ON THE FUNDAMENTAL GROUP SCHEMES OF CERTAIN QUOTIENT VARIETIES

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Abstract. In [Ar68], M. Armstrong proved a beautiful result describing fundamental groups of quotient spaces. In this paper we prove an analogue of Armstrong's theorem in the setting of F-divided [dS07] and essentially finite [No76] fundamental group schemes.

1. Introduction

The goal here is to establish an analogue, in the theory of fundamental group schemes, of a beautiful topological theorem found by M. A. Armstrong, which according to R. Geoghegan [Ge] "is the kind of basic material that ought to have been in standard textbooks on fundamental groups for the last fifty years":

Theorem 1.1 ([Ar68, p. 299, Theorem])**.** *Let X be a path connected, simply connected, locally compact metric space. Given a group G acting discontinuously on X, the fundamental group of the quotient space $G\backslash X$ is isomorphic with the quotient G/I, where $I < G$ is the (necessarily normal) subgroup generated by all elements having at least one fixed point.*

The setting we have in mind is the following. Let X be a variety over k on which a certain *finite* abstract group G acts (for unexplained notation, see the end of this introduction). Under a mild condition on the orbits of G, it is a fact that a reasonable quotient of X by G exists in the category of varieties [Mu, § 7, Theorem]; denote the quotient by $G\backslash X$. Now we can ask how the different fundamental groups of X relate to those of $G\backslash X$. In the present work, we address this question in two theories of the fundamental group scheme: the F-divided [dS07] and the essentially finite [No76]. See Theorem 5.1 and Theorem 8.1 for precise statements in these directions.

The mechanism behind these results is worth being made conspicuous as everything hinges on two main ideas. The first one is very elementary and commonplace in number theory: all ramification of an extension is concentrated on the inertia field. We present a clear geometric picture of this in Section 4. The second one is more sophisticated and based on the fact that the etale fundamental group has, in many cases, a mysterious control over other fundamental group schemes. Here this is manifested through the fact that as soon as a certain morphism $Y \longrightarrow X$ realizes $\pi_1(X)$ as a quotient of $\pi_1(Y)$, then the same is true for the F-divided and for the essentially finite fundamental groups. We offer a careful explanation in Theorem 5.2 and Theorem 8.2.

Let us now review the remaining sections of the paper.

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Section 2 explains how to develop the F-divided fundamental group scheme [dS07] beyond the realm of algebraic k-schemes. This is essentially well-known as the technique behind the main point (Lemma 2.2) works in more generality than stated in its first appearance [dS07, Lemma 6].

Section 3 exists to fill a gap in the literature and deals with an unsurprising result expressing the F-divided fundamental group of a quotient in the case of a *free* action; see Proposition 3.3. As a recompense for the reader of this section, we offer a slight variation of the criterion for exact sequences first presented in [EHS08, Appendix], see Proposition 3.4. (This variation is again used in developing the essentially finite theory.)

Section 4 introduces the concept of genuinely ramified finite morphisms, a terminology due to Balaji and Parameswaran, and demonstrates that this very useful notion easily connects to a number of other elementary ideas from the theory of coverings (in the broad sense), see Proposition 4.3.

Section 5 offers one of our main results, an analogue of Armstrong's theorem in the setting of the F-divided fundamental group; see Theorem 5.1.

Starting from Section 6 on, we concentrate on the theory of the essentially finite fundamental group scheme [No76]. But the structure is very much the same as in the theory of the F-divided group: we begin by proposing a slight generalization of Nori's theory to the case of non-proper varieties (Section 6) and conclude, in Section 8, by proving an analogue of Armstrong's theorem for this fundamental group scheme; see Theorem 8.1.

It should be pointed out that many results in this paper are consequences of the recent work of Tonini and Zhang, [TZ17], [TZ17a], [TZ17b] (we have made these connections explicit at the proper places). These authors study, in considerable generality, Tannakian categories of sheaves and, in doing so, they produce not only broader conclusions, but also more comprehensive frameworks.

As a final comment, we would like to call attention to the fact that Armstrong's theorem (Theorem 1.1) finds a very attractive description using groupoids, as R. Brown pertinently points out in [Bro06]. Since this formalism is closely related to the theory of Tannakian categories, we hope to re-examine our findings under this light in the future.

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Notations and standard terminology.

(1) We fix once and for all an algebraically closed field k of characteristic $p > 0$.
(2) A *variety* is an integral k-scheme of finite type. A *curve* is a one dimensional variety.

(3) To avoid repetitions, a point in a scheme X, unless otherwise said, means a *closed* point of X.
(4) If X is a variety, x_0 is a closed point in it, then $\pi_1(X, x_0)$ is the etale fundamental group of [SGA1].
(5) The absolute Frobenius morphism of a k-scheme X, respectively a k-algebra A, shall be denoted by F_X, respectively F_A. If no confusion is likely, the subscript will be suppressed.
(6) A *finite-Galois* morphism of integral schemes $f : Y \longrightarrow X$ is a finite morphism for which the associated extension of function fields is Galois. We refer to the pertinent Galois group by $\mathrm{Gal}(f)$.
(7) A locally free coherent sheaf is called a *vector-bundle*. If $\mathcal{E} \subset \mathcal{F}$ is an inclusion of coherent sheaves which are also vector bundles, we say that $\mathcal{E}$ is a *sub-bundle* of $\mathcal{F}$ if $\mathcal{F}/\mathcal{E}$ is also a vector bundle.
(8) An open and dense subset of a scheme X is called *big* if its complement has codimension at least two.
(9) A *quotient morphism* between affine group schemes over k is a faithfully flat morphism (this terminology comes from [Wa79, 15.1]). We will employ many times the Theorem of 14.1 from [Wa79].
(10) Our conventions on representations of group schemes follow [Jan87, Chapter 2, Part 1].

Part I – The F-divided fundamental group schemes

2. Preliminaries on F-divided sheaves

In what follows, X stands for a noetherian k-scheme; we do *not* assume X to be of finite type.

We wish to develop in the following lines some bases for a theory of F-divided sheaves on X by employing the method of [dS07].

Definition 2.1. The category of F-divided sheaves is the category $\mathbf{Fdiv}(X)$ such that:

Objects are sequences $\{\mathcal{E}_n, \sigma_n\}_{n\in\mathbb{N}}$ where $\mathcal{E}_n$ is a coherent $\mathcal{O}_X$-module and

$$\sigma_n : F_X^*\mathcal{E}_{n+1} \xrightarrow{\sim} \mathcal{E}_n$$

is an isomorphism.

Arrows between $\{\mathcal{E}_n, \sigma_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{E}'_n, \sigma'_n\}_{n\in\mathbb{N}}$ are families of morphisms $\alpha_n : \mathcal{E}_n \longrightarrow \mathcal{E}'_n$ such that $\sigma'_n \circ (F_X^*\alpha_{n+1}) = \alpha_n \circ \sigma_n$.

The construction of $\mathbf{Fdiv}$ is evidently functorial, and if $f : Y \longrightarrow X$ is an arrow of k-schemes, then the obvious functor $\mathbf{Fdiv}(X) \longrightarrow \mathbf{Fdiv}(Y)$ constructed from the pull-back functor $f^* : \mathbf{Coh}(X) \longrightarrow \mathbf{Coh}(Y)$ is denoted by $f^\#$.

As expected, if X happens to be the spectrum of a noetherian k-algebra A, we shall write $\mathbf{Fidv}(A)$ instead of $\mathbf{Fdiv}(X)$, and speak about F-divided *modules.*

Lemma 2.2. *For any F-divided module $\mathcal{E} = \{\mathcal{E}_n\}_{n\in\mathbb{N}}$ over X, the $\mathcal{O}_X$-module $\mathcal{E}_0$ is locally free.*

Proof. It suffices to show that any F-divided module $E = \{E_n, \sigma_n\}_{n=0}^\infty$ over a noetherian local ring A is free. Let $\mathfrak{m}$ stand for the maximal ideal of A. If M is an A-module of finite type, we write $\mathrm{Fitt}_i(M)$ for the i-th Fitting ideal of M [Ei94, 20.2, 492ff]. Then, for each

$n \in \mathbb{N}$, we know that $\mathrm{Fitt}_i(E_n)^{[p^m]} = \mathrm{Fitt}_i(E_0)$, where for an ideal $\mathfrak{a} \subset A$, the notation $\mathfrak{a}^{[p^n]}$ stands for the ideal of A generated by the image of $\mathfrak{a}$ under the homomorphism $a \longmapsto a^{p^n}$ [Ei94, Corollary 20.5, p. 494]. We assume that $\mathrm{Fitt}_i(E_0) \neq (1)$. In this case, $\mathrm{Fitt}_i(E_n) \neq (1)$, so that $\mathrm{Fitt}_i(E_n) \subset \mathfrak{m}$. But then

$$\mathrm{Fitt}_i(E_0) \subset \mathfrak{m}^{[p^n]} \subset \mathfrak{m}^{p^n}.$$

As the algebra A is separated with respect to the $\mathfrak{m}$-adic topology [Mat89, 8.10, p. 60], we conclude that $\mathrm{Fitt}_i(E_0) = (0)$. Hence E_0 is projective [Ei94, Proposition 20.8, p. 495], which implies that E_0 is free because A is local. □

In [Gi75, Proposition 1.5], one finds a rather meaningful statement concerning F-divided sheaves on formal schemes which, as a consequence, asserts that, up to isomorphism, only the direct sums of the unit object appear in the category of F-divided modules over $k[\![x_1,,\ldots,x_d]\!]$ (cf. Corollary 1.6 of [Gi75]). We believe that another explanation, in a simpler setting, might be useful (as it will be in the proof of Lemma 5.6 further ahead).

Lemma 2.3. *Let $(A,\mathfrak{m},k)$ be a complete local k-algebra. Then, any F-divided module over A is isomorphic to a trivial F-divided module $\{A^r, \text{canonical}\}$.*

Proof. Let $\{E_n,\sigma_n\}_{n\in\mathbb{N}}$ be an F-divided module over A. Let $\beta_n : A^r \longrightarrow E_n$ be an isomorphism (see Lemma 2.2) and let $\varphi_n : A^r \longrightarrow A^r$ correspond to σ_n under it; said differently, each diagram

$$\begin{array}{ccc} A\otimes_{F,A} E_{n+1} & \xrightarrow{\ \sigma_n\ } & E_n \\ {\scriptstyle F^*(\beta_{n+1})}\uparrow & & \uparrow{\scriptstyle \beta_n} \\ A^r & \xrightarrow[\ \varphi_n\ ]{} & A^r \end{array}$$

commutes. Clearly, proceeding inductively, it is possible to choose β_n so that $\varphi_n \equiv \mathrm{id}$ mod $\mathfrak{m}$. This being so, it follows that $F^n(\varphi_n) \equiv \mathrm{id} \mod \mathfrak{m}^{[p^n]}$, and hence the difference

$$\varphi_0 \cdot F(\varphi_1)\cdots F^m(\varphi_m) - \varphi_0 \cdot F(\varphi_1)\cdots F^n(\varphi_n) =$$
$$= \varphi_0 \cdot F(\varphi_1)\cdots F^m(\varphi_m)\cdot \left[\mathrm{id} - F^{m+1}(\varphi_{m+1})\cdots F^n(\varphi_n)\right]$$

is congruent to 0 modulo $\mathfrak{m}^{[p^{m+1}]}$, and a fortiori modulo $\mathfrak{m}^{p^{m+1}}$. Since $\mathrm{End}(A^r)$ is complete for the $\mathfrak{m}$-adic topology, the limit

$$\Phi_0 := \lim_{n\to\infty} \varphi_0 \cdot F(\varphi_1)\cdots F^n(\varphi_n)$$

exists in $\mathrm{End}(A^r)$. It is not hard to see that in fact Φ_0 belongs to $\mathrm{Aut}(A^r)$. More generally, write

$$\Phi_m := \lim_{n\to\infty} \varphi_m \cdot F(\varphi_{m+1})\cdots F^n(\varphi_{m+n});$$

this is an element of $\mathrm{Aut}(A^r)$. Then $\varphi_m \cdot F(\Phi_{m+1}) = \Phi_m$, which gives an arrow in $\mathbf{Fdiv}(A)$:

$$\{\beta_n\Phi_n\} : \mathbb{1}^r \longrightarrow \{E_n,\sigma_n\}.$$

It only takes a moments thought to see that $\{\beta_n\Phi_n\}$ is in fact an isomorphism. □

Let $\{\varphi_n\} : \{\mathcal{E}_n,\sigma_n\} \longrightarrow \{\mathcal{E}'_n,\sigma'_n\}$ be a morphism of $\mathbf{Fdiv}(X)$ and write $\mathcal{C}_n = \mathcal{C}oker(\varphi_n)$, $\mathcal{I}_n = \mathcal{I}m(\varphi_n)$ and $\mathcal{K}_n = \mathcal{K}er(\varphi_n)$. Then, the family $\mathcal{C}_n$ is also F-divided, that is, there exist unique isomorphisms

$$\tau_n : F_X^*\mathcal{C}_{n+1} \longrightarrow \mathcal{C}_n$$

rendering the diagrams

$$\begin{array}{ccccccc} F_X^*\mathcal{E}_{n+1} & \xrightarrow{\varphi_{n+1}} & F_X^*\mathcal{E}'_{n+1} & \longrightarrow & F_X^*\mathcal{C}_{n+1} & \longrightarrow & 0 \\ \downarrow{\sigma_n} & & \downarrow{\sigma'_n} & & \downarrow{\tau_n} & & \\ \mathcal{E}_n & \xrightarrow{\varphi_n} & \mathcal{E}'_n & \longrightarrow & \mathcal{C}_n & \longrightarrow & 0 \end{array}$$

commutative. It then follows that for every $n \in \mathbb{N}$, the coherent sheaf $\mathcal{C}_n$ is locally free. This and the fact that every $\mathcal{E}'_n$ is locally free together imply that $\mathcal{I}_n$ is locally free (and this shows that $\mathcal{K}_n$ is also locally free). Consequently, the pull-back sequences

$$0 \longrightarrow F_X^*\mathcal{K}_n \longrightarrow F_X^*\mathcal{E}_n \longrightarrow F_X^*\mathcal{E}'_n$$

are *exact*; because of this, each $\sigma_n : F_X^*\mathcal{E}_{n+1} \longrightarrow \mathcal{E}_n$ induces an isomorphism $F_X^*\mathcal{K}_{n+1} \xrightarrow{\sim} \mathcal{K}_n$. This proves:

Proposition 2.4. *The category of F-divided modules on X is abelian, and the forgetful functor*

$$\mathbf{Fdiv}(X) \longrightarrow \mathbf{Coh}(X), \quad \{\mathcal{E}_n\} \longmapsto \mathcal{E}_0$$

is faithful and exact. □

Now, on $\mathbf{Fidv}(X)$ we have a tensor product defined term-by-term and also a unit element. In view of Lemma 2.2, we can define the dual of each object $\mathcal{E}$ of $\mathbf{Fdiv}(X)$ in a simple fashion. This being so, $\mathbf{Fdiv}(X)$ is a abelian rigid tensor category [DM82, Definition 1.15]. In addition, as explained in [dS07, 2.2], this $\mathbf{Fdiv}(X)$ is canonically k-linear. We therefore have the following:

Theorem 2.5. *For any k-rational point x_0 of X, the functor $\{\mathcal{E}_n\} \longmapsto x_0^*\mathcal{E}_0$ defines an equivalence between $\mathbf{Fdiv}(X)$ and the category of representations of a certain affine group scheme $\Pi^{\mathrm{FD}}(X, x_0)$.*

The affine group scheme $\Pi^{\mathrm{FD}}(X, x_0)$ in Theorem 2.5 is called *the F-divided fundamental group scheme* of X based at x_0. But even in the absence of a k-point on X, we can say something by verifying the infamous condition (cf. [Del90, 1.10])

$$\mathrm{End}(\mathbb{1}) = k. \tag{2.1}$$

(This specialty will be used in proving Theorem 5.2.) We note that, in our particular case, the ring $\mathrm{End}(\mathbb{1})$ is

$$\mathcal{O}(X)^{\S} := \{a \in \mathcal{O}(X) : a \in \cap_{n\geq 1}\mathcal{O}(U)^{p^n} \text{ for each affine open } U\}.$$

Lemma 2.6. *Let $(A, \mathfrak{m})$ be a local noetherian k-algebra with residue field k and field of fractions K. If $X = \operatorname{Spec} A$, then $\mathcal{O}(X)^{\S} = k$. If $U = \operatorname{Spec} K$ and A is normal, then $\mathcal{O}(U)^{\S} = \mathcal{O}(X)^{\S} = k$.*

Proof. The equality $\mathcal{O}(X)^{\S} = k$ follows immediately from Krull's intersection theorem [Mat89, 8.10]. Now take any $a \in \mathcal{O}(U)^{\S}$. Let $v : K^* \longrightarrow \mathbb{Z}$ be a discrete valuation. Then $v(a) = 0$. In particular, if $a_n^{p^n} = a$, we also conclude that $v(a_n) = 0$. Combining this with the fact that A is the intersection of a family of discrete valuation rings in K dominating A [Mat89, Theorem 11.5], we conclude that $a \in \cap_n A^{p^n}$, which is k. □

Let us note in passing the following:

Corollary 2.7. *Let A and K be as before, and write $f : \operatorname{Spec} K \longrightarrow \operatorname{Spec} A$ for the obvious morphism. Then*

$$f^{\#} : \mathrm{Hom}_{\mathbf{Fdiv}(A)}(\mathbb{1}^r, \mathbb{1}) \longrightarrow \mathrm{Hom}_{\mathbf{Fdiv}(K)}(\mathbb{1}^r, \mathbb{1})$$

is bijective.

Remark 2.8. The results in this section follow from the work of Tonini and Zhang: Compare Theorem 2.5 to [TZ17, Theorem I] and Lemma 2.6 to [TZ17, Proposition 6.19]. It is also useful to note that [ES16, Section 3] also presents a swift development of the basis of the theory of F-divided sheaves on schemes of finite type over a field.

3. The F-divided fundamental group scheme of a quotient: the case of a free action

The main ideas leading to the findings of this section are folkloric and many of its forms have already been published (see [Ka87, Proposition 1.4.4] or [EHS08, Theorem 2.9] for example).

Let $f : Y \longrightarrow X$ be a finite and etale morphism of k-varieties. We wish to compare $\Pi^{\mathrm{FD}}(Y)$ and $\Pi^{\mathrm{FD}}(X)$, and the method relies on constructing a *right* adjoint $f_{\#} : \mathbf{Fdiv}(Y) \longrightarrow \mathbf{Fdiv}(X)$ to the functor $f^{\#} : \mathbf{Fdiv}(X) \longrightarrow \mathbf{Fdiv}(Y)$. As expected, $f_{\#}$ is built up from the usual adjoint f_*. Let us prepare the terrain.

Given a k-algebra, we write A' to denote the k-algebra whose underlying commutative ring is A, but whose multiplication by an element $\lambda \in k$ is given through the formula $\lambda \bullet a = \lambda^{p^{-1}} a$. With this definition, the Frobenius morphism $F_A : A' \longrightarrow A$ is k-linear.

Lemma 3.1. *Let A be a k-algebra and $A \longrightarrow B$ a finite and etale morphism.The following claims are true.*

(1) *Tthe natural morphism of A-algebras*

$$\theta : B' \underset{A',F_A}{\otimes} A \longrightarrow B, \qquad b' \otimes a \longmapsto b'^p a$$

is bijective. (The reader is required to write down the associated cartesian diagram of affine schemes.)

(2) *Let N' be a finitely presented B'-module. Then the natural morphism*

$$\theta_{N'} : N' \underset{A',F_A}{\otimes} A \longrightarrow N' \underset{B',F_B}{\otimes} B, \qquad n' \otimes a \longmapsto n' \otimes a.$$

is an isomorphism of A-modules.

Proof. Let $x : A \longrightarrow K$ be a morphism of k-algebras, where K is a field, and consider

$$\theta \otimes_A K : (B' \otimes_{A',F_A} A) \otimes_A K \longrightarrow B \otimes_A K.$$

Under the identification

$$(B' \otimes_{A',F_A} A) \otimes_A K \xrightarrow{\sim} (B' \otimes_{A'} K') \otimes_{K',F_K} K,$$

it is clear that $\theta \otimes_A K$ corresponds to

$$(B' \otimes_{A'} K') \otimes_{K',F_K} K \longrightarrow B \otimes_A K, \qquad (b' \otimes 1) \otimes r \longmapsto b'^p \otimes r.$$

Now $B' \otimes_{A'} K'$ is an etale K'-algebra and hence the arrow above is an isomorphism of K-algebras due to [BA, V.6.7, Corollary on p. V.35]. We conclude that θ is an isomorphism by means Nakayama's Lemma.

(2) Part (1) shows that if N' is a free B'-module, then $\theta_{N'}$ is an isomomorphism. Let

$$V' \longrightarrow W' \longrightarrow N' \longrightarrow 0$$

be a presentation of N' with V' and W' free B'-modules of finite rank. Then we arrive at a commutative diagram with exact rows

$$\begin{array}{ccccccc} V'\otimes_{A'} A & \longrightarrow & W'\otimes_{A'} A & \longrightarrow & N'\otimes_{A'} A & \longrightarrow & 0 \\ \downarrow{\scriptstyle \theta_{V'}} & & \downarrow{\scriptstyle \theta_{W'}} & & \downarrow{\scriptstyle \theta_{N'}} & & \\ V'\otimes_{B'} B & \longrightarrow & W'\otimes_{B'} B & \longrightarrow & N'\otimes_{B'} B. & \longrightarrow & 0 \end{array}$$

Because $\theta_{V'}$ and $\theta_{W'}$ are isomorphisms, so is $\theta_{N'}$. □

Corollary 3.2. *Let $f : Y \longrightarrow X$ be as above. Then, for any coherent $\mathcal{O}_Y$-module $\mathcal{N}$, the natural base-change arrow*

$$\theta_{\mathcal{N}} : F_X^* f_*(\mathcal{N}) \longrightarrow f_*(F_Y^*(\mathcal{N}))$$

is an isomorphism of $\mathcal{O}_X$-modules. □

Now let $\{\mathcal{N}_n, \sigma_n\} \in \mathbf{Fdiv}(Y)$. Then

$$\sigma_n^f := f_*(\sigma_n) \circ \theta_{\mathcal{N}_{n+1}} : F_X^*(f_*(\mathcal{N}_{n+1})) \xrightarrow{\sim} f_*(\mathcal{N}_n)$$

defines an object $\{f_*(\mathcal{N}_n), \sigma_n^f\} \in \mathbf{Fdiv}(X)$, and in this way we arrive at an adjunction

$$(f^\#, f_\#) : \mathbf{Fdiv}(X) \longrightarrow \mathbf{Fdiv}(Y).$$

As $\mathbf{Fdiv}(X) \longrightarrow \mathbf{Coh}(X)$ reflects isomorphisms, the construction shows that $f_\#$ is an exact and faithful functor and the counit

$$f^\# f_\#(\mathcal{E}_n, \sigma_n) \longrightarrow (\mathcal{E}_n, \sigma_n)$$

is always an epimorphism (the third property is in fact a consequence of the second [ML71, IV.3, Theorem 1, p.90]).

In addition, $f^\#(f_\#(\mathbb{1}))$ is the *trivial* object $\mathbb{1} \otimes_k \mathcal{O}(G)$ of $\mathbf{Fdiv}(Y)$.

Proposition 3.3. *Let $f : Y \longrightarrow X$ be a finite-Galois etale covering with group G. We have an exact sequence of group schemes*

$$1 \longrightarrow \Pi^{\mathrm{FD}}(Y, y_0) \longrightarrow \Pi^{\mathrm{FD}}(X, x_0) \longrightarrow G \longrightarrow 1\,.$$

Proof. We wish to apply Theorem A.1(iii) of [EHS08]. In fact, since our situation has an extra structure — the existence of the right-adjunct $f_\#$ — we find it opportune to take [EHS08, Theorem A.1(iii)] from a slightly different perspective, which we present as Proposition 3.4 below. As we observed above, $f_\#$ is faithful and $f^\# f_\#(\mathbb{1})$ is trivial (as an object of $\mathbf{Fdiv}(Y)$) so that Proposition 3.4 can be applied to show that $f_\natural : \Pi^{\mathrm{FD}}(Y, y_0) \longrightarrow \Pi^{\mathrm{FD}}(X, x_0)$ is a closed and normal embedding. In addition, the cokernel of $f_\natural$ is defined by the category $\mathbf{Fdiv}(Y/X) := \{\mathcal{V} \in \mathbf{Fdiv}(X) : f^\#\mathcal{V} \text{ is trivial}\}$. Now the functor $\mathrm{Rep}_k(G) \longrightarrow \mathbf{Fdiv}(Y/X)$ defined by $V \longmapsto Y \times^G V$ induces an equivalence, and $\mathbf{Fdiv}(Y/X) \longrightarrow \mathrm{Rep}_k(G)$, given by $\mathcal{E} \longmapsto \mathrm{Hom}_{\mathbf{Fdiv}}(\mathbb{1}, f^\#\mathcal{E})$, is an inverse. □

Proposition 3.4. *Let $\varphi : H \longrightarrow G$ be a morphism of affine group schemes over a field k. Assume that the functor $\varphi^\# : \mathrm{Rep}_k(G) \longrightarrow \mathrm{Rep}_k(H)$ has a* faithful *right adjoint $\varphi_\# : \mathrm{Rep}_k(H) \longrightarrow \mathrm{Rep}_k(G)$, which, in addition, is such that $\varphi^\#(\varphi_\#(\mathbb{1}))$ is a trivial object of* $\mathrm{Rep}_k(H)$. *Then φ is a closed and normal immersion.*

Moreover, if Q is the cokernel of φ, then $\mathrm{Rep}_k(Q)$ is the full subcategory of $\mathrm{Rep}_k(G)$ consisting of those objects on which H acts trivially.

Proof. Since the co-unit $\varphi^{\#}\varphi_{\#}(V) \longrightarrow V$ is an epimorphism [ML71, Theorem 1, p.90], φ is a closed immersion. Using the full sub-category $\{V \in \mathrm{Rep}_k(G) : H \text{ acts trivially on } V\}$ of $\mathrm{Rep}_k(G)$ we define a quotient morphism $G \longrightarrow R$. We set out to prove that (a), (b) and (c) of [EHS08, Theorem A.1(iii)], henceforth called simply conditions (a), (b) and (c), are satisfied for the diagram $H \longrightarrow G \longrightarrow R$. Condition (a) is assured by the construction of R from its category of representations. Condition (c) is guaranteed by the fact that the counit $\varphi^{\#}\varphi_{\#}(V) \longrightarrow V$ is an epimorphism for each V. We only need to show that (b) holds. Let $V \in \mathrm{Rep}_k(G)$ and write D for its dual; we want to show that $D^H = \mathrm{Hom}_H(V, \mathbb{1})$, considered as a subspace of D, is stable under G. The validity of condition (b) is then immediately confirmed.

For a proof, we employ the results and notations of [Jan87, Part I, Ch. 2]. We wish to prove that for any k-algebra A, the set $(D^H)_a(A) = (D_a)^H(A)$ is invariant under $G(A)$.

Let $\eta \in \mathrm{Hom}_H(\varphi^{\#}\varphi_{\#}(\mathbb{1}), \mathbb{1})$ be the counit so that

$$\mathrm{Hom}_G(V, \varphi_{\#}(\mathbb{1})) \longrightarrow \mathrm{Hom}_H(\varphi^{\#}V, \mathbb{1}), \qquad \mu \longmapsto \eta\mu,$$

is bijective. Let $\lambda : V \longrightarrow \mathbb{1}$ lie in D^H and $\mu \in \mathrm{Hom}_G(V, \varphi_{\#}(\mathbb{1}))$ be such that $\eta\mu = \lambda$. Given $g \in G(A)$, we obtain an element of $\mathrm{Hom}_A(V \otimes A, A) = D \otimes A$ defined by $(\lambda \otimes \mathrm{id}_A) \circ g^{-1} = (\eta \otimes \mathrm{id}_A) \circ (\mu \otimes \mathrm{id}_A) \circ g^{-1}$. Then, for any A-algebra B and any $h \in G(B)$, we conclude that

$$\begin{aligned}
(\lambda \otimes \mathrm{id}_B) \circ g^{-1} \circ h^{-1} &= (\eta \otimes \mathrm{id}_B) \circ (\mu \otimes \mathrm{id}_B) \circ g^{-1} \circ h^{-1} && \text{(definition of } \lambda) \\
&= (\eta \otimes \mathrm{id}_B) \circ g^{-1} \circ h^{-1} \circ (\mu \otimes \mathrm{id}_B) && \text{(equivariance of } \mu) \\
&= (\eta \otimes \mathrm{id}_B) \circ g^{-1} \circ (\mu \otimes \mathrm{id}_B) && (H \text{ acts trivially on } \varphi_{\#}(\mathbb{1})) \\
&= (\eta \otimes \mathrm{id}_B) \circ (\mu \otimes \mathrm{id}_B) \circ g^{-1} && \\
&= (\lambda \otimes \mathrm{id}_B) \circ g^{-1}. &&
\end{aligned}$$

This shows that the action of $G(A)$ takes $\left(D^H\right)_a(A)$ into $(D_a)^H(A) = (D^H)_a(A)$ (cf. Part I, 2.10 in [Jan87]). ☐

4. Genuinely ramified finite morphisms

We reinterpret a theme brought to our attention by the work [BP11, § 6]. Since the underlying assumptions in [BP11] are much too restrictive, and since a fundamental point goes unmentioned (which is Proposition 4.3 below), we think it well to interpose this section.

We remind the reader that if $\varphi : N \longrightarrow M$ is a Galois-finite morphism of normal k-varieties, then $\mathrm{Aut}_N(M) = \mathrm{Gal}(\varphi)$.

Definition 4.1 ([BP11, § 6]). Let $f : Y \longrightarrow X$ be a finite surjective morphism of k-varieties. We say that f is genuinely ramified if f is generically etale and if the only possible factorization of f as a composition

$$Y \longrightarrow X' \xrightarrow{\text{etale}} X$$

is

$$Y \xrightarrow{f} X \xrightarrow{\mathrm{id}} X.$$

Under this terminology, we can reinterpret a (probably well-known) exercise as follows:

Lemma 4.2. *Let $f : Y \longrightarrow X$ be a finite-Galois morphism between normal k-varieties. The following statements hold:*

(1) *Assume that* $\mathrm{Gal}(f)$ *is generated by elements having at least one fixed point. Then f is genuinely ramified.*
(2) *Write $I \lhd \mathrm{Gal}(f)$ for the subgroup generated by all elements of* $\mathrm{Gal}(f)$ *fixing at least one point. If $\chi : Y \longrightarrow M$ is the quotient of Y by I with $u : M \longrightarrow X$ being the canonical arrow, then χ is genuinely ramified while u is etale.*
(3) *If f is genuinely ramified, then* $\mathrm{Gal}(f)$ *is generated by the elements having at least one fixed point.*

Proof. (1) Let $Y \xrightarrow{\chi} M \xrightarrow{u} X$ be a factorization of f with u finite and etale. Let us firstly suppose that f is Galois. Then, the canonical homomorphism $\mathrm{Gal}(f) \longrightarrow \mathrm{Gal}(u)$ is surjective, so that $\mathrm{Gal}(u)$ is generated by the elements having at least one fixed point. Since no $\alpha \in \mathrm{Aut}_X(M) \setminus \{\mathrm{id}\}$ can have a fixed point [SGA1, I, Corollary 5.4], we conclude that $\mathrm{Gal}(u) = \{e\}$. Let us now deal with the general case; for that we shall require the construction of the "Galois closure" of u [SGA1, V, § 4(g)].

Let Ω be an algebraic closure of $k(Y)$ and let $\mathbf{y} \in Y(\Omega)$, $\mathbf{m} \in M(\Omega)$, and $\mathbf{x} \in X(\Omega)$ be the associated Ω-points. If $\mathbf{m}_1 = \mathbf{m}$, and $\{\mathbf{m}_1, \ldots, \mathbf{m}_d\}$ is the set of Ω-points of M above $\mathbf{x}$, basic Galois theory and the normality of Y allow us to find, for each i, an X-*automorphism* $g_i : Y \longrightarrow Y$ such that $\chi(g_i(\mathbf{y})) = \mathbf{m}_i$. (In particular, $g_1 = \mathrm{id}_Y$.) Consequently, we have constructed a morphism of X-schemes

$$\widetilde{\chi} : Y \longrightarrow \underbrace{M \times_X \cdots \times_X M}_{d}$$

satisfying:

- The composition $\mathrm{pr}_1 \circ \widetilde{\chi}$ is none other than χ, and
- the image of $\mathbf{y}$ is $(\mathbf{m}_1, \ldots, \mathbf{m}_d)$.

Now, the *connected component* of the above fiber product containing $(\mathbf{m}_1, \ldots, \mathbf{m}_d)$, call it $\widetilde{M}$, is a finite-Galois etale covering of X. (This is explained in [SGA1, V, §4(g)], but the reader should do an exercise on groups acting on finite sets). Using the factorization $Y \xrightarrow{\widetilde{\chi}} \widetilde{M} \xrightarrow{\widetilde{u}} X$, we can apply our previous result to conclude that $\widetilde{u}$ is an isomorphism, so that u is also an isomorphism.

(2) Let $I \lhd \mathrm{Gal}(\chi)$ be the (normal) subgroup generated by the automorphisms fixing at least one point, and write $\chi : Y \longrightarrow M$ for the quotient of Y by I [Mu, § 7, Theorem]. It follows that the canonical morphism $u : M \longrightarrow X$ is finite-Galois, and $\mathrm{Gal}(u) \simeq \mathrm{Gal}(f)/I$. We contend that

(i) the morphism u is etale, and
(ii) χ is genuinely ramified.

To prove (i), it is suffices to show that $\mathrm{Gal}(u)$ acts freely on the points of M [Mu, § 7, Theorem]. Since the set of points of M is just the quotient of the set of points of Y (loc. cit.), the verification is quite straight-forward. So let $I \cdot y = m \in M$, and let $s \in \mathrm{Gal}(f)$ be such that $s(m) = m$. This means that $s(y) = i(y)$, where $i \in I$. Hence, we conclude that $s \in I$, so that $s : M \longrightarrow M$ is just the identity map, which completes the verification of (i).

To establish (ii), we note that by construction, $\mathrm{Gal}(\chi) = I$, so that (1) can be directly employed.

(3): This is in fact a consequence of (2). □

The next result is an exercise from the theory of covering spaces.

Proposition 4.3. *Let $f : Y \longrightarrow X$ be a finite and surjective morphism of k-varieties. The morphism f is genuinely ramified if and only if the morphism between etale fundamental groups $f_\natural : \pi_1(Y) \longrightarrow \pi_1(X)$ is surjective.*

Proof. We shall only prove the "only if" clause; the verification of the other one is very simple. Hence we suppose that f is genuinely ramified. Let $\pi_1(X) \longrightarrow \mathfrak{g}$ be a finite quotient, and let $\mathfrak{h} < \mathfrak{g}$ be the image of $\pi_1(Y)$. We endow $E := \mathfrak{g}/\mathfrak{h}$ with the canonical left action of $\pi_1(X)$. Let $X' \longrightarrow X$ be the etale covering associated to E. Since $\pi_1(Y)$ leaves one point of E fixed, it follows that the etale covering $Y' := X' \times_X Y \longrightarrow Y$ (which is associated to the $\pi_1(Y)$-set E) must have a connected component isomorphic to Y. In this way, we obtain a section $\sigma : Y \longrightarrow Y'$, and then a lifting of $f : Y \longrightarrow X$ to X'. But this forces $X' \longrightarrow X$ to be an isomorphism, and we conclude that $\#E = 1$. Since $\mathfrak{g}$ is arbitrary, we conclude that $f_\natural \pi_1(Y) = \pi_1(X)$. □

5. The F-divided group scheme of a quotient

Our aim in this section is to prove the following:

Theorem 5.1. *Let G be a finite group acting on the normal variety Y with*

$$f : Y \longrightarrow X$$

being the quotient [Mu, Theorem, § 7]. *Choose $y_0 \in Y(k)$ above $x_0 \in X(k)$. Then, the cokernel of the induced homomorphism*

$$f_\natural : \Pi^{\mathrm{FD}}(Y, y_0) \longrightarrow \Pi^{\mathrm{FD}}(X, x_0)$$

is identified with G/I, where I is the subgroup (necessarily normal) generated by all elements of G having at least one fixed point.

The proof is a simple consequence of Theorem 5.2 below, which then becomes the main object of our efforts.

Proof of Theorem 5.1. Let $\chi : Y \longrightarrow M$ be the quotient of Y by I and $u : M \longrightarrow X$ the morphism induced by f. Then u is finite-Galois and *etale* [Mu, Theorem, §7], $\mathrm{Gal}(u) \simeq G/I$, and χ is genuinely ramified (see Lemma 4.2-(2)). Now Theorem 5.2 guarantees that the homomorphism

$$\chi_\natural : \Pi^{\mathrm{FD}}(Y, y_0) \longrightarrow \Pi^{\mathrm{FD}}(M, \chi(y_0))$$

is faithfully flat, so that the cokernel of $f_\natural$ is just the cokernel of $u_\natural$, which is G/I as guaranteed by Proposition 3.3. □

Theorem 5.2. *Let*

$$f : Y \longrightarrow X$$

be a finite Galois, genuinely ramified morphism between normal k-varieties. If $y_0 \in Y(k)$ is taken by f to $x_0 \in X(k)$, then

$$f_\natural : \Pi^{\mathrm{FD}}(Y, y_0) \longrightarrow \Pi^{\mathrm{FD}}(X, x_0)$$

is a quotient morphism.

The proof of Theorem 5.2 will rely on several subsidiary results. These are: *Lemma 5.3*, which is a minor simplification of the standard criterion [DM82, Proposition 2.12, p. 139] for a morphism of affine group schemes to be faithfully flat. *Proposition 5.4*, which notes that condition (1) in Lemma 5.3 was already explained in [Gi75]. *Proposition 5.5*, which gives us the means to deduce Theorem 5.2 from the case of etale morphisms. *Notation and assumptions* are those of Theorem 5.2.

Lemma 5.3. *Let $\varphi : \Pi' \longrightarrow \Pi$ be a homomorphism of affine group schemes over k. Then φ is faithfully flat if and only if*

(1) *the functor $\varphi^{\#} : \mathrm{Rep}(\Pi) \longrightarrow \mathrm{Rep}(\Pi')$ is fully faithful,*

and one among the following two equivalent conditions holds:

(2) *Let V be a representation of Π and $L \subset \varphi^{\#}(V)$ a Π'–submodule of rank one. Then L is also invariant under the action of Π.*

(2bis) *Let V be a representation of Π and $q : \varphi^{\#}(V) \longrightarrow L$ a quotient Π'–module of rank one. Then L also has the structure of a Π-module and q is equivariant.*

In addition, if Π is pro-finite, then condition (1) is already sufficient for φ to be faithfully flat.

Proof. We start by noting that (2) and (2bis) are equivalent: all that is needed is to take duals. In view of [DM82, p. 139, Proposition 2.21], we only need to show that (1) and (2) together imply that φ is faithfully flat. Take any $V \in \mathrm{Rep}_k(\Pi)$. The aforementioned result of [DM82] guarantees that it is sufficient to show that any Π'–submodule $W \subset V$ is also invariant under Π. Now, if $r = \dim_k W$, then $\bigwedge^r W \subset \bigwedge^r V$ is, by hypothesis, invariant under Π. This means that for all k–algebras R, the rank one R–submodule $\bigwedge^r W \otimes R \subset \bigwedge^r V \otimes R$ is invariant under all $g \in \mathrm{Aut}_R(V \otimes R)$ belonging to the image of $\Pi(R)$. The standard argument in the last paragraph in Appendix 2 on page 152 of [Wa79] proves that this is only possible if $W \otimes R$ is invariant under all $g \in \Pi(R)$.

Let us now deal with the case where Π and Π' are pro-finite and show that condition (1) is enough to show faithful flatness. Let $u : \Pi \to G$ be a finite quotient of Π and consider the following commutative diagram where u and u' are quotient morphisms and i is a closed immersions:

$$\begin{array}{ccc} \Pi' & \xrightarrow{\varphi} & \Pi \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ G' & \xrightarrow[i]{} & G. \end{array}$$

Now, if we endow $\mathcal{O}(G)$ with its right regular action and use (1), we conclude that $\dim \mathcal{O}(G)^{G'} = \dim_k \mathcal{O}(G)^G = 1$. But this is only possible if i is an isomorphism (say, because $\mathcal{O}(G)$ is locally free over $\mathcal{O}(G)^{G'}$ of rank $\dim_k \mathcal{O}(G')$ [Mu, §12, Theorem 1]) and the rest of the proof follows effortlessly. □

We now head towards a proof of Theorem 5.2 by employing Lemma 5.3. The first step was already taken by Gieseker:

Proposition 5.4. *The pull-back functor*

$$f^{\#} : \mathbf{Fdiv}(X) \longrightarrow \mathbf{Fdiv}(Y)$$

is full.

Proof. The proof of [Gi75, p. 11, Lemma 2.8] contains a proof of this claim. Let us give more details. It is enough to show that for any $\mathcal{E} \in \mathbf{Fdiv}(X)$, any arrow $\{\sigma_n\} : \mathbb{1} \longrightarrow f^{\#}\mathcal{E}$ in $\mathbf{Fdiv}(Y)$ comes from an arrow $\mathbb{1} \longrightarrow \mathcal{E}$ in $\mathbf{Fdiv}(X)$. Let $V = \operatorname{Spec} B$ be an affine open subset of Y above the affine open subset $U = \operatorname{Spec} A$ of X. We assume that $\mathcal{E}_0|_U$ is free on $\{e_1, \ldots, e_r\}$ so that we can write $\sigma_0 = \sum b_i \otimes e_i$, with $b_i \in B$. Let $y \in V$ be above $x \in U$ and let $\varphi_1, \ldots, \varphi_r \in \widehat{\mathcal{O}}_x \otimes \mathcal{E}_0$ be a trivializing frame. Then $f^*(\varphi_i)$ is a trivializing frame for $\widehat{\mathcal{O}}_y \otimes \mathcal{E}_0$. Hence $\sigma = \sum_i \lambda_i f^*(\varphi_i)$. We conclude that b_i actually belongs to the image of $\widehat{\mathcal{O}}_x$. It is easy to see that the choice of the basis is unimportant. □

From this point on we work to verify that condition (2) of Lemma 5.3 holds for the morphism $f_\natural : \Pi^{\mathrm{FD}}(Y, y_0) \longrightarrow \Pi^{\mathrm{FD}}(X, x_0)$ of Theorem 5.2. The strategy consists in concentrating on the open subset of X above which f is a principal bundle and employ:

Proposition 5.5. *Let X be as before (a normal variety). Then, for each open and dense subset $U \subset X$, and each $u_0 \in U(k)$, the homomorphism*

$$\Pi^{\mathrm{FD}}(U, u_0) \longrightarrow \Pi^{\mathrm{FD}}(X, u_0)$$

induced by the inclusion is a quotient morphism.

Proof. We first *assume* that the proposition is true for big open subsets and deduce the general case from it.

Let U be an open subset of X. Let X^{reg} (respectively, U^{reg}) be the open subset of regular points of X (respectively, U). We note that X is normal because Y is so. So X^{reg} and U^{reg} are big open subsets of X and U respectively. Consider the commutative diagram of affine group schemes

$$\begin{array}{ccc} \Pi^{\mathrm{FD}}(X^{\mathrm{reg}}) & \xrightarrow{\alpha} & \Pi^{\mathrm{FD}}(X) \\ {\scriptstyle \xi}\uparrow & & \uparrow{\scriptstyle \zeta} \\ \Pi^{\mathrm{FD}}(U^{\mathrm{reg}}) & \xrightarrow[\beta]{} & \Pi^{\mathrm{FD}}(U). \end{array}$$

Using differential operators, one comfortably shows that ξ is faithfully flat [Kin15, p. 6465, Lemma 2.8]. Since $X^{\mathrm{reg}} \subset X$ and $U^{\mathrm{reg}} \subset U$ are big open subsets, the assumption implies that α and β are faithfully flat. Consequently, ζ is also faithfully flat as a simple application of [Wa79, Chapter 14] demonstrates.

We now assume that $U \subset X$ is a *big* open subset and set out to verify that the conditions (1) and (2bis) appearing in Lemma 5.3 hold. As the restriction $\mathbf{VB}(X) \longrightarrow \mathbf{VB}(U)$ is fully faithful [SGA2, III, §3], so is the restriction $\mathbf{Fdiv}(X) \longrightarrow \mathbf{Fdiv}(U)$; condition (1) of Lemma 5.3 is then readily verified.

To verify the condition (2bis), we need to show that for $\mathcal{E} \in \mathbf{Fdiv}(X)$ and each quotient morphism

$$\varphi : \mathcal{E}|_U \longrightarrow \mathcal{L} \,,$$

where $\mathcal{L}$ is an object of rank one in $\mathbf{Fdiv}(U)$, there exists some $\widetilde{\mathcal{L}} \in \mathbf{Fdiv}(X)$ that extends $\mathcal{L}$ and furthermore there is a morphism $\widetilde{\varphi} : \mathcal{E} \longrightarrow \widetilde{\mathcal{L}}$ of $\mathbf{Fdiv}(X)$ extending φ. Two lemmas will be proved for that purpose.

Lemma 5.6. *Let X, U, $\mathcal{E}$, $\mathcal{L}$ and φ be as before. Then, there exists a line bundle $\widetilde{\mathcal{L}}_0$ extending $\mathcal{L}_0$, and furthermore there is a quotient morphism*

$$\widetilde{\varphi}_0 : \mathcal{E}_0 \longrightarrow \widetilde{\mathcal{L}}_0$$

extending φ_0.

Proof. Let $\mathbb{P}(\mathcal{E}_0) \longrightarrow X$ be the projective bundle associated to $\mathcal{E}_0$. The existence of $\mathcal{L}_0$ implies that there is a section

$$\begin{array}{ccc} & & \mathbb{P}(\mathcal{E}_0) \\ & \overset{\sigma}{\nearrow} & \downarrow \\ U & \longrightarrow & X. \end{array}$$

Let $x \in X \setminus U$ and write D (respectively, D°) for the spectrum of the complete local ring $\widehat{\mathcal{O}}_x$ (respectively, field of fractions of $\widehat{\mathcal{O}}_x$). It should be emphasized that $\widehat{\mathcal{O}}_x$ is a *normal domain*; see [Mat80, Theorem 79, p.258]. We then arrive at a commutative diagram

$$\begin{array}{ccccc} D^\circ & \longrightarrow & U & \xrightarrow{\sigma} & \mathbb{P}(\mathcal{E}_0) \\ \downarrow & & \downarrow & & \\ D & \longrightarrow & X. & & \end{array}$$

Since $\mathcal{E}|_D$ is isomorphic to $\mathbb{1}_D^{\oplus r}$, it follows that $\mathcal{E}|_{D^o}$ is isomorphic to $\mathbb{1}_{D^\circ}^{\oplus r}$; hence $\mathcal{L}|_{D^\circ}$ is a quotient of $\mathbb{1}_{D^\circ}^{\oplus r}$, which implies that $\mathcal{L}|_{D^\circ}$ is isomorphic to $\mathbb{1}_{D^o}$ (see Corollary 2.7). Consequently, we obtain the dotted arrow in the following commutative diagram:

$$\begin{array}{ccccc} D^\circ & \longrightarrow & U & \xrightarrow{\sigma} & \mathbb{P}(\mathcal{E}_0) \\ \downarrow & & \downarrow & \nearrow & \\ D & \longrightarrow & X. & & \end{array} \qquad (D \dashrightarrow \mathbb{P}(\mathcal{E}_0))$$

In view of Lemma 5.7 below, there is some open neighborhood $x \in V \subset X$ and a morphism $V \longrightarrow \mathbb{P}(\mathcal{E}_0)$ which extends $D \longrightarrow \mathbb{P}(\mathcal{E}_0)$. Therefore, it is possible to find an extension of the morphism

$$\varphi : \mathcal{E}_0|_U \longrightarrow \mathcal{L}_0$$

to V. As x is arbitrary, we arrive at a desired conclusion. □

Lemma 5.7. *Let $(A, \mathfrak{m})$ be a noetherian local ring whose completion $\widehat{A}$ is a domain (note that in this case A is also a domain). If K (respectively, $\widehat{K}$), stands for the fractions field of A (respectively, $\widehat{A}$), and if $h \in K$ is such that its image in $\widehat{K}$ belongs to $\widehat{A}$, then $h \in A$.*

In particular, if R is any ring and $u : R \longrightarrow K$ is a morphism whose image belongs to $\widehat{A}$, then $u(R) \subset A$.

Proof. We write $h = h_1 \cdot h_2^{-1}$, where $h_1 \in A$ and $h_2 \in \mathfrak{m}$ (otherwise $h \in A^\times$ and there is nothing to be done). The fact that $h \in \widehat{A}$ means simply that $\widehat{A}h_2 \supset \widehat{A}h_1$. Since $A \longrightarrow \widehat{A}$ is faithfully flat [Mat89, p. 62, 8.14], we conclude that $\widehat{A}a \cap A = Aa$ for each $a \in A$ [Mat89, p. 49, 7.5(ii)], an hence $Ah_2 \supset Ah_1$. □

Continuing with the proof of Proposition 5.5, we employ Lemma 5.6 to find epimorphisms

$$\widetilde{\varphi}_n : \mathcal{E}_n \longrightarrow \widetilde{\mathcal{L}}_n$$

extending $\varphi_n : \mathcal{E}_n|_U \longrightarrow \mathcal{L}_n$. From the fact that

$$\mathbf{VB}(X) \longrightarrow \mathbf{VB}(U)$$

is fully faithful [SGA2, III.3], the isomorphisms

$$\tau_n : F_U^*(\mathcal{L}_{n+1}) \xrightarrow{\sim} \mathcal{L}_n$$

used in the definition of $\mathcal{L} \in \mathbf{Fdiv}(U)$ can be extended to isomorphisms $\widetilde{\tau}_n : F_X^*\widetilde{\mathcal{L}}_{n+1} \longrightarrow \widetilde{\mathcal{L}}_n$. Another application of the fully faithfulness of $\mathbf{VB}(X) \longrightarrow \mathbf{VB}(U)$ shows that the arrows $\widetilde{\varphi}_n$ give rise to an epimorphisms of $\mathbf{Fdiv}(X)$. □

We can now present our

Proof of Theorem 5.2. We put $G := \mathrm{Gal}(f)$ and set out to verify that condition (2) of Lemma 5.3 is valid. Take $\mathcal{E} \in \mathbf{Fdiv}(X)$, and let $\mathcal{L} \longrightarrow f^\#\mathcal{E}$ be a sub-object of rank one. Let now $\mathcal{I} \subset f^\#\mathcal{E}$ be the image of $\bigoplus_g g^\#\mathcal{L}$ in $f^\#\mathcal{E}$ (this is an object of $\mathbf{Fdiv}(Y)$). Then, there exist an open dense subscheme U of X with pre-image V in Y, an object $\mathcal{M} \in \mathbf{Fdiv}(U)$, and an isomorphism

$$(f|_V)^\#\mathcal{M} \xrightarrow{\sim} \mathcal{I}|_V\,.$$

We now consider the commutative diagram of affine group schemes obtained by choosing $y_0 \in V(k)$ and $x_0 := f(y_0)$:

$$\begin{array}{ccc}
\Pi^{\mathrm{FD}}(V, y_0) & \longrightarrow & \Pi^{\mathrm{FD}}(Y, y_0) \\
\downarrow & & \downarrow \\
\Pi^{\mathrm{FD}}(U, x_0) & \longrightarrow & \Pi^{\mathrm{FD}}(X, x_0).
\end{array}$$

Since the horizontal arrows are all faithfully flat (Proposition 5.5), the subspace $y_0^*(\mathcal{I}) = x_0^*(\mathcal{M})$ remains fixed under the action of $\Pi^{\mathrm{FD}}(X, x_0)$ on $x_0^*(\mathcal{E})$ because it remains fixed under the action of $\Pi^{\mathrm{FD}}(U, x_0)$. It then follows that there exits $\overline{\mathcal{M}} \in \mathbf{Fdiv}(X)$ and an isomorphism

$$f^\#\overline{\mathcal{M}} \xrightarrow{\sim} \mathcal{I}\,.$$

We now observe that $y_0^*(\mathcal{I})$ is a semisimple $\Pi^{\mathrm{FD}}(Y, y_0)$-module, and hence there is an epimorphism of $\Pi^{\mathrm{FD}}(Y, y_0)$-modules:

$$\psi : \mathcal{I} \longrightarrow \mathcal{L}\,.$$

Consequently, $\mathcal{L} = \mathcal{I}m(\psi)$; since ψ is an arrow from $\mathcal{I} = f^\#\overline{\mathcal{M}}$ to $f^\#\mathcal{E}$, Proposition 5.4 tells us that ψ is the image of an arrow $\overline{\mathcal{M}} \longrightarrow \mathcal{E}$, so that $\mathcal{L}$ belongs to the essential image of $f^\#$. This completes the proof of Theorem 5.2. □

Remark 5.8. A more general result than Proposition 5.5 appears as Theorem 3.1 in [TZ17a].

Remark 5.9. It is essential that X be a *normal* variety for Proposition 5.5 to hold. Indeed, pick x_1, x_2 distinct closed points of $\mathbb{P}_k^2$ and let $\psi : \mathbb{P}_k^2 \longrightarrow X$ be the identification of them [Fe03, Theorem 5.4, p.570]. We know that on $\mathbb{P}^2 \setminus \{x_1, x_2\}$, which is thought of as an open and dense subset of X, there are no non-trivial F-divided sheaves (see Proposition 2.7 and then Theorem 2.2 in [Gi75]). On the other hand, we now show that $\Pi^{\mathrm{FD}}(X) \neq 0$ by exhibiting a non-trivial F-divided sheaf of rank one.

Take the trivial line bundle $\mathbb{L} := \mathcal{O}_{\mathbb{P}^2_k}$ and any $\lambda \in k^* = k \setminus \{0\}$. Identify the fiber $\mathbb{L}_{x_1} = k$ with $\mathbb{L}_{x_2} = k$ by sending any $c \in \mathbb{L}_{x_1}$ to $c\lambda \in \mathbb{L}_{x_2}$. This produces a line bundle on X, which we will be denoted by $\mathbb{L}(\lambda)$. Then we have $F^*\mathbb{L}(\lambda) = \mathbb{L}(\lambda^p)$, where p is the characteristic of k. Since k is perfect, the line bundle $\mathbb{L}(\lambda)$ admits an F-divided structure. On the other hand, $\mathbb{L}(\lambda)$ is nontrivial if $\lambda \neq 1$.

Part II – The essentially finite fundamental group scheme

6. Extending the Tannakian interpretation of the essentially finite fundamental group

In this section we set out to extend Nori's theory of essentially finite vector-bundles [No76] to a slightly larger class of k-algebraic schemes other than the proper ones. It should be noted that such an enterprise has received attention from other geometers in recent times (cf. [BV15] and [TZ17b]) and our modest contribution is to throw light on a variation of Nori's initial method: connect points via projective curves. It is a simple alternative to the more formal condition on global sections of vector bundles [BV15, Definition 7.1].

Definition 6.1. Let X be an algebraic k-scheme. A chain of proper curves on X is a family of morphisms from proper curves $\{\gamma_i : C_i \longrightarrow X\}_{i=1}^m$ such that the associated closed subset $\cup_i \operatorname{Im}(\gamma_i)$ is connected. For the sake of brevity, we shall refer to the closed subset $\cup_i \operatorname{Im}(\gamma_i)$ as "a chain of projective curves".

The algebraic k-scheme is said to be connected by proper chains (CPC for short) if any two points belong to a chain of proper curves.

As Ramanujam's Lemma guarantees (see the Lemma of Section 6, p. 56 of [Mu]), each proper k-scheme is connected by proper chains (CPC). Another easily available way to produce CPC algebraic schemes is to note:

Lemma 6.2. *Let $f : Y \longrightarrow X$ be a surjective morphism of algebraic k-schemes. Then, if Y is CPC, is X likewise.* □

Also, with a bit more work, we can say:

Lemma 6.3. *Let $U \longrightarrow X$ be an open embedding of algebraic k-schemes. Assume that X is projective and that U is big in X. Then, if U is connected, it is CPC.*

Proof. We give a proof by assuming U irreducible; the general case can easily be obtained from this. As U is assumed to be dense in X, this implies that X is also irreducible. Let d stand for the dimension of X. If $d = 1$, then U_{red} is a proper curve, so that nothing needs to be demonstrated. We now proceed by induction on d, which is then ≥ 2.

Let us denote by Z the reduced closed subscheme of X whose underlying topological space is $X \setminus U$. Given distinct points x, y of U, let $\varphi : X' \longrightarrow X$ stand for the blowup of $\{x, y\}$; the inverse image of U (respectively, Z) is denoted by U' (respectively, Z'). Clearly, $\varphi : Z' \longrightarrow Z$ is an isomorphism and $\operatorname{codim}(Z', X') \geq 2$. This being so, U' is big in X'. Let us now choose a closed immersion $X' \longrightarrow \mathbb{P}^N$ and then a hyperplane $H \subset \mathbb{P}^N$ enjoying the following properties

a) both and $H \cap U'$ and $H \cap X'$ are irreducible of dimension $d - 1$. (Apply (1) and (3) of [Jou83, Corollary 6.11].)

b) The intersections $H \cap \varphi^{-1}(x)$ and $H \cap \varphi^{-1}(y)$ are non-empty. (Apply (1b) of loc. cit.)
c) If $W' \subset Z'$ has dimension zero, then $H \cap W' = \emptyset$, and if $W' \subset Z'$ is an irreducible component of dimension ≥ 1, we have $\dim H \cap W' = \dim Z' - 1$. (Apply (1a) and then (3) and (1b) of loc. cit.)

We then see that $H \cap U' \subset H \cap X'$ is big and of dimension $d-1$. Hence, if $x' \in H \cap \varphi^{-1}(x)$ and $y' \in H \cap \varphi^{-1}(y)$, there exists a chain of proper curves containing them both and a fortiori there exits a chain of proper curves containing $x = \varphi(x')$ and $y = \varphi(y')$. □

Definition 6.4. Let $\mathcal{E}$ be a vector-bundle on an algebraic k-scheme X. We say that $\mathcal{E}$ is Nori-semistable if, for each morphism from a smooth projective curve $\gamma : C \longrightarrow X$, the vector-bundle $\gamma^*\mathcal{E}$ is semistable and of degree zero. The category of Nori-semistable vector-bundles on X is the full subcategory of $\mathbf{VB}(X)$ having as objects the Nori-semistable ones. This category will be denoted by $\mathbf{NS}(X)$.

Remark 6.5. It is not (yet) universally accepted to call the above defined vector-bundles *Nori-semistable*. This is because in [No76], the author introduces a slightly different category.

The next result is a straightforward application of Nori's method in [No76, § 3].

Lemma 6.6. *Let X be a reduced algebraic k-scheme which is connected by proper chains. Let $\varphi : \mathcal{E} \longrightarrow \mathcal{F}$ be a morphism of $\mathbf{NS}(X)$. Then, both $\mathcal{K}er(\varphi)$ and $\mathcal{I}m(\varphi)$ are vector-bundles. Furthermore, they are Nori-semistable.*

Proof. Let $\gamma : C \longrightarrow X$ be a morphism from a *smooth* proper curve and let x, y be points of C. Since $\gamma^*\mathcal{E}$ and $\gamma^*\mathcal{F}$ are semistable vector-bundles on C, we know that

$$\operatorname{rank} \gamma^*(\varphi)(x) = \operatorname{rank} \gamma^*(\varphi)(y)$$

(this is a simple exercise [Se82, Proposition 8, p. 18]). But one easily shows that $\gamma^*(\varphi)(x) = \varphi(\gamma(x))$ and $\gamma^*(\varphi)(y) = \varphi(\gamma(y))$, so that the function $u \longmapsto \operatorname{rank} \varphi(u)$ is constant on $\operatorname{Im}(\gamma)$. Now, if instead of assuming C to be smooth we simply take it to be projective, the same conclusion can be achieved by considering the normalization.

Since by definition any two points of X can be joined by a chain of projective curves, $u \longmapsto \operatorname{rank} \varphi(u)$ is allover constant. Consequently, $\mathcal{C}oker(\varphi)$ is a vector-bundle [Har77, Exercise 5.8, Chapter II] and employing the exact sequence

$$0 \longrightarrow \mathcal{I}m(\varphi) \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}oker(\varphi) \longrightarrow 0\,,$$

we see that $\mathcal{I}m(\varphi)$ is a vector-bundle. An analogous reasoning shows that $\mathcal{K}er(\varphi)$ is a vector-bundle. □

Theorem 6.7. *Let X be a reduced algebraic k-scheme which is connected by proper chains. Then, once a point $x_0 \in X(k)$ is chosen, the category $\mathbf{NS}(X)$, together with the functor $x_0^* : \mathbf{NS}(X) \longrightarrow k$-$\mathbf{mod}$, is neutral Tannakian.*

Definition 6.8. The affine group scheme associated to $\mathbf{NS}(X)$ via the fibre functor x_0^* is denoted by $\pi^{\mathrm{S}}(X, x_0)$.

The category $\mathbf{NS}(X)$ is rather large and its understanding is less sound than that of its "largest pro-finite quotient", now our main topic of interest. In order to present the theory in a different light, we shall make a brief digression which is certainly well-known to the cognoscenti (see the discussion on page 331 of [BV15]).

Let $\mathcal{T}$ be a small tensor category over k; its set of isomorphisms classes carries an evident structure of commutative semi-ring, with addition and multiplication constructed from direct sums and tensor products. Let $\mathrm{K}(\mathcal{T})$ stand for the associated commutative ring [BA, I.2.4]. We say that $V \in \mathcal{T}$ is *finite* if its class in $\mathrm{K}(\mathcal{T})$ is integral over the prime sub-ring (which is $\mathbb{Z}$). By standard knowledge from the theory of integral extensions [Mat89, Theorem 9.1], we see that the set of finite objects is stable by tensor products and direct sums. With a bit more effort, one sees also that the dual of a finite object is necessarily finite. We then define the *essentially finite category*, $\mathbf{EF}(\mathcal{T})$, as the *full* subcategory of $\mathcal{T}$ having

$$\left\{ V \in \mathcal{T} : \begin{array}{c} \text{there exists a finite object } \Phi \text{ and sub-objects} \\ V'' \subset V' \subset \Phi \text{ such that } V = V'/V'' \end{array} \right\}$$

for set of objects. One easily sees that $\mathbf{EF}(\mathcal{T})$ is an abelian subcategory which is stable by tensor products and duals.

Proposition 6.9. *Let G be an affine group scheme over k and $\omega : \mathrm{Rep}_k(G) \longrightarrow k$-$\mathbf{mod}$ the forgetful functor. If ω^{EF} denotes the restriction of ω to the the essentially finite category, then $\mathrm{Aut}^{\otimes}(\omega^{\mathrm{EF}})$ is a pro-finite affine group scheme, and the composite morphism*

$$G \stackrel{\sim}{\longrightarrow} \mathrm{Aut}^{\otimes}(\omega) \longrightarrow \mathrm{Aut}^{\otimes}(\omega^{\mathrm{EF}})$$

is universal [ML71, III.1] *from G to the category of pro-finite affine group schemes. In particular, it is faithfully flat.*

Sketch of proof. If H is a finite group scheme over k and R is its right-regular representation, then $R \otimes R \simeq R^{\oplus r}$. Since any representation of H is a sub-object of a certain direct sum $R \oplus \cdots \oplus R$ [Wa79, 3.5, Lemma], we conclude that $\mathbf{EF}(\mathrm{Rep}_k(H)) = \mathrm{Rep}_k(H)$. The case of a profinite group scheme $\mathfrak{H}$ is proved in the same manner since any representation $\mathfrak{H} \longrightarrow \mathbf{GL}_n$ must factor through a finite quotient $\mathfrak{H} \longrightarrow H$. □

Given Proposition 6.9, we can put forward the

Definition 6.10. The category of essentially finite vector-bundles is, in the above notation, $\mathbf{EF}(\mathbf{NS}(X))$. We shall abuse terminology and write $\mathbf{EF}(X)$ instead. If $x_0 \in X(k)$, then $\pi^{\mathrm{EF}}(X, x_0)$ stands for the pro-finite group scheme constructed from Proposition 6.9.

Let us end this section with comments on simple structural results which shall prove useful further ahead. Let X be a CPC variety and $\mathcal{V}$ an essentially finite vector bundle on it. We then derive the existence of a finite group scheme G, a representation $V \in \mathrm{Rep}_k(G)$ and a G-torsor $P \to X$ such that $P \times^G V = \mathcal{V}$. Now, if $V^{(1)}$ is the Frobenius twist of V [Jan87, Part I, 2.16], we know that $F^*(P \times^G V) \simeq P \times^G V^{(1)}$. Hence, for a certain $h \in \mathbb{N}$, the vector bundle $F^{*h}(\mathcal{V})$ is of the form $P^{\mathrm{et}} \times^{G^{\mathrm{et}}} V^{(h)}$ for a certain etale covering $P^{\mathrm{et}} \longrightarrow X$. The least integer h enjoying this property is to be called the *height* of $\mathcal{V}$.

7. The essentially finite fundamental group scheme of a quotient: the case of a free action

This section functions as did Section 3 and we show that the exact analogue of Proposition 3.3 holds true:

Proposition 7.1. *Let $f : Y \longrightarrow X$ be a finite-Galois etale covering of CPC varieties over k. Let $G := \mathrm{Gal}(f)$. We then have exact sequences of affine group schemes*

$$1 \longrightarrow \pi^{\mathrm{S}}(Y, y_0) \longrightarrow \pi^{\mathrm{S}}(X, x_0) \longrightarrow G \longrightarrow 1$$

and

$$1 \longrightarrow \pi^{\mathrm{EF}}(Y, y_0) \longrightarrow \pi^{\mathrm{EF}}(X, x_0) \longrightarrow G \longrightarrow 1\,.$$

Proof. We start by observing that if $\mathcal{E} \in \mathbf{NS}(Y)$, then

(a) $f_*\mathcal{E}$ belongs to $\mathbf{NS}(X)$ and,
(b) $f^*f_*\mathcal{E}$ lies in $\mathbf{EF}(Y)$.

(One *copies* the proof of [EHS08, Lemma 2.8]). Hence, $f^* : \mathbf{NS}(X) \longrightarrow \mathbf{NS}(Y)$ has an exact and faithful right adjoint $f_* : \mathbf{NS}(Y) \longrightarrow \mathbf{NS}(X)$ such that f^*f_* takes values in $\mathbf{EF}(Y)$. Clearly, $f^*f_*(\mathcal{O}_Y)$ is trivial (isomorphic to $\mathcal{O}_Y \otimes_k \mathcal{O}(G)$) and the twisting functor $V \longrightarrow Y \times^G V$ defines an equivalence between $\mathrm{Rep}_k(G)$ and $\{\mathcal{V} \in \mathbf{NS}(X) : f^*\mathcal{V} \text{ is trivial}\}$; an inverse is $\mathcal{V} \longmapsto H^0(Y, f^*(\mathcal{V}))$. We then conclude, applying Proposition 3.4, that $\pi^{\mathrm{S}}(Y, y_0) \longrightarrow \pi^{\mathrm{S}}(X, x_0)$ is a normal closed immersion and that the cokernel is isomorphic to G.

To prove that the sequence concerning essentially finite fundamental group schemes is exact, we only need minor adjustments: first we note that if $\mathcal{F} \in \mathbf{NS}(X)$ is such that $f^*(\mathcal{F}) \in \mathbf{EF}(Y)$, then $\pi^{\mathrm{S}}(Y, y_0) \longrightarrow \mathbf{GL}(x_0^*\mathcal{F})$ has a finite image, which proves that $\pi^{\mathrm{S}}(X, x_0) \longrightarrow \mathbf{GL}(x_0^*\mathcal{F})$ has finite image, and this means that $\mathcal{F}$ actually belongs to $\mathbf{EF}(X)$. Consequently, because of (b) above, f_* takes objects in $\mathbf{EF}(Y)$ to objects in $\mathbf{EF}(X)$, which allows us to see that the requirements of Proposition 3.4 are fulfilled. □

8. The essentially finite fundamental group of a quotient

In this section we wish to prove the following theorem.

Theorem 8.1. *Let Y be a normal CPC variety. Let G be a finite group of automorphisms of Y, and write*

$$f : Y \longrightarrow X$$

for the quotient of Y by G [Mu, § 7, Theorem]. *Choose $y_0 \in Y(k)$ above $x_0 \in X(k)$. Then, the cokernel of the induced homomorphism*

$$f_\natural : \pi^{\mathrm{EF}}(Y, y_0) \longrightarrow \pi^{\mathrm{EF}}(X, x_0)$$

is identified with G/I, where I is the subgroup (necessarily normal) generated by all elements of G having at least one fixed point.

The *proof* of Theorem 8.1 follows precisely the same path as that of Theorem 5.1, except that in place of Theorem 5.2 and Proposition 3.3 (the etale case), we apply Theorem 8.2 below and Proposition 7.1. We shall then concentrate on proving Theorem 8.2.

Theorem 8.2. *Let $f : Y \longrightarrow X$ be a genuinely ramified morphism between CPC normal varieties taking the k-point $y_0 \in Y(k)$ to the k-point $x_0 \in X(k)$. Then*

$$f_\natural : \pi^{\mathrm{EF}}(Y, y_0) \longrightarrow \pi^{\mathrm{EF}}(X, x_0)$$

is a quotient morphism.

Proof. We employ the criterion explained by Lemma 5.3, so that it is enough to show that for each $\mathcal{E} \in \mathbf{EF}(X)$, the natural morphism

$$H^0(\mathcal{E}) \longrightarrow H^0(f^*\mathcal{E})$$

is bijective. We begin by proving the lemma in the special case where $\mathcal{E}$ is of height zero. This being so, let $\mathcal{E} = T \times^{\mathfrak{g}} E$, where $\mathfrak{g}$ is finite and etale, $T \longrightarrow X$ is a *connected* $\mathfrak{g}$-torsor, and E is a representation of $\mathfrak{g}$. Then, by definition,

$$H^0(X, T \times^{\mathfrak{g}} E) = \{\mathfrak{g}\text{-equivariant morphisms } \varphi : T \longrightarrow E_a\}.$$

Using proper chains in X, we see that for any $\varphi \in H^0(X, T \times^{\mathfrak{g}} E)$ and any two points t_1, t_2 in T, $\varphi(t_1)$ and $\varphi(t_2)$ lie on the same $\mathfrak{g}$-orbit. As T is connected, this is impossible unless $\varphi(t_1) = \varphi(t_2)$ and we see that $E^{\mathfrak{g}} = H^0(X, T \times^{\mathfrak{g}} E)$. Since $f^{-1}T \longrightarrow Y$ is also connected (by Proposition 4.3), the same argument shows that $E^{\mathfrak{g}} = H^0(Y, f^*(T \times^{\mathfrak{g}} E))$, so that the proof of the special case is complete.

We now introduce the following set of vector-bundles on X:

$$\mathfrak{S} := \left\{ \mathcal{V} : \begin{array}{c} \text{for any } \tau \in H^0(f^*\mathcal{V}), \text{ its image } F_Y^*(\tau) \text{ in } H^0(F_Y^* f^* \mathcal{V}) \\ \text{belongs to the image of } f^* : H^0(F_X^*\mathcal{V}) \longrightarrow H^0(F_Y^* f^* \mathcal{V}) \end{array} \right\}.$$

We note that if $\mathcal{V} \in \mathbf{EF}(X)$ is of height one—in the sense of the discussion after Definition 6.10—then $\mathcal{V} \in \mathfrak{S}$ because in this case $F_X^*\mathcal{V}$ is of height zero and therefore $H^0(F_X^*\mathcal{V}) \xrightarrow{\sim} H^0(f^* F_X^* \mathcal{V})$. This last observation, together with a simple induction argument, shows that the lemma is a consequence of the:

Claim. If $\mathcal{E} \in \mathfrak{S}$, then the pull-back $H^0(\mathcal{E}) \longrightarrow H^0(f^*\mathcal{E})$ is bijective.

Let then τ be a section of $f^*\mathcal{E}$; by assumption, we have

$$F_Y^*(\tau) = f^*(\sigma), \quad \text{with } \sigma \in H^0(F_X^*\mathcal{E}). \tag{8.1}$$

If $X_0 \subset X$ stands for the smooth locus of X (so that $\operatorname{codim}(X \setminus X_0; X) \geq 2$), we contend that

$$\sigma|_{X_0} = F_{X_0}^*(\rho), \quad \text{for some } \rho \in H^0(\mathcal{E}|_{X_0}). \tag{8.2}$$

Let ∇ be the canonical connection on $F_X^*(\mathcal{E})|_{X_0} = F_{X_0}^*(\mathcal{E}|_{X_0})$ [Ka70, Theorem 5.1, p. 370]. Denote by X_1 and open dense subset of X such that the restriction of f to $Y_1 := f^{-1}(X_1)$ is etale. Then,

$$\nabla(\sigma|_{X_1}) = 0$$

since $f^*(\sigma|_{X_1}) = F_Y^*(\tau)|_{Y_1}$ and $f^*\Omega^1_{X_1} \simeq \Omega^1_{Y_1}$. Hence, $\nabla(\sigma|_{X_0}) = 0$, and Cartier's theorem [Ka70, Theorem 5.1, p.370] guarantees the existence of ρ as in (8.2). Since X is normal, ρ extends to $\overline{\rho} \in H^0(\mathcal{E})$ [SGA2, III, Corollary 3.5], and (8.2) gives

$$\sigma|_{X_0} = F_{X_0}^*(\overline{\rho}|_{X_0}). \tag{8.3}$$

From this point, letting $Y_0 = f^{-1}(X_0)$, equations (8.1) and (8.3) show that

$$F_Y^*(\tau)|_{Y_0} = F_Y^*(f^*(\overline{\rho}))|_{Y_0}. \tag{8.4}$$

As $Y_0 \subset Y$ is dense in the variety Y, equation (8.4) guarantees that

$$F_Y^*(\tau) = F_Y^*(f^*(\overline{\rho})).$$

As $F_Y^* : H^0(\mathcal{V}) \longrightarrow H^0(F_Y^*\mathcal{V})$ is always injective for a vector-bundle $\mathcal{V}$, we conclude that $f^*(\overline{\rho}) = \tau$. This finishes the proof of the Claim. □

Remark 8.3. The reader should compare Theorem 8.2 with Theorem II of [TZ17b].

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