GENERATION OF NONLOCAL FRACTIONAL DYNAMICAL SYSTEMS BY FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We show that any two trajectories of solutions of a one-dimensional fractional differential equation (FDE) either coincide or do not intersect each other. However, in the higher dimensional case, two different trajectories can meet. Furthermore, one-dimensional FDEs and triangular systems of FDEs generate nonlocal fractional dynamical systems, whereas a higher dimensional FDE does not, in general, generate a nonlocal dynamical system.

1. Introduction. In recent years, fractional differential equations (FDEs) have attracted increasing interest due to the fact that they can model many mathematical problems in science and engineering [**11, 15, 21**]. In this paper, we consider a d-dimensional, fractional differential equation involving the *Caputo derivative* ${}^{C}D^{\alpha}_{0+}$ *of order* $\alpha \in (0,1)$, for t in a finite interval $J := [0,T]$ or in the real half-line $J := [0,\infty)$:

$$ {}^{C}D^{\alpha}_{0+}x(t) = f(t,x(t)). \tag{1} $$

Here, $f : J \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous vector-valued function. A continuous function $x : J \to \mathbb{R}^d$ is called a *solution* of (1) if this equation is satisfied for all $t \in J \setminus \{0\}$, in which case $x(0)$ is called the *initial value* of the solution $x(\cdot)$.

We are interested to know whether (1) generates a dynamical system so that the tools and methods of the classical theory of dynamical systems are applicable in the investigation of FDEs. Another important problem in the theory of FDEs, which is closely related to the problem of generation of dynamical systems by FDEs, is the question of whether two different trajectories of a FDE can intersect. We solve both these

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problems. Namely, we show that a one-dimensional FDE or a triangular FDE generates a nonlocal fractional dynamical system, whereas, in general, a higher dimensional FDE does not. Correspondingly, two different trajectories of a one-dimensional or a triangular FDE cannot meet, whereas different trajectories of a higher dimensional FDE may intersect each other. As a byproduct of our investigation, we get lower bounds for the solutions of one-dimensional FDEs and of triangular linear FDEs.

The question of whether solutions of (1) can intersect was treated by Diethelm [**10, 11**], Diethelm and Ford [**12**], Agarwal *et al.* [**1**], Hayek *et al.* [**14**], and Bonilla, Rivero and Trujillo [**4**]. Note that, in the case of ordinary differential equations (ODEs), it is well known that two trajectories either coincide or they do not intersect; the authors mentioned above proved that similar results hold for fractional differential equations of order $\alpha \in (0,1)$. The main difficulty of the problem for FDEs is the nonlocal nature (or history-dependence) of solutions of FDEs. The above authors used various tools to deal with the FDE case. However, several flaws make their proofs incomplete. We will present some discussion about this matter in Section 3.

The paper is organized as follows. Section 2 is a preparatory section, where we present some basic notions from fractional calculus and the theory of FDEs. Section 3 is devoted to results on separation of solutions of one-dimensional FDEs; we also discuss flaws in the proofs of results from the papers mentioned above. In Section 4, we study the generation of nonlocal fractional dynamical systems by one-dimensional FDEs. Section 5 is devoted to high dimensional triangular systems of FDEs, where, based on the results in Section 4, we show that a triangular system of FDEs does generate a nonlocal fractional dynamical system. In Section 6, we show that a higher dimensional FDE does not, in general, generate a nonlocal dynamical system: two different trajectories of a high dimensional FDE may intersect each other in finite time.

2. Preliminaries.

We start this section by briefly recalling a framework of fractional calculus and fractional differential equations. For more details, we refer to the books of Diethelm [**11**] and by Kilbas, Srivastava and Trujillo [**15**].

Let $\mathbb{R}^d$ be the standard d-dimensional Euclidean space equipped with usual Euclidean norm. We denote by $\mathbb{R}_+$ the set of all nonnegative real numbers, by $C([0,\infty);\mathbb{R}^d)$ the space of continuous functions from $[0,\infty)$ to $\mathbb{R}^d$, and by $\big(C_\infty(\mathbb{R}^d), \|\cdot\|_\infty\big) \subset C([0,\infty);\mathbb{R}^d)$ the space of all continuous functions $\xi : \mathbb{R}_+ \to \mathbb{R}^d$ that are uniformly bounded on $\mathbb{R}_+$, i.e.,

$$\|\xi\|_\infty := \sup_{t\in\mathbb{R}_+} \|\xi(t)\| < \infty.$$

It is well known that $\big(C_\infty(\mathbb{R}^d), \|\cdot\|_\infty\big)$ is a Banach space.

Let $\alpha > 0$, $[a,b] \subset \mathbb{R}$ and $x : [a,b] \to \mathbb{R}$, with $x \in L^1([a,b])$, i.e., $\int_a^b |x(\tau)|\, d\tau < \infty$. Then, the Riemann–Liouville integral of order α of the function $x(\cdot)$ is defined by

$$I^\alpha_{a+}x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau)\, d\tau \quad \text{for } t \in (a,b],$$

see, e.g., Diethelm **[11]**, where the Gamma function is defined by

$$\Gamma(\alpha) := \int_0^\infty \tau^{\alpha-1} \exp(-\tau)\, d\tau \quad \text{for } \alpha > 0.$$

The corresponding Riemann–Liouville fractional derivative of order α of an absolutely continuous function $x(\cdot) : [a,b] \to \mathbb{R}$ is given by

$${}^R D^\alpha_{a+}x(t) := (D^m I^{m-\alpha}_{a+} x)(t) \quad \text{for } t \in (a,b],$$

where $D = d/dt$ is the usual derivative and $m := \lceil \alpha \rceil$ is the smallest integer greater than or equal to α. On the other hand, the *Caputo fractional derivative* ${}^C D^\alpha_{a+}x$ of a function $x \in C^m([a,b])$ is defined by

$${}^C D^\alpha_{a+}x(t) := (I^{m-\alpha}_{a+} D^m x)(t) \quad \text{for } t \in (a,b].$$

The Caputo fractional derivative of a d–dimensional vector function $x(t) = (x_1(t), \dots, x_d(t))^{\mathrm{T}}$ is defined component-wise as

$${}^C D^\alpha_{0+}x(t) = ({}^C D^\alpha_{0+}x_1(t), \dots, {}^C D^\alpha_{0+}x_d(t))^{\mathrm{T}}.$$

It is well-known that the initial-value problem of the FDE (1) is equivalent to a Volterra integral equation of the second kind. Namely, we have the following result.

Lemma 1. A continuous function $x : J \to \mathbb{R}$ is a solution of the FDE (1) with the initial value condition $x(0) = x_0$ if and only if it is a solution of the Volterra integral equation of the second kind

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f\big(\tau, x(\tau)\big)\, d\tau. \tag{2}$$

Proof. For a proof when $d = 1$, we refer to Diethelm [**11,** Lemma 6.2, p. 86] and Kilbas *et al.* [**15,** Theorem 3.24, p. 199]; the multidimensional case $d > 1$ follows by componentwise application of the one-dimensional case. □

3. Separation of trajectories of solutions of one-dimensional FDE.

3.1. Two different trajectories of a one-dimensional FDE do not meet. In this section, we consider the one-dimensional case of the system (1), i.e., the FDE

$$^{C}D^{\alpha}_{0+}x(t) = f(t, x(t)), \tag{3}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Assume that f satisfies the following Lipschitz condition on the second variable: there exists a nonnegative continuous function $L : J \to \mathbb{R}_+$ such that

$$|f(t,x) - f(t,y)| \le L(t)|x-y| \quad \text{for all } t \in J \text{ and all } x,\, y \in \mathbb{R}. \tag{4}$$

It is well known that, under the Lipschitz condition (4), the initial-value problem for (3) has a unique solution, defined on the whole interval J, for any given initial value; see, e.g., Baleanu and Mustafa [**2,** Theorem 2], Tisdell [**22,** Theorem 6.4] and Diethelm [**11,** Theorem 6.8]. We will show that any two solutions of (3) either coincide or do not intersect on J. To do so, we need two useful technical tools for investigating FDEs: the variation-of-constants formula and the comparison principle. We refer the reader to the books of Kilbas *et al.* [**15,** Chapter 5] and Diethelm [**11,** Chapter 7] for a more detailed discussion about the using Laplace transforms to obtain the variation-of-constants formula for solutions to FDEs.

Lemma 2 (Variation-of-constants formula for FDEs)**.** Consider the FDE (3) on the finite interval $J = [0, T]$. Assume that the function $f(\cdot,\cdot)$ in the equation (3) satisfies the condition (4). If the function

$f(\cdot,\cdot)$ is of the form

$$f(t,x) = Mx + g(t,x)$$

for some fixed $M \in \mathbb{R}$ and all $t \in J$ and $x \in \mathbb{R}$, then the solution $x(\cdot)$ of (3) with the initial value $x(0) = x_0$ satisfies, for all $t \in J$, the formula

$$x(t) = E_\alpha(Mt^\alpha)x_0 + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(M(t-\tau)^\alpha) g\big(\tau, x(\tau)\big)\, d\tau,$$

where, for $z \in \mathbb{C}$, the Mittag-Leffler functions are defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \text{and} \quad E_\alpha(z) := E_{\alpha,1}(z).$$

Proof. We define a function $\hat{g}(t,x)$ on $\mathbb{R}_{\geq 0} \times \mathbb{R}$ by

$$\hat{g}(t,x) := \begin{cases} g(t,x), & \text{if } t \in J \text{ and } x \in \mathbb{R}, \\ g(T,x), & \text{if } t \geq T \text{ and } x \in \mathbb{R}. \end{cases}$$

Then, there exists a positive constant $\hat{L}$ such that

$$|\hat{g}(t,x) - \hat{g}(t,y)| \leq \hat{L}|x-y| \quad \text{and} \quad |\hat{g}(t,0)| \leq \hat{L}$$

for all x, $y \in \mathbb{R}$ and all $t \in \mathbb{R}_{\geq 0}$.

We now consider, on the half real line $[0,\infty)$, the FDE

$$\begin{aligned} {}^{C}D^{\alpha}_{0+}\hat{x}(t) &= M\hat{x}(t) + \hat{g}(t,\hat{x}(t)) \\ &= M\hat{x}(t) + \hat{g}(t,\hat{x}(t)) - \hat{g}(t,0) + \hat{g}(t,0). \end{aligned} \tag{5}$$

It is obvious that the function $\hat{g}(t,x) - \hat{g}(t,0)$ is Lipschitz continuous with respect to the second variable and satisfies the condition

$$|\hat{g}(t,x) - \hat{g}(t,0)| \leq \hat{L}|x|, \quad \text{for all } t \in \mathbb{R}_{\geq 0} \text{ and } x \in \mathbb{R}.$$

Moreover, $\hat{g}(t,0)$ is a bounded continuous function on $\mathbb{R}_{\geq 0}$. By virtue of Lemma 1 and the Lipschitz property of $\hat{g}$, for any $x_0 \in \mathbb{R}$, the equation (5) with the initial condition $\hat{x}(0) = x_0$ has a unique solution which satisfies the Volterra integral equation

$$\hat{x}(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[M\hat{x}(\tau) + \hat{g}(\tau,\hat{x}(\tau)) - \hat{g}(\tau,0)\right] d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\hat{g}(\tau,0)\, d\tau,$$

for all $t \geq 0$. Therefore, for any $t \geq 0$, we have

$$|\hat{x}(t)| \leq |x_0| + \frac{\hat{L}t^\alpha}{\Gamma(\alpha+1)} + \frac{M+\hat{L}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|\hat{x}(\tau)|\, d\tau,$$

and consequently,

$$\frac{|\hat{x}(t)|}{\exp(t)} \leq \frac{1}{\exp(t)}\left(|x_0| + \frac{\hat{L}t^\alpha}{\Gamma(\alpha+1)}\right) + \frac{M+\hat{L}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\frac{|\hat{x}(\tau)|}{\exp(\tau)}\, d\tau. \tag{6}$$

Put $v(t) = \hat{x}(t)/\exp(t)$ for $t \geq 0$ and set

$$K := \sup_{t\geq 0}\frac{1}{\exp(t)}\left(|x_0| + \frac{\hat{L}t^\alpha}{\Gamma(\alpha+1)}\right) < \infty;$$

from (6) we obtain the estimate

$$v(t) \leq K + \frac{M+\hat{L}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}v(\tau)\, d\tau, \quad t \geq 0,$$

which, by a Gronwall-type inequality [**11,** Lemma 6.19, p. 111], implies that $v(t) \leq KE_\alpha((M+\hat{L})t^\alpha)$ for $t \geq 0$. Thus,

$$|\hat{x}(t)| \leq K\exp(t)E_\alpha((M+\hat{L})t^\alpha) \quad \text{for } t \geq 0,$$

which shows that, for any $x_0 \in \mathbb{R}$, the solution $\hat{x}(\cdot)$ of the equation (5) with the initial value $\hat{x}(0) = x_0$ is exponentially bounded on $[0,\infty)$. Hence, we can apply the Laplace transform to both sides of the equation (5) and get, for some positive constant c,

$$s^\alpha\mathcal{L}\{\hat{x}(t)\}(s) - s^{\alpha-1}x_0 = M\mathcal{L}\{\hat{x}(t)\}(s) + \mathcal{L}\{\hat{g}(t,\hat{x}(t))\}(s), \quad \Re(s) > c;$$

see Diethelm [**11,** Theorem 7.1, p. 134]. Therefore,

$$\mathcal{L}\{\hat{x}(t)\}(s) = \frac{s^{\alpha-1}}{s^\alpha - M}x_0 + \frac{1}{s^\alpha - M}\mathcal{L}\{\hat{g}(t,\hat{x}(t))\}(s),$$

assuming that $\Re(s) > \max\{c, |M|^{1/\alpha}\}$, so applying the inverse Laplace transform to both sides gives

$$\hat{x}(t) = \mathcal{L}^{-1}\Big\{\frac{s^{\alpha-1}}{s^\alpha - M}\Big\}(t)x_0 + \mathcal{L}^{-1}\Big\{\frac{1}{s^\alpha - M}\mathcal{L}\{\hat{g}(t,\hat{x}(t))\}(s)\Big\}(t),$$

for all $t \geq 0$. Using the known formula [**21,** formula (1.80), p. 21]

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(Mt^\alpha)\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha - M}, \quad \Re(s) > |M|^{1/\alpha},$$

with $\beta = \alpha$ or $\beta = 1$, and using the properties of the Laplace transform [**11,** Theorem D.11, p. 231], we conclude that, for all $t \geq 0$,

$$\hat{x}(t) = E_\alpha(Mt^\alpha)x_0 + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(M(t-\tau)^\alpha)\hat{g}(\tau,\hat{x}(\tau))\,d\tau.$$

For a given initial value x_0, equation (3) has a unique solution on J, and likewise equation (5) has a unique solution on $\mathbb{R}_{\geq 0}$. These two solutions coincide on J because $\hat{g}(t,x) = g(t,x)$ for all $x \in \mathbb{R}$ and $t \in J$, and therefore $x(\cdot)$ satisfies

$$x(t) = E_\alpha(Mt^\alpha)x_0 + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(M(t-\tau)^\alpha)g(\tau,x(\tau))\,d\tau,$$

for all $t \in J$. □

Remark 3. It is easily seen that, in the setting of Lemma 2, if J is not compact but is instead the real half-line $[0,\infty)$, then the variation-of-constants formula holds true on the whole of $[0,\infty)$.

Remark 4. One can see that the proof of Lemma 2 can be easily carried out for the higher dimensional case with M changed to a constant matrix and with the functions $x(\cdot)$, f, g changed to vector functions; cf. Diethelm [**11,** Remark 7.1, p. 135] for both the one dimensional and the higher dimensional cases, where f and g depend only on t but not on x.

To prove the main result in this section, we need a preparatory lemma that is a small modification of a result of Lakshmikantham [**16,** Theorem 2.1]. For convenience of the reader, we present a proof.

Lemma 5 (Comparision Principle)**.** Let $0 < q < 1$, and assume that the continuous functions v, v_1, w ,$w_1 \in C(J, \mathbb{R})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$ satisfy

$$v(t) \leq v_1(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, v(s))\, ds$$

and

$$w(t) \geq w_1(t) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, w(s))\, ds$$

for all $t \in J$. Suppose further that $g(t, x)$ is nondecreasing in x for each $t \in J$. If

$$v_1(t) < w_1(t) \quad \text{for all } t \in J, \tag{7}$$

then

$$v(t) < w(t) \quad \text{for all } t \in J. \tag{8}$$

Proof. Suppose for a contradiction that (8) is not true. Then, because of the continuity of $v(\cdot)$ and $w(\cdot)$, there exists $t_1 \in J \setminus \{0\}$ such that

$$v(t_1) = w(t_1) \quad \text{and} \quad v(t) < w(t) \quad \text{for } 0 \leq t < t_1. \tag{9}$$

Using (7), (9) and the nondecreasing nature of g, we find that

$$\begin{aligned} v(t_1) &\leq v_1(t_1) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} g(s, v(s))\, ds \\ &\leq v_1(t_1) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} g(s, w(s))\, ds \\ &< w_1(t_1) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} g(s, w(s))\, ds \leq w(t_1), \end{aligned}$$

contradicting the condition $v(t_1) = w(t_1)$ in (9). Hence, (8) holds and the proof is complete. □

Now we are in a position to prove our main theorem of this section.

Theorem 6 (Different trajectories do not meet)**.** Assume that f satisfies the Lipschitz condition (4). Then, for any two different initial values $x_{10} \neq x_{20}$ in $\mathbb{R}$, the trajectories of the corresponding solutions of the FDE (3) do not meet on J, i.e., the solutions $x_1(\cdot)$ and $x_2(\cdot)$ of (3) starting from $x_{10} = x_1(0)$ and $x_{20} = x_2(0)$ satisfy $x_1(t) \neq x_2(t)$ for all $t \in J$.

Proof. For definiteness, we assume that

$$x_1(0) = x_{10} < x_{20} = x_2(0). \tag{10}$$

To prove the theorem, we show that $x_1(t) < x_2(t)$ for all $t \in J$. Suppose that this is not true, then there is $T_1 \in J \setminus \{0\}$ such that $x_1(T_1) \geq x_2(T_1)$. By the continuity of $x_1(\cdot)$ and $x_2(\cdot)$, and by (10), there is a $T_2 > 0$ such that

$$x_1(T_2) = x_2(T_2) \quad \text{and} \quad x_1(t) < x_2(t) \quad \text{for all } 0 \leq t < T_2.$$

Put $M := \max_{0 \leq t \leq T_2} L(t)$ and define

$$g(t,x) := f(t,x) + Mx \quad \text{for all } 0 \leq t \leq T_2 \text{ and } x \in \mathbb{R}. \tag{11}$$

Then $g : [0, T_2] \times \mathbb{R} \to \mathbb{R}$ is nondecreasing in x for each $t \in [0, T_2]$. In fact, by (4), (11) and the choice of M, if $t \in [0, T_2]$ and $x \leq y$, then

$$g(t,y) - g(t,x) = M(y-x) + f(t,y) - f(t,x) \geq (M - L(t))(y-x) \geq 0.$$

By virtue of Lemma 2, since $x_1(\cdot)$ and $x_2(\cdot)$ are solutions of (3), on the interval $[0, T_2]$ we have

$$x_1(t) = E_\alpha(-Mt^\alpha)x_{10} + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-M(t-\tau)^\alpha) g\big(\tau, x_1(\tau)\big)\, d\tau, \tag{12}$$

and

$$x_2(t) = E_\alpha(-Mt^\alpha)x_{20} + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-M(t-\tau)^\alpha) g\big(\tau, x_2(\tau)\big)\, d\tau. \tag{13}$$

Since $E_{\alpha,\alpha}(s) > 0$ for all $s \in \mathbb{R}$ (see, e.g., Cong *et al.* [**5,** Lemma 2]), the function $E_{\alpha,\alpha}(-M(t-\tau)^\alpha)g(\tau,x)$ is nondecreasing in the variable x for $0 \leq \tau \leq t \leq T_2$. Therefore, Lemma 5 is applicable to the pair of integral equations (12)–(13) on $[0, T_2]$, and gives $x_1(T_2) < x_2(T_2)$. Thus, we arrive at a contradiction, and consequently the conclusion of the theorem is true. □

Remark 7. Theorem 6 provides a full solution to Conjecture 1.2 of Diethelm **[10]**. In that paper, Diethelm provides a partial solution to the problem of separation of trajectories: he proved Conjecture 1.2 under some restrictive conditions **[10,** Theorem 2.2].

3.2. Discussion on the problem of separation of solutions of a general FDE. Diethelm **[11,** Theorem 6.12] formulated and proved a theorem on separation of solutions of a FDE that is the same as our Theorem 6. For the proof, he used a fixed point theorem on a short interval of time, and then used induction to extend the result to the whole (long) interval of time under consideration. However, his proof contains a flaw in the induction part: the passage to the next step from $N = 1$ to $N = 2$ does not work, because the argument leading to a contractive mapping on the first subinterval of time fails on the second subinterval, owing to the fact that the FDE is history dependent.

In a subsquent joint work with Ford **[12,** Theorems 3.1 and 4.1], Diethelm gave an alternative proof of his result **[11,** Theorem 6.12] on separation of solutions of a FDE. Instead of induction forward in time from one subinterval to the next, they used induction backward in time from one subinterval to the foregoing interval. However, their argument in the first step of this induction does not work. Using the numbering and notation of Diethelm and Ford **[12]**, their equation (13) is equivalent to their equation (8) only if we consider (8) and (13) in the whole interval $[0, b]$. Actually, the solution $y(t)$, considered as the solution of the terminal problem (13), depends on the future value $y(b) = c$, and hence we cannot say that the function $g(t)$ on pages 29–30 is independent of the restriction of the function y to $[T_{N-1}, T_N]$. Therefore, the contraction property of the map $\mathcal{F}$ in their formula (14) does not lead to the existence and uniqueness of the solution of (14), as claimed. Consequently, their proof of Theorem 3.1 **[12]**, and hence also of Theorem 4.1, is incomplete.

Note that the earlier proof **[11,** Theorem 6.12] is correct for a first "short" interval of time: the smallness of time combined with the bounded Lipschitz condition for f make a certain operator contractive, and hence two different solutions cannot meet. By the way, we note that continuity alone can also assure the non-intersection of different trajectories in a short time interval.

Hayek, Trujillo, Rivero, Bonilla and Boreno [**14,** Theorem 3.1] have proved the separation theorem on a "short" interval of time. However, Section 4 of that paper is invalid because of the history dependence of solutions of the FDE, which prevents us from applying the "usual method of prolongation" (for ODEs) as claimed by the authors.

Bonilla, Rivero and Trujillo [**4**] treated higher dimensional linear systems of FDEs and, in Section 3 of that paper, they relied on the above mentioned result [**14,** Theorem 3.1]. Hence, there are gaps in their proofs of some results [**4,** Theorem 1 on page 71 and Propositions 1 and 2 on page 72]. For a counter example, see Section 6, which, by the way, also shows that the arguments of Diethelm and Ford [**11, 12**] cannot work: otherwise, as is easily seen, those arguments would work for the higher dimensional case as well, leading to a contradiction.

4. One-dimensional FDEs generate nonlocal dynamical systems.

In this section, based on the results on separation of trajectories presented in Section 3, we show that one-dimensional FDEs generate nonlocal dynamical systems. Hence, tools and methods from the classical theory of dynamical systems are applicable.

4.1. Bounds for solutions of FDEs.

First, we formulate and prove a lower bound for solutions of a one-dimensional FDE, which provides us with a better understanding of the geometry of the solutions.

Theorem 8 (Convergence rate for solutions of one-dimensional FDEs)**.** Assume that f satisfies the Lipschitz condition (4), and put

$$L^*(t) = \max_{0\le\tau\le t} L(\tau). \tag{14}$$

Then, for any two solutions $x_1(\cdot)$ and $x_2(\cdot)$ of the FDE (3), and for any $t \in J$,

$$|x_2(t) - x_1(t)| \ge |x_2(0) - x_1(0)| E_\alpha\big(-L^*(t)t^\alpha\big).$$

Proof. For definiteness, we assume that $x_2(0) \ge x_1(0)$. Then by Theorem 6, we have $x_2(t) \ge x_1(t)$ for any $t \in J$. For an arbitrary but fixed $t \geq 0$, we repeat the arguments in the proof of Theorem 6 on the interval $[0,t]$ to conclude that

$$x_2(s) - x_1(s) = E_\alpha(-L^*(t)s^\alpha)(x_2(0) - x_1(0))$$
$$+ \int_0^s (s-\tau)^{\alpha-1} E_{\alpha,\alpha}(-L^*(t)(s-\tau)^\alpha)\big(g(\tau, x_2(\tau)) - g(\tau, x_1(\tau))\big)\, d\tau$$
$$\geq E_\alpha(-L^*(t)s^\alpha)(x_2(0) - x_1(0)).$$

Now take $s = t$ to complete the proof. □

Corollary 9 (Lower bound for solutions of one-dimensional FDEs)**.** Assume that f satisfies the Lipschitz condition (4). Assume additionally that $f(t,0) = 0$ for all $t \in J$. Then, for any solution $x(\cdot)$ of the FDE (3) and any $t \in J$,

$$|x(t)| \geq |x(0)| E_\alpha\big(-L^*(t)t^\alpha\big).$$

Proof. Since $f(t,0) = 0$, the FDE (3) has the trivial solution. Apply Theorem 8 to the pair consisting of $x(\cdot)$ and the trivial solution of (3). □

For the divergence rate and upper bound for solutions of the FDEs, the following statements are easy modifications of well known results.

Theorem 10 (Divergence rate for solutions of one-dimensional FDEs)**.** Assume that f satisfies the Lipschitz condition (4), and recall the notation (14). Then, for any two solutions $x_1(\cdot)$ and $x_2(\cdot)$ of the FDE (3), and for any $t \in J$,

$$|x_2(t) - x_1(t)| \leq |x_2(0) - x_1(0)| E_\alpha\big(L^*(t)t^\alpha\big).$$

Proof. By Lemma 1, for all $t \in J$ we have

$$x_1(t) = x_1(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x_1(\tau))\, d\tau$$

and

$$x_2(t) = x_2(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x_2(\tau))\, d\tau.$$

Therefore, if $0 \leq s \leq t$, then

$$\begin{aligned}|x_2(s) - x_1(s)| &\leq |x_2(0) - x_1(0)| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} \big(f(\tau, x_2(\tau)) - f(\tau, x_1(\tau)) \big)\, d\tau \\ &\leq |x_2(0) - x_1(0)| + \frac{1}{\Gamma(\alpha)} \int_0^s (s-\tau)^{\alpha-1} L^*(t) |x_2(\tau) - x_1(\tau)|\, d\tau.\end{aligned}$$

By virtue of the Gronwall-type inequality for FDEs (see Diethelm [**11,** Theorem 6.19, p. 111] and Tisdell [**22,** Lemma 3.1, p. 288]), this estimate implies that

$$|x_2(s) - x_1(s)| \leq |x_2(0) - x_1(0)| E_\alpha\big(L^*(t) s^\alpha\big).$$

Since $t \in J$ is arbitrary, the theorem follows by putting $s = t$. □

Corollary 11 (Upper bound for solutions of one-dimensional FDEs)**.** Assume that f satisfies the Lipschitz condition (4). Assume additionally that $f(t, 0) = 0$ for all $t \in J$. Then, for any solution $x(\cdot)$ of the FDE (3), and for any $t \in J$,

$$|x(t)| \leq |x(0)| E_\alpha\big(L^*(t) t^\alpha\big).$$

Proof. Since $f(t, 0) = 0$, the FDE (3) has the trivial solution. Apply Theorem 10 to the pair consisting of $x(\cdot)$ and the trivial solution of (3). □

Remark 12. It is easily seen that Theorem 10 and Corollary 11 hold true also for the case of a higher dimensional system of FDEs.

4.2. One-dimensional FDEs generate two-parameter flows. Now we are in a position to show that one-dimensional FDEs generate two-parameter flows. First we define the evolution mappings of (3).

Definition 13. The *evolution mapping of* (3) is given by

$$\Phi_{0,T_1} : \mathbb{R} \to \mathbb{R}, \quad x_0 \mapsto x(T_1),$$

where $x_0 \in \mathbb{R}$ is an arbitrary initial value of (3), $x(\cdot)$ is the solution of (3) starting from $x(0) = x_0$, and $x(T_1)$ is the evaluation of $x(\cdot)$ at T_1.

Definition 14. A two-parameter family of mappings

$$\varphi_{s,t}(\cdot) : \mathbb{R} \to \mathbb{R}, \quad s, t \in J,$$

is called a *two-parameter flow in* $\mathbb{R}$ if it has the following three properties:

(i) $\varphi_{s,t}(x)$ is continuous as a function of the three variables s, $t \in J$ and $x \in \mathbb{R}$;
(ii) for any fixed s, $t \in J$, the mapping $\varphi_{s,t}$ is a homeomorphism of $\mathbb{R}$; and
(iii) the flow property $\varphi_{s,t} \circ \varphi_{u,s} = \varphi_{u,t}$ holds for all u, s, $t \in J$.

Theorem 15 (One-dimensional FDEs generate two-parameter flows in $\mathbb{R}$)**.** The following statements hold for the one-dimensional FDE (3).

(i) The evolution mapping $\Phi_{0,t}$ generated by (3) is a bijection for each $t \in J$.
(ii) The FDE (3) generates a two-parameter family of bijections on J by its evolution mappings, as follows,

$$\Phi_{s,t} := \Phi_{0,t} \circ \Phi_{0,s}^{-1} \quad \text{for all } s,\ t \in J, \tag{15}$$

where $\Phi_{0,\cdot}$ is the evolution mapping of (3) from Definition 13.
(iii) The family $\Phi_{s,t}$, for s, $t \in J$, generated by the FDE (3), is a two-parameter flow in $\mathbb{R}$.
(iv) If f is linear in x, then the two-parameter flow generated by the FDE (3) is a flow of linear operators.

Proof. (i) Fix an arbitrary $T_1 \in J$. By Theorem 6, Φ_{0,T_1} is injective. To show that Φ_{0,T_1} is surjective, it suffices to show that, for an arbitrary $x^* \in \mathbb{R}$, the terminal-value problem

$$^{C}D^{\alpha}_{0+}x(t) = f(t, x(t)) \quad \text{for } t \in [0, T_1], \tag{16}$$

$$x(T_1) = x^*, \tag{17}$$

has a continuous solution, assuming that the function f is continuous and satisfies the Lipschitz condition (4). Put

$$M_1 := L^*(T_1) = \max_{0 \le t \le T_1} L(t),$$

where $L(t)$ is determined from (4). Let us denote by $\hat{x}(\cdot)$ the solution of the FDE (16) satisfying the initial condition $\hat{x}(0) = 0$. Put

$$M_2 := \max_{0\le t\le T_1} |\hat{x}(t)|, \qquad M_3 := |x^* - \hat{x}(T_1)| + M_2,$$
$$M_4 := \frac{M_3}{E_\alpha(-M_1 T_1^\alpha)}, \qquad M_5 := M_2 + M_4 E_\alpha(M_1 T_1^\alpha) + 1.$$

Clearly $M_2 < M_3 < M_4 < M_5$. Define the function $\hat{f}$ on $[0, T_1] \times \mathbb{R}$ by

$$\hat{f}(t,x) = \begin{cases} f(t,x), & \text{if } |x| \le M_5, \\ f(t, M_5 x/|x|), & \text{if } |x| > M_5, \end{cases} \tag{18}$$

and consider the terminal value problem

$$^{C}D^{\alpha}_{0+} x(t) = \hat{f}(t, x(t)), \tag{19}$$
$$x(T_1) = x^*. \tag{20}$$

Clearly $\hat{f}$ is Lipschitz continuous and bounded on $[0, T_1] \times \mathbb{R}$, with Lipschitz constant $L(t)$. Hence, the problem (19)–(20) has at least one solution, say $x_1(\cdot)$ **[3,** Theorem 8]. We show that $x_1(\cdot)$ is the required solution of (16)–(17). To this end, notice that for all $t \in [0, T_1]$ we have

$$|\hat{x}(t)| \le M_2 < M_5 \quad \text{and hence} \quad f(t, \hat{x}(t)) = \hat{f}(t, \hat{x}(t)).$$

Therefore, $\hat{x}(\cdot)$ is the solution of the FDE (19) satisfying the initial condition $\hat{x}(0) = 0$. By applying Theorem 8 to the solutions $\hat{x}(\cdot)$ and $x_1(\cdot)$ of the FDE (19), we see that, for any $t \in [0, T_1]$,

$$|\hat{x}(t) - x_1(t)| \ge |\hat{x}(0) - x_1(0)| E_\alpha\big(-L^*(t) t^\alpha\big) \ge |x_1(0)| E_\alpha(-M_1 T_1^\alpha).$$

Substituting $t = T_1$, we get

$$|\hat{x}(T_1) - x_1(T_1)| \ge |x_1(0)| E_\alpha(-M_1 T_1^\alpha),$$

and hence

$$|x_1(0)| \le \frac{|\hat{x}(T_1) - x^*|}{E_\alpha(-M_1 T_1^\alpha)} \le M_4.$$

Applying Theorem 10 to the solutions $\hat{x}(\cdot)$ and $x_1(\cdot)$ of the FDE (19) shows that, for any $t \in [0, T_1]$,

$$|x_1(t)| \le |\hat{x}(t)| + |x_1(0)| E_\alpha(M_1 T_1^\alpha) \le M_5,$$

and hence $f(t, x_1(t)) = \hat{f}(t, x_1(t))$. Therefore, $x_1(\cdot)$ is a solution of the FDE (16), and (i) is proved.

(ii) By (i), the evolution mappings of (3) are bijective, and hence $\Phi_{s,t}$ is well defined by (15). The flow property is easily verified.

(iii) By (ii), the FDE (3) generates a two-parameter family of bijections $\Phi_{s,t}$ of $\mathbb{R}$ for all $s, t \in J$. From Theorems 8 and 10, it follows that the bijections $\Phi_{s,t}$ are homeomorphisms and Φ depends continuously on three variables s, t and x.

(iv) Obvious. □

Definition 16. The two-parameter flow $\Phi_{s,t}$, specified in Theorem 15 and generated by the FDE (3), is called the *nonlocal dynamical system generated by* (3).

Remark 17. Two distinguished features of the two-parameter flow generated by the FDE (3) are as follows.

(i) The flow has a historical memory. Although the past has impact on the behavior of the solutions, the solutions form a two-parameter flow of homeomorphisms.
(ii) The flow is in general α-Hölder, but not C^1.

Remark 18. Li and Ma [**18,** Theorem 2] claimed that they constructed a dynamical system from a FDE. However, their construction is false. For a counter example, see Cong, Son and Tuan [**8,** Remark 12].

5. Triangular systems of FDEs generate nonlocal dynamical systems. In this section, using the results of Section 4 we show that a higher dimensional, triangular system of FDEs also generates a nonlocal dynamical system.

Let us consider a d-dimensional triangular system of (not necessarily linear) FDEs,

$$\begin{aligned} {}^{C}D^{\alpha}_{0+}x_1(t) &= f_1(t, x_1(t)),\\ {}^{C}D^{\alpha}_{0+}x_2(t) &= f_2(t, x_1(t), x_2(t)),\\ &\cdots\\ {}^{C}D^{\alpha}_{0+}x_d(t) &= f_d(t, x_1(t), x_2(t), \ldots, x_d(t)), \end{aligned} \tag{21}$$

for $t \in J$, where $x(\cdot) = \big(x_1(\cdot), \dots, x_d(\cdot)\big)^{\mathrm{T}} \in \mathbb{R}^d$ and where the vector-valued function $f = (f_1, \dots, f_d)^{\mathrm{T}}$ is Lipschitz in the x variables, i.e., there exists a continuous function $L : J \to [0, \infty)$ such that, for all $i = 1, \dots, d$ and all $t \in J$, we have

$$(22) \quad |f_i(t, x_1, \dots, x_i) - f_i(t, y_1, \dots, y_i)| \\ \le L(t)\sqrt{(x_1 - y_1)^2 + \cdots + (x_i - y_i)^2}.$$

This triangular system has a distinguished property: it can be solved successively coordinate-wise and each time we have to solve only a one-dimensional FDE. Hence, the triangular system inherits many features of the one-dimensional FDEs.

Proposition 19 (Convergence rate for solutions of a triangular system of FDEs)**.** Assume that the Lipschitz condition (22) is satisfied, and define $L^*(t)$ as before in (14). Then, for any two solutions $x(\cdot)$ and $y(\cdot)$ of the triangular FDE (21), and for any $t \in J$,

$$\|x(t) - y(t)\| \ge \|x(0) - y(0)\| E_\alpha\big(-L^*(t)t^\alpha\big).$$

Proof. Let $x(\cdot) = (x_1(\cdot), \dots, x_d(\cdot))^{\mathrm{T}}$ and $y(\cdot) = (y_1(\cdot), \dots, y_d(\cdot))^{\mathrm{T}}$ be two arbitrary solutions of the triangular FDE (21). Consider the first equation in (21): it is a 1-dimensional FDE for the first coordinate. Applying Theorem 8 to this equation, we get

$$|x_1(t) - y_1(t)| \ge |x_1(0) - y_1(0)| E_\alpha\big(-L^*(t)\, t^\alpha\big).$$

Since the first coordinate is solvable from the first equation, we can substitute it into the second equation of the system (21) and get a 1-dimensional FDE for the second coordinate,

$${}^{C}D^{\alpha}_{0+} u(t) = f_2(t, x_1(t), u(t)) =: \hat{f}_2(t, u(t)),$$

where, due to (22), the function $\hat{f}_2(\cdot, \cdot) : J \times \mathbb{R} \to \mathbb{R}$ is $L(t)$-Lipschitz continuous with respect to the second variable. Applying Theorem 8 to the solutions $x_2(\cdot)$ and $y_2(\cdot)$ of this 1-dimensional FDE, we get

$$|x_2(t) - y_2(t)| \ge |x_2(0) - y_2(0)| E_\alpha\big(-L^*(t)\, t^\alpha\big).$$

Continuing this process, we get, for $i = 1, \ldots, d$ and $t \in J$, the inequality

$$|x_i(t) - y_i(t)| \geq |x_i(0) - y_i(0)| E_\alpha\big(-L^*(t)\, t^\alpha\big). \tag{23}$$

The conclusion of the proposition follows at once. □

An important particular case of the triangular system of FDEs (21) is a *linear* triangular system,

$$^{C}D^\alpha_{0+} x(t) = A(t)x(t), \tag{24}$$

where $t \in J, x \in \mathbb{R}^d$ and $A : J \times \mathbb{R}^{d\times d}$ is a bounded continuous triangular $(d \times d)$ matrix-valued function. Thus, $A(\cdot) = [a_{ij}(\cdot)]_{1\leq i,j\leq d}$ with either $a_{ij} = 0$ for all $i > j$ (upper triangular) or else $a_{ij} = 0$ for all $i < j$ (lower triangular), and there exists a continuous function $L : J \to [0,\infty)$ such that

$$\|A(t)\| \leq L(t) \quad \text{for all } t \in J. \tag{25}$$

Clearly, Proposition 19 is applicable to the linear triangular system (24). Moreover, we also have a lower bound for solutions of (24).

Proposition 20 (Lower bound for solutions of a linear triangular system of FDEs)**.** Assume that the triangular matrix function A satisfies (25), and recall the notation (14). Then, for any solution $x(\cdot)$ of the FDE (24), and for any $t \in J$,

$$\|x(t)\| \geq \|x(0)\| E_\alpha\big(-L^*(t)\, t^\alpha\big).$$

Proof. Since the system (24) has the trivial solution 0, we can apply Proposition 19 to the two solutions $x(\cdot)$ and 0 of (24) and arrive at the desired conclusion. □

Similar to the one-dimensional case, for any $T_1 \in J$, the *evolution mapping of* (21) is given by

$$\varphi_{0,T_1} : \mathbb{R}^d \to \mathbb{R}^d, \quad x_0 \mapsto x(T_1), \tag{26}$$

where $x_0 \in \mathbb{R}^d$ is an arbitrary initial value of (21), $x(\cdot)$ is the solution of (21) starting from $x(0) = x_0$, and $x(T_1)$ is the evaluation of $x(\cdot)$ at T_1. By the same arguments as in Section 4, we can show that the evolution mapping $\varphi_{0,t}$ of the triangular FDE (21) is a bijection for any $t \in J$.

Definition 21. A two-parameter family of mappings

$$\varphi_{s,t}(\cdot) : \mathbb{R}^d \to \mathbb{R}^d, \quad s, t \in J,$$

is called a *two-parameter flow in* $\mathbb{R}^d$ if it has the following three properties:

(i) $\varphi_{s,t}(x)$ is continuous as a function of the three variables s, $t \in J$ and $x \in \mathbb{R}^d$;
(ii) for any fixed s, $t \in J$, the mapping $\varphi_{s,t}(\cdot)$ is a homeomorphism of $\mathbb{R}^d$; and
(iii) the flow property $\varphi_{s,t} \circ \varphi_{u,s} = \varphi_{u,t}$ holds for all u, s, $t \in J$.

Theorem 22 (Triangular systems of FDEs generate nonlocal dynamical systems)**.** The following statements hold.

(i) The triangular system of FDEs (21) generates a two-parameter flow in $\mathbb{R}^d$, namely

$$\varphi_{s,t} := \varphi_{0,t} \circ \varphi_{0,s}^{-1} : \mathbb{R}^d \to \mathbb{R}^d \quad \text{for } s,\ t \in J,$$

where $\varphi_{0,t}$ is the evolution mapping of (21).
(ii) The linear triangular system of FDEs (24) generates a two-parameter flow of d-dimensional linear nonsingular operators.

Proof. Similar to Theorem 15. □

Definition 23. The two-parameter flow $\varphi_{s,t}$ generated by the triangular FDE (21) is called the *nonlocal dynamical system generated by* (21).

6. A general high dimensional system of FDEs does not generate a dynamical system. Finally, we show that in the high dimensional case different trajectories of a FDE *can* intersect each other. Thus, a high dimensional system of FDEs does *not*, in general, generate a dynamical system. In order for a high dimensional FDE to generate a nonlocal dynamical system we need an additional property of the FDE, such as triangularity. The results of this section also provide a counter example to the assertions of Bonilla, Rivero and Trujillo [**4,** Theorem 1, Propositions 1 and 2].

Theorem 24 (Different trajectories of a high dimensional system of FDEs can meet)**.** For any $d \geq 2$, there exists a system of type (1) with the property that it has two different solutions, $x_1(\cdot)$ and $x_2(\cdot)$, with $x_1(0) \neq x_2(0)$ but which intersect each other at some finite time moment $T \in (0,\infty)$, i.e., $x_1(T) = x_2(T)$.

Proof. It suffices to construct a two-dimensional system of type (1) having the desired property. In fact, the system we construct will also be linear and autonomous.

Since $\alpha \in (0,1)$, the complex-valued Mittag-Leffler function $E_\alpha(\cdot)$ has infinitely many zeros in $\mathbb{C}$ [**13,** Corollary 3.10, p. 30]. Fix $z^* \in \mathbb{C}$ such that $E_\alpha(z^*) = 0$. Let

$$\varphi := \arg(z^*) \in (-\pi, \pi] \quad \text{and} \quad \lambda := \cos\varphi + i\sin\varphi,$$

where $i = \sqrt{-1} \in \mathbb{C}$. Note that since $\alpha \in \mathbb{R}$, we have $E_\alpha(\overline{z^*}) = \overline{E_\alpha(z^*)} = 0$, where $\overline{w}$ denotes the complex conjugate of the complex number w. Since $\alpha \in (0,1)$ we have $z^* \notin \mathbb{R}$, and hence $\lambda \notin \mathbb{R}$. Consider the matrix

$$A := \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix},$$

which has two (complex) eigenvalues, namely λ and $\overline{\lambda}$. We show that the associated linear and autonomous FDE,

$$^{C}D^{\alpha}_{0+}x(t) = Ax(t) \quad \text{for } t \in \mathbb{R}_+, \tag{27}$$

has the desired property. Indeed, it is known [**11,** Theorem 7.15, p. 152] that this FDE has two linearly independent solutions of the form

$$x_1(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \quad \text{and} \quad x_2(t) = \begin{bmatrix} -v(t) \\ u(t) \end{bmatrix},$$

where u, $v : \mathbb{R}_+ \to \mathbb{R}$ are given by

$$u(t) := E_\alpha(\lambda t^\alpha) + E_\alpha(\overline{\lambda} t^\alpha) \quad \text{and} \quad v(t) := i\big(E_\alpha(\lambda t^\alpha) - E_\alpha(\overline{\lambda} t^\alpha)\big).$$

Since $u(0) = 2$ and $v(0) = 0$, it follows that

$$x_1(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

The general solution of (27) is

$$x(t) = ax_1(t) + bx_2(t), \tag{28}$$

where a, $b \in \mathbb{R}$ are arbitrary real constants. Let $T > 0$ be the unique finite positive number satisfying

$$\lambda T^\alpha = z^*;$$

a unique such T exists due to the definitions of z^* and λ. Clearly $u(T) = v(T) = 0$, and hence $x_1(T) = x_2(T) = (0, 0)^{\mathrm{T}}$. From (28), it follows that for *any* solution $x(t)$ of (27) we have $x(T) = 0$. □

From Theorem 24 we obtain immediately the following corollary.

Corollary 25 (A higher dimensional FDE does not generate a nonlocal dynamical system)**.** For $d \geq 2$, the FDE (1) does not, in general, generate a two-parameter flow in $\mathbb{R}^d$. Hence it does not, in general, generate a nonlocal dynamical system.

Remark 26. We note the following.

(i) Actually, for the FDE (27), *all* the solutions are equal to $(0, 0)^{\mathrm{T}}$ at time T; hence, *all* of them meet each other.
(ii) Theorem 24 shows that, in contrast to the initial-value problem, the terminal-value problem for FDEs is not always solvable.
(iii) Theorem 24 allows us to understand better the dynamics of FDEs, by revealing a distinguished feature in comparision with ODEs: different trajectories of an FDE may meet, whereas for an ODE they cannot.
(iv) By a small modification of A in the proof of Theorem 24, we can make the time of intersection T small.
(v) By a small modification of the proof, one can show that Theorem 24 also holds for any positive real $\alpha \neq 1$.

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REFERENCES

1. R. P. Agarwal, M. Benchohra and S. Hamani. A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Math Appl.*, **109** (2010), 973–1033.

2. D. Baleanu and O. Mustafa. On the global existence of solutions to a class of fractional differential equations. *Computers and Mathematics with Applications*, **59** (2010), 1835–1841.

3. M. Benchohra, S. Hamani and S. K. Ntouyas. Boundary value problems for differential equations with fractional order. *Surveys in Mathematics and Its Applications*, **3** (2008), 1–12.

4. B. Bonilla, M. Rivero and J. J. Trujillo. On systems of linear fractional differential equations with constant coefficients. *Applied Mathematics and Computation*, **187** (2007), 68–78.

5. N. D. Cong, T. S. Doan, S. Siegmund and H. T. Tuan. On stable manifolds for planar fractional differential equations. *Applied Mathematics and Computation,* **226** (2014), 157-168.

6. Nguyen Dinh Cong, Doan Thai Son and Hoang The Tuan. On fractional Lyapunov exponents for solutions of linear fractional differential equations. *Fractional Calculus and Applied Analysis*, **17** (2014), No. 2, 285–306.

7. Nguyen Dinh Cong, Doan Thai Son, Hoang The Tuan and Stefan Siegmund. Structure of the fractional Lyapunov spectrum for linear fractional differential equations. *Advances in Dynamical Systems and Applications*, **9** (2014), 149-159.

8. N. D. Cong, T. S. Doana and H. T. Tuan. An instability theorem of nonlinear fractional differential systems. To appear in *Discrete and Continuous Dynamical Systems, Series B.*

9. W. H. Deng, C. P. Li and J. H. Lu. Stability analysis of linear fractional differential systems with multiple time delays. *Nonlinear Dynamics*, **48** (2007), 409–416.

10. K. Diethelm. On the separation of solutions of fractional differential equations. *Fractional Calculus and Applied Analysis,* **11** (2008), No. 3, 259–268.

11. K. Diethelm. The analysis of fractional differential equations. An application–oriented exposition using differential operators of Caputo type. *Lecture Notes in Mathematics,* **2004**. Springer-Verlag, Berlin, 2010.

12. K. Diethelm and N. J. Ford. Volterra integral equations and fractional calculus: do neighboring solutions intersect? *Journal of Integral Equations and Applications*, **24** (2012), No. 1, 25–37.

13. R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin. *Mittag-Leffler Functions, Related Topics and Applications.* Springer-Verlag, Berlin Heidelberg, 2014.

14. N. Hayek, J. Trujillo, M. Rivero, B. Bonilla and J. C. Moreno. An extension of the Picard–Lindeloff theorem for fractional differential equations. *Applied Analysis*, **74** (1999), 347–361.

15. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, **204**. Elsevier Science B.V., Amsterdam, 2006.

16. V. Lakshmikantham and A. S. Vatsala. Basic theory of fractional differential equations. *Nonlinear Analysis,* **69** (2008), 2677–2682.

17. P. Lancaster and M. Tismenetsky. *The Theory of Matrices. Second Edition: With Applications (Computer Science and Scientific Computing).* Academic Press, San Diego, 1985.

18. C. Li and Y. Ma. Fractional dynamical system and its linearization theorem. *Nonlinear Dynamics*, **71** (2013), 621–633.

19. P. Linz. *Analytical and Numerical Methods for Volterra Equations.* SIAM Studies in Applied Mathematics. Philadelphia, 1985.

20. D. Matignon. Stability results for fractional differential equations with applications to control processing. *Computational Eng. in Sys. Appl.,* **2** (1996), 963–968.

21. I. Podlubny. *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications.* Mathematics in Science and Engineering, **198**. Academic Press, Inc., San Diego, CA, 1999.

22. C. C. Tisdell. On the application of sequential and fixed-point methods to fractional differential equations of arbitrary order. *Journal of Integral Equations and Applications*, **24** (2012), No. 2, 283–319.

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