A Sternberg theorem for nonautonomous differential equations

L.V. Cuong[*], T.S. Doan[†] and S. Siegmund[‡]

Dedicated to the memory of George Sell

Abstract

We show that a hyperbolic nonautonomous differential equation can be smoothly linearized if the associated Sacker-Sell spectrum satisfies a non-resonance condition. This result extends the classical Sternberg theorem to nonautonomous differential equations.

1 Introduction

The Hartman-Grobman theorem for nonautonomous differential equations [Pal73] states that if f is a C^0-small map from $\mathbb{R}\times\mathbb{R}^d$ into $\mathbb{R}^d$ and also small Lipschitzian with respect to the second argument, then there is a continuous family of homeomorphisms $H(t,\cdot)$, $t\in\mathbb{R}$, of $\mathbb{R}^d$, with $\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^d}\|H(t,x)-x\|<\infty$ and sending solutions $t\mapsto\mu(t)$ of the differential equation

$$\dot{x} = A(t)x + f(t,x), \tag{1}$$

onto solutions $t\mapsto H(t,\mu(t))$ of the linear differential equation

$$\dot{x} = A(t)x, \tag{2}$$

[*]kalkolai@gmail.com, Department of Information Technology, National University of Civil Engineering, 55 Giai Phong str., Hanoi, Vietnam

[†]dtson@math.ac.vn, Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Ha Noi, Viet Nam

[‡]stefan.siegmund@tu-dresden.de, Center for Dynamics, Department of Mathematics, Technische Universität Dresden, Zellescher Weg 12-14, 01069 Dresden, Germany

and vice versa with its inverse $H^{-1}(t,\cdot)$ with respect to the second argument. In this paper we show that $H(t,\cdot)$, $t \in \mathbb{R}$, can be chosen as a C^ℓ diffeomorphism for any given $\ell \in \mathbb{N}$, provided the linear differential equation (2) satisfies a non-resonance condition which is formulated in terms of the Sacker-Sell spectrum, the larger the ℓ the more non-resonance conditions have to be satisfied (Theorem 5). This result generalizes the classical Sternberg theorem [Ste57, Ste58, Bru95] to nonautonomous differential equations.

More precisely, for $k \in \mathbb{N}$ we consider C^k Carathéodory differential equations of the form (1), with a locally integrable matrix function $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ and a C^k Carathéodory function $f\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, i.e. for a.a. $t \in \mathbb{R}$ $f(t,\cdot)$ is continuous, for all $x \in \mathbb{R}^d$ $f(\cdot,x)$ is measurable, for a.a. $t \in \mathbb{R}$ and all $x \in \mathbb{R}^d$ the partial derivative $D_x^k f(t,x)$ exists and $D_x^j f$ is a Carathéodory function for every $j \in \{1,\dots,k\}$ [Sie02a, Definition 2.1]. In order to linearize (1) smoothly along its trivial solution we assume the following conditions throughout the paper.

(A1) (1) is Taylor expanded along its trivial solution:

$$f(t,0) = 0 \quad \text{and} \quad D_x f(t,0) = 0 \qquad \text{for a.a. } t \in \mathbb{R}.$$

(A2) Linearity: (2) has bounded growth [Cop78], i.e. with the evolution operator $\Phi\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{d\times d}$ of (2) there exist $K, a > 0$ such that

$$\|\Phi(t,s)\| \le K e^{a|t-s|} \qquad \text{for all } t,s \in \mathbb{R}.$$

(A3) Nonlinearity: There exists $M > 0$ such that

$$\|D_x^j f(t,x)\| \le M \qquad \text{for } j = 0,\dots,k, \text{ a.a. } t \in \mathbb{R} \text{ and all } x \in \mathbb{R}^d.$$

The global boundedness condition (A3) can be achieved by cutting of outside of a tubular neighorhood $\mathbb{R} \times B_r(0)$, $r > 0$, of the trivial solution or can be replaced by a local condition (see e.g. [Sie02a] or [BDDS15, Example 6]).

The existence of a smooth linearization of (1) will depend on resonances of uniform exponential growth rates of solutions of its linearization (2) which can be formulated with the *Sacker Sell* or *dichotomy spectrum* [SS78, Sie02b].

Definition 1 (Sacker Sell spectrum)**.**

(a) Invariant projector: An *invariant projector of* (2) is defined to be a function $P : \mathbb{R} \to \mathbb{R}^{d\times d}$ of projections $P(t)$, $t \in \mathbb{R}$, such that

$$P(t)\Phi(t,s) = \Phi(t,s)P(s) \qquad \text{for } t,s \in \mathbb{R}.$$

Note that P is continuous due to the identity $P(t) = \Phi(t,s)P(s)\Phi(s,t)$.

(b) Exponential dichotomy: (2) admits an *exponential dichotomy* (ED) if there is an invariant projector P and constants $K \geq 1$ and $\alpha > 0$ such that

$$\begin{aligned} \|\Phi(t,s)P(s)\| &\leq Ke^{-\alpha(t-s)} && \text{for } t \geq s, \\ \|\Phi(t,s)[I - P(s)]\| &\leq Ke^{\alpha(t-s)} && \text{for } t \leq s. \end{aligned}$$

(c) Sacker Sell spectrum: The *Sacker Sell spectrum* of (2) is the set

$$\Sigma_{\mathrm{SS}}(A) := \Big\{\gamma \in \mathbb{R} \colon \dot{x} = [A(t) - \gamma I]x \text{ does not admit an ED}\Big\},$$

where $I \in \mathbb{R}^{d\times d}$ denotes the identity matrix.

Example 2. If (2) is autonomous and γ is not in $\Sigma_{\mathrm{SS}}(A)$, then the projector $P : \mathbb{R} \to \mathbb{R}^{d\times d}$ of an ED of $\dot{x} = [A - \gamma I]x$ is a time-independent projection $P \equiv P(t)$ that yields the decomposition $\mathbb{R}^d = \operatorname{im} P \oplus \ker P$ where $\operatorname{im} P$ is the sum of all generalized eigenspaces corresponding to those eigenvalues of A with real part less than γ and $\ker P$ is the sum of all generalized eigenspaces corresponding to the eigenvalues of A with real part greater than γ. The Sacker Sell spectrum $\Sigma_{\mathrm{SS}}(A)$ is then given by

$$\Sigma_{\mathrm{SS}}(A) = \{\operatorname{Re}\lambda \colon \lambda \text{ is an eigenvalue of } A\}.$$

Sacker and Sell [SS78] proved that $\Sigma_{\mathrm{SS}}(A)$ is a non-empty union of at most d compact intervals (see also [Sie02b] for systems (2) which do not necessarily satisfy (A2)).

Theorem 3 (Sacker Sell spectrum and block diagonalization)**.** There exist $n \in \{1, \dots, d\}$ and $\underline{\lambda}_n \leq \overline{\lambda}_n < \cdots < \underline{\lambda}_2 \leq \overline{\lambda}_2 < \underline{\lambda}_1 \leq \overline{\lambda}_1$ such that with $\lambda_i := [\underline{\lambda}_i, \overline{\lambda}_i]$

$$\Sigma_{\mathrm{SS}}(A) = \bigcup_{i=1}^{n} \lambda_i.$$

Moreover, there exists an absolutely continuous function $S \colon \mathbb{R} \to \mathbb{R}^{d\times d}$ of invertible matrices $S(t)$ with $\sup_{t\in\mathbb{R}} \|S(t)\|, \sup_{t\in\mathbb{R}} \|S(t)^{-1}\| < \infty$ such that $y(t) = S(t)x(t)$ transforms (2) into a block diagonal system

$$\dot{y} = \operatorname{diag}(A_1(t), \dots, A_n(t))y \tag{3}$$

with locally integrable matrix functions $A_i\colon \mathbb{R} \to \mathbb{R}^{d_i \times d_i}$, $d_1 + \cdots + d_n = d$ and

$$\Sigma_{\mathrm{SS}}(A_i) = \lambda_i.$$

S is called *kinematic similarity* between (2) and (3). (2), and equivalently (3), are called *hyperbolic* if $0 \notin \Sigma_{\mathrm{SS}}(A)$.

Proof. See, e.g. [Sie02b, Spectral Theorem] and [Sie02c, Theorem 3.2]. $\square$

We will linearize (1) smoothly by first transforming (1) with a kinematic similarity from Theorem 3 which will block diagonalize the linear part and then composing a finite number of smooth transformations each of which preserves the linear part but simplifies the nonlinearity until we arrive at (3), then the inverse of the kinematic similarity transforms (3) to (2).

For $k \in \mathbb{N}$ and a locally integrable matrix function $A : \mathbb{R} \to \mathbb{R}^{d \times d}$ satisfying (A2) we define the set of admissible differential equations

$$\begin{aligned} \mathcal{O}^k(A) := \{\dot{x} = A(t)x + f(t,x)\colon f \text{ is a } C^k \text{ Carathéodory} \\ \text{function which satisfies (A1) and (A3)}\} \end{aligned}$$

and a notion of local equivalence between systems in $\mathcal{O}^k(A)$ (see e.g. [Pal73, Sie02a] for a generalization).

Definition 4 (C^k equivalence)**.** Two systems

$$\dot{x} = A(t)x + f(t,x) \tag{4}$$

and

$$\dot{y} = A(t)y + g(t,y) \tag{5}$$

in $\mathcal{O}^k(A)$ are called C^k *equivalent*, if there exist $p, \widetilde{p} > 0$ and $r \in (0,p), \widetilde{r} \in (0, \widetilde{p})$ together with continuous functions

$$H\colon \mathbb{R} \times B_r(0) \to \mathbb{R}^d \qquad \text{and} \qquad H^{-1}\colon \mathbb{R} \times B_{\widetilde{r}}(0) \to \mathbb{R}^d,$$

called *(local)* C^k *equivalence between* (4) *and* (5) and its inverse, respectively, such that the following statements hold:

(i) For each $t \in \mathbb{R}$ the mappings

$$\begin{aligned} H(t,\cdot)\colon B_r(0) \to H(t, B_r(0)) \subset B_{\widetilde{p}}(0) \\ H^{-1}(t,\cdot)\colon B_{\widetilde{r}}(0) \to H^{-1}(t, B_{\widetilde{r}}(0)) \subset B_p(0) \end{aligned}$$

are C^k diffeomorphisms with

$$H(t, H^{-1}(t,y)) = y \quad \text{for all } y \in B_{\widetilde{r}}(0) \text{ with } H^{-1}(t,y) \in B_r(0),$$
$$H^{-1}(t, H(t,x)) = x \quad \text{for all } x \in B_r(0) \text{ with } H(t,x) \in B_{\widetilde{r}}(0).$$

(ii) If ν is a solution of (4) in $B_r(0)$ then $H(\cdot, \nu(\cdot))$ is a solution of (5). If $\widetilde{\nu}$ is a solution of (5) in $B_{\widetilde{r}}(0)$ then $H^{-1}(\cdot, \widetilde{\nu}(\cdot))$ is a solution of (4).

(iii) The trivial solutions are mapped uniformly onto each other:

$$\lim_{x\to 0} H(t,x) = \lim_{x\to 0} H^{-1}(t,x) = 0 \qquad \text{uniformly in } t \in \mathbb{R}.$$

If (2) is hyperbolic, i.e. if $0 \notin \Sigma_{\mathrm{SS}}(A)$, then all systems in $\mathcal{O}^1(A)$ are topologically equivalent (C^0 equivalent) to (3), see [Pal73]. Our main result is that if (2) is hyperbolic, then for any given $\ell \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ (which depends on ℓ and the Sacker Sell spectrum $\Sigma_{\mathrm{SS}}(A)$) such that the following implication holds: $\langle$ (2) is non-resonant up to order k $\rangle \Rightarrow \langle$ all systems in $\mathcal{O}^{k+2}(A)$ are C^ℓ equivalent to (3) $\rangle$.

Theorem 5 (Sternberg theorem for nonautonomous differential equations)**.** Let $\ell \in \mathbb{N}$ and let $\Sigma_{\mathrm{SS}}(A) = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_n$ denote the Sacker Sell spectrum of the linear system (2). Assume that $0 \notin \Sigma_{\mathrm{SS}}(A)$. Then with

$$\Sigma_s := \Sigma_{\mathrm{SS}}(A) \cap \mathbb{R}_{<0} \qquad \text{and} \qquad \Sigma_u := \Sigma_{\mathrm{SS}}(A) \cap \mathbb{R}_{>0}$$

there exists a minimal $k \in \mathbb{N}$, $k \geq \ell$, such that

$$(k-\ell)\Sigma_u > \Sigma_u - \ell\Sigma_s \qquad \text{and} \qquad (k-\ell)\Sigma_s < \Sigma_s - \ell\Sigma_u. \tag{6}$$

If the non-resonance condition

$$\lambda_j \cap \sum_{i=1}^n k_i\lambda_i = \emptyset \qquad \text{for all } (k_1,\dots,k_n) \in \mathbb{N}_0^n \text{ with } 2 \leq \sum_{i=1}^n k_i \leq k \tag{7}$$

holds, then all systems in $\mathcal{O}^{k+2}(A)$ are C^ℓ equivalent to their linearization (3).

The structure of the paper is as follows: In Section 2, we establish a result on flatting invariant manifolds and eliminating non-resonant Taylor terms for systems in $\mathcal{O}^{k+2}(A)$. Section 3 is devoted to study discrete-time systems associated with systems in $\mathcal{O}^{k+2}(A)$. Using preparatory results in Section 2 and Section 3, the main result of the paper (Theorem 5) is proved in Section 4. An important technique in the proof of the main theorem, namely the method of path, is established in the Appendix.

2 Flattening invariant manifolds and eliminating non-resonant Taylor terms

Let $A\colon \mathbb{R} \to \mathbb{R}^{d\times d}$ be locally integrable with $0 \notin \Sigma_{\mathrm{SS}}(A)$. By Theorem 3 there exists a kinematic similarity $S\colon \mathbb{R} \to \mathbb{R}^{d\times d}$ such that $y(t) = S(t)x(t)$ transforms (2) into a block diagonal system (3). Since

$$H\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \quad (t,x) \mapsto H(t,x) := S(t)x,$$

satisfies conditions (i), (ii) and (iii) of Definition 4 with inverse $H^{-1}(t,\cdot) = S(t)^{-1}$, we can assume w.l.o.g. that

$$A(t) = \operatorname{diag}(A_1(t), \dots, A_n(t))$$

with $\Sigma_{\mathrm{SS}}(A_i) = \lambda_i$ for $i = 1, \dots, n$, $\Sigma_{\mathrm{SS}}(A) = \cup_{i=1}^n \lambda_i$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. With the unique $m \in \{0, \dots, n\}$ such that

$$\lambda_1 > \cdots > \lambda_m > 0 > \lambda_{m+1} > \cdots > \lambda_n,$$

define

$$\begin{aligned} A^u(t) &:= \operatorname{diag}(A_1(t), \dots, A_m(t)) \in \mathbb{R}^{d_u \times d_u}, \\ A^s(t) &:= \operatorname{diag}(A_{m+1}(t), \dots, A_n(t)) \in \mathbb{R}^{d_s \times d_s}, \end{aligned} \tag{8}$$

where $d_u + d_s = d$, and if $m = 0$ then $d_u = 0$, $A^s = A$, if $m = d$ then $d_s = 0$, $A^u = A$. Let π_u and π_s denote the canonical projections of $\mathbb{R}^d = \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}$ into $\mathbb{R}^{d_u}$ and $\mathbb{R}^{d_s}$, respectively. Rewriting a differential equation (1) in $\mathcal{O}^k(A)$ in (x^u, x^s)-coordinates, with $x^u := \pi_u x$, $x^s := \pi_s x$, gives

$$\begin{aligned} \dot{x}^u &= A^u(t)x^u + \pi_u f(t, x^u, x^s), \\ \dot{x}^s &= A^s(t)x^s + \pi_s f(t, x^u, x^s). \end{aligned} \tag{9}$$

Let $\varphi(\cdot, t_0, x_0^u, x_0^s)$ denote the solution of the initial value problem (9), $x(t_0) = (x_0^u, x_0^s)$. Then nonautonomous invariant manifold theory (also called integral manifold theory) yields *unstable* and *stable manifolds*

$$\mathcal{U} = \Big\{(t, x^u, x^s)\colon \lim_{s\to-\infty} \varphi(s, t, x^u, x^s) = 0\Big\}$$
$$\mathcal{S} = \Big\{(t, x^u, x^s)\colon \lim_{s\to\infty} \varphi(s, t, x^u, x^s) = 0\Big\}$$

of the zero solution of (9), respectively (see e.g. [AW96] or [Sie99]). We say that $\mathcal{U}$ and $\mathcal{S}$ are *flat*, if they coincide with the unstable and stable manifolds of the linearization

$$\begin{aligned} \dot{x}^u &= A^u(t)x^u, \\ \dot{x}^s &= A^s(t)x^s, \end{aligned}$$

which are the subspaces (or, more precisely, the trivial vector bundles over the base space $\mathbb{R}$) $\mathbb{R} \times \mathbb{R}^{d_u} \times \{0\} \subseteq \mathbb{R}^{1+d}$ and $\mathbb{R} \times \{0\} \times \mathbb{R}^{d_s} \subseteq \mathbb{R}^{1+d}$, respectively. Using the invariance of $\mathcal{U}$ and $\mathcal{S}$, one can characterize flatness by

$$\begin{aligned}\mathcal{U} = \mathbb{R} \times \mathbb{R}^{d_u} \times \{0\} &\Leftrightarrow \pi_s f(t, x^u, 0) = 0 \text{ for all } (t, x^u) \in \mathbb{R} \times \mathbb{R}^{d_u},\\ \mathcal{S} = \mathbb{R} \times \{0\} \times \mathbb{R}^{d_u} &\Leftrightarrow \pi_u f(t, 0, x^s) = 0 \text{ for all } (t, x^s) \in \mathbb{R} \times \mathbb{R}^{d_s}.\end{aligned}$$

We refer the reader to [BDDS15, Theorem 8] for a construction of a C^k equivalence between an $\mathcal{O}^{k+2}(A)$ system (9) and a system with flat stable and unstable manifolds in case $d_u = d_s = 1$. We will use the same construction for not necessarily planar systems. Moreover, using [Sie02a] we will also eliminate the higher-order Taylor terms $D_x^i f(t, 0)$ for $i = 2, \dots, k$ with a C^k equivalence, provided $\Sigma_{\mathrm{SS}}(A)$ satisfies certain non-resonance conditions, in order to arrive at a system which additionally satisfies the following two conditions.

(A4) Stable and unstable manifolds are flat: $\pi_u f(t, 0, x^s) = 0, \pi_s f(t, x^u, 0) = 0$ for a.a. $t \in \mathbb{R}$ and all $(x^u, x^s) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}$,

(A5) Taylor terms up to order k vanish: $D_x^i f(t, 0) = 0$ for $i = 2, \dots, k$ and a.a. $t \in \mathbb{R}$. (Recall that $D_x f(t, 0) = 0$ by (A1).)

Define $\mathcal{O}^k_{\text{flat}}(A) \subseteq \mathcal{O}^k(A)$ by

$$\mathcal{O}^k_{\text{flat}}(A) := \mathcal{O}^k(A) \cap \{\dot{x} = A(t)x + f(t, x) \colon f \text{ satisfies (A4) and (A5)}\}.$$

In the following proposition we show that under the non-resonance condition of Theorem 5 any system in $\mathcal{O}^{k+2}(A)$ is C^k equivalent to a system in $\mathcal{O}^k(A)_{\text{flat}}$.

Proposition 6 (Flattening of invariant manifolds and elimination of non-resonant terms)**.** Let $\Sigma_{\mathrm{SS}}(A) = \lambda_1 \cup \lambda_2 \cup \dots \cup \lambda_n$ denote the Sacker Sell spectrum of the linear system (3). Assume that $0 \notin \Sigma_{\mathrm{SS}}(A)$. Let $k \in \mathbb{N}$. If the non-resonance condition

$$\lambda_j \cap \sum_{i=1}^n k_i \lambda_i = \emptyset \qquad \text{for all } (k_1, \dots, k_n) \in \mathbb{N}_0^n \text{ with } 2 \le \sum_{i=1}^n k_i \le k \tag{10}$$

holds, then any system in $\mathcal{O}^{k+2}(A)$ is C^k equivalent to a system in $\mathcal{O}^k_{\text{flat}}(A)$.

Proof. Consider an arbitrary system

$$\dot{x} = A(t)x + f(t,x) \tag{11}$$

in $\mathcal{O}^{k+2}(A)$ with solution $\varphi(\cdot, t_0, x_0^u, x_0^s)$ satisfying $\varphi(t_0, t_0, x_0^u, x_0^s) = (x_0^u, x_0^s)$. By definition (8) of A^u and A^s, we have $\Sigma_{\mathrm{SS}}(A^u) > 0 > \Sigma_{\mathrm{SS}}(A^s)$. Let Φ^u and Φ^s denote the evolution operators of $\dot{x}^u = A^u(t)x^u$ and $\dot{x}^s = A^s(t)x^s$, respectively. Then by Definition 1 of the Sacker Sell spectrum, there exist $K, \alpha > 0$ such that

$$\begin{aligned} \|\Phi^u(t,s)\| &\leq Ke^{\alpha(t-s)} && \text{for } t \leq s, \\ \|\Phi^s(t,s)\| &\leq Ke^{-\alpha(t-s)} && \text{for } t \geq s. \end{aligned}$$

Step 1 (Flattening of stable and unstable manifolds): Using [AW96, Theorem 4.1] there exists an $L = L(K,\alpha) > 0$ such that, if $\sup_{t\in\mathbb{R}, x\in\mathbb{R}^d} \|D_x f(t,x)\| \leq L$, then there exist stable and unstable manifolds of the trivial solution of (11). Using assumption (A3) and the mean value theorem, it follows that

$$\sup_{t\in\mathbb{R}, x\in\mathbb{R}^d} \|D_x f(t,x)\| \leq \sup_{s\in\mathbb{R}, y\in\mathbb{R}^d} \|D_x^2 f(s,y)\| \|x\| \leq M\|x\|$$

and replacing f by a cut-off $\tilde{f}$ (see e.g. [BDDS15, Example 6]) which agrees with f on a tubular neighborhood $\mathbb{R} \times B_r(0) \subset \mathbb{R} \times \mathbb{R}^d$ for some sufficiently small $r > 0$, and still satisfies (A3), we get $\sup_{t\in\mathbb{R}, x\in\mathbb{R}^d} \|D_x \tilde{f}(t,x)\| \leq L$. The cut-off procedure is a C^k equivalence ([BDDS15, Example 6]) and for ease of notation we omit the tilde and write again f instead of $\tilde{f}$.

By [AW96, Theorem 4.1] there exists a continuous map $\mathfrak{u}\colon \mathbb{R} \times \mathbb{R}^{d_u} \to \mathbb{R}^{d_s}$ whose graph

$$\mathcal{U} := \big\{(t, x^u, \mathfrak{u}(t,x^u))\colon (t,x^u) \in \mathbb{R} \times \mathbb{R}^{d_u}\big\}$$

can be characterized dynamically as

$$\mathcal{U} = \big\{(t, x^u, x^s) : \lim_{s\to-\infty} \varphi(s, t, x^u, x^s) = 0\big\}.$$

$\mathcal{U}$ is the unstable manifold of the trivial solution of (11), $\mathfrak{u}(\cdot, 0) = 0$ and the invariance equation

$$\pi_s \circ \varphi(t, s, x^u, \mathfrak{u}(s,x^u)) = \mathfrak{u}\big(t, \pi_u \circ \varphi(t, s, x^u, \mathfrak{u}(s,x^u))\big) \tag{12}$$

holds for $t, s \in \mathbb{R}$, $(x^u, x^s) \in \mathbb{R}^d$. By [Sie99, Theorem 5.20] $\mathfrak{u}$ is a C^{k+2} Carathéodory function with globally bounded derivatives and $\frac{\partial \mathfrak{u}}{\partial x^u}(t,0) = 0$ for $t \in \mathbb{R}$. Define $H\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$H(t, x^u, x^s) := (x^u, x^s - \mathfrak{u}(t,x^u)).$$

Then $H(\cdot,0)=0$, $H(t,\cdot)$ are C^{k+2} diffeomorphisms with inverse $H^{-1}(t,x)=(x^u, x^s+\mathfrak{u}(t,x^u))$ and $\lim_{x\to 0} H(t,x)=\lim_{x\to 0} H^{-1}(t,x)=0$ uniformly in $t\in\mathbb{R}$ due to the fact that $\sup_{t\in\mathbb{R},x^u\in\mathbb{R}^{d_u}}\|\frac{\partial\mathfrak{u}}{\partial x^u}(t,x^u)\|<\infty$.

Taking derivatives of both sides of the invariance equation (12) yields

$$\begin{aligned}\frac{\partial\mathfrak{u}}{\partial t}(t,x^u) &= A^s(t)\mathfrak{u}(t,x^u)+\pi_s f(t,x^u,\mathfrak{u}(t,x^u))\\ &\quad -\frac{\partial\mathfrak{u}}{\partial x^u}(t,x^u)\big(A^u(t)x^u+\pi_u f(t,x^u,\mathfrak{u}(t,x^u))\big)\end{aligned}$$

which implies that H transforms (11) to

$$\dot{x}=A(t)x+g(t,x) \tag{13}$$

with

$$\begin{aligned}\pi_u g(t,x^u,x^s) &= \pi_u f(t,x^u,x^s+\mathfrak{u}(t,x^u)),\\ \pi_s g(t,x^u,x^s) &= \pi_s f(t,x^u,x^s+\mathfrak{u}(t,x^u))-\pi_s f(t,x^u,\mathfrak{u}(t,x^u))\\ &\quad -\frac{\partial\mathfrak{u}}{\partial x^u}(t,x^u)\big(\pi_s f(t,x^u,x^s+\mathfrak{u}(t,x^u))-\pi_s f(t,x^u,\mathfrak{u}(t,x^u))\big).\end{aligned}$$

System (13) is in $\mathcal{O}^{k+1}(A)$, it satisfies additionally that $\pi_s g(t,x^u,0)=0$, i.e. its unstable manifold is flat and (11) is C^{k+1} equivalent to (13). Similarly one can flatten the stable manifold $\mathcal{S}:=\{(t,\mathfrak{s}(t,x^s),x^s)\colon (t,x^s)\in\mathbb{R}\times\mathbb{R}^{d_s}\}$ of (13) with a C^k equivalence $(t,x^u,x^s)\mapsto(x^u-\mathfrak{s}(t,x^s),x^s)$. By composing both transformations, (11) is C^k equivalent to a system in $\mathcal{O}^k(A)$ which satisfies condition (A4), i.e. the stable and unstable manifolds are flat.

Step 2 (Elimination of non-resonant terms): If $k=1$ no terms need to be eliminated. If $k\geq 2$, the normal form theorem for nonautonomous differential equations [Sie02a] implies that all non-resonant Taylor terms up to order k can be eliminated by a C^k equivalence. By assumption (10), all Taylor terms up to order k are non-resonant and therefore the simplified system from Step 1 is C^k equivalent to a system in $\mathcal{O}^k_{\text{flat}}(A)$. $\square$

3 Smooth equivalence of discrete time systems

Nonautonomous difference equations (also called discrete time systems) of the form

$$x_{n+1}=B_n x_n+g_n(x_n) \tag{14}$$

with a matrix-valued function $B\colon \mathbb{Z} \to \mathbb{R}^{d\times d}$, $n \mapsto B_n$, and a function $g\colon \mathbb{Z}\times \mathbb{R}^d \to \mathbb{R}^d$, $(n,x) \mapsto g_n(x)$, such that g_n is C^k, arise e.g. as time-1 maps of (1) by setting $B_n := \Phi(n+1,n)$ and $g_n(x) := \varphi(n+1,n,x) - \Phi(n+1,n)x$ with the evolution operator Φ of (2) and the solution φ of (1). Then obviously every solution $n \mapsto x_n$ of (14) satisfies $x_n = \varphi(n,m,x_m)$.

For discrete time systems (14) with $g_n(0) = 0$ and $g_n'(0) = 0$ ($n \in \mathbb{Z}$), a Hartman-Grobman topological linearization result and a smooth Poincaré normal form theory are already established (see e.g. [AW06] and the references therein, and [Sie03], respectively). To the best of our knowledge, a Sternberg linearization theorem for (14) was not proven yet. For an autonomous version of Sternberg's linearization theorem for difference equations, see [BD84] and [Nei05] and the references therein.

In this section we utilize a result on the 'continuous dependence of the smooth change of coordinates in parametrized normal form theorems' for autonomous difference equations [Bon93] which we extend to the nonautonomous discrete time case (14) and then relate to the construction of smooth linearizations of (1). According to Proposition 6, in order to prove Theorem 5, it remains to show that all non-resonant systems in $\mathcal{O}^k_{\mathrm{flat}}(A)$ are C^r equivalent to their linearization (3). As a main step in the proof we discuss discrete time systems (14) which arise as time-κ maps ($\kappa > 0$ large) of systems (1) in $\mathcal{O}^k_{\mathrm{flat}}(A)$.

3.1 Associated discrete-time systems

Definition 7 (Associated time-κ system)**.** For $k \in \mathbb{N}$ and a locally integrable matrix function $A : \mathbb{R} \to \mathbb{R}^{d\times d}$ satisfying (A2), consider a differential equation

$$\dot{x} = A(t)x + f(t,x) \tag{15}$$

in $\mathcal{O}^k(A)$ with the solution $\varphi(\cdot, t_0, x_0)$ of (15), $x(t_0) = x_0$. Let $\kappa \in \mathbb{R}$, $\kappa > 0$. For $n \in \mathbb{Z}$, $x \in \mathbb{R}^d$, define

$$\begin{aligned} &F_n^{(\kappa)}(x) := \varphi((n+1)\kappa, n\kappa, x) \quad \text{and} \\ &A_n^{(\kappa)} := \Phi((n+1)\kappa, n\kappa), \quad f_n^{(\kappa)}(x) := F_n^{(\kappa)}(x) - A_n^{(\kappa)}x \end{aligned}$$

with the evolution operator Φ of (2). Then

$$x_{n+1} = F_n^{(\kappa)}(x_n) \quad \text{and equivalently} \quad x_{n+1} = A_n^{(\kappa)}x_n + f_n^{(\kappa)}(x_n) \tag{16}$$

are called the *associated time-κ system* of (15).

For $k \in \mathbb{N}$ and a C^k map $f\colon \mathbb{R}^d \to \mathbb{R}^d$ let $\|f\|_k := \max_{0\le i\le k} \sup_{x\in\mathbb{R}^d} \|D^i f(x)\|$. Define

$$
\begin{aligned}
C^k(\mathbb{R}^d) &:= \{f\colon \mathbb{R}^d \to \mathbb{R}^d\colon f \text{ is } C^k \text{ with } \|f\|_k < \infty\},\\
C^k_1(\mathbb{R}^d) &:= \{f \in C^k(\mathbb{R}^d)\colon f(0) = 0, Df(0) = 0\},\\
C^k_{\text{flat}}(\mathbb{R}^d) &:= \{f \in C^k(\mathbb{R}^d)\colon D^i f(0) = 0 \text{ for } i = 0,\dots,k\}.
\end{aligned}
$$

Lemma 8. For an associated time-κ system (16) of (15) the following statements hold:

(a) $\sup_{n\in\mathbb{Z}} \|A_n^{(\kappa)}\| < \infty$ and $\sup_{n\in\mathbb{Z}} \|(A_n^{(\kappa)})^{-1}\| < \infty$.

(b) If (15) is in $\mathcal{O}^k_{\text{flat}}(A)$, then $f_n^{(\kappa)} \in C^k_{\text{flat}}(\mathbb{R}^d)$ for $n \in \mathbb{N}$.

(c) If (15) is in $\mathcal{O}^k_{\text{flat}}(A)$, then $\pi_u f_n^{(\kappa)}(0, x^s) = 0$ and $\pi_s f_n^{(\kappa)}(x^u, 0) = 0$ for $n \in \mathbb{Z}$ and $(x^u, x^s) \in \mathbb{R}^d$.

Proof. (a) By (A2), $\|A_n^{(\kappa)}\| = \|\Phi((\kappa+1)n, \kappa n)\| \le Ke^{a\kappa}$ and similarly for $\|(A_n^{(\kappa)})^{-1}\|$.

(b) Using $f_n^{(\kappa)}(x) = \varphi((n+1)\kappa, n\kappa, x) - \Phi((n+1)\kappa, n\kappa)x$, this follows for $i = 0$ from $\varphi(\cdot,\cdot,0) = 0$, for $i = 1$ from $\frac{\partial\varphi}{\partial x}((n+1)\kappa, n\kappa, 0) = \Phi((n+1)\kappa, n\kappa)$ and for $i = 2,\dots k$ from Taylor expanding the C^k function $x \mapsto \varphi((n+1)\kappa, n\kappa, x)$ in $x = 0$.

(c) Using $f_n^{(\kappa)}(x) = \varphi((n+1)\kappa, n\kappa, x) - \Phi((n+1)\kappa, n\kappa)x$, the claim follows from the invariance equation (12) for stable und unstable manifolds of (15) which are flat by (A4), i.e. $\pi_u f(t, 0, x^s) = 0$ and $\pi_s f(t, x^u, 0) = 0$ for a.a. $t \in \mathbb{R}$ and all $(x^u, x^s) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}$, or equivalently, $\mathfrak{u}(\cdot, 0) = 0$ and $\mathfrak{s}(\cdot, 0) = 0$. ∎

3.2 Discrete time characterizaton of C^k equivalence

Theorem 9 (Time-κ characterization of C^k equivalence)**.** Two systems

$$\dot{x} = A(t)x + f(t, x) \tag{17}$$

and

$$\dot{y} = A(t)y + g(t, y) \tag{18}$$

in $\mathcal{O}^k(A)$ are C^k equivalent, if there exists a $\kappa > 0$ such that their associated time-κ systems

$$x_{n+1} = F_n^{(\kappa)}(x_n) \tag{19}$$

and

$$y_{n+1} = G_n^{(\kappa)}(y_n) \tag{20}$$

are C^k equivalent, i.e. if there exist $p, \widetilde{p} > 0$ and $r \in (0, p), \widetilde{r} \in (0, \widetilde{p})$ and maps

$$T_n \colon B_r(0) \to \mathbb{R}^d \qquad \text{and} \qquad S_n \colon B_{\widetilde{r}}(0) \to \mathbb{R}^d,$$

such that the following statements hold:

(i) For each $n \in \mathbb{Z}$ the maps

$$\begin{aligned} T_n &\colon B_r(0) \to T_n(B_r(0)) \subset B_{\widetilde{p}}(0), \\ S_n &\colon B_{\widetilde{r}}(0) \to S_n(B_{\widetilde{r}}(0)) \subset B_p(0), \end{aligned}$$

are C^k diffeomorphisms with

$$\begin{aligned} T_n(S_n(y)) = y &\quad \text{for all } y \in B_{\widetilde{r}}(0) \cap S_n^{-1}(B_r(0)), \\ S_n(T_n(x)) = x &\quad \text{for all } x \in B_r(0) \cap T_n^{-1}(B_{\widetilde{r}}(0)). \end{aligned}$$

(ii) For $x \in B_r(0)$, $y \in B_{\widetilde{r}}(0)$ with $F_n^{(\kappa)}(x) \in B_r(0)$, $G_n^{(\kappa)}(y) \in B_{\widetilde{r}}(0)$

$$T_{n+1}(F_n^{(\kappa)}(x)) = G_n^{(\kappa)}(T_n(x)) \quad \text{and} \quad S_{n+1}(G_n^{(\kappa)}(y)) = F_n^{(\kappa)}(S_n(y)).$$

(iii) The trivial solutions are mapped uniformly onto each other:

$$\lim_{x \to 0} T_n(x) = 0 \quad \text{and} \quad \lim_{y \to 0} S_n(y) = 0 \qquad \text{uniformly in } n \in \mathbb{Z}.$$

Proof. With the solutions φ and ψ of (17) and (18), respectively,

$$F_n^{(\kappa)}(x) = \varphi((n+1)\kappa, n\kappa, x) \quad \text{and} \quad G_n^{(\kappa)}(y) = \psi((n+1)\kappa, n\kappa, y).$$

Since (17) and (18) are in $\mathcal{O}^k(A)$, the functions f and g satisfy the global boundedness condition (A3). By the Gronwall inequality [Har82, Chapter 3, Theorem 1.1], there exist $r^* > 0$ and $\tilde{r}^* > 0$ such that for all $n \in \mathbb{Z}$

$$\begin{aligned} \varphi(\kappa n, t, x) \in B_r(0) &\qquad \text{for } t \in [\kappa n, \kappa(n+1)], x \in B_{r^*}(0), \\ \psi(\kappa n, t, y) \in B_{\tilde{r}}(0) &\qquad \text{for } t \in [\kappa n, \kappa(n+1)], y \in B_{\tilde{r}^*}(0). \end{aligned} \tag{21}$$

By assumption (iii), and by shrinking r^* and $\tilde{r}^*$ if necessary, we can assume additionally that

$$\begin{aligned} T_n(\varphi(\kappa n, t, x)) &\in B_r(0) \qquad \text{for } t \in [\kappa n, \kappa(n+1)], x \in B_{r^*}(0), \\ S_n(\psi(\kappa n, t, y)) &\in B_{\tilde{r}}(0) \qquad \text{for } t \in [\kappa n, \kappa(n+1)], y \in B_{\tilde{r}^*}(0). \end{aligned} \tag{22}$$

Define $H : \mathbb{R} \times B_{r^*}(0) \to \mathbb{R}^d$ and $H^{-1} : \mathbb{R} \times B_{\tilde{r}^*}(0) \to \mathbb{R}^d$ by

$$\begin{aligned} H(t,x) &:= \psi\big(t, \kappa n, T_n(\varphi(\kappa n, t, x))\big) \qquad \text{for } x \in B_{r^*}(0), \\ H^{-1}(t,y) &:= \varphi\big(t, \kappa n, S_n(\psi(\kappa n, t, y))\big) \qquad \text{for } y \in B_{\tilde{r}^*}(0), \end{aligned} \tag{23}$$

for $t \in [\kappa n, \kappa(n+1))$. By (21) and (22), the functions H and H^{-1} are well-defined and to conclude the proof, we show that H is a C^k equivalence between (17) and (18) by verifying that H and H^{-1} satisfy conditions (i), (ii) and (iii) in Definition 4.

Step 1 (Verification of Definition 4(i)): The functions $H(t,\cdot)$ and $H^{-1}(t,\cdot)$ are C^k, since T_n and S_n are C^k diffeomorphisms and the solutions φ and ψ are C^k by the classical result on the smooth dependence of solutions of ordinary differential equations on the initial value, see e.g. [Har82, Chapter V, Theorem 4.1]. Now let $y \in B_{\tilde{r}^*}(0)$ satisfy that $H^{-1}(t,y) \in B_{r^*}(0)$. By definition (23) of H and H^{-1}, we have

$$H(t, H^{-1}(t,y)) = \psi\big(t, \kappa n, T_n \circ S_n(\psi(\kappa n, t, y))\big).$$

By (21), (22) and assumption (i), $T_n \circ S_n(\psi(\kappa n, t, y)) = \psi(\kappa n, t, y)$. Therefore, $H(t, H^{-1}(t,y)) = y$. Similarly, for $x \in B_{r^*}(0)$ with $H(t,x) \in B_{\tilde{r}^*}(0)$ we also have $H^{-1}(t, H(t,x)) = x$.

Step 2 (Verification of Definition 4(ii)): Let $\mu(t)$ be an arbitrary solution of (17) in $B_{r^*}(0)$. By definition (23) of H, for $n \in \mathbb{Z}$ and $t \in [\kappa n, \kappa(n+1))$

$$\begin{aligned} H(t, \mu(t)) &= \psi\big(t, \kappa n, T_n(\varphi(\kappa n, t, \mu(t)))\big) \\ &= \psi(t, \kappa n, T_n(\mu(\kappa_n))), \end{aligned}$$

where we use the fact that $\mu(t)$ is a solution of (4) to obtain the last equality. Hence $H(t, \mu(t))$ is a solution of (18) on $[\kappa n, \kappa(n+1))$ and, by continuity, also on $\mathbb{R}$. Similarly, $H^{-1}(t, \nu(t))$ is a solution of (17) for any solution $\nu(t)$ of (18) in $B_{\tilde{r}^*}(0)$.

Step 3 (Verification of Definition 4(iii)): Since (17) and (18) are in $\mathcal{O}^k(A)$, the functions f and g satisfy the global boundedness condition (A3) with bound $M > 0$ and therefore by the mean value theorem

$$\|f(t,x)\|, \|g(t,x)\| \le M\|x\| \qquad \text{for a.a. } t \in \mathbb{R}, x \in \mathbb{R}^d.$$

Using the variation of constants formula, we obtain for all $t \in [\kappa n, \kappa(n+1)]$

$$\psi(t,\kappa n, x) = \Phi(t,\kappa n)x + \int_{\kappa n}^{t} \Phi(t,s) f(s, \psi(s,\kappa n, x))\, ds,$$

with the evolution operator Φ of $\dot{x} = A(t)x$, which together with (A2) implies that

$$\|\psi(t,\kappa n,x)\| \le K e^{a\kappa}\|x\| + MKe^{a\kappa} \int_{\kappa n}^{t} \|\psi(s,\kappa n,x)\|\, ds.$$

Then, using the Gronwall inequality [Har82, Chapter 3, Theorem 1.1], there exists $C_1 > 0$ such that

$$\|\psi(t,\kappa n,x)\| \le C_1\|x\| \qquad \text{for } t \in [\kappa n, \kappa(n+1)).$$

Similarly, there exists $C_2 > 0$ such that

$$\|\varphi(\kappa n,t,x)\| \le C_2\|x\| \qquad \text{for } t \in [\kappa n, \kappa(n+1)).$$

As a consequence, by (23) we get for $t \in [\kappa n, \kappa(n+1))$

$$\|H(t,x)\| \le C_1 \sup_{z \in B_{C_2\|x\|}(0)} T_n(z)$$

and by taking the supremum over $t \in \mathbb{R}$, we have

$$\sup_{t\in\mathbb{R}} \|H(t,x)\| \le C_1 \sup_{n\in\mathbb{Z}} \sup_{z \in B_{C_2\|x\|}(0)} T_n(z).$$

Assumption (iii) implies that $\lim_{x\to 0} H(t,x) = 0$ uniformly in $t \in \mathbb{R}$. Similarly, we also have $\lim_{y\to 0} H^{-1}(t,y) = 0$ uniformly in $t \in \mathbb{R}$. □

3.3 Associated discrete time systems with spectral gap and flat nonlinearity

Recall that $A\colon \mathbb{R} \to \mathbb{R}^{d\times d}$ is of the form (8), i.e. $A(t) = \operatorname{diag}(A^u(t), A^s(t))$ with $0 \notin \Sigma_{\mathrm{SS}}(A) = \Sigma^u \cup \Sigma^s$, $\Sigma^u = \Sigma_{\mathrm{SS}}(A) \cap \mathbb{R}_{>0}$, $\Sigma^s = \Sigma_{\mathrm{SS}}(A) \cap \mathbb{R}_{<0}$ and corresponding coordinates $x = (x^u, x^s) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}$. In this subsection we discuss discrete-time systems

$$x_{n+1} = B_n x_n + g_n(x_n), \qquad (n \in \mathbb{Z})$$

with $B_n = \operatorname{diag}(B_n^u, B_n^s) \in \mathrm{Gl}(d,\mathbb{R})$, $B^u\colon \mathbb{Z} \to \mathbb{R}^{d_u\times d_u}$, $B^s\colon \mathbb{Z} \to \mathbb{R}^{d_s\times d_s}$, $g\colon \mathbb{Z}\times\mathbb{R}^n \to \mathbb{R}^n$, which satisfy the following conditions for fixed $k,\ell \in \mathbb{N}$:

(B1) Linearity: B is bounded, i.e. there exists $M > 0$ such that

$$\sup_{n\in\mathbb{Z}} \|B_n\| \le M \quad \text{and} \quad \sup_{n\in\mathbb{Z}} \|B_n^{-1}\| \le M, \tag{24}$$

and satisfies the following gap conditions for a $\gamma \in (0,1)$

$$\begin{gathered} \sup_{n\in\mathbb{Z}} \|(B_n^u)^{-1}\| < \gamma, \quad \sup_{n\in\mathbb{Z}} \|B_n\| \|(B_n^u)^{-1}\|^{k-\ell} \|B_n^{-1}\|^\ell < \gamma, \\ \sup_{n\in\mathbb{Z}} \|B_n^s\| < \gamma, \quad \sup_{n\in\mathbb{Z}} \|B_n^{-1}\| \|B_n^s\|^{k-\ell} \|B_n\|^\ell < \gamma. \end{gathered} \tag{25}$$

(B2) Nonlinearity: $g_n \in C^k_{\text{flat}}(\mathbb{R}^d)$ and additionally satisfies

$$\pi_u g_n(0, x^s) = 0 \quad \text{and} \quad \pi_s g_n(x^u, 0) = 0 \qquad (n \in \mathbb{Z}).$$

For $k, \ell \in \mathbb{N}$ and $B\colon \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$, $(n, x) \mapsto B_n x$, with $B_n = \operatorname{diag}(B_n^u, B_n^s) \in \mathrm{Gl}(d, \mathbb{R})$, $B^u\colon \mathbb{Z} \to \mathbb{R}^{d_u \times d_u}$, $B^s\colon \mathbb{Z} \to \mathbb{R}^{d_s \times d_s}$, which satisfies (B1), define

$$\mathcal{D}^{k,\ell}_{\text{flat}}(B) := \{x_{n+1} = B_n x_n + g_n(x_n)\colon g \text{ satisfies (B2)}\}.$$

Proposition 10. If a system

$$\dot{x} = A(t)x + f(t, x) \tag{26}$$

in $\mathcal{O}^k_{\text{flat}}(A)$, $k \ge 2$, satisfies the spectral gap condition (6) for an $\ell \in \mathbb{N}$, $\ell \le k$, i.e.

$$(k-\ell)\Sigma_u > \Sigma_u - \ell\Sigma_s \qquad \text{and} \qquad (k-\ell)\Sigma_s < \Sigma_s - \ell\Sigma_u, \tag{27}$$

then there exists a $\kappa > 0$ such that for the associated time-κ system

$$x_{n+1} = A_n^{(\kappa)} x_n + f_n^{(\kappa)}(x_n) \tag{28}$$

of (26), $A^{(\kappa)}$ satisfies (B1) and (28) is in $\mathcal{D}^{k,\ell}_{\text{flat}}(A^{(\kappa)})$.

Proof. By Lemma 8, all time-κ systems

$$x_{n+1} = A_n^{(\kappa)} x_n + f_n^{(\kappa)}(x_n) \qquad (\kappa > 0)$$

associated with (26) satisfy (B2) and condition (24) of (B1). To conclude the proof, it is sufficient to find $\kappa > 0$ for which the linear part

$$A_n^{(\kappa)} = \Phi_A((n+1)\kappa, n\kappa) \tag{29}$$

satisfies condition (25) of (B1), where Φ_A denotes the evolution operator of the linear differential equation $\dot{\xi} = A(t)\xi = \operatorname{diag}(A^u(t), A^s(t))\xi$. Then, $\Phi_A(\cdot,\cdot) = \operatorname{diag}(\Phi_{A^u}(\cdot,\cdot), \Phi_{A^s}(\cdot,\cdot))$ and since $\Sigma_{\mathrm{SS}}(A^u) = \Sigma_u > 0$ and $\Sigma_{\mathrm{SS}}(A^s) = \Sigma_s < 0$, for any $\varepsilon > 0$ there exists $K > 0$ such that for all $t \geq s$

$$\begin{aligned} \|\Phi_A(t,s)\| &\leq K e^{(\max \Sigma_u + \varepsilon)(t-s)}, \\ \|\Phi_A(s,t)\| &\leq K e^{(-\min \Sigma_s + \varepsilon)(t-s)}, \\ \|\Phi_{A^u}(s,t)\| &\leq K e^{(-\min \Sigma_u + \varepsilon)(t-s)}. \end{aligned} \tag{30}$$

By assumption (27), there exists $\varepsilon > 0$ such that

$$\begin{aligned} (k-\ell)(\Sigma_u - \varepsilon) &> \Sigma_u + \varepsilon - \ell(\Sigma_s - \varepsilon), \\ (k-\ell)(\Sigma_s + \varepsilon) &< \Sigma_s - \varepsilon - \ell(\Sigma_u + \varepsilon). \end{aligned}$$

This, together with the estimates (30), implies for $t \geq s$

$$\|\Phi_A(t,s)\| \|\Phi_{A^u}(s,t)\|^{k-\ell} \|\Phi_A(s,t)\|^{\ell} \leq K^3 e^{-\beta(t-s)},$$

where

$$\beta := (\max \Sigma_u + \varepsilon) + (k-\ell)(-\min \Sigma_u + \varepsilon) + \ell(-\min \Sigma_s + \varepsilon) < 0.$$

Therefore, for $\gamma := \frac{1}{2}$ there exists $\kappa_1 > 0$ such that

$$\|\Phi_A(t,s)\| \|\Phi_{A^u}(s,t)\|^{k-\ell} \|\Phi_A(s,t)\|^{\ell} \leq \gamma \quad \text{for } t - s \geq \kappa_1. \tag{31}$$

Similarly, there exists $\kappa_2 > 0$ such that

$$\|\Phi_A(s,t)\| \|\Phi_{A^s}(t,s)\|^{k-\ell} \|\Phi_A(t,s)\|^{\ell} \leq \gamma \qquad \text{for } t - s \geq \kappa_2. \tag{32}$$

With $\kappa := \max\{\kappa_1, \kappa_2\} > 0$ the linear part (29) of the associated time-κ system of (26) satisfies condition (25) of (B1) and hence the associated time-κ system of (26) is in $\mathcal{D}^{k,\ell}_{\mathrm{flat}}(A^{(\kappa)})$. $\square$

4 Nonautonomous Sternberg theorem

In this section we prove the Sternberg Theorem 5 for nonautonomous differential equations. Recall that $A \colon \mathbb{R} \to \mathbb{R}^{d \times d}$ is of the form (8), i.e. $A(t) = \operatorname{diag}(A^u(t), A^s(t))$ with $0 \notin \Sigma_{\mathrm{SS}}(A) = \Sigma^u \cup \Sigma^s$, $\Sigma^u = \Sigma_{\mathrm{SS}}(A) \cap \mathbb{R}_{>0}$,

$\Sigma^s = \Sigma_{\mathrm{SS}}(A) \cap \mathbb{R}_{<0}$ and corresponding coordinates $x = (x^u, x^s) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}$. Define two closed subsets of $C^k_{\mathrm{flat}}(\mathbb{R}^d)$:

$$C^k_{\mathrm{flat,u}}(\mathbb{R}^d) := \{f \in C^k_{\mathrm{flat}}(\mathbb{R}^d) \colon D^i f(0, x_s) = 0 \text{ for } i = 0, \dots, k\},$$
$$C^k_{\mathrm{flat,s}}(\mathbb{R}^d) := \{f \in C^k_{\mathrm{flat}}(\mathbb{R}^d) \colon D^i f(x_u, 0) = 0 \text{ for } i = 0, \dots, k\}.$$

Remark 11. In [BD84, Lemma 3] it is shown that for any $f \in C^k_{\mathrm{flat}}(\mathbb{R}^d)$ there exists $\varphi \in C^k_{\mathrm{flat,u}}(\mathbb{R}^d)$ and $\psi \in C^k_{\mathrm{flat,s}}(\mathbb{R}^d)$ such that $f = \varphi + \psi$.

To prove the main Theorem 5, we show several preparatory results.

Proposition 12 (C^k equivalence of discrete time systems with close nonlinearities)**.** Let $\ell \in \mathbb{N}$, $1 \leq \ell \leq k$, and let B satisfy (B1) with $B_n = \mathrm{diag}(B^u_n, B^s_n) \in \mathrm{Gl}(d, \mathbb{R})$, $B^u \colon \mathbb{Z} \to \mathbb{R}^{d_u \times d_u}$, $B^s \colon \mathbb{Z} \to \mathbb{R}^{d_s \times d_s}$. Then two systems

$$x_{n+1} = B_n x_n + f_n(x_n), \tag{33}$$
$$y_{n+1} = B_n y_n + g_n(y_n), \tag{34}$$

in $\mathcal{D}^{k,\ell}_{\mathrm{flat}}(B)$ are C^k equivalent, if either

$$f_n - g_n \in C^k_{\mathrm{flat,u}}(\mathbb{R}^d) \qquad (n \in \mathbb{N}), \tag{35}$$

or

$$f_n - g_n \in C^k_{\mathrm{flat,s}}(\mathbb{R}^d) \qquad (n \in \mathbb{N}). \tag{36}$$

Note that we prove Proposition 12 only under the assumption that (35) holds. The proof under the assumption (36) is completely analogous. The main ingredient in the proof of Proposition 12 is the nonautonomous method of path which is developed in Appendix 5. To apply the method of path, we define the function $P \colon \mathbb{Z} \times \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$, $(n, x, \tau) \mapsto P^\tau_n(x)$, by

$$P^\tau_n(x) := B_n x + (1 - \tau) f_n(x) + \tau g_n(x) \tag{37}$$

and consider the homotopy of nonautonomous difference equations

$$z_{n+1} = P^\tau_n(z_n), \qquad \tau \in [0, 1]$$

between (33) and (34). Actually, we do not apply the method of path directly to (33) and (34), but first replace the nonlinearities f_n and g_n by cut-off nonlinearities $\widetilde{f}_n := \psi \circ f_n$ and $\widetilde{g}_n := \psi \circ g_n$ where $\psi\colon \mathbb{R}^d \to \mathbb{R}^d$ is the C^∞ bump function

$$\psi(x) = \begin{cases} x, & \text{for } 0 \le \|x\| \le r, \\ \exp\left(1 - \frac{1}{1-(\|x\|-r)^2}\right)x, & \text{for } r < \|x\| < r+1, \\ 0, & \text{for } \|x\| \ge r, \end{cases} \tag{38}$$

for some $r > 0$. This technical step does not change the differential equations (33) and (34) on a tubular neighborhood $\mathbb{R} \times B_r(0)$ of the zero solutions and ensures that P_n^τ is invertible. We use the notation

$$\begin{aligned} \operatorname{Diff}^k(\mathbb{R}^d) &:= \{f \in C^k(\mathbb{R}^d)\colon f^{-1} \text{ exists and } f^{-1} \in C^k(\mathbb{R}^d)\}, \\ \operatorname{Diff}_0^k(\mathbb{R}^d) &:= \{f \in \operatorname{Diff}^k(\mathbb{R}^d)\colon f(0) = 0\}. \end{aligned}$$

Lemma 13 (Invertibility of homotopy difference equations)**.** For every $\delta > 0$ there exists an $r > 0$ such that, with (33) and (34) from Proposition 12 and ψ from (38), the function $P\colon \mathbb{Z} \times \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$, $(n, x, \tau) \mapsto P_n^\tau(x)$, defined by

$$P_n^\tau(x) := B_n x + (1-\tau)\psi(f_n(x)) + \tau\psi(g_n(x)), \tag{39}$$

satisfies the following statements.

(i) $P_n^\tau \in \operatorname{Diff}_0^k(\mathbb{R}^d)$ for $n \in \mathbb{N}$ and the map $\mathbf{P}\colon [0,1] \to \operatorname{Diff}_0^k(\mathbb{R}^d)^{\mathbb{Z}}$, $\tau \mapsto \mathbf{P}^\tau := P_\cdot^\tau(\cdot)$ is continuous,

(ii) $\sup_{n\in\mathbb{Z},x\in\mathbb{R}^d} \|DP_n^\tau(x) - B_n\| \le \delta$, $\sup_{n\in\mathbb{Z},x\in\mathbb{R}^d} \|D(P_n^\tau)^{-1}(x) - B_n^{-1}\| \le \delta$.

To prove Lemma 13, we need the following lemma.

Lemma 14 (Identity plus C^k contraction)**.** Let $\eta : \mathbb{R}^d \to \mathbb{R}^d$ be C^k with

$$\rho := \sup_{x\in\mathbb{R}^d} \|D\eta(x)\| < 1. \tag{40}$$

Then, the map $f : \mathbb{R}^d \to \mathbb{R}^d$ defined by $f(x) := x + \eta(x)$ is a C^k diffeomorphism. Furthermore, for all $r > 0$ we have $B_{(1-\rho)r}(0) \subset f(B_r(0))$ and therefore

$$\|f^{-1}(x)\| \le \frac{1}{1-\rho}\|x\| \qquad \text{for all } x \in \mathbb{R}^d.$$

Proof. To show that f is injective, let $x, y \in \mathbb{R}^d$, $x \ne y$. By (40),

$$\|\eta(x) - \eta(y)\| < \|x - y\|,$$

which implies $\|f(x)-f(y)\| = \|x-y+\eta(x)-\eta(y)\| \geq \|x-y\|-\|\eta(x)-\eta(y)\| > 0$, i.e. $f(x) \neq f(y)$.

To show surjectivity of f, let $x \in \mathbb{R}^d$. Consider $T : \mathbb{R}^d \to \mathbb{R}^d$, $y \mapsto T(y)$, defined by

$$T(y) := x - \eta(y).$$

By (40),

$$\|T(y) - T(\widehat{y})\| = \|\eta(y) - \eta(\widehat{y})\| \leq \rho\|y - \widehat{y}\|.$$

Then, by the Banach Contraction Mapping Theorem there exists $y \in \mathbb{R}^d$ such that $T(y) = y$, or equivalently, $f(y) = x$. So far, we have shown that f is bijective.

Furthermore, by smoothness of η the function f is also C^k. On the other hand, using the Inverse Function Theorem, the inverse f^{-1} of f is also C^k.

To conclude the proof, let $r > 0$ and $y \in B_{(1-\rho)r}(0)$. Let $x \in \mathbb{R}^d$ satisfy that $f(x) = y$. Hence, $\|x + \eta(x)\| = \|y\| < (1 - \rho)r$ and therefore

$$\|x\| < (1 - \rho)r + \|\eta(x)\| \leq (1 - \rho)r + \rho\|x\|,$$

which implies $\|x\| < r$, proving that $B_{(1-\rho)r}(0) \subset f(B_r(0))$. □

Proof of Lemma 13. By (B1), there exists $M > 1$ such that

$$\sup_{n\in\mathbb{Z}} \|B_n\| \leq M \quad \text{and} \quad \sup_{n\in\mathbb{Z}} \|B_n^{-1}\| \leq M. \tag{41}$$

Let $\delta > 0$. Then there exists $\delta^* \in (0, \min\{\frac{\delta}{M}, \frac{1}{M^2}\})$ which additionally satisfies $\frac{M^3\delta^*}{1-M^2\delta^*} \leq \delta$. Since $f_n, g_n \in C^k_{\text{flat}}(\mathbb{R}^d)$ for $n \in \mathbb{N}$, there exists an $r > 0$ such that with the bump function ψ from (38)

$$\sup_{n\in\mathbb{Z},x\in\mathbb{R}^d} \|D\psi(f_n(x))\| \leq \delta^* \quad \text{and} \quad \sup_{n\in\mathbb{Z},x\in\mathbb{R}^d} \|D\psi(g_n(x))\| \leq \delta^*. \tag{42}$$

For $\tau \in [0, 1]$ define

$$\eta_n^\tau(x) := (1 - \tau)B_n^{-1}\psi(f_n(x)) + \tau B_n^{-1}\psi(g_n(x)).$$

Then, from (42) we derive that $\|D\eta_n^\tau(x)\| \leq M\delta^* < 1$. Therefore, by virtue of Lemma 14, the map $x \mapsto x + \eta_n^\tau(x)$ is a C^k diffeomorphism. By (39), we have

$$P_n^\tau(x) = B_n(x + \eta_n^\tau(x)),$$

which implies that $P_n^\tau \in \mathrm{Diff}^k(\mathbb{R}^d)$. The continuous dependence of P_n^τ on τ follows directly from the definition (39) of P_n^τ and therefore (i) is proved.

(ii) For $\tau \in [0,1]$ define

$$\Delta_n^\tau(x) := DP_n^\tau(x) - B_n = D\eta_n^\tau(x).$$

As in (i), from (41) and (42) we derive that

$$\|DP_n^\tau(x) - B_n\| \leq M\delta^* \leq \delta.$$

Also, $\|\Delta_n^\tau(x)B_n^{-1}\| \leq M^2\delta^* < 1$. Then

$$\begin{aligned} DP_n^\tau(x)^{-1} &= (B_n + \Delta_n^\tau(x))^{-1} = B_n^{-1}(\mathrm{id} + \Delta_n^\tau B_n^{-1})^{-1} \\ &= B_n^{-1}\sum_{k=0}^{\infty}(-\Delta_n^\tau(x)B_n^{-1})^k \\ &= B_n^{-1}\big(\mathrm{id} + \sum_{k=1}^{\infty}(-1)^k(\Delta_n^\tau(x)B_n^{-1})^k\big), \end{aligned}$$

which implies that

$$\|DP_n^\tau(x)^{-1} - B_n^{-1}\| \leq M\frac{M^2\delta^*}{1-M^2\delta^*} \leq \delta,$$

proving (ii). $\square$

To show Proposition 12, choose $r > 0$ such that the statements of Lemma 13 hold for P_n^τ and additionally, using (B1), also the following inequalities hold for $n \in \mathbb{Z}$:

$$\left\|\frac{\partial(\pi_u \circ (P_n^\tau)^{-1})}{\partial x_u}\right\|_\infty < 1 \quad \text{and} \quad \|D_x(P_n^\tau)^{-1}\|_\infty > 1 \tag{43}$$

and

$$\|D_xP_n^\tau\|_\infty \left\|\frac{\partial(\pi_u \circ (P_n^\tau)^{-1}}{\partial x_u}\right\|_\infty^{k-\ell} \|D_x(P_n^\tau)^{-1}\|_\infty^\ell \quad < \quad \gamma, \tag{44}$$

$$\big\|D_x(P_n^\tau)^{-1}\big\|_\infty \left\|\frac{\partial(\pi_s \circ P_n^\tau)}{\partial x_u}\right\|_\infty^{k-\ell} \|D_xP_n^\tau\|_\infty^\ell \quad < \quad \gamma, \tag{45}$$

for a $\gamma \in (0,1)$.

According to Theorem 17, to prove Proposition 12 it is sufficient to show the existence of $Z_n^\tau \in C_1^k(\mathbb{R}^d)$ such that

$$Z_{n+1}^\tau(x) - \frac{\partial P_n^\tau}{\partial x}\big((P_n^\tau)^{-1}(x)\big) Z_n^\tau\big((P_n^\tau)^{-1}(x)\big) = \frac{\partial P_n^\tau}{\partial \tau}\big((P_n^\tau)^{-1}(x)\big) \tag{46}$$

for $n \in \mathbb{N}$. In the following lemma, some properties of the right hand side of (46) are established. For this purpose, let $\ell \in \mathbb{N}$, $1 \le \ell \le k$, $R > 0$, and define

$$\mathcal{M}_R^{k,\ell} := \big\{W \in C([0,1], C_{\text{flat,u}}^\ell(\mathbb{R}^d)) \colon \big\|\tfrac{\partial^\ell W}{\partial x^\ell}(\tau, x)\big\| \le R\|\pi_u x\|^{k-\ell}\big\}.$$

Lemma 15. The right hand side

$$X_n(\tau, x) := \frac{\partial P_n^\tau}{\partial \tau}\big((P_n^\tau)^{-1}(x)\big) \qquad (\tau \in [0,1], n \in \mathbb{N}) \tag{47}$$

of (46) satisfies

(i) $X_n(\tau, \cdot) \in C_{\text{flat,u}}^k(\mathbb{R}^d)$ for $n \in \mathbb{Z}$,

(ii) there exists $R > 0$ such that $X_n \in \mathcal{M}_R^{k,\ell}$ for $n \in \mathbb{Z}$.

Proof. From (39) and (47), we derive that

$$X_n(\tau, x) = g_n\big((P_n^\tau)^{-1}(x)\big) - f_n\big((P_n^\tau)^{-1}(x)\big). \tag{48}$$

Since $f_n - g_n \in C_{\text{flat}}^k(\mathbb{R}^d)$, it follows that $X_n(\tau, \cdot) \in C_{\text{flat}}^k(\mathbb{R}^d)$. Furthermore, using (B2) the subspace $\{(0, x^s) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}\}$ is invariant under f_n and g_n. This together with (37) and the fact that $B_n = \operatorname{diag}(B_n^u, B_n^s)$ implies that

$$\begin{aligned} \pi_u P_n^\tau(0, x^s) &= \pi_u B_n (0, x^s)^{\mathrm{T}} + (1-\tau)\pi_u f_n(0, x^s) + \tau \pi_u g_n(0, x^s) \\ &= 0. \end{aligned}$$

Consequently, the subspace $\{(0, x^s) \in \mathbb{R}^{d_u} \times \mathbb{R}^{d_s}\}$ is invariant under P_n^τ, i.e. $\pi_u (P_n^\tau)^{-1}(0, x^s) = 0$. By (35) we have $f_n - g_n \in C_{\text{flat,u}}^k(\mathbb{R}^d)$. Therefore,

$$D^i f_n\big((P_n^\tau)^{-1}(0, x^s)\big) - D^i g_n\big((P_n^\tau)^{-1}(0, x^s)\big) = 0.$$

Thus, from (48) we derive that $D_x^i X_n(\tau, 0, x^s) = 0$ and thus $X_n(\tau, \cdot) \in C_{\text{flat,u}}^k(\mathbb{R}^d)$.

(ii) Let

$$R := \sup_{x \in \mathbb{R}^d} \left\| \frac{\partial^k X_n}{\partial x^k}(\tau, x) \right\|.$$

Hence, for each $\tau \in [0,1]$ using the Taylor expansion of the function $\frac{\partial^\ell X_n}{\partial x^\ell}(\tau, x)$ up to order $k-\ell$ at $(0, x^s)$ leads to

$$\left\|\frac{\partial^\ell X_n}{\partial x^\ell}(\tau, x)\right\| = \left\|\frac{\partial^\ell X_n}{\partial x^\ell}(\tau, x^u, x^s) - \frac{\partial^\ell X_n}{\partial x^\ell}(\tau, 0, x^s)\right\| \leq R\|\pi_u x\|^{k-\ell},$$

which completes the proof. □

Next, we define for $n \in \mathbb{Z}$ the map $L_n \colon C\big([0,1], C^\ell(\mathbb{R}^d)\big) \to C\big([0,1], C^\ell(\mathbb{R}^d)\big)$ by

$$(L_n W)(\tau, x) := D_x P_n^\tau\big((P_n^\tau)^{-1}(x)\big)\, W\big(\tau, (P_n^\tau)^{-1}(x)\big). \tag{49}$$

Obviously, $\mathcal{M}_R^{k,\ell} \subset C([0,1], C^\ell(\mathbb{R}^d))$ and in the following lemma, we provide some useful estimates on the restriction of the product of linear operators $L_n^j := L_{n+j-1} \cdots L_n$ on $\mathcal{M}_R^{k,\ell}$, where $n \in \mathbb{Z}$ and $j \in \mathbb{N}$.

Lemma 16. For each $W \in \mathcal{M}_R^{k,\ell}$ the following statements hold for all $(\tau, x) \in [0,1] \times \mathbb{R}^d$ and $n \in \mathbb{Z}, j \in \mathbb{N}$.

(i) $\|L_n^j W(\tau, x)\| \leq R\gamma^j \|\pi_u x\|^k$.

(ii) There exists $\varepsilon > 0$ such that for $\|\pi_u x\| \leq \varepsilon$

$$\left\|\frac{\partial^i L_n^j W}{\partial x^i}(\tau, x)\right\| \leq R\gamma^j \|\pi_u x\|^{k-i} \qquad \text{for } i = 1, \dots, \ell.$$

Proof. (i) By (49), we have

$$L_n^j W(\tau, x) = \prod_{i=0}^{j-1} D_x P_{n+i}^\tau((P_{n+i}^\tau)^{-1}(x))\big(W(\tau, (P_{n+j-1}^\tau)^{-1} \circ \cdots \circ (P_n^\tau)^{-1}(x))\big), \tag{50}$$

which implies that

$$\|L_n^j W(\tau, x)\| \leq \prod_{i=0}^{j-1} \|D_x P_{n+i}^\tau\|_\infty \|W(\tau, (P_{n+j-1}^\tau)^{-1} \circ \cdots \circ (P_n^\tau)^{-1}(x))\|. \tag{51}$$

Since $W \in \mathcal{M}_R^{k,\ell}$, it follows with the mean value theorem that

$$\|W(\tau, x)\| \leq \frac{R}{(k-r+1)\cdots k}\|\pi_u x\|^k.$$

On the other hand, using the invariance of $\{0\}\times\mathbb{R}^{d_s}$ under P_n^τ and the mean value theorem, we obtain that

$$\|\pi_u \circ (P_{n+j-1}^\tau)^{-1} \circ \cdots \circ (P_n^\tau)^{-1}(x)\| \le \left(\prod_{i=0}^{j-1} \left\| \frac{\partial(\pi_u \circ (P_{n+i}^\tau)^{-1})}{\partial x_u} \right\|_\infty \right) \|\pi_u x\|.$$

Therefore, from (51) we derive that

$$\|L_n^j W(\tau, x)\| \le \frac{R}{(k-r+1)\cdots k} \|\pi_u x\|^k \prod_{i=0}^{j-1} \|D_x P_{n+i}^\tau\|_\infty \left\| \frac{\partial(\pi_u \circ (P_{n+i}^\tau)^{-1})}{\partial x_u} \right\|_\infty^k,$$

which together with (43) and (44) completes the proof of this part.

(ii) By induction, it is sufficient to show this assertion for $j = 1$, i.e. there exists $\varepsilon > 0$ such that for $\|\pi_u x\| \le \varepsilon$

$$\left\| \frac{\partial^i L_n W}{\partial x^i}(\tau, x) \right\| \le R\gamma \|\pi_u x\|^{k-i} \quad \text{ for } i = 1, \dots, \ell.$$

Note that this assertion follows directly from [Bon93, Lemma 4]. □

Proof of Proposition 12. By virtue of Lemma 16(i), the series

$$\sum_{j=0}^{\infty} L_{n-j}^j X_{n-1-j}(\tau, x) =: Z_n^\tau(x), \tag{52}$$

with $X_n(\tau, x)$ as in (47), converges for each $n \in \mathbb{Z}$. By definition of Z_n, we have

$$\begin{aligned} Z_{n+1}^\tau(x) &= \sum_{j=0}^{\infty} L_{n+1-j}^j X_{n-j}(\tau, x) \\ &= \sum_{j=1}^{\infty} L_{n+1-j}^j X_{n-j}(\tau, x) + X_n(\tau, x) \\ &= L_n Z_n^\tau(x) + X_n(\tau, x), \end{aligned}$$

which shows that $(Z_n^\tau(x))_{n\in\mathbb{Z}}$ is a solution of (46). Furthermore, using Lemma 16(ii), we get

$$\left\| \frac{\partial^i L_n^j X}{\partial x^i}(\tau, x) \right\| \le L\gamma^j \|\pi_u x\|^{k-i} \quad \text{ for } i = 1, \dots, \ell \tag{53}$$

if $\|x\| \leq \varepsilon$ for some $\varepsilon > 0$, and by cutting-off f_n and g_n, this estimate also holds for all $x \in \mathbb{R}^d$. Consequently, $Z_n^\tau(\cdot)$ is a C^ℓ function and in particular, $Z_n^\tau(0) = 0$ and $DZ_n^\tau(0) = 0$. Hence, $Z_n^\tau \in C_1^\ell(\mathbb{R}^d)$. Finally, by (53) the series $\sum_{j=0}^\infty L_{n+1-j}^j X_{n-j}(\tau, x)$ converges uniformly in τ to Z_n^τ in $\|\cdot\|_\ell$. Therefore, the map $\mathbf{Z}\colon [0,1] \to C_1^\ell(\mathbb{R}^d)^{\mathbb{Z}}$, $\tau \mapsto \mathbf{Z}^\tau := Z_\cdot^\tau(\cdot)$, is continuous. By virtue of Theorem 17, the systems (33) and (34) are C^ℓ-equivalent. $\square$

We are now in a position to prove the main theorem.

Proof of Theorem 5. By Proposition 6, an arbitrary system in $\mathcal{O}^{k+2}(A)$ is C^k equivalent to a system

$$\dot{x} = A(t)x + g(x) \tag{54}$$

in $\mathcal{O}_{\text{flat}}^k(A)$. According to Proposition 10, there exists a $\kappa > 0$ such that the associated time-κ system of (54)

$$x_{n+1} = A_n^{(\kappa)} x_n + f_n^{(\kappa)}(x_n) \tag{55}$$

is an element of $\mathcal{D}_{\text{flat}}^{k,\ell}(A)$. By Remark 11, there exist $\varphi_n \in C_{\text{flat,u}}^k(\mathbb{R}^d)$ and $\psi_n \in C_{\text{flat,s}}^k(\mathbb{R}^d)$ for $n \in \mathbb{N}$ such that (55) can be rewritten as

$$x_{n+1} = A_n^{(\kappa)} x_n + \varphi_n(x_n) + \psi_n(x_n). \tag{56}$$

Since $\varphi_n \in C_{\text{flat,u}}^k(\mathbb{R}^d)$ for $n \in \mathbb{N}$, it follows together with Proposition 12 that system (56) is C^ℓ equivalent to the following system

$$x_{n+1} = A_n^{(\kappa)} x_n + \psi_n(x_n).$$

Applying Proposition 12 once again, this system is C^ℓ equivalent to the linear system

$$x_{n+1} = A_n^{(\kappa)} x_n.$$

Therefore, by Proposition 9 system (54) is C^ℓ equivalent to its linearization $\dot{x} = A(t)x$. $\square$

5 Appendix

In this section the *method of path* [DRR81] is extended from autonomous to nonautonomous discrete time systems. It provides a way to construct a C^k

equivalence between two nonautonomous difference equations

$$x_{n+1} = F_n(x_n), \tag{57}$$
$$y_{n+1} = G_n(y_n), \tag{58}$$

with $F_n, G_n \in \mathrm{Diff}^k(\mathbb{R}^d)$ for $n \in \mathbb{N}$, by considering the homotopy

$$P\colon \mathbb{Z} \times \mathbb{R}^d \times [0,1] \to \mathbb{R}^d, \quad (n, x, \tau) \mapsto P_n^\tau(x) := (1-\tau)F_n(x) + \tau G_n(x)$$

between F_n and G_n or, equivalently, the homotopy of difference equations

$$z_{n+1} = P_n^\tau(z_n), \qquad (\tau \in [0,1]) \tag{59}$$

between (57) and (58). Define $\mathbf{P}\colon [0,1] \to \mathrm{Diff}^k(\mathbb{R}^d)^{\mathbb{Z}}, \tau \mapsto \mathbf{P}^\tau := P_\cdot^\tau(\cdot)$.

Theorem 17 (Method of the path)**.** Assume that

(i) $\mathbf{P}^\tau \in \mathrm{Diff}_0^k(\mathbb{R}^d)^{\mathbb{Z}}$ for $\tau \in [0,1]$,

(ii) $\mathbf{P}\colon [0,1] \to \mathrm{Diff}_0^k(\mathbb{R}^d)^{\mathbb{Z}}$ is continuous,

(iii) For every $\tau \in [0,1]$ and $n \in \mathbb{N}$ there exists $Z_n^\tau \in C_1^k(\mathbb{R}^d)$ with

$$Z_{n+1}^\tau(x) - \frac{\partial P_n^\tau}{\partial x}\big((P_n^\tau)^{-1}(x)\big) Z_n^\tau\big((P_n^\tau)^{-1}(x)\big) = \frac{\partial P_n^\tau}{\partial \tau}\big((P_n^\tau)^{-1}(x)\big) \tag{60}$$

for $n \in \mathbb{N}$ and $\mathbf{Z}\colon [0,1] \to C_1^k(\mathbb{R}^d)^{\mathbb{Z}}$, $\tau \mapsto \mathbf{Z}^\tau := Z_\cdot^\tau(\cdot)$, is continuous.

Then (57) and (58) are C^k equivalent.

Proof. Since $\mathbf{Z} \in C([0,1], C_1^k(\mathbb{R}^d)^{\mathbb{Z}})$, there exists $K > 0$ such that

$$\|Z_n^\tau(x)\| \le K\|x\| \qquad \text{for all } n \in \mathbb{Z}, \tau \in [0,1], x \in \mathbb{R}^d,$$

Consequently, for $n \in \mathbb{Z}$, $x \in \mathbb{R}^d$ the initial value problem

$$\frac{d\xi}{d\tau}(\tau) = Z_n^\tau(\xi(\tau)), \qquad \xi(0) = x,$$

is uniquely solvable on $[0,1]$. Let $T_n(\tau, x)$ denote its unique solution, i.e.

$$\frac{\partial T_n}{\partial \tau}(\tau, x) = Z_n^\tau(T_n(\tau, x)), \qquad T_n(0, x) = x. \tag{61}$$

Then,

$$\|T_n(\tau, x)\| \leq \|x\| + M \int_0^\tau \|T_n(s, x)\| \, ds.$$

Using Gronwall's inequality [Har82, Chapter 3, Theorem 1.1], we obtain that $\|T_n(\tau, x)\| \leq e^{K\tau}\|x\|$ and thus $\lim_{x\to 0} T_n(\tau, x) = 0$ uniformly in $n \in \mathbb{Z}$.

To show that $(n, x) \mapsto T_n(\tau, x)$ is a C^k equivalence between (57) and (59), we first show that

$$T_{n+1}(\tau, F_n(x)) = P_n^\tau(T_n(\tau, x)) \qquad \text{for } n \in \mathbb{Z}, x \in \mathbb{R}^d. \tag{62}$$

For this purpose, let $n \in \mathbb{Z}$, $x \in \mathbb{R}^d$ and define $V : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$V(\tau, x) := P_n^\tau\big(T_n(\tau, F_n^{-1}(x))\big).$$

Taking derivatives on both sides with respect to τ yields

$$\frac{\partial V}{\partial \tau}(\tau, x) = D_x P_n^\tau\big(T_n(\tau, F_n^{-1}(x))\big)\frac{\partial T_n}{\partial \tau}(\tau, F_n^{-1}(x)) + \frac{\partial P_n^\tau}{\partial \tau}\big(T_n(\tau, F_n^{-1}(x))\big),$$

which, together with the fact that $T_n(\tau, F_n^{-1}(x)) = (P_n^\tau)^{-1}(V(\tau, x))$, implies

$$\frac{\partial V}{\partial \tau}(\tau, x) = D_x P_n^\tau\big((P_n^\tau)^{-1}(V(\tau, x))\big)\frac{\partial T_n}{\partial \tau}(\tau, F_n^{-1}(x)) + \frac{\partial P_n^\tau}{\partial \tau}\big((P_n^\tau)^{-1}(V(\tau, x))\big).$$

Using (61), we obtain

$$\begin{aligned}\frac{\partial V}{\partial \tau}(\tau, x) = {} & D_x P_n^\tau\big((P_n^\tau)^{-1}(V(\tau, x))Z_n(\tau, (P_n^\tau)^{-1}(V(\tau, x))) + \\ & + \frac{\partial P_n^\tau}{\partial \tau}((P_n^\tau)^{-1}(V(\tau, x)))\big),\end{aligned}$$

which, together with (60), gives

$$\frac{\partial V}{\partial \tau}(\tau, x) = Z_{n+1}(\tau, V(\tau, x)).$$

Furthermore, $V(0, x) = x$ and by the uniqueness of the solution of (61) for $n + 1$, we have $V(\tau, x) = T_{n+1}(\tau, x)$ and (62) follows.

To conclude the proof, it remains to show that $T_n(\tau, \cdot)$ is a local C^k diffeomorphism on $B_r(0)$ for some $r > 0$ which is independent of $n \in Z$. For this purpose, we define for $n \in \mathbb{N}$ the function $\eta_n(\tau, \cdot)\colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$\eta_n(\tau, x) := T_n(\tau, x) - x.$$

Since $T_n(0,x) = x$, it follows that $\eta_n(0,x) = 0$. Since $Z_n(\tau,x)$ is C^k in x, also $T_n(\tau,x)$ and $\eta_n(\tau,x)$ are C^k in x. Furthermore, from (61) we derive that

$$\frac{\partial D_x T_n}{\partial \tau}(\tau,x) = D_x Z_n(\tau, T_n(\tau,x)) D_x T_n(\tau,x),$$

which implies that

$$\frac{\partial D_x \eta_n}{\partial \tau}(\tau,x) = D_x Z_n(\tau, T_n(\tau,x))(\mathrm{id} + D_x \eta_n(\tau,x)),$$

or equivalently,

$$D_x \eta_n(\tau,x) = \int_0^\tau D_x Z_n(s, x + \eta_n(s,x))(\mathrm{id} + D_x \eta_n(s,x))\, ds. \tag{63}$$

Since $\mathbf{Z} \in C([0,1], C_1^k(\mathbb{R}^d)^{\mathbb{Z}})$, there exists $L > 0$ such that $\|D_x Z_n(\tau,x)\| \leq L\|x\|$. This, together with the fact that $\|T_n(\tau,x)\| \leq e^{K\tau}\|x\| \leq e^K\|x\|$, implies that

$$\|D_x Z_n(\tau, T_n(\tau,x))\| \leq Le^K\|x\|.$$

Hence, from (63) we derive that

$$\begin{aligned}
\|D_x \eta_n(\tau,x)\| &\leq Le^K \int_0^\tau \|x\|(1 + \|D_x \eta_n(s,x)\|)\, ds \\
&\leq Le^K\|x\| + Le^K\|x\| \int_0^\tau \|D_x \eta_n(s,x)\|\, ds.
\end{aligned}$$

Using Gronwall's inequality [Har82, Chapter 3, Theorem 1.1], we obtain that

$$\|D_x \eta_n(\tau,x)\| \leq Le^K\|x\| e^{Le^K\|x\|\tau}.$$

Consequently, there exists $r > 0$ such that

$$\rho := \sup_{n\in\mathbb{Z}} \max_{\tau\in[0,1]} \max_{\|x\|\leq r} \|\eta_n(\tau,x)\| < 1.$$

In view of Lemma 14, the map $T_n(\tau,\cdot) : B_r(0) \to T_n(\tau, B_r(0))$ is a C^k diffeomorphism and $B_{(1-\rho)r}(0) \subset T_n(\tau, B_r(0))$. Furthermore, by Lemma 14 the inverse function of $T_n(\tau,\cdot)$ denoted by $S_n(\tau,\cdot) : B_{(1-\rho)r}(0) \to B_r(0)$ satisfies $\lim_{x\to 0} S_n(\tau,x) = 0$ uniformly in $n \in \mathbb{Z}$. Consequently, $(n,x) \mapsto T_n(\tau,x)$ is a C^k equivalence between (57) and (59) and, in particular, between (57) and (58) for $\tau = 1$. $\square$

References

[AW96] B. Aulbach and T. Wanner. Integral manifolds for Carathéodory type differential equations in Banach spaces, in: B. Aulbach, F. Colonius (Eds.), Six Lectures on Dynamical Systems, World Scientific, Singapore, 1996, pp. 45-119.

[AW06] B. Aulbach and T. Wanner. Topological simplification of nonautonomous difference equations. *Journal of Difference Equations and Applications* **12** (2006), 283–296.

[BD84] P. Bonckaert and F. Dumortier. On a linearization theorem of Sternberg for germs of diffeomorphisms. *Math.Z.* **185** (1984), no. 1, 115–135.

[Bon93] P. Bonckaert. On the continuous dependence of the smooth change of coordinates in parametrized normal form theorems. *Journal of Differential Equations* **106** (1993), 107–120.

[BDDS15] P. Bonckaert, P. De Maesschalck, T.S. Doan and S. Siegmund. Partial linearization for planar nonautonomous differential equations. *Journal of Differential Equations* **258** (2015), 1618–1652.

[Bru95] F. Bruhat. Travaux de Sternberg. (French) [Works of Sternberg], *Séminaire Bourbaki* **6**, Exp. No. 217, 179–196, Soc. Math. France, Paris, 1995.

[Cop78] W.A. Coppel. *Dichotomies in Stability Theory*. Springer Lecture Notes in Mathematics **629**. Springer-Verlag, Berlin, 1978.

[DRR81] F. Dumortier, P.R. Rodrigues and R. Roussarie. *Germs of Diffeomorphisms in the Plane*. Springer Lecture Notes in Mathematics **902**. Springer-Verlag, 1981.

[Har82] P. Hartman. *Ordinary differential equations*. Birkhäuser, Boston, 1982.

[Nei05] K. Neirynck. *Local equivalence and conjugacy of families of vector fields and diffeomorphisms*. Diss. UHasselt Diepenbeek, 2005.

[Pal73] K. Palmer. A generalization of Hartman's linearization theorem. *J. Math. Anal. Appl.* **41** (1973), 753-758.

[SS78] R.J. Sacker and G.R. Sell. A spectral theory for linear differential systems. *J. Differential Equations* **27** (1978), no. 3, 320-358.

[Sie99] S. Siegmund. *Spektraltheorie, glatte Faserungen und Normalformen fr Differentialgleichungen vom Carathéodory-Typ.* Dissertation, University of Augsburg, 1999.

[Sie02a] S. Siegmund. Normal forms for nonautonomous differential equations. *Journal of Differential Equations* **178** (2002), 541–573.

[Sie02b] S. Siegmund. Dichotomy spectrum for nonautonomous differential equations. *Journal of Dynamics and Differential Equations* **14** (2002), no. 1, 243–258.

[Sie02c] S. Siegmund. Reducibility of nonautonomous linear differential equations. *J. London Math. Soc.* **65** (2002), no. 2, 397-410.

[Sie03] S. Siegmund. Normal forms for nonautonomous difference equations. *Computers & Mathematics with Applications* **45** (2003), 1059–1073.

[Ste57] S. Sternberg. Local contractions and a theorem of Poincaré. *Amer. J. Math.* **79** (1957), 809–824.

[Ste58] S. Sternberg. On the structure of local homeomorphisms of Euclidian n-space, I. Amer. J. Math. **80** (1958), 623–631.

[Ste59] S. Sternberg. On the structure of local homeomorphisms of Euclidian n-space, II. Amer. J. Math. **81** (1959), 578–605.