varepsilon \in S(U^*, \varepsilon) \cap (\mathbb{R}^*)^n, \Lambda(U_\varepsilon) < 0$. That is, $U_\varepsilon \in \widetilde{K} \cap (\mathbb{R}^*)^n$. Let $\varepsilon \longrightarrow 0, U_\varepsilon \longrightarrow U^*$. Hence, $U^* \in \overline{\widetilde{K} \cap (\mathbb{R}^*)^n}$. □

Proof of Corollary 3.2. By Theorem 3.1, it remains to prove that $\widetilde{K}$ is equal to $\overline{\widetilde{K} \cap (\mathbb{R}^*)^n}$. Suppose on the contrary that the closure of $\widetilde{K} \cap (\mathbb{R}^*)^n$ is not the same as $\widetilde{K}$. There exists a point $U^* \in \widetilde{K}$ and a positive number ε such that the ball $S(U^*, \varepsilon)$ centered at U^* with radius ε does not intersect $\overline{\widetilde{K} \cap (\mathbb{R}^*)^n}$. By Lemma 3.2, $\Lambda(U) \geq 0$ for every U in $S(U^*, \varepsilon)$. This means that U^* is a local minimal point of Λ and $\Lambda(U^*) = 0$ is a local minimal value, which contradicts the hypothesis. □

3.1. **Checking a polynomial f bounded on K and finding its infimum.** Let $K = [F \leq r]$ be a basic closed semi-algebraic set as above. Assume that F is W- asymptotic, $C(W) = C(F)$ is unimodular generated by the column vectors of matrix $A = [\alpha^1, \dots, \alpha^n]$, where $\det(A) = 1$.

Step 1: A polynomial f is bounded on K if and only if $supp(f) \subset C(W) = C(\alpha^1, \dots, \alpha^n)$. This is equivalent to

$$\forall b \in supp(f), \exists c_i \in \mathbb{N};; i = 1, 2, \dots, n : b = \sum_{i=1}^{n} c_i \alpha^i,$$

which, in turn, is equivalent to the systems of equations $Ac = b$ having solution $c \in \mathbb{N}^n$ for every $b \in supp(f)$.

Step 2: We make change of variables $U = X^A$ to obtain $\widetilde{f}$ such that $\widetilde{f}(X^A) = f(X)$, where $f(X) = \sum_b f_b X^b$. If we write $\widetilde{f} = \sum_{c \in supp(\widetilde{f})} \widetilde{f}_c U^c$, then

$$supp(\widetilde{f}) = \{A^{-1}b \mid b \in supp f\} \text{ and } \widetilde{f}_c = f_b, \forall c = A^{-1}b.$$

Step 3: Using SOStools, we can find the infimum of $\widetilde{f}$ on $\widetilde{K}$. Then

$$\inf_{\widetilde{K}} \widetilde{f} \leq \inf_K f. \tag{4}$$

If $\widetilde{K} = \overline{\widetilde{K} \cap (\mathbb{R}^*)^n}$ holds then the equality in (4) holds by Theorem 3.1.

Example 2. *Consider the following two-dimensional optimization problem*

$$f^* = \inf_K f,$$

where $f(x,y) = x^6y^2 + x^6y^4 - x^4y^2; K = [F \le 4] = \{(x,y) \in \mathbb{R}^2 \; | F(x,y) = x^2 + x^2y^2 \le 4\}.$

We see that F is $W = \{(1,0);(1,1)\} -$ asymptotic and $C(F) = C(W)$ is unimodular since $\det(A) = 1$, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

By Proposition 2.2 and Proposition 2.1, $B(K) = A(C(W)) = \mathbb{R}[x, xy]$.
We have $supp(f) = \{(6,2);(6,4);(4,2)\}$. We can use Matlab to check $f \in B(K) = A(C(F))$ by the following simple code

```
A = [1   1; 0   1]
b1 = [6; 2]
b2 = [6; 4]
b3 = [4; 2]
c1 = A \ b1
c2 = A \ b2
c3 = A \ b3
```

We obtain $c1 = [4;2], c2 = [2;4], c3 = [2;2]$. So $f \in B(K)$. We change the variables

$$(u,v) = (x,y)^A = (x, xy).$$

Since $\widetilde{f}((x,y)^A) = f(x,y)$, we have $supp(\widetilde{f}) = \{c1, c2, c3\}$, so $\widetilde{f}(u,v) = u^4v^2 + u^2v^4 - u^2v^2 = u^2v^2(u^2+v^2-1)$. $\widetilde{F}((x,y)^A) = F(x,y)$, so $\widetilde{F}(u,v) = u^2 + v^2$ and $\widetilde{K} = [\widetilde{F} \le 4] = \{(u,v) \in \mathbb{R}^2 \mid u^2 + v^2 \le 4\}$ sastifying $\widetilde{K} = \overline{\widetilde{K} \cap (\mathbb{R}^*)^n}$. Now the problem is transformed into the optimization problem: Find $f^* = \min_{\widetilde{K}} \widetilde{f}$. According to [4, Example 5.3], we obtain the optimal value is $f^* = -\dfrac{1}{27}$ with global minimizers $(u^*, v^*) = (\pm\frac{\sqrt{3}}{3}; \pm\frac{\sqrt{3}}{3})$. Furthermore, the global minimizers of the initial problem are $(x^*, y^*) = (u^*, v^*)^{A^{-1}} = (u^*, \frac{u^*}{v^*}) = (\pm\frac{\sqrt{3}}{3}; \pm 1)$. Hence

$$\min_K f = f(\pm\frac{\sqrt{3}}{3}; \pm 1) = -\frac{1}{27}.$$

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University of Transport and Communications, 3 Cau Giay Street, Lang Thuong Ward, Dong Da District, Hanoi, Vietnam

E-mail address: dutrang1986@utc.edu.vn

Institute of Mathematics, VAST, 18 Hoang Quoc Viet, Ha Noi, Vietnam

E-mail address: hmtoan@math.ac.vn