# A variation of constant formula for Caputo fractional stochastic differential equations

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**Abstract**

We establish and prove a variation of constant formula for Caputo fractional stochastic differential equations whose coefficients satisfy a standard Lipschitz condition. The main ingredient in the proof is to use Ito's representation theorem and the known variation of constant formula for deterministic Caputo fractional differential equations. As a consequence, for these systems we point out the coincidence between the notion of classical solutions introduced in [13] and mild solutions introduced in [12].

**keyword** Fractional stochastic differential equations, Classical solution, Mild solution, Inhomogeneous linear systems, A variation of constant formula.

## 1 Introduction

Fractional differential equations have recently been received an increasing attention due to their applications in a variety of disciplines such as mechanics, physics, electrical engineering, control theory, etc. We refer the interested reader to the monographs [1, 6, 11] and the references therein for more details.

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In contrast to the well development in the qualitative theory of deterministic fractional differential equations, there have been only a few papers contributing to the qualitative theory of stochastic differential equations involving with a Caputo fractional time derivative and most of these articles have limited to the existence and uniqueness of solutions, see [13, 7].

It is undoubtable that a variation of constant formula for deterministic fractional systems, see [8], is an important tool in the qualitative theory including the stability theory and the invariant manifold theory built in [3, 4, 5]. In this paper, a stochastic version of variation of constant formula for Caputo fractional systems whose coefficients satisfy a standard Lipschitz condition is established. Roughly speaking, this formula indicates that a solution of nonlinear system can be given as a fixed point of the corresponding Lyapunov-Perron operator. A direct application is that an explicit formula for solutions of inhomogeneous linear fractional stochastic differential equations is formed. Concerning more potential applications, we refer the readers to the conclusion section.

It is also worth mentioning that the established variation of constant formula in this paper also points out the coincidence between the notion of classical solutions introduced in [13] and mild solutions introduced in [12] for fractional stochastic differential equations without impulsive effects in a finite-dimensional space. It is interesting to know whether this result can be extended to systems involving impulsive effects and in an infinite dimensional systems. Another question is to weaken the Lipschitz assumption on the coefficients of the systems (cf. [2]). We leave these problems as open questions for the further research.

The paper is structured as follows: In Section 2, we introduce briefly about Caputo fractional stochastic differential equations and state the main results of the paper. The first part of Section 3 is devoted to show the result on the existence and uniqueness of mild solutions (Theorem 3.2). The main result (Theorem 2.3) concerning a variation of constants formula for fractional stochastic differential equations is proved in the second part of Section 3.

# 2 Preliminaries and the statement of the main results

## 2.1 Fractional calculus and fractional differential equations

Let $\alpha \in (0,1]$, $[a,b] \subset \mathbb{R}$ and $x : [a,b] \to \mathbb{R}^d$ be a measurable function such that $\int_a^b \|x(\tau)\| \, d\tau < \infty$. The *Riemann–Liouville integral operator of order* $\alpha$ is defined by

$$(I_{a+}^\alpha x)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) \, d\tau, \text{ where } \Gamma(\alpha) := \int_0^\infty t^{\alpha-1} \exp(-t) \, dt,$$

see [6]. The *Caputo fractional derivative* of order $\alpha$ of a function $x \in C^1([a,b])$ is defined by ${}^C D_{a+}^\alpha x(t) := (I_{a+}^{1-\alpha} D x)(t)$, where $D = \frac{d}{dt}$ is the usual derivative. Analog to the case of integer derivative, a variation of constant formula is used to derive an explicit solution for inhomogenous linear systems involving fractional derivatives. More concretely, consider an inhomogenous linear fractional differential equation on a bounded interval $[0,T]$

$${}^C D_{a+}^\alpha x(t) = Ax(t) + f(t), \quad x(0) = \eta, \tag{1}$$

where $A \in \mathbb{R}^{d\times d}$ and $f : [0,T] \to \mathbb{R}^d$ is measurable and bounded. Then, an explicit formula for solution of (1) is given in the following theorem and its proof can be found in [8].

**Theorem 2.1** (A variation of constant formula for Caputo fractional differential equations)**.** *The unique solution of* (1) *on* $[0,T]$ *is given by*

$$x(t) = E_\alpha(t^\alpha A)\eta + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) f(\tau) \, d\tau,$$

*where* $E_\alpha(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+1)}, \quad E_{\alpha,\alpha}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+\alpha)}$.

## 2.2 Fractional stochastic differential equation and the main results

Consider a Caputo fractional stochastic differential equation (for short Caputo fsde) of order $\alpha \in (\frac{1}{2}, 1)$ on a bounded interval $[0,T]$ of the following form

$$ {}^{C}D^{\alpha}_{0+}X(t) = AX(t) + b(t, X(t)) + \sigma(t, X(t))\,\frac{dW_t}{dt}, \tag{2} $$

where $(W_t)_{t\in[0,\infty)}$ is a standard scalar Brownian motion on an underlying complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,\infty)}, \mathbb{P})$, $A \in \mathbb{R}^{d\times d}$ and $b, \sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions satisfying the following conditions:

(H1) There exists $L > 0$ such that for all $x, y \in \mathbb{R}^d$, $t \in [0,T]$

$$ \|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \le L\|x - y\|. $$

(H2) $\int_0^T \|b(\tau,0)\|^2\, d\tau < \infty, \quad \operatorname{esssup}_{\tau\in[0,T]} \|\sigma(\tau,0)\| < \infty.$

For each $t \in [0,\infty)$, let $\mathfrak{X}_t := \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ denote the space of all mean square integrable functions $f : \Omega \to \mathbb{R}^d$ with $\|f\|_{\text{ms}} := \sqrt{\mathbb{E}(\|f\|^2)}$. A process $\xi : [0,\infty) \to \mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathbb{F}$-adapted if $\xi(t) \in \mathfrak{X}_t$ for all $t \ge 0$. Now, we recall the notion of classical solution of Caputo fsde, see e.g. [13, p. 209] and [7].

**Definition 2.2** (Classical solution of Caputo fsde)**.** For each $\eta \in \mathfrak{X}_0$, a $\mathbb{F}$-adapted process $X$ is called a solution of (2) with the initial condition $X(0) = \eta$ if the following equality holds for $t \in [0,T]$

$$ \begin{aligned} X(t) \;&=\; \eta + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(AX(\tau) + b(\tau, X(\tau)))\, d\tau \\ &\quad + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\sigma(\tau, X(\tau))\, dW_\tau. \end{aligned} \tag{3} $$

It was proved in [7] that for any $\eta \in \mathfrak{X}_0$, there exists a unique solution which is denoted by $\varphi(t,\eta)$ of (3). In the following main result of this paper, we establish a variation of constant formula for (2) which gives a special presentation of the solution $\varphi(t,\eta)$.

**Theorem 2.3** (A variation of constant formula for Caputo fsde)**.** *Let $\eta \in \mathfrak{X}_0$ arbitrary. Then, the following statement*

$$ \begin{aligned} \varphi(t,\eta) = \;& E_\alpha(t^\alpha A)\eta + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) b(\tau, \varphi(\tau,\eta))\, d\tau \\ & + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A)\sigma(\tau, \varphi(\tau,\eta))\, dW_\tau \end{aligned} \tag{4} $$

*holds for all $t \in [0,T]$.*

**Remark 2.4.** (i) If the noise in (2) vanishes, i.e. $\sigma(t, X(t)) = 0$, then (4) becomes the variation of constant formula for deterministic fractional differential equations (cf. Theorem 2.1). (ii) Note that $E_1(M) = E_{1,1}(M) = e^M$ for $M \in \mathbb{R}^{d\times d}$. Letting $\alpha \to 1$, (4) formally becomes

$$\varphi(t,\eta) \;=\; e^{tA}\eta + \int_0^t e^{(t-\tau)A} b(\tau, \varphi(\tau,\eta))\, d\tau + \int_0^t e^{(t-\tau)A}\sigma(\tau,\varphi(\tau,\eta))\, dW_\tau,$$

which is a variation of constant formula for solutions of stochastic differential equation

$$dX(t) = (AX(t) + b(t, X(t)))\, dt + \sigma(t, X(t))\, dW_t,$$

see [9, Theorem 3.1].

As an application of the preceding theorem, we obtain an explicit representation of the solution of inhomogeneous linear fsde of the form

$$^{C}D^{\alpha}_{0+}X(t) = AX(t) + b(t) + \sigma(t)\,\frac{dW_t}{dt}, \quad X(0) = \eta. \tag{5}$$

**Corollary 2.5.** *Suppose that* $b \in \mathbb{L}^2([0,T],\mathbb{R}^d), \sigma \in \mathbb{L}^\infty([0,T],\mathbb{R}^d)$*, where* $T > 0$*. Then, the explicit solution for* (5) *on* $[0,T]$ *is given by*

$$\begin{aligned} X(t) \;&=\; E_\alpha(t^\alpha A)\eta + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) b(\tau)\, d\tau \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A)\sigma(\tau)\, dW_\tau. \end{aligned}$$

# 3 Proof of the main results

We will fix the following notions through this section. Let $\mathbb{R}^d$ be endowed with the standard Euclidean norm. For $T > 0$, let $\mathbb{H}^2([0,T],\mathbb{R}^d)$ denote the space of all processes $\xi$ which are measurable, $\mathbb{F}_T$-adapted, where $\mathbb{F}_T := \{\mathcal{F}_t\}_{t\in[0,T]}$, and satisfies that $\|\xi\|_{\mathbb{H}^2} := \sup_{0\le t\le T}\|\xi(t)\|_{\rm ms} < \infty$. Obviously, $(\mathbb{H}^2([0,T],\mathbb{R}^d), \|\cdot\|_{\mathbb{H}^2})$ is a Banach space.

## 3.1 Existence and uniqueness of mild solutions

We are now recalling the notion of mild solutions of (2), see [12].

**Definition 3.1** (Mild solutions of Caputo fsdes)**.** A $\mathbb{F}$-adapted process $Y$ is called a mild solution of (2) with the initial condition $Y(0) = \eta$ if the following equality holds for $t \in [0,T]$

$$\begin{aligned} Y(t) \;&=\; E_\alpha(t^\alpha A)\eta + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) b(\tau, Y(\tau))\, d\tau \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A)\sigma(\tau, Y(\tau))\, dW_\tau. \end{aligned} \tag{6}$$

Now, we establish a result on the existence and uniqueness of mild solutions for equation (2). In this result, we require that the coefficients of the system satisfy (H1) and (H2). The main ingredient of the proof is to introduce a suitable weighted norm (cf. [7]). Note that in [12], a result of the existence and uniqueness of mild solutions for a larger class of systems was also given. However, the assumption of the coefficients of these systems is stronger than (H1) and (H2).

**Theorem 3.2** (Existence and uniqueness of mild solutions)**.** *Suppose that (H1) and (H2) hold. For any $\eta \in \mathfrak{X}_0$, there exists a unique mild solution $Y$ of* (2) *satisfying $Y(0) = \eta$, which is denoted by $\psi(t, \eta)$.*

*Proof.* Let $\mathbb{H}^2_\eta([0,T], \mathbb{R}^d) := \{\xi \in \mathbb{H}^2([0,T], \mathbb{R}^d) : \xi(0) = \eta\}$. Define the corresponding *Lyapunov-Perron operator* $\mathcal{T}_\eta : \mathbb{H}^2_\eta([0,T], \mathbb{R}^d) \to \mathbb{H}^2_\eta([0,T], \mathbb{R}^d)$ by

$$\begin{aligned} \mathcal{T}_\eta Y(t) = \;& E_\alpha(t^\alpha A)\eta + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) b(\tau, Y(\tau))\, d\tau \\ & + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A)\sigma(\tau, Y(\tau))\, dW_\tau. \end{aligned}$$

It is easy to see that operator $\mathcal{T}_\eta$ is well-defined. To complete the proof, it is sufficient to show that $\mathcal{T}_\eta$ is contractive with respect to a suitable metric on $\mathbb{H}^2_\eta([0,T], \mathbb{R}^d)$. For this purpose, let $\mathbb{H}^2([0,T], \mathbb{R}^d)$ be endowed with a weighted norm $\|\cdot\|_\gamma$, where $\gamma > 0$, defined as follows

$$\|\xi\|_\gamma := \sup_{t\in[0,T]} \sqrt{\frac{\mathbb{E}(\|\xi(t)\|^2)}{E_{2\alpha-1}(\gamma t^{2\alpha-1})}} \qquad \text{for all } \xi \in \mathbb{H}^2([0,T], \mathbb{R}^d). \tag{7}$$

Obviously, two norms $\|\cdot\|_{\mathbb{H}^2}$ and $\|\cdot\|_\gamma$ are equivalent. Thus, $(\mathbb{H}^2([0,T], \mathbb{R}^d), \|\cdot\|_\gamma)$ is also a Banach space. Therefore, the set $\mathbb{H}^2_\eta([0,T], \mathbb{R}^d)$ with the metric induced by $\|\cdot\|_\gamma$ is complete. By compactness of $[0,T]$ and continuity of the function $t \mapsto E_{\alpha,\alpha}(t^\alpha A)$, there exists $M_T := \max_{t\in[0,T]} \|E_{\alpha,\alpha}(t^\alpha A)\| > 0$. Choose and fix a positive constant $\gamma$ such that

$$2L^2 M_T^2 (T+1) \frac{\Gamma(2\alpha-1)}{\gamma} < 1. \tag{8}$$

Now, by definition of $\mathcal{T}_\eta$, (H1), Ito's isometry and $M_T$ we have

$$\begin{aligned}\|\mathcal{T}_\eta X(t) - \mathcal{T}_\eta Y(t)\|_{\text{ms}}^2 \;\leq\; & 2L^2 M_T^2 \left\| \int_0^t (t-\tau)^{\alpha-1} \|X(\tau) - Y(\tau)\| \, d\tau \right\|_{\text{ms}}^2 \\ & + 2L^2 M_T^2 \int_0^t (t-\tau)^{2\alpha-2} \|X(\tau) - Y(\tau)\|_{\text{ms}}^2 \, d\tau.\end{aligned}$$

Using Hölder inequality, we obtain that

$$\|\mathcal{T}_\eta X(t) - \mathcal{T}_\eta Y(t)\|_{\text{ms}}^2 \leq 2L^2 M_T^2 (T+1) \int_0^t (t-\tau)^{2\alpha-2} \|X(\tau) - Y(\tau)\|_{\text{ms}}^2 \, d\tau.$$

Hence, by definition of $\|\cdot\|_\gamma$ we have

$$\begin{aligned}& \frac{\|\mathcal{T}_\eta X(t) - \mathcal{T}_\eta Y(t)\|_{\text{ms}}^2}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \\ \leq\; & 2L^2 M_T^2 (T+1) \frac{\int_0^t (t-\tau)^{2\alpha-2} E_{2\alpha-1}(\gamma \tau^{2\alpha-1}) \, d\tau}{E_{2\alpha-1}(\gamma t^{2\alpha-1})} \|X - Y\|_\gamma^2.\end{aligned}$$

Note that for all $t > 0$

$$\frac{\gamma}{\Gamma(2\alpha-1)} \int_0^t (t-\tau)^{2\alpha-2} E_{2\alpha-1}\left(\gamma \tau^{2\alpha-1}\right) d\tau \leq E_{2\alpha-1}\left(\gamma t^{2\alpha-1}\right),$$

see [7, Lemma 5]. Thus,

$$\|\mathcal{T}_\eta X - \mathcal{T}_\eta Y\|_\gamma \leq \sqrt{2L^2 M_T^2 (T+1) \frac{\Gamma(2\alpha-1)}{\gamma}} \|X - Y\|_\gamma,$$

which together with (8) implies that $\mathcal{T}_\eta$ is contractive on $\mathbb{H}_\eta^2([0,T], \mathbb{R}^d)$. By contraction mapping principle, $\mathcal{T}_\eta$ has a unique fixed point and the proof is complete. $\square$

### 3.2 Proof of Theorem 2.3

By virtue of Theorem 3.2, to prove Theorem 2.3 it is sufficient to show that

$$\varphi(t,\eta) = \psi(t,\eta) \qquad \text{for all } \eta \in \mathfrak{X}_0, t \in [0,T]. \tag{9}$$

For a clearer presentation, we give here the motivation and the structure of this proof:

Using the Ito's representation theorem, for any function $f \in \mathfrak{X}_T$ there exists a unique adapted process $\Xi \in \mathbb{H}^2([0,T],\mathbb{R}^d)$ such that $f = \mathbb{E}f + \int_0^T \Xi(\tau)\, dW_\tau$, see e.g. [10, Theorem 4.3.3]. Then, to prove (9) it is sufficient to show that the following statement

$$\left\langle \varphi(t,\eta), C + \int_0^T \Xi(\tau)\, dW_\tau \right\rangle = \left\langle \psi(t,\eta), C + \int_0^T \Xi(\tau)\, dW_\tau \right\rangle$$

holds for all $C \in \mathbb{R}^d$ and $\Xi \in \mathbb{H}^2([0,T],\mathbb{R}^d)$. To do this, we establish in Proposition 3.5 an estimate on $\left|\left\langle \varphi(t,\eta) - \psi(t,\eta), C + \int_0^T \Xi(\tau)\, dW_\tau \right\rangle\right|$. Before going to state and prove this estimate, we need a preparatory result in which we examine the components of the above term, i.e. we estimate

$$\left\|\mathbb{E}(\varphi(t,\eta) - \psi(t,\eta))(c + \int_0^T \xi(\tau)\, dW_\tau)\right\| \quad \text{where } c \in \mathbb{R}, \xi \in \mathbb{H}^2([0,T],\mathbb{R}).$$

Define functions $\chi_{\xi,\eta,c}, \kappa_{\xi,\eta,c}, \widehat{\chi}_{\xi,\eta,c}, \widehat{\kappa}_{\xi,\eta,c} : [0,T] \to \mathbb{R}^d$ by

$$\chi_{\xi,\eta,c}(t) := \mathbb{E}\varphi(t,\eta)\left(c + \int_0^T \xi(\tau)\, dW_\tau\right), \tag{10}$$

$$\kappa_{\xi,\eta,c}(t) := \mathbb{E}b(t,\varphi(t,\eta))\left(c + \int_0^T \xi(\tau)\, dW_\tau\right), \tag{11}$$

$$\widehat{\chi}_{\xi,\eta,c}(t) := \mathbb{E}\psi(t,\eta)\left(c + \int_0^T \xi(\tau)\, dW_\tau\right), \tag{12}$$

$$\widehat{\kappa}_{\xi,\eta,c}(t) := \mathbb{E}b(t,\psi(t,\eta))\left(c + \int_0^T \xi(\tau)\, dW_\tau\right). \tag{13}$$

**Remark 3.3.** In the proof of the existence and uniqueness of classical solution and mild solution, we have $\varphi(\cdot,\eta), \psi(\cdot,\eta) \in \mathbb{H}^2([0,T],\mathbb{R}^d)$. Thus, $\chi_{\xi,\eta,c}, \kappa_{\xi,\eta,c}, \widehat{\chi}_{\xi,\eta,c}, \widehat{\kappa}_{\xi,\eta,c}$ is measurable and bounded on $[0,T]$.

**Lemma 3.4.** *For all $t \in [0,T]$, the following statements hold:*

$$
\begin{aligned}
\chi_{\xi,\eta,c}(t) &= c\, E_\alpha(t^\alpha A)\mathbb{E}\eta \\
&+ \int_0^t (t-\tau)^\alpha E_{\alpha,\alpha}((t-\tau)^\alpha A)\big(\kappa_{\xi,\eta,c}(t) + \mathbb{E}\xi(\tau)\sigma(\tau,\varphi(\tau,\eta))\big)\, d\tau, \quad (14)\\
\widehat{\chi}_{\xi,\eta,c}(\tau) &= c\, E_\alpha(t^\alpha A)\mathbb{E}\eta \\
&+ \int_0^t (t-\tau)^\alpha E_{\alpha,\alpha}((t-\tau)^\alpha A)\big(\widehat{\kappa}_{\xi,\eta,c}(\tau) + \mathbb{E}\xi(\tau)\sigma(\tau,\psi(\tau,\eta))\big)\, d\tau. \quad (15)
\end{aligned}
$$

*Proof.* Since $\varphi(t,\eta)$ is a solution of (2) it follows that

$$
\begin{aligned}
\varphi(t,\eta) = \eta \;&+\; \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(A\varphi(\tau,\eta) + b(\tau,\varphi(\tau,\eta)))\, d\tau \\
&+\; \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,\varphi(\tau,\eta))\, dW_\tau.
\end{aligned}
$$

Taking product of both sides of the preceding equality with $c + \int_0^T \xi(\tau)\, dW_\tau$ and then taking the expectation of both sides give that

$$
\begin{aligned}
\chi_{\xi,\eta,c}(t) &= c\,\mathbb{E}\eta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(A\chi_{\xi,\eta,c}(\tau) + \kappa_{\xi,\eta,c}(\tau))\, d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)}\left\langle \int_0^t (t-\tau)^{\alpha-1}\sigma(\tau,\varphi(\tau,\eta))\, dW_\tau, \int_0^T \xi(\tau)\, dW_\tau \right\rangle.
\end{aligned}
$$

Using Ito's isometry, we obtain that

$$
\begin{aligned}
\chi_{\xi,\eta,c}(t) &= c\,\mathbb{E}\eta \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\big(A\chi_{\xi,\eta,c}(\tau) + \kappa_{\xi,\eta,c}(\tau) + \mathbb{E}\xi(\tau)\sigma(\tau,\varphi(\tau,\eta))\big)\, d\tau.
\end{aligned}
$$

In the other words, $\chi_{\xi,\eta,c}(t)$ is the solution of the following fractional differential equation

$$
{}^{C}D^\alpha_{0+}x(t) = Ax(t) + \kappa_{\xi,\eta,c}(t) + \mathbb{E}\xi(t)\sigma(t,\varphi(t,\eta)), \qquad x(0) = c\,\mathbb{E}\eta.
$$

Then, by virtue of Remark 3.3 and Theorem 2.1, the equality (14) is verified. Next, by Definition 3.1 we have

$$
\begin{aligned}
\psi(t,\eta) &= E_\alpha(t^\alpha A)\eta + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)b(\tau,\psi(\tau,\eta))\, d\tau \\
&\quad + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)\sigma(\tau,\psi(\tau,\eta))\, dW_\tau.
\end{aligned}
$$

Taking product of both sides of the above equality with $c+\int_0^T \xi(\tau)\, dW_\tau$ and then taking the expectation of both sides give that

$$\begin{aligned}\widehat{\chi}_{\xi,\eta,c}(t) &= c\, E_\alpha(t^\alpha A)\mathbb{E}\eta + \int_0^t (t-\tau)^\alpha E_{\alpha,\alpha}((t-\tau)^\alpha A)\widehat{\kappa}_{\xi,\eta,c}(\tau)\, d\tau \\ &\quad + \left\langle \int_0^t (t-\tau)^\alpha E_{\alpha,\alpha}((t-\tau)^\alpha A)\sigma(\tau,\psi(\tau,\eta))\, dW_\tau, \int_0^T \xi(\tau)\, dW_\tau \right\rangle.\end{aligned}$$

Thus, by Ito's isometry (15) is proved. □

**Proposition 3.5.** *Let* $M_T := \max_{t\in[0,T]} \|E_{\alpha,\alpha}(t^\alpha A)\|$. *Then, for any* $C \in \mathbb{R}^d$ *and* $\Xi \in \mathbb{H}^2([0,T],\mathbb{R}^d)$ *we have*

$$\begin{aligned}&\left|\left\langle \varphi(t,\eta)-\psi(t,\eta), C + \int_0^T \Xi(\tau)\, dW_\tau \right\rangle\right|^2 \\ \leq\ & 2dM_T^2L^2\frac{T^{2\alpha-1}}{2\alpha-1}\left\|C+\int_0^T \xi(\tau)\, dW_\tau\right\|_{\mathrm{ms}}^2 \int_0^t \|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\mathrm{ms}}^2\, d\tau \\ & +2dM_T^2L^2\left\|C+\int_0^T \xi(\tau)\, dW_\tau\right\|_{\mathrm{ms}}^2 \int_0^t (t-\tau)^{2\alpha-2}\|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\mathrm{ms}}^2\, d\tau.\end{aligned}$$

*Proof.* Let $C=(c_1,\dots,c_d)^{\mathrm{T}}$ and $\Xi=(\xi_1,\dots,\xi_d)^{\mathrm{T}}$, where $\xi_i \in \mathbb{H}^2([0,T],\mathbb{R})$, $c_i\in\mathbb{R}$. Then,

$$\begin{aligned}&\left|\left\langle \varphi(t,\eta)-\psi(t,\eta), C + \int_0^T \Xi(\tau)\, dW_\tau \right\rangle\right| \\ \leq\ & \sqrt{d\sum_{i=1}^d \left|\left\langle \varphi_i(t,\eta)-\psi_i(t,\eta), c_i + \int_0^T \xi_i(\tau)\, dW_\tau \right\rangle\right|^2} \\ \leq\ & \sqrt{d\sum_{i=1}^d \left\|\mathbb{E}(\varphi(t,\eta)-\psi(t,\eta))\left(c_i+\int_0^T \xi_i(\tau)\, dW_\tau\right)\right\|^2} \\ =\ & \sqrt{d\sum_{i=1}^d \|\chi_{\xi_i,\eta,c_i}(t)-\widehat{\chi}_{\xi_i,\eta,c_i}(t)\|^2}. \qquad (16)\end{aligned}$$

Next, we are estimating $\|\chi_{\xi_i,\eta,c_i}(t)-\widehat{\chi}_{\xi_i,\eta,c_i}(t)\|$. In light of Lemma 3.4, we arrive at

$$\begin{aligned}\|\chi_{\xi_i,\eta,c_i}(t)-\widehat{\chi}_{\xi_i,\eta,c_i}(t)\| &\leq M_T\int_0^t (t-\tau)^{\alpha-1}\|\kappa_{\xi_i,\eta,c_i}(\tau)-\widehat{\kappa}_{\xi_i,\eta,c_i}(\tau)\|\,d\tau\\ &\quad +M_T L\int_0^t (t-\tau)^{\alpha-1}\|\xi_i(\tau)\|_{\text{ms}}\|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\text{ms}}\,d\tau.\end{aligned}$$

Consequently, applying Hölder inequality yields that

$$\begin{aligned}&\|\chi_{\xi_i,\eta,c_i}(t)-\widehat{\chi}_{\xi_i,\eta,c_i}(t)\|\\ \leq\ & M_T\left(\int_0^t (t-\tau)^{2\alpha-2}\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|\kappa_{\xi_i,\eta,c_i}(\tau)-\widehat{\kappa}_{\xi_i,\eta,c_i}(\tau)\|^2\,d\tau\right)^{\frac{1}{2}}\\ &+M_T L\left(\int_0^t \|\xi_i(\tau)\|_{\text{ms}}^2\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t (t-\tau)^{2\alpha-2}\|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\text{ms}}^2\,d\tau\right)^{\frac{1}{2}}\\ \leq\ & M_T\sqrt{\frac{T^{2\alpha-1}}{2\alpha-1}}\left(\int_0^t \|\kappa_{\xi_i,\eta,c_i}(\tau)-\widehat{\kappa}_{\xi_i,\eta,c_i}(\tau)\|^2\,d\tau\right)^{\frac{1}{2}}\\ &+M_T L\left(\int_0^t \|\xi_i(\tau)\|_{\text{ms}}^2\,d\tau\right)^{\frac{1}{2}}\left(\int_0^t (t-\tau)^{2\alpha-2}\|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\text{ms}}^2\,d\tau\right)^{\frac{1}{2}}.\end{aligned}$$

On the other hand, by definition of $\kappa$ and $\widehat{\kappa}$ we have for all $\tau\in[0,T]$

$$\begin{aligned}&\|\kappa_{\xi_i,\eta,c_i}(\tau)-\widehat{\kappa}_{\xi_i,\eta,c_i}(\tau)\|^2\\ =\ & \sum_{j=1}^d \left|\left\langle b_j(\tau,\varphi(\tau,\eta))-b_j(\tau,\psi(\tau,\eta)),\left(c_i+\int_0^T \xi_i(\tau)\,dW_\tau\right)\right\rangle\right|^2\\ \leq\ & L^2\|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\text{ms}}^2\left\|c_i+\int_0^T \xi_i(\tau)\,dW_\tau\right\|_{\text{ms}}^2,\end{aligned}$$

where we use (H1) to obtain the preceding inequality. Thus,

$$\begin{aligned}&\|\chi_{\xi_i,\eta,c_i}-\widehat{\chi}_{\xi_i,\eta,c_i}\|^2\\ \leq\ & 2M_T^2L^2\frac{T^{2\alpha-1}}{2\alpha-1}\left\|c_i+\int_0^T \xi_i(\tau)\,dW_\tau\right\|_{\text{ms}}^2\int_0^t \|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\text{ms}}^2\,d\tau\\ &+2M_T^2L^2\int_0^t \|\xi_i(\tau)\|_{\text{ms}}^2\,d\tau\int_0^t (t-\tau)^{2\alpha-2}\|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\text{ms}}^2\,d\tau,\end{aligned}$$

which together with (16) implies that

$$
\begin{aligned}
&\left|\left\langle \varphi(t,\eta)-\psi(t,\eta), C+\int_0^T \Xi(\tau)\, dW_\tau \right\rangle\right|^2\\
\leq\ & 2dM_T^2L^2\frac{T^{2\alpha-1}}{2\alpha-1}\left\|C+\int_0^T \xi(\tau)\, dW_\tau\right\|_{\rm ms}^2 \int_0^t \|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\rm ms}^2\, d\tau\\
& +2dM_T^2L^2\int_0^t \|\xi(\tau)\|_{\rm ms}^2\, d\tau \int_0^t (t-\tau)^{2\alpha-2}\|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\rm ms}^2\, d\tau.
\end{aligned}
$$

Furthermore, by Ito's isometry

$$
\left\|C+\int_0^T \xi(\tau)\, dW_\tau\right\|_{\rm ms}^2 = \|C\|^2+\int_0^T \|\xi(\tau)\|_{\rm ms}^2\, d\tau \geq \int_0^t \|\xi(\tau)\|_{\rm ms}^2\, d\tau,
$$

which completes the proof. □

*Proof of Theorem 2.3.* Let $T^* := \inf\{t\in[0,T]: \varphi(t,\eta)\neq\psi(t,\eta)\}$. Then, it is sufficient to show that $T^*=T$. Suppose the contrary, i.e. $T^*<T$. Choose and fix an arbitrary $\delta>0$ satisfing the following inequality

$$
2dM_T^2L^2\frac{T^{2\alpha-1}}{2\alpha-1}\delta+2dM_T^2L^2\frac{\delta^{2\alpha-1}}{2\alpha-1}<1. \tag{17}
$$

To lead a contradiction, we show that $\varphi(t,\eta)=\psi(t,\eta)$ for all $t\in[T^*,T^*+\delta]$. For this purpose, choose and fix an arbitrary $t\in[T^*,T^*+\delta]$. Using Ito's representation theorem, there exists a unique $C_t\in\mathbb{R}^d$ and $\xi_t^*\in\mathbb{H}^2([0,t],\mathbb{R}^d)$ such that $\varphi(t,\eta)-\psi(t,\eta)=C_t+\int_0^t\xi_t^*(\tau)\, dW_\tau$. We extend $\xi_t$ to the whole interval [0,T] by letting $\xi_t^*(\tau)=0$ for all $\tau\in(t,T]$. For such a $\xi_t^*$, we have

$$
\left\|C_t+\int_0^T \xi_t^*(\tau)\, dW_\tau\right\|_{\rm ms}^2 = \|\varphi(t,\eta)-\psi(t,\eta)\|_{\rm ms}^2.
$$

Thus, using Proposition 3.5 for $C=C_t, \Xi=\xi_t^*$ we obtain that

$$
\begin{aligned}
\|\varphi(t,\eta)-\psi(t,\eta)\|_{\rm ms}^2 \ \leq\ & 2dM_T^2L^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_{T^*}^t \|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\rm ms}^2\, d\tau\\
& +2dM_T^2L^2\int_{T^*}^t (t-\tau)^{2\alpha-2}\|\varphi(\tau,\eta)-\psi(\tau,\eta)\|_{\rm ms}^2\, d\tau.
\end{aligned}
$$

Consequently,

$$\begin{aligned}
&\sup_{t\in[T^*,T^*+\delta]}\|\varphi(t,\eta)-\psi(t,\eta)\|^2_{\mathrm{ms}}\\
\le\;& 2dM_T^2L^2\frac{T^{2\alpha-1}}{2\alpha-1}\delta\sup_{t\in[T^*,T^*+\delta]}\|\varphi(t,\eta)-\psi(t,\eta)\|^2_{\mathrm{ms}}\\
&+2dM_T^2L^2\frac{\delta^{2\alpha-1}}{2\alpha-1}\sup_{t\in[T^*,T^*+\delta]}\|\varphi(t,\eta)-\psi(t,\eta)\|^2_{\mathrm{ms}}.
\end{aligned}$$

By a choice of $\delta$ as in (17), we have $\sup_{t\in[T^*,T^*+\delta]}\|\varphi(t,\eta)-\psi(t,\eta)\|_{\mathrm{ms}}=0$. This leads to a contradiction and the proof is complete. □

# 4 Conclusion

In this paper, a variation of constant formula for stochastic fractional differential equations of order $\alpha\in(\frac{1}{2},1)$ is established. This formula is a natural extension of the one for fractional differential equations and stochastic differential equations. In the forthcoming paper, we apply this formula to achieve a linearized stability theory for stochastic fractional differential equations.


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