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Global attractivity for some classes of Riemann–Liouville fractional differential systems

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Abstract

In this paper, we present some results for existence of global solutions and attractivity for mulidimensional fractional differential equations involving Riemann-Liouville derivative. First, by using a Bielecki type norm and Banach fixed point theorem, we prove a Picard-Lindelöf type theorem on the existence and uniqueness of solutions. Then, applying the properties of Mittag-Leffler functions, we describe the attractivity of solutions to some classes of Riemann–Liouville linear fractional differential systems.

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1 Introduction

In recent years, fractional-order differential equations have attracted increasing interests due to their applications in modeling anomalous diffusion, time-dependent materials and processes with long range dependence, allometric scaling laws, and complex networks. For more details, we refer the reader to the monographs e.g. [9, 12, 7, 5]. In this paper we consider Riemann-Liouville differential systems

$$D^{\alpha}_{0+}x(t) = f(t, x(t)), \tag{1}$$

where $\alpha \in (0, 1)$ and D^{α}_{0+} is the Riemann-Liouvile derivative of order α, $f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is a given function and $x : (0, \infty) \to \mathbb{R}^d$ is the solution. The initial value problem for (1) we define as a problem of finding a solution that fulfills the condition

$$\lim_{t\to 0^+} t^{1-\alpha}x\,(t) = x_0$$

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for an a priori given $x_0 \in \mathbb{R}^d$. We investigate two fundamental problems connected to this equation: existence of a unique global on $(0,\infty)$ solution of the initial value problem and the attractivity of these solutions understood as the property of tending to zero of each solution. To prove the existence and uniqueness of the initial value problem, in the space $C_{1-\alpha}([0,\infty),\mathbb{R}^s)$ consisting of all continuous functions $f:(0,\infty)\to\mathbb{R}^s$ such that there exists the limit

$$\lim_{t\to 0^+} t^{1-\alpha} f(t),$$

we assume that there exists a bounded (or continuous) nonnegative function $L:[0,\infty)\to[0,\infty)$ such that

$$\|f(t,x)-f(t,y)\| \le L(t)\|x-y\| \tag{2}$$

for all $t\in[0,\infty)$ and $x,y\in\mathbb{R}^d$. For the attractivity problem we assume that the function f has the following form

$$f(t,x) = Ax(t) + Q(t)x(t) + g(t).$$

It seems that the problem for existence of solution of (1) was for the first time considered in [11] in case of $d=1$ and zero initial condition under assumption that f is bounded, continuous and Lipschitzian in the second variable. This result has been extended to arbitrary initial condition in [3]. Next paper dealing with the initial value problem for global solution of Riemann-Liouville differential systems is [4]. In this paper the authors consider the one-dimensional case ($d=1$) and they prove that for a stationary system

$$D^{\alpha}_{0+}x(t) = f(x(t))$$

the condition (2) with a constant function L is a sufficient condition for the existence and uniqueness of the solution of the initial value problem. After this paper the problem of existence of a global solution of Riemann-Liouville differential systems has been also considered in [8, 6, 16]. The results of the first paper are still about scalar equation and provide sufficient conditions for existence of at least one solution. The conditions are as follows:

1) $f(t,x(t))\in C_{1-\alpha}([0,\infty),\mathbb{R})$ for any $x(t)\in C_{1-\alpha}([0,\infty),\mathbb{R})$;

2) there exist three nonnegative continuous functions $p(t), w(t)$ and $q(t)$ defined on $[0,\infty)$, $p(t)$ and $q(t)$ are bounded, such that

$$|f(t,x)| \le p(t)w\left(\frac{|x|}{1+t^2}\right) + q(t),$$
$$w(t) < t$$

for all $t\in[0,\infty)$ and $x\in\mathbb{R}$;

3)

$$\sup_{t\ge 0}\int_0^t \frac{t^{1-\alpha}(t-s)^{\alpha-1}s^{\alpha-1}}{1+t^2}p(s)ds < \Gamma(\alpha).$$

Even these conditions guarantee only existence not uniqueness of a solution of the initial value problem it is difficult to compare them with our ones. Also for the one-dimensional case there are the results presented in [16], where the author proved existence and uniqueness of the global solution of the initial value problem if the function f has the following form

$$f(t,x) = p(t)x + q(t),$$

where $p \in C_{\beta}([0,\infty),\mathbb{R})$, $q \in C_{1-\alpha}([0,\infty),\mathbb{R})$, p and q are nonnegative and $0 \leq \beta < \alpha$. The multi-dimensional version of existence and uniqueness of the global solution of the initial value problem has been discussed in [6]. In this paper the authors, motivated by control theory applications, study this problem in the space of summable function and show that if the condition (2) with a constant function L is satisfied and the function $f(\cdot,0)$ is summable, then there exists a unique global solution of the initial value problem which is summable. In the above mentioned papers also the results for local solution are presented and further results of this kind are published in [5] and [17].

The problem of attractivity sometimes also called incorrectly asymptotic stability of solutions of nonlinear Riemann-Liouville differential systems has been considered in [13, 14, 1, 18]. The authors of [13] consider the right-hand side of (1) in the form

$$f(t,x) = Ax(t) + Q(t)x(t)$$

and claim that the stability of the matrix A and boundedness of the function Q implies attractivity. As it was noticed in [2] this is not true (it is enough to consider $Q(t) = C - A$, where C is any d by d matrix). In [1] a one-dimensional version of the equation (1) is investigated. The presented in this paper conditions are however difficult to check since they are expressed in the terms of properties of the function f as a functional on the set $(0,\infty) \times C((0,\infty),\mathbb{R})$, where $C((0,\infty),\mathbb{R})$ is the set of all continuous functions $g : (0,\infty) \to \mathbb{R}$. The main result of [14] (Theorem 1) is obtained under the assumptions that

$$\left\| t^{\alpha-1} E_{\alpha,\alpha}\left(At^{\alpha}\right) \right\| \leq M e^{-\gamma t}$$

for all $t \in [0,\infty)$ and certain positive M and γ. However, this assumption is never satisfied since

$$\lim_{t\to 0^+} \left\| t^{\alpha-1} E_{\alpha,\alpha}\left(At^{\alpha}\right) \right\| = \infty,$$

whereas

$$\lim_{t\to 0^+} M e^{-\gamma t} = 1.$$

Finally, in [18] the case of Riemann-Liouville differential equation in a Banach space $(X, \|\cdot\|)$ is considered. The author shows that the hypotheses:

1) $\|f(t,x)\| \leq L t^{-\beta} \|x\|^{\delta}$ for all $t \in (0,\infty)$, $x \in X$ and certain $L \geq 0$, $\alpha < \beta < 1$ and $\delta \in \mathbb{R}$;

2) there exists a constant $\kappa > 0$ such that for any bounded set $E \subset X$,

$$\sigma\left(f(t,E)\right) \leq \kappa \sigma(E),$$

where σ is the Hausdorf measure of noncompactness, guarantee existence of at least one attractive solution for the initial value problem.

2 Preliminaries

Consider the equation

$$D_{0_+}^{\alpha} x(t) = Ax(t) + Q(t)x(t) + g(t), \tag{3}$$

where $\alpha \in (0,1)$, and the Riemann–Liouville fractional derivative

$$D_{0+}^{\alpha} x(t) := \frac{d}{dt} I_{0+}^{1-\alpha} x(t)$$

is defined with the Riemann–Liouville fractional integral

$$I^{\beta}_{0+}x(t) := \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{1-\beta}x(\tau)d\tau,$$

for $\beta > 0$ and $I^0_{0+}x(t) := x(t)$, see [5, p. 13 & p. 27] and [10, p. 2]. Denote by $C_{1-\alpha}\left([0,\infty),\mathbb{R}^s\right)$ the set of all continuous functions $f:(0,\infty)\to\mathbb{R}^s$ such that there exists the limit

$$\lim_{t\to 0^+} t^{1-\alpha}f(t)$$

and $C^0_{1-\alpha}\left([0,\infty),\mathbb{R}^s\right)$ the subset of $C_{1-\alpha}\left(([0,\infty),\mathbb{R}^s\right)$ consisting of all functions $f\in C_{1-\alpha}\left([0,\infty),\mathbb{R}^s\right)$ satisfying

$$\sup_{t\ge 0} t^{1-\alpha}\left\|f(t)\right\| < \infty,$$

where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^s$. For $f\in C^0_{1-\alpha}\left([0,\infty),\mathbb{R}^s\right)$ denote

$$\|f\|_{1-\alpha} = \sup_{t\ge 0} t^{1-\alpha}\left\|f(t)\right\|.$$

It is obvious to see that $\|\cdot\|_{1-\alpha}$ is a norm and the space $\left(C^0_{1-\alpha}\left([0,\infty),\mathbb{R}^s\right),\|\cdot\|_{1-\alpha}\right)$ is a Banach space. In through this paper, we define

$$\Lambda^s_\alpha := \left\{\lambda\in\mathbb{C}\setminus\{0\} : |\arg(\lambda)| > \frac{\alpha\pi}{2}\right\}.$$

For any matrix $A\in\mathbb{R}^{s\times s}$, the set $\sigma(A)$ is the spectrum of A, i.e.

$$\sigma(A) := \{\lambda\in\mathbb{C} : \lambda \text{ is an eigenvalue of the matrix } A\}.$$

Furthermore, we use $|||A|||$ to denote the norm of A respect to the norm $\|\cdot\|$ on $\mathbb{R}^s$.

In our further consideration we will use the following result.

Lemma 2.1. *Let $A\in\mathbb{R}^{s\times s}$ and suppose that $\sigma(A)\subset\Lambda^s_\alpha$. Then the following statements are valid.*

(i) There exists $t_0>0$ and a positive constant M which depends on parameters t_0, α, A such that

$$t^{\alpha-1}|||E_{\alpha,\alpha}(t^\alpha A)||| \le \frac{M}{t^{\alpha+1}}, \quad \forall t\ge t_0.$$

(ii) The quantity

$$t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}d\tau$$

is bounded on $[0,\infty)$, i.e.

$$\sup_{t\ge 0} t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}d\tau < \infty.$$

Proof. (i) The proof is obtained by using [15, Lemma 4] and arguments as in [15, Lemma 5].

(ii) Let $T > 2t_0$ be an arbitrary constant. First, we consider the case $t\in[0,T]$. Due to the fact that $E_{\alpha,\alpha}(t^\alpha A)$ is continuous on $[0,T]$, we have

$$t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}d\tau \le C_1 t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}d\tau$$

$$
\begin{aligned}
&= C_1 t^{1-\alpha} t^{2\alpha-1} B(\alpha,\alpha) \\
&= C_1 t^{\alpha} B(\alpha,\alpha) \\
&\le C_1 T^{\alpha} B(\alpha,\alpha),
\end{aligned}
$$

where $C_1 := \sup_{t\in[0,T]} \|E_{\alpha,\alpha}(t^\alpha A)\|$, and $B(\alpha,\alpha)$ is the Beta function.

For the case $t > T > 2t_0$, we have

$$
\begin{aligned}
t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}d\tau &\le t^{1-\alpha}\int_0^{t-t_0} (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}d\tau \\
&\quad + t^{1-\alpha}\int_{t-t_0}^{t} (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}d\tau \\
&\le t^{1-\alpha}(I_1(t)+I_2(t)),
\end{aligned}
$$

where

$$
I_1(t) := \int_0^{t-t_0} (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}d\tau
$$

and

$$
I_2(t) := \int_{t-t_0}^{t} (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}d\tau.
$$

From (i), we see that

$$
\begin{aligned}
I_1(t) &\le M\int_0^{t-t_0} \frac{1}{(t-\tau)^{\alpha+1}}\tau^{\alpha-1}d\tau \\
&= M\int_0^{t/2} \frac{1}{(t-\tau)^{\alpha+1}}\tau^{\alpha-1}d\tau + M\int_{t/2}^{t-t_0} \frac{1}{(t-\tau)^{\alpha+1}}\tau^{\alpha-1}d\tau \\
&\le M\frac{2^{\alpha+1}}{t^{\alpha+1}}\int_0^{t/2}\tau^{\alpha-1}d\tau + M\left(\frac{t}{2}\right)^{\alpha-1}\int_{t_0}^{t/2}\frac{1}{u^{\alpha+1}}du \\
&\le M\left(\frac{2}{\alpha t} + \frac{2^{1-\alpha}}{\alpha t_0^\alpha t^{1-\alpha}}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
t^{1-\alpha}I_1(t) &\le M\left(\frac{2}{\alpha t^\alpha} + \frac{2^{1-\alpha}}{\alpha t_0^\alpha}\right) \\
&\le M\left(\frac{2}{\alpha T^\alpha} + \frac{2^{1-\alpha}}{\alpha t_0^\alpha}\right). \qquad (4)
\end{aligned}
$$

On the other hand, for $t > T > 2t_0$, we see that

$$
\begin{aligned}
t^{1-\alpha}I_2(t) &\le t^{1-\alpha}(t-t_0)^{\alpha-1}\int_0^{t_0} u^{\alpha-1}|||E_{\alpha,\alpha}(u^\alpha A)|||\,du \\
&\le 2^{1-\alpha}t_0^\alpha E_{\alpha,\alpha+1}(t_0^\alpha|||A|||). \qquad (5)
\end{aligned}
$$

From (4) and (5), we have

$$
\sup_{t\ge 0} t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}\,d\tau < \infty.
$$

The proof is complete. □

3 Existence and uniqueness of solutions to Riemann–Liouville fractional differential systems

The next theorem contains the main result of this paper regarding the existence and uniqueness of solution.

Theorem 3.1. *Suppose that the function $f : [0, \infty) \times \mathbb{R}^s \to \mathbb{R}^s$ is continuous and there exists a bounded (or continuous) nonnegative function $L : [0, \infty) \to [0, \infty)$ such that*

$$\|f(t, x) - f(t, y)\| \leq L(t)\|x - y\|$$

for all $t \in [0, \infty)$ and $x, y \in \mathbb{R}^s$, then the equation

$$D^{\alpha}_{0_+} x(t) = f(t, x) \tag{6}$$

with the initial condition

$$\lim_{t \to 0^+} t^{1-\alpha} x(t) = x_0, \tag{7}$$

has a unique solution in the space $C_{1-\alpha}([0, \infty); \mathbb{R}^s)$ for all $x_0 \in \mathbb{R}^s$.

Proof. To complete the proof of this theorem we only need proving that for any $T > 0$ the following integral equation

$$x(t) = \frac{x_0}{t^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, x(\tau))\, d\tau \tag{8}$$

has a unique solution on the interval $[0, T]$ for every $x_0 \in \mathbb{R}^s$.

For any $x_0 \in \mathbb{R}^s$ let us define an operator $\mathcal{T}_{x_0} : C_{1-\alpha}([0, T], \mathbb{R}^s) \to C_{1-\alpha}([0, T], \mathbb{R}^s)$ by the following formula

$$(\mathcal{T}_{x_0}\xi)(t) = \frac{x_0}{t^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, \xi(\tau))\, d\tau, \quad \forall t > 0.$$

This operator is well-defined. Indeed, for any $\xi \in C_{1-\alpha}([0, T], \mathbb{R}^s)$, we see

$$\begin{aligned}
t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} \|f(\tau, \xi(\tau))\| d\tau &\leq L t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} \|\xi(\tau)\| d\tau + t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} \|f(\tau, 0)\| d\tau \\
&\leq L t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} \tau^{\alpha-1} d\tau \times \|\xi(\tau)\|_{1-\alpha, T} \\
&\quad + t^{1-\alpha} \sup_{t \in [0,T]} \|f(t, 0)\| \int_0^t (t-\tau)^{\alpha-1} d\tau \\
&= L t^{\alpha} B(\alpha, \alpha) \times \|\xi\|_{1-\alpha} + \frac{\sup_{t\in[0,T]} \|f(t, 0)\|}{\alpha} t,
\end{aligned}$$

where $L := \sup_{t \in [0,T]} \|L(t)\|$, and $\|\xi\|_{1-\alpha,T} := \sup_{t\in[0,T]} t^{1-\alpha} \|\xi(t)\|$. This shows that

$$\lim_{t \to 0^+} t^{1-\alpha} (\mathcal{T}_{x_0}\xi)(t) = x_0.$$

Now we consider $t_0 \in (0, T)$ arbitrarily. For $h > 0$ small enough, we have

$$\left\| \int_0^{t_0+h} (t_0 + h - \tau)^{\alpha-1} f(\tau, \xi(\tau)) d\tau - \int_0^{t_0} (t_0 - \tau)^{\alpha-} f(\tau, \xi(\tau)) d\tau \right\|$$

$$\leq \int_{t_0}^{t_0+h} (t_0+h-\tau)^{\alpha-1}\|f(\tau,\xi(\tau))\|d\tau + \int_0^{t_0} \left((t_0-\tau)^{\alpha-1} - (t_0+h-\tau)^{\alpha-1}\right)\|f(\tau,\xi(\tau))\|d\tau$$
$$= I_1(h) + I_2(h),$$

where

$$I_1(h) := \int_{t_0}^{t_0+h} (t_0+h-\tau)^{\alpha-1}\|f(\tau,\xi(\tau))\|d\tau,$$
$$I_2(h) := \int_0^{t_0} \left((t_0-\tau)^{\alpha-1} - (t_0+h-\tau)^{\alpha-1}\right)\|f(\tau,\xi(\tau))\|d\tau.$$

By direct computation,

$$\begin{aligned} I_1(h) &\leq \sup_{t\in[t_0,t_0+h]} \|f(t,\xi(t))\| \int_{t_0}^{t_0+h} (t_0+h-\tau)^{\alpha-1}\, d\tau \\ &= \sup_{t\in[t_0,t_0+h]} \|f(t,\xi(t))\| \times \frac{h^\alpha}{\alpha}. \end{aligned} \tag{9}$$

Furthermore,

$$\begin{aligned} I_2(h) &\leq L\|\xi\|_{1-\alpha,T} \int_0^{t_0} \left((t_0-\tau)^{\alpha-1} - (t_0+h-\tau)^{\alpha-1}\right)\tau^{\alpha-1}d\tau \\ &\qquad + \sup_{t\in[0,t_0]} \|f(t,0)\| \int_0^{t_0} \left((t_0-\tau)^{\alpha-1} - (t_0+h-\tau)^{\alpha-1}\right) d\tau \\ &\leq I_{2,1}(h) + I_{2,2}(h). \end{aligned}$$

Note that

$$\begin{aligned} I_{2,1}(h) &= L\|\xi\|_{1-\alpha,T} \int_0^{t_0} \left((t_0-\tau)^{\alpha-1} - (t_0+h-\tau)^{\alpha-1}\right)\tau^{\alpha-1}d\tau \\ &= L\|\xi\|_{1-\alpha,T} \int_0^{t_0/2} \left((t_0-\tau)^{\alpha-1} - (t_0+h-\tau)^{\alpha-1}\right)\tau^{\alpha-1}d\tau \\ &\qquad + L\|\xi\|_{1-\alpha,T} \int_{t_0/2}^{t_0} \left((t_0-\tau)^{\alpha-1} - (t_0+h-\tau)^{\alpha-1}\right)\tau^{\alpha-1}d\tau \\ &\leq \frac{L\|\xi\|_{1-\alpha,T}}{\alpha} \left((t_0-\tau^*)^{\alpha-1} - (t_0+h-\tau^*)^{\alpha-1}\right)\left(\frac{t_0}{2}\right)^\alpha \\ &\qquad + \left(\frac{t_0}{2}\right)^\alpha \frac{L\|\xi\|_{1-\alpha,T}}{\alpha} \int_{t_0/2}^{t_0} \left((t_0-\tau)^{\alpha-1} - (t_0+h-\tau)^{\alpha-1}\right) d\tau \\ &\leq \frac{L\|\xi\|_{1-\alpha,T}}{\alpha} \left((t_0-\tau^*)^{\alpha-1} - (t_0+h-\tau^*)^{\alpha-1}\right)\left(\frac{t_0}{2}\right)^\alpha \\ &\qquad + \left(\frac{t_0}{2}\right)^\alpha \frac{L\|\xi\|_{1-\alpha,T}}{\alpha} h^\alpha, \end{aligned}$$

for some $\tau^* \in (0, t_0/2)$. Furthermore,

$$I_{2,2}(h) = \sup_{t\in[0,t_0]} \|f(t,0)\| \int_0^{t_0} \left((t_0-\tau)^{\alpha-1} - (t_0+h-\tau)^{\alpha-1}\right) d\tau$$

$$\leq \sup_{t\in[0,t_0]} \|f(t,0)\| \times \frac{h^\alpha}{\alpha}.$$

Thus the function $\mathcal{T}_{x_0}\xi(t)$ is continuous on the right-hand side at t_0. The left-hand side continuity at t_0 is proved similarly, so for any $\xi \in C^0_{1-\alpha}([0,\infty),\mathbb{R}^s)$, the function $\mathcal{T}_{x_0}\xi(t)$ is continuous on $(0,T]$.

Let $\gamma > 0$ be arbitrary but fixed, we have the estimates

$$\begin{aligned}
\frac{\left\|t^{1-\alpha}\left(\mathcal{T}_{x_0}\xi - \mathcal{T}_{x_0}\tilde{\xi}\right)(t)\right\|}{e^{\gamma t}} &\leq \frac{t^{1-\alpha}}{\Gamma(\alpha)e^{\gamma t}}\int_0^t (t-\tau)^{\alpha-1}\frac{L(\tau)\|\xi(\tau)-\tilde{\xi}(\tau)\|\tau^{1-\alpha}\tau^{\alpha-1}e^{\gamma\tau}}{e^{\gamma\tau}}d\tau \\
&\leq \frac{Lt^{1-\alpha}}{\Gamma(\alpha)e^{\gamma t}}\|\xi-\tilde{\xi}\|_{w,T}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}e^{\gamma\tau}d\tau, \qquad (10)
\end{aligned}$$

where $\xi, \tilde{\xi} \in C_{1-\alpha}([0,T],\mathbb{R}^s)$ and

$$\|\xi\|_{w,T} := \sup_{t\in[0,T]} \frac{t^{1-\alpha}\|\xi(t)\|}{e^{\gamma t}}.$$

Moreover,

$$\begin{aligned}
\frac{1}{e^{\gamma t}}\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}e^{\gamma\tau}d\tau &= \int_0^{t/2}(t-\tau)^{\alpha-1}\tau^{\alpha-1}e^{-\gamma(t-\tau)}d\tau + \int_{t/2}^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}e^{-\gamma(t-\tau)}d\tau \\
&\leq \frac{2^{1-\alpha}}{t^{1-\alpha}}\int_0^{t/2}\tau^{\alpha-1}e^{-\gamma\tau}d\tau + \frac{2^{1-\alpha}}{t^{1-\alpha}}\int_0^{t/2}\tau^{\alpha-1}e^{-\gamma\tau}d\tau \\
&\leq \frac{2^{2-\alpha}}{t^{1-\alpha}\gamma^\alpha}\Gamma(\alpha).
\end{aligned}$$

The last estimate together with (10) implies that

$$\|\mathcal{T}_{x_0}\xi - \mathcal{T}_{x_0}\hat{\xi}\|_{w,T} \leq \frac{L2^{2-\alpha}}{\gamma^\alpha} \times \|\xi - \hat{\xi}\|_{w,T}.$$

Choosing γ such that

$$\frac{L2^{2-\alpha}}{\gamma^\alpha} < 1$$

implies that the operator $\mathcal{T}_{x_0}$ is contractive. By Banach fixed point theorem we see that the operator $\mathcal{T}_{x_0}$ has a unique fixed point in $C_{1-\alpha}([0,T];\mathbb{R}^s)$ which is also the unique solution of the integral equation (8). The proof is complete. □

4 Attractivity of solutions to Riemann–Liouville fractional differential systems

Consider the equation

$$D^\alpha_{0+}x(t) = Ax(t) + Q(t)x(t) + g(t), \qquad (11)$$

where $A \in \mathbb{R}^{s\times s}$, $Q : [0,\infty) \to \mathbb{R}^{s\times s}$ and $g : [0,\infty) \to \mathbb{R}^s$ are continuous functions.

From Theorem 3.1, we see that for any $x_0 \in \mathbb{R}^s$, the equation (11) with the initial condition

$$\lim_{t\to 0^+} t^{1-\alpha}x(t) = x_0$$

has a unique global solution in $C_{1-\alpha}([0,\infty),\mathbb{R}^s)$. In this section, we will study the global attractivity of (11). We recall here this important definition, see e.g. [1, Definition 2.4].

Definition 4.1. *The equation* (11) *is called globally attractive if for any* $x_0 \in \mathbb{R}^s$ *the solution* $\varphi(\cdot, x_0)$ *of* (11) *such that*

$$\lim_{t\to 0^+} t^{1-\alpha}\varphi(t,x_0) = x_0$$

tends to zero at infinity, i.e.

$$\lim_{t\to\infty} \varphi(t,x_0) = 0.$$

Theorem 4.2. *Consider the equation* (11)*. Suppose that* $\sigma(A) \subset \Lambda^s_\alpha$*, the matrix valued function* $Q : [0,\infty) \to \mathbb{R}^{s\times s}$ *satisfies*

$$\sup_{t\geq 0} t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} |||E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) Q(\tau)||| \, d\tau < 1,$$

and $g : [0,\infty) \to \mathbb{R}^s$ *is continuous such that*

$$\sup_{t\geq 0} t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} \|E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) g(\tau)\| \, d\tau < \infty.$$

Then for any $x_0 \in \mathbb{R}^s$*, we have* $\varphi(\cdot, x_0) \in C^0_{1-\alpha}([0,\infty),\mathbb{R}^s)$*. In particular, the equation* (11) *is globally attractive.*

Proof. It is easy to chack that the conditions of this Theorem implies that the condition of Theorem 3.1 in particular there exists a unique global solution of the initial problem for all initial conditions. Using the variation-of-constants formula [6, Theorem 4.2], we see that the solution $\varphi(\cdot, x_0)$ satisfies the following equation

$$\varphi(t,x_0) = t^{\alpha-1}E_{\alpha,\alpha}(At)\,x_0 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) Q(\tau)\varphi(\tau,x_0)\,d\tau$$
$$+ \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)g(\tau)\, d\tau$$

for all $t > 0$. For $x_0 \in \mathbb{R}^s$ let us define an operator

$$\mathcal{T}_{x_0} : C^0_{1-\alpha}([0,\infty),\mathbb{R}^s) \to C^0_{1-\alpha}([0,\infty),\mathbb{R}^s)$$

by the following formula

$$(\mathcal{T}_{x_0}\xi)(t) = t^{\alpha-1}E_{\alpha,\alpha}(At)\,x_0 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) Q(\tau)\xi(\tau)\,d\tau$$
$$+ \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha A)g(\tau)\, d\tau, \quad \forall t > 0.$$

It is obvious that $\mathcal{T}_{x_0}\xi \in C^0_{1-\alpha}([0,\infty),\mathbb{R}^s)$ for each $\xi \in C^0_{1-\alpha}([0,\infty),\mathbb{R}^s)$. We will show that $\mathcal{T}_{x_0}$ is a contraction mapping, i.e. there exists a constant $q \in (0,1)$ such that

$$\|\mathcal{T}_{x_0}\xi - \mathcal{T}_{x_0}\tilde{\xi}\|_{1-\alpha} \leq q\|\xi - \tilde{\xi}\|_{1-\alpha}$$

for all ξ, $\tilde{\xi} \in C^0_{1-\alpha}([0,\infty),\mathbb{R}^s)$. Indeed,

$$\left\|\left(\mathcal{T}_{x_0}\xi - \mathcal{T}_{x_0}\tilde{\xi}\right)(t)\right\| \leq \int_0^t (t-\tau)^{\alpha-1} |||E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) Q(\tau)||| \times \|\xi(\tau) - \tilde{\xi}(\tau)\|$$

$$\leq \int_0^t (t-\tau)^{\alpha-1} \, |||E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) Q(\tau)||| \, \tau^{\alpha-1} \, d\tau \times \|\xi - \tilde{\xi}\|_{1-\alpha}$$

and therefore

$$\begin{aligned}\| \, \mathcal{T}_{x_0}\xi - \mathcal{T}_{x_0}\tilde{\xi}\|_{1-\alpha} &\leq \|\xi - \tilde{\xi}\|_{1-\alpha} t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} \, |||E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) Q(\tau)||| \, \tau^{\alpha-1} d\tau \\ &\leq q\|\xi - \tilde{\xi}\|_{1-\alpha},\end{aligned}$$

where

$$q := \sup_{t\geq 0} \, t^{1-\alpha} \int_0^t ||| \, (t-\tau)^{\alpha-1} \, E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) Q(\tau)||| \, \tau^{\alpha-1} d\tau.$$

By Banach fixed point theorem there exists a unique fixed point ξ of $\mathcal{T}_{x_0}$. It is easy to check that this fixed point is the solution $\varphi(\cdot, x_0)$ of the equation (11) and, since $\xi \in C^0_{1-\alpha}\left([0,\infty), \mathbb{R}^s\right)$, in particular

$$\lim_{t\to\infty} \varphi\left(t, x_0\right) = 0.$$

The proof is complete. □

Using Theorem 4.2, we obtain immediately the following corollary.

Corollary 4.3. *Consider the equation* (11)*. Suppose that* $\sigma(A) \subset \Lambda^s_\alpha$*, the function* $Q : [0,\infty) \to \mathbb{R}^{s\times s}$ *satisfies*

$$\sup_{t\geq 0} |||Q(t)||| < \frac{1}{\sup\limits_{t\geq 0} t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} \, |||E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right)||| \, d\tau},$$

and $g : [0,\infty) \to \mathbb{R}^s$ *is continuous such that*

$$\sup_{t\geq 0} \, t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} \, ||E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) g(\tau)|| \, d\tau < \infty.$$

Then,

$$\varphi\left(\cdot, x_0\right) \in C^0_{1-\alpha}([0,\infty), \mathbb{R}^s), \quad \forall x_0 \in \mathbb{R}^s.$$

Theorem 4.4. *Assume that* $\sigma(A) \subset \Lambda^s_\alpha$*,* $Q : [0,\infty) \to \mathbb{R}^{s\times s}$ *is continuous and satisfies*

$$\lim_{t\to\infty} |||Q(t)||| = 0, \tag{12}$$

and $g : [0,\infty) \to \mathbb{R}^s$ *is continuous such that*

$$\sup_{t\geq 0} \, t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} \, \|E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) g(\tau)\| \, d\tau < \infty.$$

Then for any $x_0 \in \mathbb{R}^s$*, the solution* $\varphi(\cdot, x_0)$ *of* (11) *satisfies*

$$\varphi(\cdot, x_0) \in C^0_{1-\alpha}([0,\infty), \mathbb{R}^s).$$

In particular, the equation (11) *is globally attractive.*

Proof. Let us choose $K > 0$ large enough such that

$$\sup_{t\geq 0} |||E_{\alpha,\alpha}(t^\alpha A)||| \times \sup_{t\geq 0} |||Q(t)||| \leq K$$

and

$$\frac{\sup_{t\geq 0} t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1} |||E_{\alpha,\alpha}((t-\tau)^\alpha)A)|||\ \tau^{\alpha-1}\, d\tau}{K} \leq \frac{1}{4}.$$

Let $\gamma > 0$ be arbitrary but fixed. On the space $C^0_{1-\alpha}([0,\infty),\mathbb{R}^s)$, we defined a functional $\|\cdot\|_w$ as below

$$\|\xi\|_w := \sup_{t\geq 0} \frac{t^{1-\alpha}\|\xi(t)\|}{\exp(\gamma t)}, \quad \forall \xi \in C^0_{1-\alpha}([0,\infty),\mathbb{R}^s).$$

It is obvious to see that $(C^0_{1-\alpha}([0,\infty),\mathbb{R}^s), \|\cdot\|_w)$ is also a Banach space.

Now for each $x_0 \in \mathbb{R}^s$, on the space $C^0_{1-\alpha}([0,\infty),\mathbb{R}^s)$, we construct an operator $\mathcal{T}_{x_0}$ by

$$\begin{aligned}\mathcal{T}_{x_0}\xi(t) := t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha A)x_0 &+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left((t-\tau)^\alpha A\right) Q(\tau)\xi(\tau)\, d\tau \\ &+ \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) g(\tau)\, d\tau\end{aligned}$$

for all $t > 0$. This operator is well-defined.

Because (12), there exists $T > 0$ (large enough) such that

$$|||Q(t)||| \leq \frac{1}{K}, \quad \forall t \geq T.$$

Consider the case $t \in [0,T]$, we obtain the estimates

$$\begin{aligned}\frac{t^{1-\alpha}\|\mathcal{T}_{x_0}\xi(t) - \mathcal{T}_{x_0}\tilde{\xi}(t)\|}{\exp(\gamma t)} &\leq \frac{t^{1-\alpha}}{\exp(\gamma t)} \int_0^t (t-\tau)^{\alpha-1} |||E_{\alpha,\alpha}((t-\tau)^\alpha A)||| \times |||Q(\tau)||| \times \|\xi(\tau) - \tilde{\xi}(\tau)\| d\tau \\ &\leq K \times t^{1-\alpha} \int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1} \exp(-\gamma(t-\tau)) d\tau \times \|\xi - \tilde{\xi}\|_w \\ &\leq \frac{K 2^{2-\alpha}\Gamma(\alpha)}{\gamma^\alpha} \times \|\xi - \tilde{\xi}\|_w\end{aligned}$$

for any $\xi, \tilde{\xi} \in C^0_{1-\alpha}([0,\infty),\mathbb{R}^s)$. On the other hand, for $t > T$, we have

$$\begin{aligned}\frac{t^{1-\alpha}\|\mathcal{T}_{x_0}\xi(t) - \mathcal{T}_{x_0}\tilde{\xi}(t)\|}{\exp(\gamma t)} &\leq \frac{t^{1-\alpha}}{\exp(\gamma t)} \int_0^t (t-\tau)^{\alpha-1} |||E_{\alpha,\alpha}((t-\tau)^\alpha A)||| \times |||Q(\tau)||| \times \|\xi(\tau) - \tilde{\xi}(\tau)\|\, d\tau \\ &\leq I_1(t) + I_2(t),\end{aligned}$$

where

$$I_1(t) := \frac{t^{1-\alpha}}{\exp(\gamma t)} \int_0^T (t-\tau)^{\alpha-1} |||E_{\alpha,\alpha}((t-\tau)^\alpha A)||| \times |||Q(\tau)||| \times \|\xi(\tau) - \tilde{\xi}(\tau)\|\, d\tau$$

and

$$I_2(t) := \frac{t^{1-\alpha}}{\exp(\gamma t)} \int_T^t (t-\tau)^{\alpha-1} |||E_{\alpha,\alpha}((t-\tau)^\alpha A)||| \times |||Q(\tau)||| \times \|\xi(\tau) - \tilde{\xi}(\tau)\|\, d\tau.$$

By using the same arguments as in the proof of Theorem 3.1, we obtain

$$I_1(t) \leq \frac{K 2^{2-\alpha}\Gamma(\alpha)}{\gamma^\alpha} \times \|\xi - \tilde{\xi}\|_w.$$

For the quantity $I_2(t)$, we estimate

$$\begin{aligned} I_2(t) &\leq t^{1-\alpha}\int_T^t (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}\exp\left(-\gamma(t-\tau)\right)\, d\tau \times \frac{1}{K} \times \|\xi - \tilde{\xi}\|_w \\ &\leq \frac{\sup_{t\geq 0} t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}\, d\tau}{K} \times \|\xi - \tilde{\xi}\|_w. \end{aligned}$$

Thus,

$$\|\mathcal{T}_{x_0}\xi - \mathcal{T}_{x_0}\tilde{\xi}\|_w \leq \left(\frac{K 2^{2-\alpha}\Gamma(\alpha)}{\gamma^\alpha} + \frac{\sup_{t\geq 0} t^{1-\alpha}\int_0^t (t-\tau)^{\alpha-1}|||E_{\alpha,\alpha}((t-\tau)^\alpha A)|||\tau^{\alpha-1}\, d\tau}{K}\right) \times \|\xi - \tilde{\xi}\|_w$$

for all $\xi, \tilde{\xi} \in C^0_{1-\alpha}([0,\infty), \mathbb{R}^s)$. This implies that $\mathcal{T}_{x_0}$ is contractive in $(C^0_{1-\alpha}([0,\infty),\mathbb{R}^s), \|\cdot\|_w)$ if we choose $\gamma > 0$ satisfies

$$\frac{K 2^{2-\alpha}\Gamma(\alpha)}{\gamma^\alpha} \leq \frac{1}{4}.$$

By Banach fixed point theorem, the operator $\mathcal{T}_{x_0}$ has a unique fixed point in $C^0_{1-\alpha}([0,\infty),\mathbb{R}^s)$ which is also the unique solution $\varphi(\cdot, x_0)$ of (11) on $[0,\infty)$. Because $\varphi(\cdot, x_0) \in C^0_{1-\alpha}([0,\infty),\mathbb{R}^s)$, the proof is complete. □

5 Example

In this section, we give some examples to illustrate for the theoretical results above.

Example 5.1. *Consider the equation*

$$D^{1/2}_{0+}x(t) = -x(t) + Q(t)x(t) + g(t), \tag{13}$$

where $Q : [0,\infty) \to \mathbb{R}$ is continuous and satisfies

$$|Q(t)| \leq \frac{1}{\sup_{t\geq 0} t^{1/2}E_{1/2}(-t^{1/2})}, \quad \forall t > 0,$$

and $g(t) = \frac{1}{1+t^{1/2}}$ for all $t \geq 0$. Then for any $x_0 \in \mathbb{R}$, the solution $\varphi(\cdot, x_0)$ of (13) with the initial condition $\lim_{t\to 0^+} t^{1/2}x(t) = x_0$ converges to zero as t tends to infinity. Indeed, from the assumption on the function $Q(\cdot)$, we have

$$\begin{aligned} &\left| t^{1/2}\int_0^t (t-\tau)^{-1/2}E_{1/2,1/2}\left(-(t-\tau)^{1/2}Q(\tau)\right)\, d\tau\right| \\ &\qquad \leq \frac{t^{1/2}}{\sup_{t\geq 0} t^{1/2}E_{1/2}(-t^{1/2})}\int_0^t (t-\tau)^{-1/2}E_{1/2,1/2}(-(t-\tau)^{1/2})\tau^{-1/2}\, d\tau \\ &\qquad \leq \frac{t^{1/2}}{\sup_{t\geq 0} t^{1/2}E_{1/2}(-t^{1/2})} \times E_{1/2}(-t^{1/2}), \end{aligned}$$

see [12, Formula (1.100), p. 25]. Hence,

$$\left| t^{1/2} \int_0^t (t-\tau)^{-1/2} E_{1/2,1/2}\left(-(t-\tau)^{1/2} Q(\tau)\right)\, d\tau \right| < 1.$$

On the other hand,

$$\begin{aligned}
&\left| t^{1/2} \int_0^t (t-\tau)^{-1/2} E_{1/2,1/2}\left(-(t-\tau)^{1/2} g(\tau)\right)\, d\tau \right| \\
&\qquad \leqslant \left| t^{1/2} \int_0^t (t-\tau)^{-1/2} E_{1/2,1/2}\left(-(t-\tau)^{1/2} \tau^{-1/2}\right)\, d\tau \right| \\
&\qquad = t^{1/2} E_{1/2}(-t^{1/2}) < \infty
\end{aligned}$$

for all $t \geq 0$. *Following Theorem 4.2, we see that for any* $x_0 \in \mathbb{R}$, *the solution* $\varphi(\cdot, x_0)$ *of* (13) *with the initial condition* $\lim_{t\to 0^+} t^{1/2} x(t) = x_0$ *converges to zero as* t *tends to infinity.*

Example 5.2. *Consider the equation*

$$D_{0+}^{1/2} x(t) = -x(t) + Q(t)x(t) + g(t), \tag{14}$$

where

$$Q(t) := \begin{cases} 1000, & t \in [0, 1000], \\ \frac{1000^2}{t}, & t \in [1000, \infty), \end{cases}$$

and $g(t) = \frac{1}{1+t^{1/2}}$ *for all* $t \geq 0$. *In this case, it is easy to see that the conditions in Theorem 4.4 are satisfied. Thus, for any* $x_0 \in \mathbb{R}$, *the solution* $\varphi(\cdot, x_0)$ *of* (14) *with the initial condition* $\lim_{t\to 0^+} t^{1/2} x(t) = x_0$ *decays at infinity.*

6 Conclusions

In this paper, we have studied the existence and attractivity of solution of nonlinear differential system with Riemann-Liouville fractional derivative. Using Banach fixed point theorem and properties of Mittag-Leffler functions we obtained some sufficient conditions for these properties.

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