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Stability of fractional-order nonlinear systems by Lyapunov direct method

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Abstract: In this paper, by using a characterization of functions having fractional derivative, we propose a rigorous fractional Lyapunov function candidate method to analyze the stability of fractional-order nonlinear systems. First, we prove an inequality concerning the fractional derivatives of convex Lyapunov functions without the assumption of the existence of derivative of pseudo-states. Second, we establish fractional Lyapunov functions to fractional-order systems without the assumption of the global existence of solutions. Our theorems fill the gaps and strengthen results in some existing papers.

1 Introduction

Fractional differential equations have attracted increasing interest in the last decade due to the fact that many mathematical problems in science and engineering can be modeled by fractional differential equations. For more details on applications of fractional differential equations, we refer the interested reader to the monographs [1], [2], [3] and the references therein.

One of the most important problems in the qualitative theory of fractional differential equations is stability theory. Following Lyapunov's seminal 1892 thesis, these two methods are expected to also work for fractional differential equations:

- Lyapunov's first method: the method of linearization of the nonlinear equation along an orbit, and the transfer of asymptotic stability from the linear to the nonlinear equation; and
- Lyapunov's second method: the method of Lyapunov candidate functions, i.e. of scalar functions on the state space such that their energy decreases along orbits.

Recently, in [4] and [5], Cong *et al.* fully developed Lyapunov's first method for fractional-order nonlinear systems. On the other hand, although several results on the Lyapunov's second method for fractional-order nonlinear systems have been published, the development of this theory is still in its infancy and requires further investigation. One of the reasons for this might be that computation and estimation of fractional derivatives of Lyapunov candidate functions are very complicated due to the fact that the well-known Leibniz rule does not hold true for such derivatives.

To the best of our knowledge, the first valuable contribution in the theory of fractional Lyapunov functions is the paper [6]. The method in [6] became applicable after effective fractional derivative inequalities were established, see e.g. [7, Inequalities (6) & (16)], [8, Inequality (24)], and [9, Inequality (10)]. In this direction, we recommend the papers [10, Theorems 2 & 3], [9, Theorems 2 & 3], [11, Theorems 3.1 & 3.3], and [12, Example 1]. However, there are some unavoidable shortcomings of this approach such as:

- Assumption of the global existence of solutions to fractional-order nonlinear systems, see e.g., [6], [7], and [9].
- The derivative of the solutions are required for the proof of the involved fractional derivative inequalities in [7, Inequalities (6) & (16)], [8, Inequality (24)], and [9, Inequality (10)].

Following another approach using the fractional derivative of the Lyapunov candidate function along the vector field, Lakshmikantham, Leela and Devi [13, Theorem 4.3.2, pp. 100–101] also attempted to prove a Lyapunov sufficient condition for fractional differential equations. However, confusion on the locality of solutions to fractional systems makes their proof incomplete, see [13, pp. 101, lines 4–5] (note that the solution to equation (4.2.1) in [13, pp. 96] starts from t_0 and its solution which starts from t_1 are different).

Motivated by the aforementioned observations, in this paper we focus on proposing a rigorous method of Lyapunov candidate functions which is suitable for fractional-order nonlinear systems. Specifically, we establish fractional Lyapunov functions without the assumption of the global existence of solutions to fractional-order nonlinear systems. We also do not require the condition on the existence of derivative to pseudo-states in the inequality concerning the fractional derivatives of convex Lyapunov functions. The rest of our paper is organized as follows. Section 2 is devoted to recalling some notations and results about fractional calculus. In Section 3, we formulate the main result which concerns the stability of the trivial solution of fractional-order nonlinear systems based on designing an effective Lyapunov candidate function.

To conclude this introductory section, we introduce some notations which are used throughout the paper. Denote by $\mathbb{N}$ the set of nature numbers, by $\mathbb{R}$ and $\mathbb{R}_+$ the set of real numbers and nonnegative numbers, respectively. For some arbitrary positive constant d, let $\mathbb{R}^d$ be the d-dimensional Euclidean space with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. In $\mathbb{R}^d$, let $B_r(0)$ be the closed ball with the center at the origin and the radius $r > 0$. For some $T > 0$, denote by $C([0, T], \mathbb{R}^d)$ the space of continuous functions $x : [0, T] \to \mathbb{R}^d$. Finally, for $\alpha \in (0, 1]$, we mean $\mathcal{H}^\alpha([0, T], \mathbb{R}^d)$ the standard Hölder space consisting of functions $v \in C([0, T], \mathbb{R}^d)$ such that

$$\|v\|_{\mathcal{H}^\alpha} := \max_{0\le t\le T} \|v(t)\| + \sup_{0\le s<t\le T} \frac{\|v(t) - v(s)\|}{(t - s)^\alpha} < \infty$$

and by $\mathcal{H}_0^\alpha([0, T], \mathbb{R}^d)$ the closed subspace of $\mathcal{H}^\alpha([0, T], \mathbb{R}^d)$ consisting of functions $v \in \mathcal{H}^\alpha([0, T], \mathbb{R}^d)$ such that $v(0) = 0$ and

$$\sup_{0\le s<t\le T, t-s\le\varepsilon} \frac{\|v(t) - v(s)\|}{(t - s)^\alpha} \to 0 \quad \text{as } \varepsilon \to 0.$$

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2 Preliminaries

We recall briefly a framework of fractional calculus and fractional differential equations.

Let $\alpha \in (0,1)$, $[0,T] \subset \mathbb{R}$ and $x : [0,T] \to \mathbb{R}$ be a measurable function such that $x \in L^1([0,T])$, i.e. $\int_0^T |x(\tau)|\, d\tau < \infty$. Then, the Riemann–Liouville integral of order α is defined by

$$I^{\alpha}_{0+}x(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau)\, d\tau \quad \text{for } t \in (0,T],$$

where the Gamma function $\Gamma : (0,\infty) \to \mathbb{R}$ is defined as

$$\Gamma(\alpha) := \int_0^{\infty} \tau^{\alpha-1} \exp(-\tau)\, d\tau,$$

see e.g., Diethelm [14]. The corresponding Riemann–Liouville fractional derivative of order α is given by

$${}^{R}D^{\alpha}_{0+}x(t) := (DI^{1-\alpha}_{0+}x)(t) \quad \text{for } t \in (0,T],$$

where $D = \frac{d}{dt}$ is the usual derivative. On the other hand, the *Caputo fractional derivative* ${}^{C}D^{\alpha}_{0+}x$ of x is defined by

$${}^{C}D^{\alpha}_{0+}x(t) :={}^{R} D^{\alpha}_{0+}(x(t)-x(0)), \qquad \text{for } t \in (0,T],$$

see [14, Definition 3.2, pp. 50]. The Caputo fractional derivative of a d-dimensional vector function $x(t) = (x_1(t), \ldots, x_d(t))^T$ is defined component-wise as

$${}^{C}D^{\alpha}_{0+}x(t) = ({}^{C}D^{\alpha}_{0+}x_1(t), \ldots, {}^{C}D^{\alpha}_{0+}x_d(t))^T.$$

Denote by $I^{\alpha}_{0+}C([0,T],\mathbb{R}^d)$ the space of functions $\varphi : [0,T] \to \mathbb{R}^d$ such that there exists a function $\psi \in C([0,T],\mathbb{R}^d)$ satisfying $\varphi = I^{\alpha}_{0+}\psi$. The following result gives a characterization of functions having Caputo fractional derivative.

Theorem 1. *For $\alpha \in (0,1)$ and a function $v \in C([0,T],\mathbb{R}^d)$, the following conditions (i), (ii), (iii) are equivalent:*

(i) the fractional derivative ${}^{C}D^{\alpha}_{0+}v \in C([0,T],\mathbb{R}^d)$ exists;
(ii) a finite limit $\lim_{t\to 0} \frac{v(t)-v(0)}{t^{\alpha}} := \gamma$ exists, and

$$\sup_{0<t\le T} \| \int_{\theta t}^{t} (t-\tau)^{-\alpha-1}(v(t)-v(\tau))\, d\tau \| \to 0 \qquad \text{as} \quad \theta \to 1,$$

i.e. the integral $\int_0^t (t-\tau)^{-\alpha-1}(v(t)-v(\tau))d\tau$ is equi-convergent for any $t \in (0,T]$;
(iii) v has the structure $v - v(0) = t^{\alpha}\gamma + v_0$, where γ is a constant vector, $v_0 \in \mathcal{H}^{\alpha}_0([0,T],\mathbb{R}^d)$, and $\int_0^t (t-\tau)^{-\alpha-1}(v(t)-v(\tau))d\tau =: w(t)$ is equi-convergent for every $t \in (0,T]$ defining a function $w \in C((0,T],\mathbb{R}^d)$ which has a finite limit $\lim_{t\to 0} w(t) =: w(0)$.

For $v \in C([0,T],\mathbb{R}^d)$ with ${}^{C}D^{\alpha}_{0+}v \in C([0,T],\mathbb{R}^d)$, it holds ${}^{C}D^{\alpha}_{0+}v(0) = \Gamma(\alpha+1)\gamma$, and

$${}^{C}D^{\alpha}_{0+}v(t) = \frac{v(t)-v(0)}{\Gamma(1-\alpha)t^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{v(t)-v(\tau)}{(t-\tau)^{\alpha+1}} d\tau, \quad 0 < t \le T.$$

Proof: See [15, Theorem 5.2, pp. 475]. □

Let $D \subset \mathbb{R}^d$ is an open set and $0 \in D$. In this paper, we consider the following equation with the fractional order $\alpha \in (0,1)$:

$${}^{C}D^{\alpha}_{0+}x(t) = f(x(t)), \quad \text{for } t \in (0,\infty), \tag{1}$$

where $f : D \to \mathbb{R}^d$ satisfies the conditions:

(f.1) $f(0) = 0$;
(f.2) the function $f(\cdot)$ is locally Lipchitz continuous in a neighborhood of the origin.

Since f is local Lipschitz continuous, [14, Theorem 6.5] implies unique existence of solutions of initial value problems (1), $x(0) = x_0$ for $x_0 \in \mathbb{R}^n$. Let $\varphi : I \times \mathbb{R}^d \to \mathbb{R}^d$, $t \mapsto \varphi(t,x_0)$ denote the solution of (1), $x(0) = x_0$, on its maximal interval of existence $I = [0, t_{\max}(x_0))$ with $0 < t_{\max}(x_0) \le \infty$. We now give the notions of stability of the trivial solution of (1).

Definition 1. *(i) The trivial solution of* (1) *is called* stable *if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $\|x_0\| < \delta$ we have $t_{\max}(x_0) = \infty$ and*

$$\|\varphi(t,x_0)\| < \varepsilon, \quad \forall t \ge 0.$$

(ii) The trivial solution is called weakly asymptotically stable *if it is stable and there exists $\widehat{\delta} > 0$ and a positive sequence $\{t_k\}_{k=1}^{\infty}$, where $t_k \to \infty$ as $k \to \infty$, such that $\lim_{k\to\infty} \varphi(t_k,x_0) = 0$ whenever $\|x_0\| < \widehat{\delta}$.*
(iii) The trivial solution is called asymptotically stable *if it is stable and there exists $\widehat{\delta} > 0$ such that $\lim_{t\to\infty} \varphi(t,x_0) = 0$ whenever $\|x_0\| < \widehat{\delta}$.*

3 Lyapunov direct method for fractional order systems

In this section, we will establish a Lyapunov candidate function for a fractional-order system to analyze the asymptotic behavior of solutions around the equilibrium points. To do this, we need the following preparatory result which gives an upper bound of the fractional derivative of a composite function.

Theorem 2. *For a given $x_0 \in \mathbb{R}^d$, let $u \in \{x_0\} + I^{\alpha}_{0+}C([0,T],\mathbb{R}^d)$ and $V : \mathbb{R}^d \to \mathbb{R}$ satisfies the following conditions:*

(V.1) the function V is convex on $\mathbb{R}^d$ and $V(0) = 0$;
(V.2) the function V is differentiable on $\mathbb{R}^d$.

Then the following inequality holds for all $t \in [0,T]$:

$${}^{C}D^{\alpha}_{0+}V(u(t)) \le \langle \nabla V(u(t)), {}^{C}D^{\alpha}_{0+}u(t)\rangle, \tag{2}$$

where ∇V is the gradient of the function V.

Proof: Due to $u \in \{x_0\} + I^{\alpha}_{0+}C([0,T],\mathbb{R}^d)$, there exists a function $\psi \in C([0,T],\mathbb{R}^d)$ such that $u = x_0 + I^{\alpha}_{0+}\psi$. From [15, Proposition 6.4, pp. 479], we see that the Caputo fractional derivative ${}^{C}D^{\alpha}_{0+}u$ exists and continuous on $[0,T]$. On the other hand, by Theorem 1, this derivative has the representation

$${}^{C}D^{\alpha}_{0+}u(0) := \Gamma(\alpha+1)\gamma, \tag{3}$$

where $\gamma = \frac{\psi(0)}{\Gamma(\alpha+1)}$, and

$${}^{C}D^{\alpha}_{0+}u(t) = \frac{u(t)-u(0)}{\Gamma(1-\alpha)t^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{u(t)-u(\tau)}{(t-\tau)^{\alpha+1}} d\tau, \quad 0 < t \le T. \tag{4}$$

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Using (V.1), (V.2) and by a direct computation, we have

$$\lim_{t\to 0}\frac{V(u(t))-V(u(0))}{t^\alpha}=\langle\nabla V(u(0)),\gamma\rangle.$$

Moreover, from (V.2) and the fact

$$\sup_{0<t\leq T}\left\|\int_{\theta t}^t (t-\tau)^{-\alpha-1}(u(t)-u(\tau))d\tau\right\|\to 0 \qquad \text{as} \quad \theta\to 1,$$

the limit below holds

$$\sup_{0<t\leq T}\left|\int_{\theta t}^t (t-\tau)^{-\alpha-1}(V(u(t))-V(u(\tau)))d\tau\right|\to 0,$$

as $\theta\to 1$, which together with Theorem 1 shows that

$$^{C}D^\alpha_{0+}V(u(0))=\Gamma(\alpha+1)\langle\nabla V(u(0)),\gamma\rangle \tag{5}$$

and for $t\in(0,T]$:

$$^{C}D^\alpha_{0+}V(u(t))=\frac{V(u(t))-V(u(0))}{\Gamma(1-\alpha)t^\alpha}+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^t\frac{V(u(t))-V(u(\tau))}{(t-\tau)^{\alpha+1}}d\tau. \tag{6}$$

From (3) and (5) we have

$$^{C}D^\alpha_{0+}V(u(0))=\langle\nabla V(u(0)),{}^{C}D^\alpha_{0+}u(0)\rangle. \tag{7}$$

For $0<t\leq T$, using the representations (4) and (6) leads to

$$\begin{aligned}&^{C}D^\alpha_{0+}V(u(t))-\langle\nabla V(u(t)),{}^{C}D^\alpha_{0+}u(t)\rangle\\ &=\frac{V(u(t))-V(u(0))-\langle\nabla V(u(t)),u(t)-u(0)\rangle}{\Gamma(1-\alpha)t^\alpha}\\ &\quad+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^t\frac{V(u(t))-V(u(\tau))}{(t-\tau)^{\alpha+1}}d\tau\\ &\quad-\frac{\alpha}{\Gamma(1-\alpha)}\int_0^t\frac{\langle\nabla V(u(t)),u(t)-u(\tau)\rangle}{(t-\tau)^{\alpha+1}}d\tau.\end{aligned} \tag{8}$$

Because V is convex and differentiable, using [16, Theorem 25.1, pp. 242], we obtain

$$V(u(t))-V(u(\tau))-\langle\nabla V(u(t)),u(t)-u(\tau)\rangle\leq 0$$

for all $0\leq\tau\leq t\leq T$, which together with (7) and (8) implies that

$$^{C}D^\alpha_{0+}V(u(t))\leq\langle\nabla V(u(t)),{}^{C}D^\alpha_{0+}u(t)\rangle,\quad \forall t\in[0,T].$$

The proof is complete. □

Remark 1. *A special case of Theorem* 2 *when* $V(x)=\|x\|^2$ *was proven by Aguila-Camacho, Duarte-Mermoud and Gallegos [7, Lemma 1 & Remark 1]. In the case V is convex and differentiable, the inequality* (2) *was formulated by Chen et al. [9, Theorem 1]. To obtain the proof of these results, the authors of [7, 9] required that the function* x *in Theorem* 2 *is differentiable. However, in general, the solutions to fractional differential equations are not always differentiable. For example, we consider the following equation*

$$^{C}D^\alpha_{0+}x(t)=f(t),\quad t>0,\quad x(0)=0, \tag{9}$$

where

$$f(t)=\begin{cases}t^\beta, & t\in[0,1],\ 0<\beta<1-\alpha,\\ 1, & t\geq 1.\end{cases}$$

It is obvious to see that the solution $\varphi(\cdot,0)$ *to* (9) *is continuous on* $[0,\infty)$ *but not differentiable at* $t=0$ *and* $t=1$. *Thus, in our opinion, the assumption in [7, 9] is restrictive which makes the inequality* (2) *unable to be directly applied to study the asymptotic behavior of solutions to fractional systems. Our result as presented in Theorem 3 now removes this restriction.*

We are now in a position to state the main theorem. It is worth noticing that we do not need the assumption on the global existence of solutions to the system (1).

Theorem 3. *Consider the equation* (1). *Assume there is a function* $V:\mathbb{R}^d\to\mathbb{R}$ *satisfying*

(V.i) the function V *is convex, differentiable on* $\mathbb{R}^d$ *and* $V(0)=0$*;*
(V.ii) there exist constants $a,b,C_1,C_2,r>0$ *such that*

$$C_1\|x\|^a\leq V(x)\leq C_2\|x\|^b$$

for all $x\in B_r(0)$*;*
(V.iii) there are constants $C_3\geq 0$ *and* $c\geq b$ *such that*

$$\langle\nabla V(x),f(x)\rangle\leq -C_3\|x\|^c$$

for all $x\in B_r(0)$.

Then,

(a) the trivial solution of (1) *is stable if* $C_3=0$*;*
(b) the trivial solution of (1) *is asymptotically stable if* $C_3>0$ *and* $c=b$*;*
(c) the trivial solution of (1) *is weakly asymptotically stable if* $C_3>0$ *and* $c>b$.

Proof: From the assumptions (f.1) and (f.2), there is a constant $r_1\in(0,r)$ such that f is Lipschitz continuous on $B_{r_1}(0)$. Let L be a Lipschitz constant to f on $B_{r_1}(0)$ and let F denote an extended Lipschitz function of f with the Lipschitz constant L, i.e. the function $F:\mathbb{R}^d\to\mathbb{R}^d$ is Lipschitz continuous with the Lipschitz constant L and $F(x)=f(x)$ for all $x\in B_{r_1}(0)$. Note that this extension always exists, see e.g. [17, Theorem 2.5]. For any $\varepsilon\in(0,r_1)$, we choose $\delta=\frac{1}{K}\left(\frac{C_1}{C_2}\right)^{1/b}\varepsilon^{a/b}$, where $K>1$ is large enough to $\delta<\varepsilon$. For any $x_0\in B_\delta(0)$, denote $\tilde\varphi(\cdot,x_0)$ the solution to the initial problem

$$\begin{cases}^{C}D^\alpha_{0+}x(t)=F(x(t)), & t>0,\\ x(0)=x_0.\end{cases} \tag{10}$$

Due to [18, Theorem 2], this solution is defined uniquely on the whole interval $[0,\infty)$. Assume that there is a time $t>0$ such that $\|\tilde\varphi(t,x_0)\|=\varepsilon$. Put $t_0:=\inf\{t>0:\|\tilde\varphi(t,x_0)\|\geq\varepsilon\}$, then $t_0>0$, $\|\tilde\varphi(t_0,x_0)\|=\varepsilon$ and $\|\tilde\varphi(t,x_0)\|<\varepsilon$ for all $t\in[0,t_0)$. Hence, $\tilde\varphi(\cdot,x_0)$ satisfies the conditions (V.ii) and (V.iii) on the interval $[0,t_0]$.

(a) Now consider the case $C_3=0$. Using Theorem 2, we have

$$^{C}D^\alpha_{0+}V(\tilde\varphi(t,x_0))\leq\langle\nabla V(\tilde\varphi(t,x_0)),F(\tilde\varphi(t,x_0))\rangle\leq 0,$$

for all $t\in[0,t_0]$. Hence, by the comparison lemma [6, Lemma 10], the following estimation holds

$$V(\tilde\varphi(t,x_0))\leq V(x_0),\quad \text{for all } t\in[0,t_0], \tag{11}$$

this combining with (V.i) implies that

$$\|\tilde\varphi(t,x_0)\|\leq\left(\frac{C_2}{C_1}\|x_0\|^b\right)^{1/a},\quad \text{for all } t\in[0,t_0]. \tag{12}$$

From (12), we see

$$\|\tilde\varphi(t_0,x_0)\|\leq\left(\frac{C_2}{C_1}\|x_0\|^b\right)^{1/a}\leq\left(\frac{C_2}{C_1}\delta^b\right)^{1/a}<\varepsilon,$$

a contradiction. Thus, $\|\tilde\varphi(t,x_0)\|<\varepsilon$ for all $t\in[0,\infty)$. However, in this case, $\tilde\varphi(\cdot,x_0)$ is also a solution to the original equation (1)

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with the initial condition $x(0) = x_0$, which shows that the trivial solution to (1) is stable.

(b) Assume that $C_3 > 0$. Using Theorem 2 and the conditions (V.ii) and (V.iii), we have

$$^C D_{0+}^{\alpha} V(\tilde{\varphi}(t,x_0)) \le -\frac{C_3}{C_2} V(\tilde{\varphi}(t,x_0)), \quad \text{for all } t \in [0,t_0].$$

As shown in [14], the function $V(x_0)E_\alpha\left(-\frac{C_3}{C_2}t^\alpha\right)$, where $E_\alpha : \mathbb{C} \to \mathbb{C}$ is the Mittag-Leffler function defined by $E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$, is the unique solution to the initial value problem

$$\begin{cases} ^C D_{0+}^{\alpha} x(t) = -\frac{C_3}{C_2} x(t), & t \in (0,t_0], \\ x(0) = V(x_0). \end{cases}$$

Then, by using the same arguments as in the proof of [6, Theorem 5] shows that

$$V(\tilde{\varphi}(t,x_0)) \le V(x_0)E_\alpha\left(-\frac{C_3}{C_2}t^\alpha\right), \quad \text{for all } t \in [0,t_0],$$

which together with (V.ii) implies

$$\|\tilde{\varphi}(t,x_0)\| \le \left(\frac{C_2}{C_1}\|x_0\|^b E_\alpha\left(-\frac{C_3}{C_2}t^\alpha\right)\right)^{1/a}, \quad \forall t \in [0,t_0].$$

In particular,

$$\|\tilde{\varphi}(t_0,x_0)\| \le \left(\frac{C_2}{C_1}\delta^b\right)^{1/a} < \varepsilon,$$

a contradiction. Hence, $\|\tilde{\varphi}(t,x_0)\| < \varepsilon$ for every $t \ge 0$ and thus the solution $\tilde{\varphi}(t,x_0)$ satisfies the conditions (V.ii) and (V.iii) for any $t \ge 0$, which leads to

$$\|\tilde{\varphi}(t,x_0)\| \le \left(\frac{C_2}{C_1}\|x_0\|^b E_\alpha\left(-\frac{C_3}{C_2}t^\alpha\right)\right)^{1/a}, \quad \forall t \in [0,\infty).$$

In other words, $\lim_{t\to\infty} \tilde{\varphi}(t,x_0) = 0$. On the other hand,

$$\begin{aligned} ^C D_{0+}^{\alpha} \tilde{\varphi}(t,x_0) &= F(\tilde{\varphi}(t,x_0)) \\ &= f(\tilde{\varphi}(t,x_0)), \quad \text{for all } t \in (0,\infty), \end{aligned}$$

i.e. $\tilde{\varphi}(t,x_0)$ is also the unique global solution to the original problem (1), $x(0) = x_0$. Thus, the trivial solution to (1) is asymptotically stable.

(c) Consider the case $C_3 > 0$ and $c > b$. By using the same arguments as above, we see that the trivial solution to (1) is stable. Hence, for $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that every solution $\varphi(t,x_0)$ to (1) with $\|x_0\| < \delta$ satisfies $\|\varphi(t,x_0)\| < \varepsilon$ for all $t \ge 0$. Suppose that there is an $\varepsilon_0 > 0$ such that

$$\|\varphi(t,x_0)\| \ge \varepsilon_0, \quad \text{for all } t \ge 0. \tag{13}$$

From (V.iii), we have

$$\begin{aligned} ^C D_{0+}^{\alpha} V(\varphi(t,x_0)) &\le -C_3\|\varphi(t,x_0)\|^c \\ &\le -\frac{C_3}{C_2}\varepsilon_0^{c-b}\|\varphi(t,x_0)\|^b, \quad \forall t \ge 0, \end{aligned}$$

which together with (V.ii) implies that

$$\|\varphi(t,x_0)\| \le \left(\frac{V(x_0)}{C_1} E_\alpha\left(-\frac{C_3}{C_2}\varepsilon_0^{c-b}t^\alpha\right)\right)^{1/a}, \quad \forall t \ge 0. \tag{14}$$

However, the inequality (14) shows that $\varphi(t,x_0)$ tends to zero as $t \to \infty$, a contradiction with the assumption (13). Thus, the assumption (13) is false and there exists a positive $\{t_k\}_{k=1}^{\infty}$, where $t_k \to \infty$ as $k \to \infty$ such that $\varphi(t_k,x_0) \to 0$ as $k \to \infty$. The proof is complete. □

Remark 2. *Recently, Chen* et al. *[9, Theorem 2, pp. 1072] proposed a fractional Lyapunov candidate for fractional systems with the same assumptions as in Theorem* 3*. However, the proof of their result is incomplete. Indeed, the approach in [9, Theorem 2] is based on the inequality (10) in [9, Theorem 1] and [6, Theorem 5, pp. 1967]. As mentioned in Remark* 1*, this inequality was established for only differentiable functions. On the other hand, solutions of fractional differential equations are generally not differentiable. Thus, it is impossible to prove [9, Theorem 2] by using the inequality (10) in [9, Theorem 1] as the authors asserted.*

Remark 3. *Many researchers have proposed fractional Lyapunov functions for fractional-order systems by combining the inequalities [7, Inequality (16), pp. 2954], or [8, Inequality (24), pp. 654], or [9, Inequality (10), pp. 8] with [6, Theorem 11], see e.g. [9, Theorem 3, pp. 1072], [19, Theorem 1, pp. 1361], [20, Theorems 2 & 3, pp. 684]. However, it should be noted that there is a gap in the proof of [6, Theorem 11]. Indeed, to prove this theorem, the authors [6] relied on the arguments as follows. Consider a function $x : [0,\infty) \to \mathbb{R}_+$. If x does not satisfy the two conditions as below:*
Case 1*: there is a constant $t_1 > 0$ such that $x(t_1) = 0$; and*
Case 2: *there exists a positive constant ε such that $x(t) \ge \varepsilon$, $\forall t \ge 0$.*
Then $\lim_{t\to\infty} x(t) = 0$. Unfortunately, this argument seems incorrect. For a counter example, we consider the function $x(t) = \frac{1}{1+t} + \sin(t) + 1$, $\forall t \ge 0$. It is obvious that $x(t) > 0$, $\forall t \ge 0$. Furthermore, there exists the sequence $\{t_k\}_{k=1}^{\infty}$, where $t_k = -\frac{\pi}{2} + 2k\pi$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} x(t_k) = 0$. Hence, there does not exist a parameter $\varepsilon > 0$ such that $x(t) \ge \varepsilon$, $\forall t \ge 0$. Thus, the function x does not satisfy both **Case 1** *and* **Case 2** *as above. On the other hand, this function does not tend to zero at infinity. This implies that the proof of [6, Theorem 11] is wrong. Hence, we recommend the reader to be careful when citing this result [6, Theorem 11].*

Remark 4. *If the "vector field" of the fractional-order systems has no linearization part, then the linearization method along the trajectories (see [4, 5]) does not work, whereas, the method in our paper can be applied. Moreover, using our method, one can show the stability without solving the solution of systems. On the other hand, the disadvantage is that there is no general method of constructing suitable candidate for the Lyapunov functions.*

Remark 5. *In the theory of ordinary differential equations, to obtain Lyapunov functions, one does not need the assumptions* (V.i)–(V.iii) *of Theorem* 3*. However, due to the fact that the fractional derivative has no geometrical property (the negativity of fractional derivative without implying the decreasing of the function), therefore in the theory of fractional systems, we have to impose the geometrical condition on the candidate for fractional Lyapunov function (the convex condition of the function V) and inequalities as* (V.ii)–(V.iii)*. In our opinion, these assumptions are restrictive but essential and they are characteristics of fractional-order differential systems.*

Finally, we illustrate the theoretical result by three examples as follows.

Example 1. *Let the equation*

$$^C D_{0+}^{\alpha} x(t) = -Ax(t), \tag{15}$$

where $A \in \mathbb{R}^{d\times d}$ is a symmetric and positive definite matrix. Choosing the Lyapunov function $V(x) = \langle x, x\rangle$ for all $x \in \mathbb{R}^d$ and using Theorem 3 *(b), we see that the trivial solution to this equation is asymptotically stable. In this example, for any initial condition $x_0 \in \mathbb{R}^d$, the solution $\varphi(\cdot,x_0)$ of* (15) *tends to zero at the infinity.*

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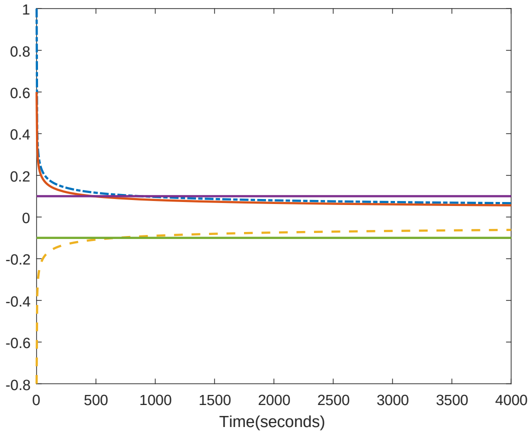

Fig. 1: Trajectory of the solutions $\varphi(\cdot, 1)$, $\varphi(\cdot, 0.6)$ and $\varphi(\cdot, -0.8)$ with $\alpha = 0.8$.

Example 2. *Consider the equation*

$$^{C}D^{\alpha}_{0+}x(t) = -x^3(t), \quad \forall t \geq 0. \tag{16}$$

It is obvious that the function $f(x) = -x^3$ is local Lipschitz. Choosing the function $V(x) = x^2$ for all $x \in \mathbb{R}$. This function satisfies the conditions (V.i), (V.ii) (with $C_1 = C_2 = 1$ and $a = b = 2$), and (V.iii) (with $C_3 = 2$ and $c = 4$). Thus, from Theorem 3(c), the trivial solution to (16) is weakly asymptotically stable. From the proof of Theorem 3, we see that the solution $\varphi(\cdot, x_0)$ of (16) is always bounded for any $x_0 \in \mathbb{R}$. Moreover, there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $\varphi(t_n, x_0) \to 0$ as $n \to \infty$. Fig. 1 depicts the trajectory of the solutions $\varphi(\cdot, 1)$, $\varphi(\cdot, 0.6)$ and $\varphi(\cdot, -0.8)$ to the equation (16) with $\alpha = 0.8$. For a small ε_0 (in this case we choose $\varepsilon_0 = 0.1$), after $t_0 = 1000s$ these solutions contain in the interval $[-0.1, 0.1]$. Note that Li et al., [6, Example 14] attempted to show that the trivial solution to (16) is asymptotically stable. Their proof was based on the following statement: Let x be a solution to (16) with $x(0) \neq 0$. If there is not a constant $\xi > 0$ to $x(t)x(0) \geq \xi$ for all $t \geq 0$, then $x(t) \to 0$ as $t \to \infty$, see the lines from -1 to -6, column 2, pp. 1968 in [6]. Unfortunately, this statement is not correct. For a counterexample, see Remark 3. In [21, Remark 11], the authors revised [6, Example 14]. However, their proof is also incomplete because they used [6, Theorem 11]. As we have shown in Remark 3, the proof of [6, Theorem 11] is wrong. Zhou et al., [22] also attempted to prove the asymptotic stability of the trivial solution to (16). However, their work was based on an incorrect result, see [22, Theorem 3.1]. To our knowledge, the question on the asymptotic stability of the trivial solution to (16) is still open.

Example 3. *Consider the following fractional-order nonlinear system*

$$^{C}D^{\alpha}_{0+}x(t) = f(x(t)), \quad t > 0, \tag{17}$$

where $f(x) = (-x_1^3 - x_1x_2^2, -x_2^3 + x_2x_1^2)^{\mathrm{T}}$ for any $x = (x_1, x_2) \in \mathbb{R}^2$. In this case the function f satisfies:

- *$f(0) = 0$;*
- *f is Lipschitz continuous in a ball $B_r(0)$ for any $r > 0$.*

We choose the Lyapunov candidate $V(x) = \|x\|^2 = x_1^2 + x_2^2$ for $x = (x_1, x_2) \in \mathbb{R}^2$. The function V satisfies the conditions (V.i), (V.ii) *and* (V.iii) *in Theorem* 3 *for $a = b = 2$, $c = 4$, $C_1 = C_2 = 1$ and $C_3 = 1$. Hence, the trivial solution of* (17) *is weakly asymptotically stable. Furthermore, from the proof of this result, we see that the stability is global, i.e., it does not depend on the initial condition $x_0 \in \mathbb{R}^2$.*

Conclusion

In this paper, we have proposed a rigorous Lyapunov type method to analyze the stability of fractional-order nonlinear systems. More precisely, we make two main contributions:

- Proving the inequality concerning the fractional derivatives of convex Lyapunov functions without the assumption on the existence of derivative of pseudo-states, see Theorem 2; and
- Establishing the fractional Lyapunov functions to fractional-order systems without the assumption of the global existence of solutions, see Theorem 3.

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