COVERINGS, MATCHINGS AND THE NUMBER OF MAXIMAL INDEPENDENT SETS OF GRAPHS

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Abstract. We determine the maximum number of maximal independent sets of an arbitrary graph in terms of its covering number and we completely characterize the extremal graphs. As an application, we give a similar result for König–Egerváry graphs in terms of their matching numbers.

1. Introduction

Throughout this paper let G be a simple (i.e. finite, undirected, loopless and without multiple edges) graph. An independent set in G is a set of vertices no two of which are adjacent to each other. An independent set in G is maximal (with respect to set inclusion) if the set cannot be extended to a larger independent set. Let $\mathrm{m}(G)$ be the number of maximal independent sets of a simple graph G. Around 1960, Erdös and Moser raised the problem of determining the largest value of $\mathrm{m}(G)$ in terms of the order of G, which we shall denote by n in this paper, and determining the extremal graphs. In 1965, Moon and Moser [14] solved this problem.

Since then, research now has been focused on investigating $\mathrm{m}(G)$ for various classes of graphs such as: connected graphs by Füredi [5]; and independently Griggs et al. [8]; triangle-free graphs by Hujter and Tuza [10] and connected triangle-free graphs by Chang and Jou [3]; graphs with at most r cycles by Sagan and Vatter [16] and Goh et al. [6]; connected unicyclic graphs by Koh et al. [11]; trees independently by Cohen [4], Griggs and Grinstead [7], Sagan [15], Wilf [17]; bipartite graphs by Liu [13] and bipartite graphs with at least one cycle by Li et al. [12].

A subset of the vertices of a graph G is called a vertex cover if every edge in G is incident to at least one vertex of the set. The *covering number* of G, denoted by $\tau(G)$, is the minimum size of a vertex cover of G. The goal of this paper is to determine the maximum value of $\mathrm{m}(G)$ for an arbitrary simple graph G in terms of its covering number, and to characterize the extremal graphs. Our results improve certain results among those mentioned above. Before stating our results, recall that a matching in G is a set of edges, no two of which meet a common vertex. The *matching number* $\nu(G)$ of G is the maximum size of matchings of G. An induced matching M in a graph G is a matching where no two edges of M are joined by an edge of G. The *induced matching number* $\nu_0(G)$ of G is the maximum size of induced matchings of

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G. We always have $\nu_0(G) \leq \nu(G)$; and if $\nu_0(G) = \nu(G)$ then we call G is a *Cameron–Walker* graph. This definition is similar to the one in Hibi et al [9] including both disconnected graphs and star graphs and star triangle graphs. The main result of the paper is as follows:

Theorem (Theorem 2.7 and Theorem 3.3)**.** *Let G be a graph. Then* $\mathrm{m}(G) \leq 2^{\tau(G)}$, *and the equality holds if and only if G is a Cameron–Walker bipartite graph.*

A graph G is called a *König–Egerváry graph* if the matching number is equal to the covering number that is $\tau(G) = \nu(G)$. As an application, we determine the maximum value of $\mathrm{m}(G)$ for König–Egerváry graphs G, and characterize the extremal graphs.

Corollary 3.4. *Let G be a König–Egerváry graph. Then*

$$\mathrm{m}(G) \leq 2^{\nu(G)},$$

and the equality holds if and only if G is a Cameron–Walker bipartite graph.

It is well-known that all bipartite graphs are König–Egerváry (see [1, Theorem 8.32]). In general, $\nu(G) \leqslant \lfloor \frac{n}{2} \rfloor$, where n is the order of G. Thus Corollary 3.4 improves the main result of Liu (see [13, Theorem 2.1]) for bipartite graphs.

2. Bounds for $m(G)$

We now recall some basic concepts and terminology from graph theory (see [1]). Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. An edge $e \in E(G)$ connecting two vertices x and y will be also written as xy (or yx). For a subset S of $V(G)$, we denote by $G[S]$ the induced subgraph of G on the vertex set S; and use $G \setminus S$ to denote $G[V(G) \setminus S]$. The neighborhood of S in G is the set

$$N_G(S) := \{y \in V(G) \setminus S \mid xy \in E(G) \text{ for some } x \in S\},$$

the closed neighborhood of S is $N_G[S] := S \cup N_G(S)$. Let $G_S := G \setminus N_G[S]$. If $S = \{x\}$, we write $N_G(x)$ (resp. $N_G[x]$, G_x, $G \setminus x$) instead of $N_G(\{x\})$ (resp. $N_G[\{x\}]$, $G_{\{x\}}$, $G \setminus \{x\}$). The number $\deg_G(x) := |N_G(x)|$ is called the *degree* of x in G. A vertex in G of degree zero is called an *isolated vertex* of G. A vertex x of G is called *leaf adjacent to* y if $\deg_G(x) = 1$ and xy is an edge of G. A complete graph with n vertices is denoted by K_n. A graph K_3 is called *triangle*. The union of two disjoint graphs G and H is the graph $G \cup H$ with vertex set $V(G \cup H) = V(G) \cup V(H)$ and edge set $E(G \cup H) = E(G) \cup E(H)$. The union of t copies of disjoint graphs isomorphic to G is denoted by tG, where t is a positive integer.

A graph is called *totally disconnected* if it is either a null graph or contains no edge. Thus, $\mathrm{m}(G) = 1$ whenever G is totally disconnected. The following basic lemmas on determining $\mathrm{m}(G)$ for arbitrary graph G will be frequently used later.

Lemma 2.1. [10, Lemma 1] *Let G be a graph. Then*

(1) $\mathrm{m}(G) \leq \mathrm{m}(G_x) + \mathrm{m}(G \setminus x)$, *for any vertex x of G.*

(2) If x is a leaf adjacent to y of G, then $\mathrm{m}(G) = \mathrm{m}(G_x) + \mathrm{m}(G_y)$.

(3) If $G_1, \dots, G_s$ are connected components of G, then

$$\mathrm{m}(G) = \prod_{i=1}^{s} \mathrm{m}(G_i).$$

Lemma 2.2. *If H is an induced subgraph of G, then $\mathrm{m}(H) \le \mathrm{m}(G)$.*

We first give an upper bound for $\mathrm{m}(G)$ in terms of $\nu(G)$, and the extremal graphs.

Proposition 2.3. *Let G be a graph. Then, $\mathrm{m}(G) \le 3^{\nu(G)}$ and the equality holds if and only if $G \cong sK_3 \cup tK_1$, where $s = \nu(G)$ and $t = |V(G)| - 3s$.*

Proof. We prove the proposition by induction on $\nu(G)$. If $\nu(G) = 0$, then G is totally disconnected, and then the assertion is trivial.

If $\nu(G) = 1$, let xy be an edge of G and let $S := V(G) \setminus \{x, y\}$. Then $G[S]$ is totally disconnected and if we have two vertices in S, say u and v, such that xu and yv are edges of G, then $\{xu, yv\}$ is a matching in G, a contradiction. Thus, there is at most one vertex in S that is adjacent to both x and y. We now consider two cases:

Case 1: There is no vertex in S which is adjacent to both x and y. In this case, G is a star union some number of isolated vertices. Thus, we have $\mathrm{m}(G) = 2$, and the proposition holds.

Case 2: There is a vertex in S, say z, that is adjacent to both x and y. In this case, every other vertex of S is not adjacent to either x or y. Thus, $G = K_3 \cup tK_1$, where $t = |V(G)| - 3$ and $\mathrm{m}(G) = 3 = 3^{\nu(G)}$. Therefore, the proposition is proved in this case.

Assume that $\nu(G) \geqslant 2$. Let xy be an edge of G. Since both x and y are not vertices of the following graphs: G_x, G_y and $G \setminus \{x, y\}$, we deduce that

$$\nu(G_x) \leqslant \nu(G) - 1,\ \nu(G_y) \leqslant \nu(G) - 1 \ \text{ and } \nu(G \setminus \{x, y\}) \le \nu(G) - 1.$$

Thus, by the induction hypothesis, we obtain

$$\mathrm{m}(G_x) \leqslant 3^{\nu(G)-1},\ \mathrm{m}(G_y) \leqslant 3^{\nu(G)-1} \text{ and } \ \mathrm{m}(G \setminus \{x, y\}) \leqslant 3^{\nu(G)-1}.$$

Note that $(G \setminus x)_y = G_y$. Combining with Lemma 2.1, we obtain

$$\begin{aligned} \mathrm{m}(G) &\le \mathrm{m}(G_x) + \mathrm{m}(G \setminus x) \\ &\le \mathrm{m}(G_x) + \mathrm{m}(G_y) + \mathrm{m}(G \setminus \{x, y\}) \\ &\le 3^{\nu(G)-1} + 3^{\nu(G)-1} + 3^{\nu(G)-1} = 3^{\nu(G)}. \end{aligned}$$

This proves the first conclusion of the proposition. The equality $\mathrm{m}(G) = 3^{\nu(G)}$ occurs if and only if

$$\begin{aligned} \mathrm{m}(G) &= \mathrm{m}(G_x) + \mathrm{m}(G \setminus x),\ \mathrm{m}(G \setminus x) = \mathrm{m}(G_y) + \mathrm{m}(G \setminus \{x, y\}), \\ \mathrm{m}(G_x) &= \mathrm{m}(G_y) = \mathrm{m}(G \setminus \{x, y\}) = 3^{\nu(G)-1}, \end{aligned}$$

and

$$\nu(G_x) = \nu(G_y) = \nu(G \setminus \{x, y\}) = \nu(G) - 1.$$

If $G = sK_3 \cup tK_1$, then $s = \nu(G)$ and $\mathrm{m}(G) = 3^{\nu(G)}$. This establishes the necessary condition of the second conclusion of the proposition. Now, it remains to prove that if $\mathrm{m}(G) = 3^{\nu(G)}$ then $G \cong sK_3 \cup tK_1$.

Indeed, by the induction hypothesis, it follows that when the isolated vertices of G_x, G_y and $G \setminus \{x, y\}$ are removed, the remaining graphs are isomorphic, namely $(s-1)K_3$, where $s = \nu(G)$. In particular, x and y are not adjacent to any vertex of $(s-1)K_3$. Let H be an induced subgraph of G on the vertex set $V(G) \setminus V((s-1)K_3)$. Then, H and $(s-1)K_3$ are disjoint subgraphs of G. By Lemma 2.1, we infer $\mathrm{m}(G) = \mathrm{m}(H)\,\mathrm{m}((s-1)K_3) = \mathrm{m}(H)3^{s-1}$. Since $\mathrm{m}(G) = 3^s$, $\mathrm{m}(H) = 3$. Note that $\nu(H) = 1$, so the induction hypothesis again yields $H = K_3 \cup tK_1$. Thus, $G = sK_3 \cup tK_1$. The proof is complete. $\square$

The following lemma gives a lower bound for $\mathrm{m}(G)$ in terms of the induced matching number $\nu_0(G)$.

Lemma 2.4. *Let G be a graph. Then,* $\mathrm{m}(G) \geq 2^{\nu_0(G)}$.

Proof. Let $\{x_1y_1, \ldots, x_ry_r\}$ be an induced matching of G, where $r = \nu_0(G)$. Set $H := G[\{x_1, \ldots, x_r, y_1, \ldots, y_r\}]$. By Lemma 2.2, $\mathrm{m}(G) \geq \mathrm{m}(H) = 2^{\nu_0(G)}$. $\square$

Recall that a vertex cover of G is a subset S of $V(G)$ such that for each $xy \in E(G)$, either $x \in S$ or $y \in S$. The following two lemmas are obvious.

Lemma 2.5. *Let H be an induced subgraph of G. Then,*

(1) If S is a vertex cover of G, then $S \cap V(H)$ is a vertex cover of H; and
(2) $\tau(H) \leq \tau(G)$.

Lemma 2.6. *Assume S is a vertex cover of G. If $U \subseteq S$, then*

(1) $S \setminus U$ is a vertex cover of $G \setminus U$; and
(2) $\tau(G \setminus U) \leq \tau(G) - |U|$.

We conclude this section by giving an upper bound for $\mathrm{m}(G)$ in terms of $\tau(G)$.

Theorem 2.7. *Let G be a graph. Then,* $\mathrm{m}(G) \leq 2^{\tau(G)}$.

Proof. We prove the theorem by induction on $\tau(G)$. If $\tau(G) = 0$, then G is totally disconnected, and so the assertion is trivial.

Assume that $\tau(G) \geqslant 1$. Let S be a vertex cover of G such that $|S| = \tau(G)$. Let $x \in S$. By Lemma 2.6, we have $\tau(G \setminus x) \leq \tau(G) - 1$. Hence, $\mathrm{m}(G \setminus x) \leq 2^{\tau(G \setminus x)}$ by the induction hypothesis.

Since G_x is an induced subgraph of $G \setminus x$, $\mathrm{m}(G_x) \leq \mathrm{m}(G \setminus x)$ by Lemma 2.2. Together with Lemma 2.1, we obtain

$$\begin{aligned} \mathrm{m}(G) &\leq \mathrm{m}(G \setminus x) + \mathrm{m}(G_x) \\ &\leq 2\,\mathrm{m}(G \setminus x) \leq 2^{\tau(G \setminus x)+1} \leq 2^{\tau(G)}, \end{aligned}$$

as required. $\square$

3. Extremal graphs

A graph G is called *bipartite* if its vertex set can be partitioned into two subsets A and B so that every edge has one end in A and one end in B; such a partition is called a *bipartition* of the graph, and denoted by (A, B). If every vertex in A is joined to every vertex in B then G is called a complete bipartite graph, which is denoted by $K_{|A|,|B|}$. A *star* is the complete bipartite graph $K_{1,m}$ $(m \geq 0)$ consisting of $m+1$ vertices. A *star triangle* is a graph consisting of some triangles joined at one common vertex.

Cameron and Walker [2] gave firstly a classification of the connected graphs G with $\nu(G) = \nu_0(G)$. Hibi et al [9] modified their result slightly and gave a full generalization with some corrections.

Lemma 3.1. ([2, Theorem 1] or [9, p.258]) *A connected graph G is Cameron–Walker if and only if it is one of the following graphs:*

(1) a star;
(2) a star triangle;
(3) a finite graph consisting of a connected bipartite graph with bipartition (A, B) such that there is at least one leaf edge attached to each vertex $i \in A$ and that there may be possibly some pendant triangles attached to each vertex $j \in B$.

Example 3.2. Let G be Cameron–Walker graph with 8 vertices in Figure 1. Then $\nu(G) = 2$ and the maximal independent sets of G are

$$\{1, 2, 5, 6, 7, 8\}; \{3, 4\}; \{3, 5, 6\}; \{4, 7, 8\}.$$

Hence, $\mathrm{m}(G) = 4$.

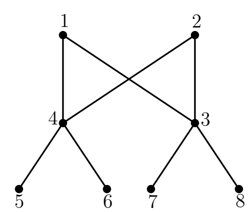

Figure 1.

Theorem 3.3. *Let G be a graph. Then $\mathrm{m}(G) = 2^{\tau(G)}$ if and only if G is a Cameron–Walker bipartite graph.*

Proof. If G is a Cameron–Walker bipartite graph, then $\nu_0(G) = \nu(G) = \tau(G)$. Together with Lemma 2.4 and Theorem 2.7, this fact yields $\mathrm{m}(G) = 2^{\tau(G)}$.

Conversely, assume that $\mathrm{m}(G) = 2^{\tau(G)}$. We will prove that G is Cameron–Walker bipartite by induction on $\tau(G)$.

If $\tau(G) = 0$, then G is totally disconnected and so the assertion is trivial. If $\tau(G) = 1$, then G is a union of a star and isolated vertices. In this case, G is a Cameron–Walker bipartite graph by Lemma 3.1.

Assume that $\tau(G) \geq 2$. Let S be a minimal vertex cover of G such that $|S| = \tau(G)$. We first prove two following claims.

Claim 1: S is an independent set of G.

Assume to the contrary that there is an edge, say xy, with $x, y \in S$. By Lemma 2.5, $S \cap V(G_x)$ is a vertex cover of G_x. Since $S \cap V(G_x) \subseteq S \setminus \{x, y\}$, we deduce that

$$\tau(G_x) \leqslant |S| - 2 = \tau(G) - 2.$$

Similarly, $S \setminus \{x\}$ is a vertex cover of $G \setminus x$. Thus $\tau(G \setminus x) \leqslant \tau(G) - 1$.

Together those inequalities with Lemma 2.1 and Theorem 2.7, we have

$$\mathrm{m}(G) \leqslant \mathrm{m}(G_x) + \mathrm{m}(G \setminus x) \leqslant 2^{\tau(G)-2} + 2^{\tau(G)-1} < 2^{\tau(G)}.$$

This inequality contradicts our assumption. Therefore, S is an independent set of G.

Claim 2: $\mathrm{m}(G_U) = 2^{\tau(G_U)}$ and $\tau(G_U) = \tau(G) - |U|$ for any $U \subseteq S$.

We prove the claim by induction on $|U|$. If $|U| = 0$, i.e., U is empty, then there is nothing to prove.

If $|U| = 1$, then $U = \{x\}$ for some vertex x. Since $x \in S$, by Lemmas 2.5 and 2.6, we have $\tau(G_x) \leq \tau(G \setminus x) \leq \tau(G) - 1$. By Theorem 2.7, $\mathrm{m}(G \setminus x) \leq 2^{\tau(G\setminus x)}$ and $\mathrm{m}(G_x) \leq 2^{\tau(G_x)}$. Together these inequalities with equality $\mathrm{m}(G) = 2^{\tau(G)}$, Lemma 2.1 gives

$$\begin{aligned} 2^{\tau(G)} = \mathrm{m}(G) &\leq \mathrm{m}(G \setminus x) + \mathrm{m}(G_x) \leq 2^{\tau(G\setminus x)} + 2^{\tau(G_x)} \\ &\leq 2^{\tau(G)-1} + 2^{\tau(G)-1} = 2^{\tau(G)}. \end{aligned}$$

Hence, $\mathrm{m}(G_x) = 2^{\tau(G_x)}$ and $\tau(G_x) = \tau(G) - 1$, and the claim holds in this case.

We now assume $|U| \geq 2$. Let $x \in U$ and let $T := U \setminus \{x\}$. Note that T is a nonempty independent set of S and $|T| = |U| - 1$. By the induction hypothesis of our claim, $\mathrm{m}(G_T) = 2^{\tau(G_T)}$ and $\tau(G_T) = \tau(G) - |T|$.

Note that, by Claim 1, S is an independent set of G. Thus $S \setminus T = S \setminus N_G[T]$. By Lemma 2.5, $S \setminus T$ is a vertex cover of G_T. Since $x \in S \setminus T$, by the same argument in the inductive step of our claim with G_T replacing by G, we have $\mathrm{m}((G_T)_x) = 2^{\tau((G_T)_x)}$ and $\tau((G_T)_x) = \tau(G_T) - 1$.

Since $G_U = (G_T)_x$, we obtain $\mathrm{m}(G_U) = 2^{\tau(G_U)}$ and

$$\tau(G_U) = \tau(G_T) - 1 = \tau(G) - (|T| + 1) = \tau(G) - |U|,$$

as claimed.

We turn back to the proof of the theorem. By Claim 1, S is both a vertex cover and an independent set of G. Therefore G is a bipartite graph with bipartition $(S, V(G) \setminus S)$. It remains to prove G is a Cameron–Walker graph.

For each $x \in S$, let $U := S \setminus \{x\}$. By Claim 2, $\tau(G_U) = \tau(G) - |U| = 1$. Hence, G_U is a union of a star with bipartition $(\{x\}, Y)$, where $\emptyset \neq Y \subseteq V(G) \setminus S$ and isolated vertices. Thus, there is a vertex $y \in Y$ such that $\deg_{G_U}(y) = 1$ and $xy \in E(G)$. Since $V(G) \setminus S$ is an independent set, the equality $\deg_{G_U}(y) = 1$ forces $\deg_G(y) = 1$. By using Lemma 3.1, we conclude that G is a Cameron–Walker graph, and the proof is complete. $\square$

If G is a König–Egerváry graph, then $\tau(G) = \nu(G)$. Together Theorems 2.7 and 3.3, this fact yields.

Corollary 3.4. *Let G be a König–Egerváry graph. Then*

$$\mathrm{m}(G) \leq 2^{\nu(G)},$$

and the equality holds if and only if G is a Cameron–Walker bipartite graph.

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