ON THE BETTI NUMBERS OF EDGE IDEAL OF SKEW FERRERS GRAPHS

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Abstract. We prove that $\beta_p(I(G)) = \beta_{p,p+r}(I(G))$ for skew Ferrers graph G, where $p := \operatorname{pd}(I(G))$ and $r := \operatorname{reg}(I(G))$. As a consequence, we confirm that Ene, Herzog and Hibi's conjecture is true for the Betti numbers in the last columm of Betti table. We also give an explicit formula for the unique extremal Betti number of binomial edge ideal for some closed graphs.

Introduction

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over an arbitrary field k. Associated to any homogeneous ideal I of R is a minimal free graded resolution

$$0 \to \bigoplus_j R(-j)^{\beta_{\ell,j}(I)} \to \bigoplus_j R(-j)^{\beta_{\ell-1,j}(I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(I)} \to I \to 0,$$

where $R(-j)$ denotes the R-module obtained by shifting the degrees of R by j, and $\ell = \operatorname{pd}(I)$ is the projective dimension of I. The number $\beta^R_{i,j}(I)$ (or write $\beta_{i,j}(I)$ if no confusion is caused) is the (i,j)*-th graded Betti number* of I and equals the number of minimal generators of degree j in the i-th syzygy module. We have $\beta_{i,j}(I) = \dim_k \operatorname{Tor}^R_{i+1}(R/I; k)_j$. The set of graded Betti numbers is represented in terms of a Betti table, in which the entry at column i and row j is $\beta_{i,i+j}(I)$. The i-th total Betti number of I is defined by $\beta_i(I) = \sum_j \beta_{i,j}(I)$. The ℓ-th column of Betti table of I is called *the last column of Betti table* of I. The regularity of I is defined by $\operatorname{reg}(I) := \max\{j - i \mid \beta_{i,j}(I) \neq 0\}$.

Let G be a simple graph on the vertex set $V(G) = \{x_1, \dots, x_n\}$ and edge set $E(G)$. We associate to the graph G a quadratic squarefree monomial ideal

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E(G)) \subseteq R,$$

which is called the *edge ideal* of G. In [4], Corso and Nagel showed that the edge ideal of Ferrers graphs has linear resolution, and furthermore they gave an explicit formula for Betti numbers of this ideals. After that, Nagel and Reiner (see [15]) showed that the Betti numbers of the edge ideal of skew Ferrers graphs are independent on the base field k. In this paper, we show that the last Betti number of the edge ideal of skew Ferrers graphs is equal to its unique extremal Betti number.

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Theorem 2.5. *Let G be a skew Ferrers graph. Then $\beta_p(I(G)) = \beta_{p,p+r}(I(G))$, where $p := \mathrm{pd}(I(G))$ and $r := \mathrm{reg}(I(G))$.*

Now we assume a vertex set of G is $V(G) = \{1, \dots, n\}$. Herzog et al. [11]; and Ohtani [16] independently introduced a *binomial edge ideal*, denoted by J_G, associated to G in polynomial ring $S := k[x_1, \dots, x_n, y_1, \dots, y_n]$ which is generated by $x_i y_j - x_j y_i$, where $\{i, j\} \in E(G)$ and $i < j$. It is known that $\beta_{i,j}(J_G) \leq \beta_{i,j}(\mathrm{in}(J_G))$ for all i, j. This fact implies in particular that $\mathrm{pd}(J_G) \leq \mathrm{pd}(\mathrm{in}(J_G))$, and $\mathrm{reg}(J_G) \leq \mathrm{reg}(\mathrm{in}(J_G))$. In [11, Theorem 1.1], J_G has a quadratic Gröbner basis with respect to the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$ if and only if the graph G is closed with respect to the given labeling, in other words, if G satisfies the following condition: whenever $\{i, j\}$ and $\{i, k\}$ are edges of G and either $i < j$, $i < k$ or $i > j$, $i > k$ then $\{j, k\}$ is also an edge of G. One calls a graph G *closed* if it is closed with respect to some labeling of its vertices. This concept is also called *PI graph* (see in [2, 9]). When G is a closed graph, Ene, Herzog and Hibi conjectured in [7] that $\beta_{i,j}(J_G) = \beta_{i,j}(\mathrm{in}(J_G))$ for all i, j. This conjecture has been confirmed to be true for Cohen-Macaulay binomial edge ideals in [7], and for closed graphs which consist at most two cliques (see [2]). Recently, Hernán and I (see [6]) proved the conjecture in some cases when J_G is not Cohen-Macaulay. Herzog and Rinaldo also considered the conjecture for the extremal Betti numbers of binomial edge ideals of block graphs (see [12]).

The second result of the paper is an affirmation that the conjecture of Ene, Herzog and Hibi is true for the Betti numbers in the last Betti table of the binomial edge ideal of closed graphs. From that, we obtain that $\mathrm{reg}(J_G) = \mathrm{reg}(\mathrm{in}(J_G))$ and $\mathrm{pd}(J_G) = \mathrm{pd}(\mathrm{in}(J_G))$ for all closed graph G. The statement on the equality of $\mathrm{reg}(J_G)$ and $\mathrm{reg}(\mathrm{in}(J_G))$ is also proved by Ene and Zarojanu [8].

Theorem 3.1. *If G is a closed graph, then $\mathrm{reg}(J_G) = \mathrm{reg}(\mathrm{in}(J_G)) =: r$, $\mathrm{pd}(J_G) = \mathrm{pd}(\mathrm{in}(J_G)) =: p$, and $\beta_p(J_G) = \beta_{p,p+r}(J_G) = \beta_p(\mathrm{in}(J_G)) = \beta_{p,p+r}(\mathrm{in}(J_G)) \neq 0$.*

The paper is organized as follows. In Section 1, we recall some basic notations and terminology about simplicial complexes. In Section 2, we study non-vanishingness of Betti numbers of binomial edge ideal of skew Ferrers graphs in the last column. Section 3 we obtain that the conjecture of Ene, Herzog and Hibi is true for the Betti numbers in the last column of Betti table. An explicit formula for the unique extremal Betti number of binomial edge ideal will be given in the last section.

1. Preliminaries

A simplicial complex Δ on the vertex set $V(\Delta) := \{1, \dots, n\}$ is a collection of subsets of $V(\Delta)$ such that $F \in \Delta$ whenever $F \subseteq F'$ for some $F' \in \Delta$. Given any field k, we attach to Δ the *Stanley-Reisner* ideal I_Δ of Δ to be the squarefree monomial ideal

$$I_\Delta = (x_{j_1} \cdots x_{j_i} \mid j_1 < \cdots < j_i \text{ and } \{j_1, \dots, j_i\} \notin \Delta) \text{ in } R = k[x_1, \dots, x_n],$$

and the *Stanley-Reisner* ring of Δ to be the quotient ring $k[\Delta] = R/I_\Delta$. This provides a bridge between combinatorics and commutative algebra (see [17]). Then, we say that Δ

is Cohen-Macaulay over k if $k[\Delta]$ has the same property. We denote $\widetilde{H}_j(\Delta; k)$ is reduced homology group of a simplicial complex Δ over k. The restriction of Δ to a subset S of $V(\Delta)$ is $\Delta[S] := \{F \in \Delta \mid F \subseteq S\}$. A very useful result to compute the graded Betti numbers of the Stanley- Reisner ideal of simplicial complex is the so-called Hochster formula (c.f. [10, Theorem 8.1.1]) as follows:

$$\beta_{i,j}(I_\Delta) := \sum_{W \subseteq V(\Delta), |W|=j} \beta_{i,W}(I_\Delta),$$

where $\beta_{i,W}(I_\Delta) := \dim_k \widetilde{H}_{|W|-i-2}(\Delta[W]; k)$. By Hochster formula, $\beta_{i,j}(I_\Delta) \geq \beta_{i,j}(I_{\Delta[S]})$ for all i, j and $S \subseteq V(\Delta)$. Thus we immediately obtain the following lemma:

Lemma 1.1. *Let $S \subseteq V(\Delta)$. Then $\operatorname{pd}(I_\Delta) \geq \operatorname{pd}(I_{\Delta[S]})$.*

Let Γ and Λ be two simplicial complexes on the disjoint vertex sets $V(\Gamma)$ and $V(\Lambda)$, respectively. Define the *join* on the vertex $V(\Gamma) \cup V(\Lambda)$ to be $\Gamma * \Lambda = \{\sigma \cup \tau \mid \sigma \in \Gamma, \tau \in \Lambda\}$. Using Künneth formula (c.f. [1, Proposition 3.2]), we can describe the reduced homology of a join of two simplicial complexes in terms of the reduced homologies of the factors as follows:

$$\widetilde{H}_i(\Gamma * \Lambda; k) \cong \bigoplus_{p+q=i-1} \widetilde{H}_p(\Gamma; k) \otimes \widetilde{H}_q(\Lambda; k), \text{ for each } i.$$

From this formula, we obtain the following lemma.

Lemma 1.2. *Let $\Delta = \Delta_1 * \ldots * \Delta_m$, where Δ_i are disjoint subcomplexes of simplicial complex Δ. Let $p_i := \operatorname{pd}(I_{\Delta_i})$, and $r_i := \operatorname{reg}(I_{\Delta_i})$ for $1 \leq i \leq m$. Then*

$$\beta_{i-1,j}(I_\Delta) = \sum_{\substack{a_1+\ldots+a_m=i,\\ b_1+\ldots+b_m=j}} \prod_{k=1}^{m} \beta_{a_k-1,b_k}(I_{\Delta_k}).$$

In particular, $\operatorname{pd}(I_\Delta) = \sum_{i=1}^m p_k + (m-1) := p$, $\operatorname{reg}(I_\Delta) = \sum_{k=1}^m r_k - (m-1) := r$, and $\beta_{p,p+r}(I_\Delta) = \prod_{i=1}^m \beta_{p_i,p_i+r_i}(I_{\Delta_i})$.

Proof. We prove the lemma by induction on m. If $m = 1$, there is nothing to prove. Now, we assume that $m \geq 2$. Let $\Gamma = \Delta_1 * \ldots * \Delta_{m-1}$ and $\Lambda = \Delta_m$. By the induction hypothesis, we have

$$\beta_{s-1,u}(I_\Gamma) = \sum_{\substack{a_1+\ldots+a_{m-1}=s,\\ b_1+\ldots+b_{m-1}=u}} \prod_{k=1}^{m-1} \beta_{a_k-1,b_k}(I_{\Delta_k}). \tag{1}$$

By Hochster formula,

$$\beta_{i-1,j}(I_\Delta) = \sum_{W \subseteq V(\Delta), |W|=j} \dim_k \widetilde{H}_{j-i-1}(\Delta[W]; k).$$

For each $W \subseteq V(\Delta)$, we have $\Delta[W] = \Gamma[W_1] * \Lambda[W_2]$, where $W_1 := W \cap V(\Gamma)$ and $W_2 := W \cap V(\Lambda)$. Using Künneth formula, we obtain that

$$\begin{aligned}\beta_{i-1,j}(I_\Delta) &= \sum_{\substack{W\subseteq V(\Delta),\\ |W|=j}} \sum_{p+q=j-i-2} \dim_k \widetilde{H}_p(\Gamma[W_1];k) \dim_k \widetilde{H}_q(\Lambda[W_2];k) \\ &= \sum_{\substack{W_1\subseteq V(\Gamma), W_2\subseteq V(\Lambda),\\ |W_1|+|W_2|=j}} \sum_{p+q=j-i-2} \dim_k \widetilde{H}_p(\Gamma[W_1];k) \dim_k \widetilde{H}_q(\Lambda[W_2];k).\end{aligned}$$

Set $u := |W_1|$, $b_m := |W_2|$, $s := u - p - 1$ and $a_m := b_m - q - 1$. Thus $s + a_m = i$ and we get

$$\beta_{i-1,j}(I_\Delta) = \sum_{\substack{s+a_m=i,\\ u+b_m=j}} \beta_{s-1,u}(I_\Gamma)\beta_{a_m-1,b_m}(I_\Lambda).$$

Using (1) for the above formula, we imply that

$$\beta_{i-1,j}(I_\Delta) = \sum_{\substack{a_1+\ldots+a_m=i,\\ b_1+\ldots+b_m=j}} \prod_{k=1}^{m} \beta_{a_k-1,b_k}(I_{\Delta_k}).$$

From the above formula, we imply the last statements of lemma. □

2. Betti numbers of edge ideal of skew Ferrers graphs

In this section, we will study non-vanishingness of the Betti numbers of edge ideal of skew Ferrers graphs in the last columm of the Betti table. In order to obtain these results, we recall a rectangular decomposition for skew Ferrers diagram (c.f. [15, Section 2.4]). First, we define a *Ferrers diagram* $D_{X,Y}$ with $\lambda = (\lambda_1 = m \geq \ldots \geq \lambda_n)$ on $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ is an array of cells doubly indexed by pairs (x_i, y_j) with $1 \leq i \leq n$, $m+1-\lambda_i \leq j \leq m$. The difference between two Ferrers diagrams is called a *skew Ferrers diagram*. On the other hand, the skew Ferrers diagram $D_{X,Y}$ on (X, Y), where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, is defined by two non-increasing sequences of integers, $\lambda = (\lambda_1 = m \geq \cdots \geq \lambda_n)$ and $\mu = (\mu_1 \geq \ldots \geq \mu_n)$ and $\lambda_i \geq \mu_i$ for all i, and having $\lambda_i - \mu_i$ cells in row i, namely $\{(x_i, y_j) \mid 1 \leq i \leq n, m+1-\lambda_i \leq j \leq m-\mu_i\}$ (see [14]). A skew Ferrers diagram such that $\mu_i = 0$ for all $i = 1, \ldots, n$ is a Ferrers diagram.

A bipartite graph G on two distinct vertex sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ corresponding to a Ferrers diagram (resp. skew Ferrers diagram) is called a *Ferrers graph* (resp. *skew Ferrers graph*) if $\{x_i, y_j\}$ is an edge of G whenever (x_i, y_j) is a cell in Ferrers diagram (resp. skew Ferrers diagram). In [4], Corso and Nagel obtained the irredundant primary decomposition and gave an explicit formula for the Betti numbers of edge ideal of Ferrers graphs.

Lemma 2.1. [4, Theorem 2.1] *Let G be a Ferrers graph. Then the minimal $\mathbb{Z}$-graded free resolution of $I(G)$ is 2-linear with i-th Betti number given by*

$$\beta_i(I(G)) = \binom{\lambda_1}{i+1} + \binom{\lambda_2+1}{i+1} + \ldots + \binom{\lambda_n+n-1}{i+1} - \binom{n}{i+2},$$

where $1 \leq i \leq \mathrm{pd}(I(G)) = \max_{1\leq j\leq n}\{\lambda_j + j - 2\}$.

For skew Ferrers graphs, it is not easy to give an explicit formula for Betti numbers of edge ideals of skew Ferrers graphs (see [6]). However, Nagel and Reiner [15, Definition 2.9] gave a rectangular decomposition to analyze the homotopy type of the associated the independence complexes of skew Ferrers graphs. A rectangular decomposition of skew Ferrers diagram $D_{X,Y}$ with $X = \{x_1, \cdots, x_n\}$ and $Y = \{y_1, \cdots, y_m\}$ is a partition

$$D_{X,Y} = D_{(x_{i_1}, y_{j_1})} \sqcup \ldots \sqcup D_{(x_{j_r}, y_{j_r})},$$

where all $D_{(x_{i_k}, y_{j_k})}$ are defined inductively as follows:

Step 1: Choose a top cell $(x_{i_1}, y_{j_1}) := (x_1, y_1)$. We denote $D_{(x_{i_1}, y_{j_1})}$ contains all cells (x_k, y_l) in $D_{X,Y}$ such that either (x_k, y_{j_1}) or (x_{i_1}, y_l) is cell in $D_{X,Y}$.

Step 2: We set

$$\begin{aligned} X' &:= \{k \mid k \geq i_1 \text{ and } (x_k, y_{j_1}) \text{ is a cell of } D_{(x_{i_1}, y_{j_1})}\}, \\ Y' &:= \{l \mid l \geq j_1 \text{ and } (x_{i_1}, y_l) \text{ is a cell of } D_{(x_{i_1}, y_{j_1})}\}. \end{aligned}$$

We set $a := |X'|$ and $b := |Y'|$. Then $X' = \{i_1, \ldots, i_1+a-1\}$ and $Y' = \{j_1, \ldots, j_1+b-1\}$. Thus, $X\backslash X' = \{i_1 + a, \ldots, n\}$ and $Y\backslash Y' = \{j_1 + b, \ldots, m\}$. We denote X'' (resp. Y'') to be a set of all $k \geq i_1 + a$ (resp. $l \geq j_1 + b$) such that k-row (resp. l-column) of the diagram $D_{X\backslash X', Y\backslash Y'}$ doesn't contain any cell. We call X'' (resp. Y'') is *empty rectangle* if $X'' \neq \emptyset$ (resp. $Y'' \neq \emptyset$). If $X\backslash(X' \cup X'') \neq \emptyset$ or $Y\backslash(Y' \cup Y'') \neq \emptyset$, then we repeat step 1 for skew Ferrers diagram $D_{X\backslash(X'\cup X''), Y\backslash(Y'\cup Y'')}$.

Finally, we get the rectangular decomposition of $D_{X,Y}$ as above. We denote $\mathrm{rect}(D_{X,Y}) := r$ is called *rectangularity number* of $D_{X,Y}$. A skew Ferrers diagram $D_{X,Y}$ is called *spherical* if in its rectangular decomposition it has no empty rectangles.

Example 2.2. The rectangular decomposition of a skew Ferrers diagram $D_{X,Y}$ with $\mu = (4, 4, 2, 2, 2, 1, 0)$ and $\lambda = (7, 6, 6, 5, 4, 3, 2)$ is

$$D_{X,Y} = D_{(x_1,y_1)} \sqcup D_{(x_3,y_4)} \sqcup D_{(x_6,y_6)},$$

	y_7	y_6	y_5	y_4	y_3	y_2	y_1
x_1					×	×	×
x_2					×	×	
x_3			×	×	×		
x_4			×	×			
x_5			×	×			
x_6		×	×				
x_7	×	×					

where $D_{(x_1,y_1)} = \{(x_1,y_1),(x_1,y_2),(x_1,y_3),(x_2,y_2),(x_2,y_3),(x_3,y_3)\}$, $D_{(x_6,y_6)} = \{(x_6,y_6),(x_7,y_6),(x_7,y_7)\}$, and $D_{(x_3,y_4)} = \{(x_3,y_4),(x_3,y_5),(x_4,y_4),(x_4,y_5),(x_5,y_5),(x_5,y_6),(x_6,y_5)\}$. Then $\mathrm{rect}(D_{X,Y}) = 3$ and $\{x_2\}$ and $\{y_7\}$ are empty rectangles, but $D_{X\setminus\{x_2\},Y\setminus\{y_7\}}$ is a spherical.

Lemma 2.3. [15, Corollary 2.15, Theorem 2.23 and Proposition 2.25] *Let G be a skew Ferrers graph with vertex set $X \sqcup Y$. Then*

(1) $\beta_{i,X'\sqcup Y'}(I(G)) = \begin{cases} 1, & \textit{if } D_{X',Y'} \textit{ is spherical with } \mathrm{rect}(D_{X',Y'}) = |X'\cup Y'| - i - 1, \\ 0 & \textit{otherwise,} \end{cases}$
where $X' \subseteq X$, $Y' \subseteq Y$.

(2) $\mathrm{pd}(I(G)) = \max\{|X'\cup Y'| - \mathrm{rect}(D_{X',Y'}) - 1\}$*, where the maximum runs over all subsets $X' \subseteq X$, $Y' \subseteq Y$ for which $D_{X',Y'}$ is spherical.*
(3) $\mathrm{reg}(I(G)) = \mathrm{rect}(D_{X,Y}) + 1$.

Let G be a simple graph. For a subset S of $V(G)$ we denote by $G[S]$ the induced subgraph of G on the vertex set S; and denote $G\setminus S$ by $G[V\setminus S]$. A matching in a graph is a set of edges, no two of which meet a common vertex. An *induced matching* M in a graph G is a matching where no two edges of M are adjacented by an edge of G. The maximum size of an induced matching in G is denoted $\nu(G)$. By [6, Theorem 4.5], $\mathrm{rect}(D_{X,Y}) = \nu(G)$ for any skew Ferrers graph G.

Lemma 2.4. *Let G be a skew Ferrers graph with vertex set $X \sqcup Y$. If there exists subsets $X_1 \subseteq X$, $Y_1 \subseteq Y$ such that D_{X_1,Y_1} is spherical and* $\mathrm{pd}(I(G)) = |X_1\cup Y_1| - \mathrm{rect}(D_{X_1,Y_1}) - 1$*, then* $\mathrm{rect}(D_{X_1,Y_1}) = \mathrm{rect}(D_{X,Y})$.

Proof. Assume on the contrary that $\mathrm{rect}(D_{X,Y}) \neq \mathrm{rect}(D_{X_1,Y_1})$. By [15, Lemma 2.24], we may assume that $\mathrm{rect}(D_{X,Y}) > \mathrm{rect}(D_{X_1,Y_1})$. Then $X_1 \neq X$ or $Y_1 \neq Y$. By the rectangular decomposition of D_{X_1,Y_1}, we set $M := \{\{x_{i_1},y_{j_1}\},\ldots,\{x_{i_{r_1}},y_{j_{r_1}}\}\}$ is a maximal induced matching of $G[X_1\cup Y_1]$, where $i_1 < \ldots < i_{r_1}$, $j_1 < \ldots < j_{r_1}$ and $r_1 := \mathrm{rect}(D_{X_1,Y_1})$. If (x_u,y_v) is a cell in $D_{X,Y} - D_{X_1,Y_1}$ such that $M\cup\{x_u,y_v\}$ is a maximal induced matching of G. Then $D_{X_1\cup\{x_u\},Y_1\cup\{y_v\}}$ is also spherical and $\mathrm{rect}(D_{X_1\cup\{x_u\},Y_1\cup\{y_v\}}) = r_1+1$. However,

$$|(X_1\cup\{x_u\})\cup(Y_1\cup\{y_v\})| - \mathrm{rect}(D_{X_1\cup\{x_u\},Y_1\cup\{y_v\}}) - 1 = |X_1\cup Y_1| - r_1,$$

a contradiction thanks to Lemma 2.3 (2). Therefore, each cell (x_u,y_v) in $D_{X,Y} - D_{X_1,Y_1}$, $M\cup\{x_u,y_v\}$ is not an induced matching of G. Without loss of generality, we assume that $x_u \notin X_1\cup Y_1$ and $x_u y_{j_t} \in E(G)$ for $1 \le t \le r_1$. Thus, $\mathrm{rect}(D_{X_1\cup\{x_u\},Y_1}) = r_1$ and $i_{t-1} < u < i_{t+1}$, $u \neq i_t$. By the rectangular decomposition of $D_{X_1\cup\{x_u\},Y_1}$, we reduce two following cases:

	$y_{j_{t+1}}$	y_{j_t}	$y_{j_{t-1}}$
$x_{i_{t-1}}$			$\times$
x_{i_t}		$\times$	
x_u		$\times$	
$x_{i_{t+1}}$	$\times$		

or

	$y_{j_{t+1}}$	y_{j_t}	$y_{j_{t-1}}$
$x_{i_{t-1}}$			$\times$
x_u		$\times$	
x_{i_t}		$\times$	
$x_{i_{t+1}}$	$\times$		

Thus, $D_{X_1\cup\{x_u\},Y_1}$ is also spherical, and so

$$|(X_1\cup\{x_u\})\cup Y_1| - \mathrm{rect}(D_{X_1\cup\{x_u\},Y_1}) - 1 = |X_1\cup Y_1| - r_1,$$

a contradiction thanks to Lemma 2.3 (2). Therefore, we conclude that $\mathrm{rect}(D_{X,Y}) = \mathrm{rect}(D_{X_1,Y_1})$, as required. □

Theorem 2.5. *Let G be a skew Ferrers graph. Then $\beta_p(I(G)) = \beta_{p,p+r}(I(G))$, where $p := \mathrm{pd}(I(G))$ and $r := \mathrm{reg}(I(G))$.*

Proof. By Lemma 2.3 (3), we have $r = \mathrm{rect}(D_{X,Y})+1$. For each $1 \le i \le r-1$, we assume $\beta_{p,p+r-i}(I(G)) \neq 0$. Since

$$\beta_{p,p+r-i}(I(G)) = \sum_{V'\subseteq V(G) \text{ and } |V'|=p+r-i} \beta_{p,V'}(I(G)),$$

so there exists $X' \subseteq X$ and $Y' \subseteq Y$ such that $|X'\cup Y'| = p+r-i$ and $\beta_{p,X'\sqcup Y'}(I(G)) \neq 0$. By Lemma 2.3 (1), we have $\beta_{p,X'\sqcup Y'}(I(G)) = 1$, $D_{X',Y'}$ is spherical, and

$$p = |X'\cup Y'| - \mathrm{rect}(D_{X',Y'}) - 1.$$

By Lemma 2.4, $\mathrm{rect}(D_{X',Y'}) = \mathrm{rect}(D_{X,Y})$, and thus, $p+r = |X'\cup Y'|$, a contradiction. We conclude that $\beta_{p,p+r-i}(I(G)) = 0$ for all $1 \le i \le r-1$. Therefore, $\beta_p(I(G)) = \beta_{p,p+r}(I(G))$, as required. □

Corollary 2.6. *Let G be a skew Ferrers graph with $p := \mathrm{pd}(I(G))$ and $r := \mathrm{reg}(I(G))$. Then $p+2 \ge r$, and moreover if the equality happens, then $\beta_p(I(G)) = \beta_{p,p+r}(I(G))$ is a number of induced subgraphs of G consisting of $\nu(G)$ disjoint edges.*

Proof. By Lemma 2.3 (3), $r = \nu(G)+1$. Let W be a set of vertices of a maximum induced matching of G. Then, $I(G[W])$ is a complete intersection ideal. Thus, by Lemma 1.1, we have $\mathrm{pd}(I(G)) \ge \mathrm{pd}(I(G[W])) = \nu(G) - 1$. Hence $p+2 \ge \nu(G)+1 = r$.

On the other hand, if $p+2 = r$, then $p+1 = \nu(G)$. By Theorem 2.5 and [13, Lemma 2.2], $\beta_p(I(G)) = \beta_{p,p+r}(I(G)) = \beta_{p,2(p+1)}(I(G))$ is number of induced subgraphs of G consisting of $\nu(G)$ disjoint edges. □

3. Application to binomial edge ideals of closed graphs

In [7, p. 67], Ene, Herzog and Hibi gave a conjecture that the graded Betti numbers of J_G and $\mathrm{in}(J_G)$ coincide for all closed graphs. This section is aimed at proving this conjecture for the Betti numbers in the last column of their Betti table. If G is a closed graph without cut vertices, the initial ideal of binomial edge ideal $\mathrm{in}(J_G)$ of G is generated by squarefree monomials of degree two (see [11, Theorem 1.1]). We set a nontrivial connected graph H, which is called *initial-closed* graph, corresponding to $\mathrm{in}(J_G)$. In [6], Hernán and I studied the structure of this initial-closed graphs, and we realize that one is a special skew Ferrers graph.

Let G be a closed graph without closed graph on vertex set $V(G) = \{1,\dots,n\}$. Define $N_G^{>}(i) := \{j \in V(G) \mid i < j \text{ and } \{i,j\} \in E(G)\}$ and $\deg_G^{>}(i) := |N_G^{>}(i)|$. We associate with G a vector $\mu(G) = (\mu_1,\dots,\mu_n) \in \mathbb{N}^n$, where $\mu_j = n - j - \deg_G^{>}(j)$ for all $1 \le j \le$

n. Then the initial-closed graph H is a bipartite graph with bipartition (X, Y), where $X = \{x_1, \dots, x_{n-1}\}$ and $Y = \{y_1, \dots, y_{n-1}\}$, and $\mu(H) = (\mu_1, \dots, \mu_{n-1}) \in \mathbb{N}^{n-1}$. By [6, p. 33], H is a skew Ferrers graph with $\lambda_i = n - i$ and $\mu_i \leq n - 2 - i$ for $1 \leq i \leq n-3$, $\mu_{n-2} = \mu_{n-1} = 0$. The following theorem confirms that conjecture of Ene, Herzog and Hibi holds for Betti numbers in the last column of Betti table.

Theorem 3.1. *If G is a closed graph, then* $\operatorname{reg}(J_G) = \operatorname{reg}(\operatorname{in}(J_G)) =: r$, $\operatorname{pd}(J_G) = \operatorname{pd}(\operatorname{in}(J_G)) =: p$, *and* $\beta_p(J_G) = \beta_{p,p+r}(J_G) = \beta_p(\operatorname{in}(J_G)) = \beta_{p,p+r}(\operatorname{in}(J_G)) \neq 0$.

Proof. If G is disconnected, we may assume $G_1, \dots, G_s$ are connected components of G. It is well-known that the (i,j)-th Betti numbers of J_G and $\operatorname{in}(J_G)$ coincide if the (i,j)-th Betti numbers of J_{G_k} and $\operatorname{in}(J_{G_k})$ coincide for all $k = 1, \dots, s$. Therefore, we now consider G is a connected graph. Then there exists m ($m \geq 0$) cut vertices of G, say $v_1, \dots, v_m$. We may write G in the form

$$G = G_1 \cup \ldots \cup G_{m+1},$$

where G_i is a subgraph without cut vertices of G, and for $1 \leq i < j \leq m+1$ either $G_i \cap G_j = \emptyset$, or $G_i \cap G_j = \{v_k\}$ for some k. By the assumption, G is a closed graph, so G_i is. Let $r_i := \operatorname{reg}(\operatorname{in}(J_{G_i}))$ and $p_i := \operatorname{pd}(\operatorname{in}(J_{G_i}))$, and $p := \operatorname{pd}(\operatorname{in}(J_G))$ and $r = \operatorname{reg}(\operatorname{in}(J_G))$.

Let H (resp. H_i) be a nontrivial graph such that $I(H) = \operatorname{in}(J_G)$ (resp. $I(H_i) = \operatorname{in}(J_{G_i})$). Thus, $\beta_p(\operatorname{in}(J_G)) = \beta_p(I(H))$ and $\beta_{p,p+r}(\operatorname{in}(J_G)) = \beta_{p,p+r}(I(H))$. First, we claim that $\beta_p(\operatorname{in}(J_G)) = \beta_{p,p+r}(\operatorname{in}(J_G)) \neq 0$. In order to prove this, we need to prove $\beta_p(I(H)) = \beta_{p,p+r}(I(H)) \neq 0$. Indeed, for each $1 \leq i \leq m+1$, by [6, Lemma 2.4], H_i is an initial-closed graph correponding to closed graph without cut vertices G_i. By Theorem 2.5, we have $\beta_{p_i}(I(H_i)) = \beta_{p_i,p_i+r_i}(I(H_i)) \neq 0$. Moreover, by [6, Lemma 2.7], $H_1, \dots, H_{m+1}$ are connected components of H, and thus $H = H_1 \sqcup \ldots \sqcup H_{m+1}$. So $p = p_1 + \ldots + p_{m+1} + m$, $r = r_1 + \ldots + r_{m+1} - m$ and

$$\Delta(H) = \Delta(H_1) * \ldots * \Delta(H_{m+1}).$$

By Lemma 1.2, we have $\beta_{p,p+r}(I(H)) = \prod_{i=1}^{m+1} \beta_{p_i,p_i+r_i}(I(H_i)) \neq 0$, and moreover for $1 \leq j \leq r-1$,

$$\beta_{p,p+r-j}(I(H)) = \sum_{u_1+\ldots+u_{m+1}=p+r-j} \prod_{k=1}^{m+1} \beta_{p_k,u_k}(I(H_k)).$$

Since $u_1 + \ldots + u_{m+1} = p + r - j$, there exists $1 \leq \ell \leq m+1$ such that $u_\ell < p_\ell + r_\ell$. This implies that $\beta_{p_\ell,u_\ell}(I(H_\ell)) = 0$, which means that $\beta_{p,p+r-j}(I(H)) = 0$. Hence, $\beta_p(I(H)) = \beta_{p,p+r}(I(H)) \neq 0$, as claimed.

On the other hand, since J_G and $\operatorname{in}(J_G)$ have the same Hilbert polynomial and together with $\beta_{p,p+r}(\operatorname{in}(J_G)) \neq 0$, we obtain $\beta_{p,p+r}(\operatorname{in}(J_G)) = \beta_{p,p+r}(J_G)$, $r = \operatorname{reg}(J_G)$ and $p = \operatorname{pd}(J_G)$. Moreover, for each $0 \leq j \leq r-1$, we have $\beta_{p,p+r-j}(J_G) \leq \beta_{p,p+r-j}(\operatorname{in}(J_G)) = 0$ which means that $\beta_{p,p+r-j}(J_G) = 0$. Therefore, $\beta_p(J_G) = \beta_{p,p+r}(J_G)$, as required. ∎

Corollary 3.2. *If G is a closed graph, then*

$$\beta_{p-1,p+r-1}(J_G) = \beta_{p-1,p+r-1}(\operatorname{in}(J_G)),$$

where $r := \operatorname{reg}(J_G)$ *and* $p := \operatorname{pd}(J_G)$.

Proof. Since J_G and $\operatorname{in}(J_G)$ have the same Hilbert polynomial,

$$\beta_{p-1,p+r-1}(J_G) - \beta_{p,p+r-1}(J_G) = \beta_{p-1,p+r-1}(\operatorname{in}(J_G)) - \beta_{p,p+r-1}(\operatorname{in}(J_G)).$$

By Theorem 3.1, we have $\beta_{p,p+r-1}(\operatorname{in}(J_G)) = \beta_{p,p+r-1}(J_G) = 0$. Therefore, $\beta_{p-1,p+r-1}(J_G) = \beta_{p-1,p+r-1}(\operatorname{in}(J_G))$, as required. ☐

4. The unique extremal Betti numbers of binomial edge ideal of certain closed graphs

To compute the unique extremal Betti numbers of binomial edge ideal of closed graphs, by the argument of Theorem 3.1, we can reduce the case closed graphs without cut vertices. Now we let G be a closed graph without cut vertices, and H be an initial-closed graph with $\mu(H) = (\mu_1, \dots, \mu_{n-1}) \in \mathbb{N}^{n-1}$ corresponding to G. By [6, Theorem 4.5], the regularity of J_G equals three if and only if $\mu_1 = \dots = \mu_s =: \mu \geq 1$ and $s \geq 1$, where $s := \min\{k - 1 \mid \mu_k = 0\}$. In this section, we will give an explicit formula for the unique extremal Betti number of binomial edge ideal of closed graph without cut vertices G whenever the regularity of J_G equals three. From there, we have an explicit formula for larger regularity case.

Firstly, we need some technical lemmas:

Lemma 4.1. *Let* $I = J + (x_1, \dots, x_s) \subseteq S := k[x_1, \dots, x_n]$ *with* $1 \leq s \leq n$, *where* $J \subseteq S/(x_1, \dots, x_s) =: S'$. *Then* $p := \operatorname{pd}(I) = \operatorname{pd}(J) + s$, *and* $\beta^S_{p,j}(I) = \beta^{S'}_{p-s,j-s}(J)$.

Proof. We will prove the lemma by induction on s. If $s = 1$, by [3, Remark 2.1], we have

$$\beta^S_{p,j}(S/I) = \beta^{S_1}_{p,j}(I) + \beta^{S_1}_{p-1,j-1}(I).$$

Since $\operatorname{pd}(J) = \operatorname{pd}(I) - 1 = p - 1$, so $\beta^{S_1}_{p,j}(I) = \beta^{S_1}_{p,j}(J) = 0$. Thus, we complete the proof in this case.

If $s \geq 2$, let $S_{s-1} := S/(x_s)$ and $J_{s-1} = J + (x_1, \dots, x_{s-1})$. Then $I = J_{s-1} + (x_s)$. By [3, Remark 2.1], we have

$$\beta^S_{p,j}(I) = \beta^{S_{s-1}}_{p,j}(J_{s-1}) + \beta^{S_{s-1}}_{p-1,j-1}(J_{s-1}).$$

However, since $\operatorname{pd}(I_{s-1}) = p - 1$, so $\beta^{S_{s-1}}_{p,j}(J_{s-1}) = 0$. This means that $\beta^S_{p,j}(I) = \beta^{S_{s-1}}_{p-1,j-1}(J_{s-1})$. By the induction hypothesis, $\beta^S_{p,j}(I) = \beta^{S'}_{p-s,j-s}(J)$, as required. ☐

For any simple graph G, the neighborhood of a vertex x of G is the set $N_G(x) := \{y \in V(G) \mid \{x, y\} \in E(G)\}$. If $S = \{x\}$, we write $G \backslash x$ (resp. G_x) instead of $G \backslash \{x\}$ (resp. $G \backslash (N_G(x) \cup \{x\})$).

Lemma 4.2. *Let* x *be a vertex of* G *with neighbors* $y_1, y_2, \dots, y_s$. *Let* $S_1 := k[V(G \backslash x)]$ *and* $S_2 := k[V(G_x)]$. *Let* $p := \operatorname{pd}(I(G))$. *For each* j,

(1) If $p = \operatorname{pd}((I(G), x))$, *then* $\beta^S_i((I(G), x)) = 0$ *for all* $i > p$, *and*

$$\beta^S_{p,j}((I(G), x)) = \beta^{S_1}_{p-1,j-1}(I(G \backslash x)).$$

(2) If $p = \mathrm{pd}((I(G):x))$*, then* $\beta^S_{p,j}((I(G):x)(-1)) = \beta^{S_2}_{p-s,j-1-s}(I(G_x))$.

Proof. In order to prove the lemma, we first use the following result by [5, Lemma 3.1]:

$$\begin{aligned} I(G):x &= I(G_x) + (y_1,\dots,y_s),\\ (I(G),x) &= I(G\backslash x) + (x). \end{aligned}$$

(1) By the definition of the projective dimension, we get $\beta^S_i((I(G),x)) = 0$ for all $i > p$. By Lemma 4.1, we have $\beta^S_{p,j}((I(G),x)) = \beta^{S_1}_{p-1,j-1}(I(G\backslash x))$. This completes the proof of the assertion.

(2) We have $\beta^S_{p,j}((I(G):x)(-1)) = \beta^S_{p,j-1}((I(G):x))$. By Lemma 4.1, $\beta^S_{p,j-1}((I(G):x)) = \beta^{S'}_{p-s,j-1-s}(I(G_x))$, where $S' := S/(y_1,\dots,y_s)$. Since $I(G_x)$ is an ideal in polynomial ring S_2, we imply that $\beta^{S'}_{p-s,j-1-s}(I(G_x)) = \beta^{S_2}_{p-s,j-1-s}(I(G_x))$. Hence, the assertion is proved. □

Lemma 4.3. *Let* H *be an initial-closed graph with* $\mu = (\mu_1,\dots,\mu_{n-1}) \in \mathbb{N}^{n-1}$*, where* $\mu_1 = \ldots = \mu_s = 1$*,* $\mu_{s+1} = \ldots = \mu_{n-1} = 0$ *and* $s \geq 1$*. Then*

$$\beta_{2n-s-4,2n-s-1}(I(H)) = s.$$

Proof. Let $p := 2n-4-s$. By [6, Lemma 3.8], we have $\mathrm{pd}(I(H)) = p$. Note that $n-s \geq 3$. Since $H_{y_{n-1}}$ is a Ferrers graph with $\lambda(H_{y_{n-1}}) = (n-2,\dots,n-1-s) \in \mathbb{N}^s$. By Lemma 2.1, $\mathrm{pd}(I(H_{y_{n-1}})) = n-3$, $\mathrm{reg}(I(H_{y_{n-1}})) = 2$ and $\beta^{S'}_{n-3,n-1}(I(H_{y_{n-1}})) = \beta^{S'}_{n-3}(I(H_{y_{n-1}})) = s$, where $S' := k[V(H_{y_{n-1}})]$. Since $N_H(y_{n-1}) = \{x_{s+1},\dots,x_{n-1}\}$,

$$\mathrm{pd}((I(H):y_{n-1}) = \mathrm{pd}(I(H_{y_{n-1}})) + (n-s-1) = (n-3) + (n-s-1) = p.$$

On the other hand, $H\backslash\{x_{n-1},y_{n-1}\}$ is also a Ferrers graph with $\lambda(H\backslash\{x_{n-1},y_{n-1}\}) = (n-2,\dots,2,1) \in \mathbb{N}^{n-2}$. By Lemma 2.1, $\mathrm{pd}(I(H\backslash\{x_{n-1},y_{n-1}\})) = n-3$. Thus,

$$\mathrm{pd}((I(H),y_{n-1})) = 1 + \mathrm{pd}(I(H\backslash x_{n-1})) = 1 + \mathrm{pd}(I(H\backslash\{x_{n-1},y_{n-1}\})) = n-2.$$

Since $p > n-2$, so $\mathrm{Tor}^S_i(S/(I(H),y_{n-1});k) = 0$ for $i \geq p$.

From a short exact following sequence:

$$0 \to S/(I(H):y_{n-1})(-1) \to S/I(H) \to S/(I(H),y_{n-1}) \to 0,$$

we get a long exact sequence of Tor-modules. This implies that the following sequence:

$$0 \to \mathrm{Tor}^S_{p+1}(S/(I(H):y_{n-1})(-1);k)_{p+2} \to \mathrm{Tor}^S_{p+1}(S/I(H);k)_{p+2} \to 0$$

is exact. Thus, $\beta^S_{p,p+3}(I(H)) = \beta^S_{p,p+3}((I(H):y_{n-1})(-1)) = \beta^S_{p,p+2}((I(H):y_{n-1}))$. Together with Lemma 4.2, we get $\beta^S_{p,p+3}(I(H)) = \beta^{S'}_{p-(n-s-1),p+2-(n-s-1)}(I(H_{y_{n-1}})) = \beta^{S'}_{n-3,n-1}(I(H_{y_{n-1}})) = s$. □

Lemma 4.4. *Let* H *be an initial-closed graph with* $\mu(H) = (\mu_1,\dots,\mu_s,0,\dots,0) \in \mathbb{N}^{n-1}$*, where* $\mu_1 = \ldots = \mu_s =: \mu > 0$ *and* $s \geq 1$*. Then*

$$\beta_{2n-\mu-s-3,2n-\mu-s}(I(H)) = s\mu.$$

Proof. By [6, Lemma 3.8], $\operatorname{pd}(I(H)) = 2n - \mu - s - 3 =: p$. We will prove by induction on μ. If $\mu = 1$, the lemma is proved by Lemma 4.3. Now we assume that $\mu \geq 2$. Since $H_{y_{n-1}}$ is a Ferrers graph with $\lambda(H_{y_{n-1}}) = (n-\mu-1, \ldots, n-\mu-s) \in \mathbb{N}^s$. By Lemma 2.1, $\operatorname{pd}(I(H_{y_{n-1}})) = n - \mu - 2$, $\operatorname{reg}(I(H_{y_{n-1}})) = 2$ and $\beta_{n-\mu-2,n-\mu}(I(H_{y_{n-1}})) = \beta_{n-\mu}(I(H_{y_{n-1}})) = s$. Then

$$\begin{aligned} \operatorname{pd}((I(H) : y_{n-1}) &= \operatorname{pd}(I(H_{y_{n-1}})) + (n - s - 1) = p, \\ \operatorname{reg}((I(H) : y_{n-1}) &= \operatorname{reg}(I(H_{y_{n-1}})) = 3. \end{aligned}$$

Thus, $\beta^S_{p-1,p+3}((I(H) : y_{n-1})(-1)) = \beta^S_{p-1,p+3}((I(H) : y_{n-1})) = 0$. Hence, we conclude that $\operatorname{Tor}^S_p(S/(I(H) : y_{n-1})(-1); k)_{p+3} = \mathbf{0}$.

On the other hand, $H\backslash\{x_{n-1}, y_{n-1}\}$ is an initial-closed graph with $\mu(H\backslash\{x_{n-1}, y_{n-1}\}) = (\mu - 1, \ldots, \mu - 1, 0, \ldots, 0) \in \mathbb{N}^{n-2}$. By [6, Lemma 3.8], $\operatorname{pd}(I(H\backslash\{x_{n-1}, y_{n-1}\})) = 2n - 3 - \mu - 1 - s = p - 1$. Then $\operatorname{pd}((I(H), y_{n-1})) = 1 + \operatorname{pd}(I(H\backslash\{x_{n-1}, y_{n-1}\})) = p$. Thus, $\operatorname{Tor}^S_{p+2}(S/(I(H), y_{n-1}); k) = \mathbf{0}$.

From a short exact sequence of S-modules:

$$0 \to S/(I(H) : y_{n-1})(-1) \to S/I(H) \to S/(I(H), y_{n-1}) \to 0,$$

we obtain a long exact sequence of Tor-modules and thus, the following sequence

$$\begin{aligned} 0 \to \operatorname{Tor}^S_{p+1}(S/(I(H) : y_{n-1})(-1); k)_{p+3} &\to \operatorname{Tor}^S_{p+1}(S/I(H); k)_{p+3} \\ &\to \operatorname{Tor}^S_{p+1}(S/(I(H), y_{n-1}); k)_{p+3} \to 0, \end{aligned}$$

is exact. Therefore, we get

$$\beta^S_{p,p+3}(I(H)) = \beta^S_{p,p+3}((I(H) : y_{n-1})(-1)) + \beta^S_{p,p+3}((I(H), y_{n-1}).$$

Let $S_1 := k[V(H\backslash\{x_{n-1}, y_{n-1}\})]$ and $S_2 := k[V(H_{y_{n-1}})]$. By Lemma 4.2, we obtain

$$\begin{aligned} \beta^S_{p,p+3}(I(H)) &= \beta^{S_2}_{n-\mu-2,n-\mu}(H_{y_{n-1}}) + \beta^{S_1}_{p-1,p+2}(I(H\backslash\{x_{n-1}, y_{n-1}\})) \\ &= s + \beta^{S_1}_{p-1,p+2}(I(H\backslash\{x_{n-1}, y_{n-1}\})). \end{aligned}$$

By the induction hypothesis, we have $\beta^{S_1}_{p-1,p+2}(I(H\backslash\{x_{n-1}, y_{n-1}\})) = (\mu - 1)s$. Hence, we conclude that $\beta_{p,p+3}(I(H)) = s + (\mu - 1)s = \mu s$, as required. □

Theorem 4.5. *Let G be a closed graph without cut vertices with $\mu(G) = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$. Then $\operatorname{reg}(J_G) = 3$ if and only if $\mu_1 = \ldots = \mu_s =: \mu \geq 1$ and $s \geq 1$, where $s := \min\{k - 1 \mid \mu_k = 0\}$. In particular,*

$$\beta_p(J_G) = \beta_{p,p+3}(J_G) = \beta_p(\operatorname{in}(J_G)) = \beta_{p,p+3}(\operatorname{in}(J_G)) = s\mu,$$

where $p = \operatorname{pd}(J_G) = \operatorname{pd}(\operatorname{in}(J_G)) = 2n - \mu - s - 3$.

Proof. The first statement is followed by [6, Corollary 4.7]. Now we prove the second statement. Let H be an initial-closed graph corresponding to closed graph without cut vertices G. Then $\beta_{p,p+3}(\operatorname{in}(J_G)) = \beta_{p,p+3}(I(H))$. By the assumption, we have $\mu(H) = (\mu_1, \ldots, \mu_{n-1}) \in \mathbb{N}^{n-1}$, where $\mu_1 = \ldots = \mu_s =: \mu \geq 1$ and $s \geq 1$. By Lemma 4.4, $\beta_{p,p+3}(I(H)) = s\mu$. From this, the theorem is proved by Theorem 4.5. □

Theorem 4.6. *Let G be a connected closed graph with m cut vertices, say $v_1, \ldots, v_m$. Then G is written in the form*

$$G = G_1 \cup \ldots \cup G_{m+1},$$

where $G_i \cap G_{i+1} = \{v_i\}$ and $G_i \cap G_j = \emptyset$ for $i = 1, \ldots, m$ and $i \neq j \neq i+1$. Let $\mu(G_i) = (\mu_{i1}, \ldots, \mu_{in_i})$ and $s_i := \min\{k-1 \mid \mu_{ik} = 0\}$, where $n_i = |V(G_i)|$. If $\mu_{i1} = \ldots = \mu_{is_i} =: \mu_i \geq 1$ and $s_i \geq 1$ for all i, then

$$\beta_p(J_G) = \beta_{p,p+r}(J_G) = \beta_p(\mathrm{in}(J_G)) = \beta_{p,p+r}(\mathrm{in}(J_G)) = \prod_{i=1}^{m+1} s_i \mu_i,$$

where $p := \mathrm{pd}(J_G) = 2n - 3 - \sum_{i=1}^{m+1}(\mu_i + s_i)$ and $r := \mathrm{reg}(J_G) = 2m + 3$.

Proof. By Theorem 3.1, we only need to prove $\beta_{p,p+r}(\mathrm{in}(J_G)) = \prod_{i=1}^{m+1} s_i \mu_i$, where $p = \mathrm{pd}(\mathrm{in}(J_G)) = 2n - 3 - \sum_{i=1}^{m+1}(\mu_i + s_i)$ and $r = \mathrm{reg}(\mathrm{in}(J_G)) = 2m + 3$. First, we call H is a nontrivial graph such that $I(H) = \mathrm{in}(J_G)$ and H_i is an initial-closed graph correspoding to the closed graph without cut vertices G_i for $1 \leq i \leq m+1$. Then $n = \sum_{i=1}^{m+1} n_i - m$. By the assumption, we have $p_i := \mathrm{pd}(I(H_i)) = \mathrm{pd}(\mathrm{in}(J_{G_i})) = 2n_i - \mu_i - s_i - 3$ and $r_i := \mathrm{reg}(I(H_i)) = \mathrm{reg}(\mathrm{in}(J_{G_i})) = 3$. Thus, $p = \mathrm{pd}(I(H)) = p_1 + \ldots + p_{m+1} + m = 2n - 3 - \sum_{i=1}^{m+1}(\mu_i + s_i)$, $r = \mathrm{reg}(I(H)) = r_1 + \ldots + r_{m+1} - m = 2m + 3$. By [6, Lemma 2.7], $H = H_1 \sqcup \ldots \sqcup H_{m+1}$. By Lemma 1.2,

$$\beta_{p,p+r}(I(H)) = \prod_{i=1}^{m+1} \beta_{p_i, p_i + r_i}(I(H_i)).$$

Together with Theorem 4.5, $\beta_{p,p+r}(\mathrm{in}(J_G)) = \beta_{p,p+r}(I(H)) = \prod_{i=1}^{m+1} s_i \mu_i$, as required. ☐

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References

[1] K. Baclawski and A. M. Grasia, *Combinatorial decompositions of a class of rings*, Adv. Math. **39** (1981) 155-184.

[2] H. Baskoroputro, *On the binomial edge ideals of proper interval graphs*, arXiv:1611.10117v1 (2016).

[3] W. Bruns, A. Conca and T. Römer, *Koszul homology and syzygies of Veronese subalgebras*, Math. Ann. **351** (2011), 761-779.

[4] A. Corso and U. Nagel, *Monomial and toric ideals associated to Ferrers graphs*, Trans. Am. Math. Soc. **361** (2009), 1371-1395.

[5] H. Dao, C. Huneke, and J. Schweig, *Bounds on the regularity and projective dimension of ideals associated to graphs*, J. Algebraic Combin. **38** (2013), 37-55.

[6] H. de Alba and D.T. Hoang, *On the extremal Betti numbers of binomial edge ideals of closed graphs*, Math. Nachr. **291** (2018), 28-40.

[7] V. Ene, J. Herzog and T. Hibi, *Cohen-Macaulay binomial edge ideals*, Nagoya Math. J. **204** (2011), 57-68.

[8] V. Ene and A. Zarojanu, *On the regularity of binomial edge ideals*, Math. Nachr. **288** (2015), 19-24.

[9] G. Hajós, *Uber eine Art von Graphen*, Internat. Math. Nachrichten, **11** (1957) page 65.

[10] J. Herzog and T. Hibi, Monomial ideals. Graduate Texts in Mathematics, 260. Springer-Verlag London, Ltd., London, 2011.

[11] J. Herzog, T. Hibi, F. Hreinsdottir, T. Kahle, and J. Rauh, *Binomial edge ideals and conditional independence statements*, Adv. Appl. Math. **45** (2010), no. 3, 317-333.

[12] J. Herzog and G. Rinaldo, *On the extremal Betti numbers of binomial edge ideals of block graphs*, arXiv: 1802.06020v1.

[13] M. Katzman, *Characteristic-independence of Betti numbers of graph ideals*, J. Combin. Theory, Ser. A **113** (2006), 435-454.

[14] I. G. Macdonald, Symmetric functions and Hall polynomials, 2nd edition, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[15] U. Nagel and V. Reiner, *Betti numbers of monomial ideals and shifted skew shapes*, Electron. J. Comb. **16** (2) (2009), Special volume in honor of Anders Björner, Research paper 3, pp. 59.

[16] M. Ohtani, *Graphs and Ideals generated by some 2-minors*, Comm. Algebra **39** (2011), no. 3, 905-917.

[17] R. Stanley, Combinatorics and Commutative Algebra, 2. Edition, Birkhäuser, 1996.

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