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Well-posedness for the Navier-Stokes equations with datum in the Sobolev spaces

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Abstract In this paper, we study local well-posedness for the Navier-Stokes equations with arbitrary initial data in homogeneous Sobolev spaces $\dot{H}^s_p(\mathbb{R}^d)$ for $d \geq 2, p > \frac{d}{2}$, and $\frac{d}{p} - 1 \leq s < \frac{d}{2p}$. The obtained result improves the known ones for $p > d$ and $s = 0$ (see [4,6]). In the case of critical indexes $s = \frac{d}{p} - 1$, we prove global well-posedness for Navier-Stokes equations when the norm of the initial value is small enough. This result is a generalization of the one in [5] in which $p = d$ and $s = 0$.

Keywords Navier-Stokes equations, existence and uniqueness of local and global mild solutions, critical Sobolev and Besov spaces.

Mathematics Subject Classification (2000) Primary 35Q30; Secondary 76D05, 76N10.

1 Introduction

We consider the Navier-Stokes equations (NSE) in d dimensions in special setting of a viscous, homogeneous, incompressible fluid which fills the entire space and is not submitted to external forces. Thus, the equations we consider are the system

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\ \nabla \cdot u = 0, \\ u(0, x) = u_0, \end{cases}$$

which is a condensed writing for

$$\begin{cases} 1 \leq k \leq d, \quad \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l (u_l u_k) - \partial_k p, \\ \sum_{l=1}^d \partial_l u_l = 0, \\ 1 \leq k \leq d, \quad u_k(0, x) = u_{0k}. \end{cases}$$

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The unknown quantities are the velocity $u(t,x) = (u_1(t,x), \dots, u_d(t,x))$ of the fluid element at time t and position x and the pressure $p(t,x)$.
A translation invariant Banach space of tempered distributions $\mathcal{E}$ is called a critical space for NSE if its norm is invariant under the action of the scaling $f(.) \longrightarrow \lambda f(\lambda.)$. One can take, for example, $\mathcal{E} = L^d(\mathbb{R}^d)$ or the smaller space $\mathcal{E} = \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$. In fact, one has the chain of critical spaces given by the continuous embeddings

$$\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}^d)_{(p<\infty)} \hookrightarrow BMO^{-1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^d). \quad (1.1)$$

It is remarkable feature that NSE are well-posed in the sense of Hadarmard (existence, uniqueness and continuous dependence on the data) when the initial datum is divergence-free and belongs to the critical function spaces (except $\dot{B}^{-1}_{\infty,\infty}$) listed in (1.1) (see [4] for $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$, $L^d(\mathbb{R}^d)$, and $\dot{B}^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}^d)$, see [22] for $BMO^{-1}(\mathbb{R}^d)$, and the recent ill-posedness result [1] for $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^d)$).
In the 1960s, mild solutions were first constructed by Kato and Fujita [9,19] that are continuous in time and take values in the Sobolev space $H^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$, say $u \in C([0,T]; H^s(\mathbb{R}^d))$. In 1992, a modern treatment for mild solutions in $H^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$ was given by Chemin [7]. In 1995, using the simplified version of the bilinear operator, Cannone proved the existence for mild solutions in $\dot{H}^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$, see [4]. Results on the existence of mild solutions with value in $L^p(\mathbb{R}^d), (p > d)$ were established in the papers of Fabes, Jones and Rivière [8] and of Giga [10]. Concerning the initial datum in the space L^∞, the existence of a mild solution was obtained by Cannone and Meyer in [4,6]. Moreover, in [4,6], they also obtained theorems on the existence of mild solutions with value in the Morrey-Campanato space $M_2^p(\mathbb{R}^d), (p > d)$ and the Sobolev space $H_p^s(\mathbb{R}^d), (p < d, \frac{1}{p} - \frac{s}{d} < \frac{1}{d})$. NSE in the Morrey-Campanato space were also treated by Kato [21] and Taylor [24]. In 1981, Weissler [25] gave the first existence result of mild solutions in the half space $L^3(\mathbb{R}^3_+)$. Then Giga and Miyakawa [11] generalized the result to $L^3(\Omega)$, where Ω is an open bounded domain in $\mathbb{R}^3$. Finally, in 1984, Kato [20] obtained, by means of a purely analytical tool (involving only the Hölder and Young inequalities and without using any estimate of fractional powers of the Stokes operator), an existence theorem in the whole space $L^3(\mathbb{R}^3)$. In [4,5], Cannone showed how to simplify Kato's proof. The idea is to take the advantage of the structure of the bilinear operator in its scalar form. In particular, the divergence ∇ and heat $e^{t\Delta}$ operators can be treated as a single convolution operator. Recently, the authors of this article have considered NSE in Sobolev spaces, Sobolev-Lorentz spaces, mixed-norm Sobolev-Lorentz spaces, and Sobolev-Fourier-Lorentz spaces, see [15], [16], [12], and [13] respectively. In [17], we prove some results on the existence and decay properties of high order derivatives in time and space variables for local and global solutions of the Cauchy problem for NSE in Bessel-potential spaces. In [18], we prove some results on the existence and space-time decay rates of global strong solutions of the Cauchy problem for NSE equations in weighed $L^\infty(\mathbb{R}^d, |x|^\beta \mathrm{d}x)$ spaces. In [14], we considered the initial value problem for the non stationary NSE on torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ and showed that NSE are well-posed when the initial datum belongs to Sobolev spaces $V_\alpha := D(-\Delta)^{\alpha/2}$ with $\frac{1}{2} < \alpha < \frac{3}{2}$. In this paper, we construct mild solutions in the spaces $C([0,T]; \dot{H}_p^s(\mathbb{R}^d))$ to the Cauchy prob-

lem for NSE when the initial datum belongs to the Sobolev spaces $\dot{H}^s_p(\mathbb{R}^d)$, with $d \geq 2, p > \frac{d}{2}$, and $\frac{d}{p} - 1 \leq s < \frac{d}{2p}$. We obtain the existence of mild solutions with arbitrary initial value when T is small enough; and existence of mild solutions for any $T < +\infty$ when the norm of the initial value in the Besov spaces $\dot{B}^{s-d(\frac{1}{p}-\frac{1}{\tilde{q}}),\infty}_{\tilde{q}}$, $(\tilde{q} > \max\{p, q\}$, where $\frac{1}{q} = \frac{1}{p} - \frac{s}{d})$ is small enough. In the case $p > d$ and $s = 0$, this result is stronger than that of Cannone and Meyer [4,6] but under a weaker condition on the initial data. In the case of critical indexes $(p > \frac{d}{2}, s = \frac{d}{p} - 1)$, we can take $T = \infty$ when the norm of the initial value in the Besov spaces $\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}(\mathbb{R}^d)$, $(\tilde{q} > \max\{d, p\})$ is small enough. This result when $s = 0$ and $p = d$ is the theorem of Cannone [5]. The content of this paper is as follows: in Section 2, we state our main theorem after introducing some notations. In Section 3, we first establish some estimates concerning the heat semigroup with differential. We also recall some auxiliary lemmas and several estimates in the homogeneous Sobolev spaces and Besov spaces. Finally, in Section 4, we will give the proof of the main theorem.

2 Statement of the results

Now, for $T > 0$, we say that u is a mild solution of NSE on $[0, T]$ corresponding to a divergence-free initial datum u_0 when u solves the integral equation

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla \cdot \big(u(\tau, .) \otimes u(\tau, .)\big)\mathrm{d}\tau.$$

Above we have used the following notation: for a tensor $F = (F_{ij})$ we define the vector $\nabla \cdot F$ by $(\nabla \cdot F)_i = \sum_{j=1}^d \partial_j F_{ij}$ and for two vectors u and v, we define their tensor product $(u \otimes v)_{ij} = u_i v_j$. The operator $\mathbb{P}$ is the Helmholtz-Leray projection onto the divergence-free fields

$$(\mathbb{P}f)_j = f_j + \sum_{1 \leq k \leq d} R_j R_k f_k,$$

where R_j is the Riesz transforms defined as

$$R_j = \frac{\partial_j}{\sqrt{-\Delta}} \quad \text{i.e.} \quad \widehat{R_j g}(\xi) = \frac{i\xi_j}{|\xi|}\hat{g}(\xi).$$

The heat kernel $e^{t\Delta}$ is defined as

$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2}e^{-|.|^2/4t} * u)(x).$$

For a space of functions defined on $\mathbb{R}^d$, say $E(\mathbb{R}^d)$, we will abbreviate it as E. We denote by $L^q := L^q(\mathbb{R}^d)$ the usual Lebesgue space for $q \in [1, \infty]$ with the norm $\|.\|_{L^q}$, and we do not distinguish between the vector-valued and scalar-valued spaces of functions. Given a Banach space E with norm $\|.\|_E$, we denote by $L^p([0, T], E), 1 \leq p \leq +\infty$, set of functions $f(t)$ defined on $(0, T)$ with values in E such that $\int_0^T \|f(t)\|_E^p \mathrm{d}t < +\infty$. Let $BC([0, T); E)$ denote the bounded continuous functions defined on $(0, T)$. For vector-valued $f = (f_1, ..., f_M)$, we define $\|f\|_E = \big(\sum_{m=1}^{m=M} \|f_m\|_E^2\big)^{\frac{1}{2}}$. We define the Sobolev space by $\dot{H}^s_q := \dot{\Lambda}^{-s}L^q$

equipped with the norm $\|f\|_{\dot{H}^s_q} := \|\dot{\Lambda}^s f\|_{L^q}$. Here $\dot{\Lambda}^s := \mathcal{F}^{-1}|\xi|^s\mathcal{F}$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse, respectively. $\dot{\Lambda} = \sqrt{-\Delta}$ is the homogeneous Calderon pseudo-differential operator. Throughout the paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \leq CB$ with a uniform constant C. The notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$. Now we can state our results

Theorem 1 *Let p and s be such that*

$$p > \frac{d}{2} \text{ and } \frac{d}{p} - 1 \leq s < \frac{d}{2p}.$$

Set

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d}.$$

(a) *For all $\tilde{q} > \max\{p, q\}$, there exists a positive constant $\delta_{q,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying*

$$T^{\frac{1}{2}(1+s-\frac{d}{p})} \sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{p}-\frac{s}{d}-\frac{1}{\tilde{q}})} \left\|\mathrm{e}^{t\Delta}u_0\right\|_{L^{\tilde{q}}} \leq \delta_{q,\tilde{q},d}, \tag{2.1}$$

NSE has a unique mild solution $u \in BC([0,T); \dot{H}^s_p)$. Moreover, we have

$$t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{r})}u(t) \in BC([0,T); L^r), \text{ for all } r > \max\{p, q\}.$$

In particular, the condition (2.1) holds for arbitrary $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ when $T = T(u_0)$ is small enough.
(b) *If $s = \frac{d}{p} - 1$ then for all $\tilde{q} > \max\{p, d\}$ there exists a constant $\sigma_{\tilde{q},d} > 0$ such that if $\left\|u_0\right\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}} \leq \sigma_{\tilde{q},d}$ and $T = +\infty$ then the condition (2.1) holds.*

In the case of critical indexes ($s = \frac{d}{p} - 1, p > \frac{d}{2}$), we obtain the following consequence.

Proposition 1 *Let $p > \frac{d}{2}$. Then for any $\tilde{q} > \max\{p, d\}$, there exists a positive constant $\delta_{\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^{\frac{d}{p}-1}_p(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying*

$$\sup_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{\tilde{q}})} \left\|\mathrm{e}^{t\Delta}u_0\right\|_{L^{\tilde{q}}} \leq \delta_{\tilde{q},d}, \tag{2.2}$$

NSE has a unique mild solution $u \in BC([0,T); \dot{H}^{\frac{d}{p}-1}_p)$. Moreover, we have

$$t^{\frac{d}{2}(\frac{1}{d}-\frac{1}{r})}u(t) \in BC([0,T); L^r), \text{ for all } r > \max\{p, d\}.$$

Denoting $w = u - \mathrm{e}^{t\Delta}u_0$ then $w \in BC([0,T); \dot{H}^{\frac{d}{\tilde{p}}-1}_{\tilde{p}})$ for all $\tilde{p} > \frac{1}{2}\max\{p, d\}$.
In particular, the condition (2.2) holds for arbitrary $u_0 \in \dot{H}^{\frac{d}{p}-1}_p(\mathbb{R}^d)$ when $T = T(u_0)$ is small enough, and there exists a positive constant $\sigma_{\tilde{q},d}$ such that if

$$\left\|u_0\right\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}} \leq \sigma_{\tilde{q},d} \text{ and } T = +\infty,$$

then the condition (2.2) holds.

Remark 1 If $p = d$ then Proposition 1 is the theorem of Canone [5].

In the case of supercritical indexes $p > \frac{d}{2}$ and $\frac{d}{p} - 1 < s < \frac{d}{2p}$, we get the following consequence.

Proposition 2 *Let $p > \frac{d}{2}$ and $\frac{d}{p} - 1 < s < \frac{d}{2p}$. Then for any $\tilde{q}$ be such that $\tilde{q} > \max\{p, q\}$, where*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d},$$

there exists a positive constant $\delta_{q,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying

$$T^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|u_0\big\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}} \leq \delta_{q,\tilde{q},d}, \tag{2.3}$$

NSE has a unique mild solution $u \in BC([0, T); \dot{H}^s_p)$. Moreover, we have

$$t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{r})}u(t) \in BC([0, T); L^r), \text{ for all } r > \max\{p, q\}.$$

Remark 2 Proposition 2 is the theorem of Canone and Meyer [4,6] if $s = 0$, $p > d$, and the condition (2.3) is replaced by the condition

$$T^{\frac{1}{2}(1-\frac{d}{p})}\big\|u_0\big\|_{L^p} \leq \delta_{p,d}.$$

Note that in the case $s = 0$ and $p > d$, the condition (2.3) is weaker than the above condition because of the following elementary imbedding maps

$$L^p(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d), (\tilde{q} > p \geq d),$$

but these two spaces are different. Indeed, we have $\big|x\big|^{-\frac{d}{p}} \notin L^p(\mathbb{R}^d)$. On the other hand by using Lemma 3, we can easily prove that $\big|x\big|^{-\frac{d}{p}} \in \dot{B}_{\tilde{q}}^{-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d)$ for all $\tilde{q} > p$.

3 Tools from harmonic analysis

In this section we prepare some auxiliary lemmas.
The main property we use throughout this paper is that the operator $\dot{\Lambda}^s e^{t\Delta}\mathbb{P}\nabla$ is a matrix of convolution operators with bounded integrable kernels.

Lemma 1 *Let $s > -1$. Then the kernel function of $\dot{\Lambda}^s e^{t\Delta}\mathbb{P}\nabla$ is the function*

$$K_t(x) = t^{-\frac{d+1+s}{2}}K\big(\frac{x}{\sqrt{t}}\big),$$

where the function K is the kernel function of $\dot{\Lambda}^s e^{\Delta}\mathbb{P}\nabla$ which satisfies the following inequality

$$|K(x)| \lesssim \frac{1}{1 + |x|^{d+1+s}}.$$

Proof See Proposition 11.1 [23], p. 107.

Lemma 2 (Sobolev inequalities).
If $s_1 > s_2,\ 1 < q_1,\ q_2 < \infty$, *and* $s_1 - \frac{d}{q_1} = s_2 - \frac{d}{q_2}$, *then we have the following embedding mapping*

$$\dot{H}^{s_1}_{q_1} \hookrightarrow \dot{H}^{s_2}_{q_2}.$$

In this paper we use the definition of the homogeneous Besov space $\dot{B}^{s,p}_q$ in [2, 3]. The following lemma will provide a different characterization of Besov spaces $\dot{B}^{s,p}_q$ in terms of the heat semigroup and will be one of the staple ingredients of the proof of Theorem 1.

Lemma 3
Let $1 \leq p, q \leq \infty$ *and* $s < 0$. *Then the two quantities*

$$\Big(\int_0^\infty (t^{-\frac{s}{2}}\big\|\mathrm{e}^{t\Delta}f\big\|_{L^q})^p\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}} \text{ and } \big\|f\big\|_{\dot{B}^{s,p}_q} \text{ are equivalent.}$$

Proof See Theorem 5.4 in ([23], p. 45).

Lemma 4 *Let* $\theta < 1$ *and* $\gamma < 1$ *then*

$$\int_0^t (t-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau = Ct^{1-\gamma-\theta},\ \text{where } C = \int_0^1 (1-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau < \infty.$$

The proof of this lemma is elementary and may be omitted. □

Let us recall following result on solutions of a quadratic equation in Banach spaces (Theorem 22.4 in [23], p. 227).

Theorem 2 *Let* E *be a Banach space, and* $B : E \times E \to E$ *be a continuous bilinear map such that there exists* $\eta > 0$ *so that*

$$\|B(x, y)\| \leq \eta\|x\|\|y\|,$$

for all x *and* y *in* E. *Then for any fixed* $y \in E$ *such that* $\|y\| \leq \frac{1}{4\eta}$, *the equation* $x = y - B(x, x)$ *has a unique solution* $\overline{x} \in E$ *satisfying* $\|\overline{x}\| \leq \frac{1}{2\eta}$.

4 Proof of Theorem 1

In this section we shall give the proof of Theorem 1.
We now need four more lemmas. In order to proceed, we define an auxiliary space $\mathcal{N}^s_{p,T}$ which is made up of the functions $u(t, x)$ such that

$$u \in C([0, T); \dot{H}^s_p), \big\|u\big\|_{\mathcal{N}^s_{p,T}} := \sup_{0<t<T}\big\|u(t, .)\big\|_{\dot{H}^s_p} < \infty,$$

and

$$\lim_{t\to 0}\big\|u(t)\big\|_{\dot{H}^s_p} = 0, \tag{4.1}$$

with $p > 1$ and $s \geq \frac{d}{p} - 1$.
We define the auxiliary space $\mathcal{K}^{\tilde{q}}_{q,T}$ which is made up of the functions $u(t, x)$ such that

$$t^{\frac{\alpha}{2}}u \in C([0, T); L^{\tilde{q}}), \big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}}\big\|u(t, .)\big\|_{L^{\tilde{q}}} < \infty,$$

and

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|u(t)\big\|_{L^{\tilde{q}}} = 0, \tag{4.2}$$

with $\tilde{q} \geq q \geq d$ and $\alpha = d(\frac{1}{q} - \frac{1}{\tilde{q}})$.

Remark 3 The auxiliary space $\mathcal{K}_{\tilde{q}} := \mathcal{K}^{\tilde{q}}_{d,T}$ ($\tilde{q} \geq d$) was introduced by Weissler and systematically used by Kato [20] and Cannone [5].

Lemma 5 *Suppose that $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ with $p > 1$ and $\frac{d}{p} - 1 \leq s < \frac{d}{p}$. Then for all $\tilde{q}$ satisfying*

$$\tilde{q} > \max\{p, q\},$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d},$$

we have

$$\mathrm{e}^{t\Delta} u_0 \in \mathcal{K}^{\tilde{q}}_{q,\infty}.$$

Proof First, we consider the case $p \leq q$. In this case $s \geq 0$, applying Lemma 2, we have $u_0 \in L^q$. We will prove that

$$\sup_{0<t<\infty} t^{\frac{\alpha}{2}} \big\|\mathrm{e}^{t\Delta} u_0\big\|_{L^{\tilde{q}}} \lesssim \big\|u_0\big\|_{L^q}, \text{ for all } \tilde{q} \geq q.$$

Set

$$\frac{1}{h} = 1 + \frac{1}{\tilde{q}} - \frac{1}{q}.$$

Applying Young's inequality we obtain

$$\begin{aligned} \big\|\mathrm{e}^{t\Delta} u_0\big\|_{L^{\tilde{q}}} &= \frac{1}{(4\pi t)^{d/2}} \big\|\mathrm{e}^{\frac{-|.|^2}{4t}} * u_0\big\|_{L^{\tilde{q}}} \lesssim \frac{1}{t^{d/2}} \big\|\mathrm{e}^{\frac{-|.|^2}{4t}}\big\|_{L^h} \big\|u_0\big\|_{L^q} \\ &= t^{-\frac{\alpha}{2}} \big\|\mathrm{e}^{\frac{-|.|^2}{4}}\big\|_{L^h} \big\|u_0\big\|_{L^q} \simeq t^{-\frac{\alpha}{2}} \big\|u_0\big\|_{L^q}. \end{aligned} \tag{4.3}$$

This proves the result. We now prove that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \big\|\mathrm{e}^{t\Delta} u_0\big\|_{L^{\tilde{q}}} = 0, \text{ for all } \tilde{q} > q.$$

Set $\mathcal{X}_n(x) = 0$ for $x \in \{x : \ |x| < n\} \cap \{x : \ |u_0(x)| < n\}$ and $\mathcal{X}_n(x) = 1$ otherwise. We have

$$t^{\frac{\alpha}{2}} \big\|\mathrm{e}^{t\Delta} u_0\big\|_{L^{\tilde{q}}} \leq C\big(t^{\frac{\alpha-d}{2}} \big\|\mathrm{e}^{\frac{-|.|^2}{4t}} * (\mathcal{X}_n u_0)\big\|_{L^{\tilde{q}}} + t^{\frac{\alpha-d}{2}} \big\|\mathrm{e}^{\frac{-|.|^2}{4t}} * ((1-\mathcal{X}_n) u_0)\big\|_{L^{\tilde{q}}}\big).$$

Applying Young's inequality, we have

$$Ct^{\frac{\alpha-d}{2}} \big\|\mathrm{e}^{\frac{-|.|^2}{4t}} * (\mathcal{X}_n u_0)\big\|_{L^{\tilde{q}}} \leq C_1 \big\|\mathrm{e}^{\frac{-|.|^2}{4}}\big\|_{L^h} \big\|\mathcal{X}_n u_0\big\|_{L^q} \leq C_2 \big\|\mathcal{X}_n u_0\big\|_{L^q}. \tag{4.4}$$

For any $\epsilon > 0$, we can take n large enough that $C_2 \big\|\mathcal{X}_n u_0\big\|_{L^q} < \frac{\epsilon}{2}$.
Fixed one of such n and applying Young's inequality, we have

$$\begin{aligned} Ct^{\frac{\alpha-d}{2}} \big\|\mathrm{e}^{\frac{-|.|^2}{4t}} * ((1-\mathcal{X}_n) u_0)\big\|_{L^{\tilde{q}}} &\leq C_3 t^{\frac{\alpha-d}{2}} \big\|\mathrm{e}^{\frac{-|.|^2}{4t}}\big\|_{L^1} \big\|(1-\mathcal{X}_n) u_0\big\|_{L^{\tilde{q}}} \\ &\leq C_3 t^{\frac{\alpha}{2}} \big\|\mathrm{e}^{\frac{-|.|^2}{4}}\big\|_{L^1} \big\|n(1-\mathcal{X}_n)\big\|_{L^{\tilde{q}}} = C_4(n) t^{\frac{\alpha}{2}} < \frac{\epsilon}{2}, \text{ for } t < t_0 \end{aligned} \tag{4.5}$$

with small enough $t_0 = t_0(n)$. From estimates (4.4) and (4.5), we have

$$t^{\frac{\alpha}{2}} \big\|\mathrm{e}^{t\Delta} u_0\big\|_{L^{\tilde{q}}} \leq C_2 \big\|\mathcal{X}_n u_0\big\|_{L^q} + C_5(n) t^{\frac{\alpha}{2}} < \epsilon, \text{ for } t < t_0.$$

Finally, we consider the case $p > q$. In this case $s < 0$, we will prove that

$$\sup_{0<t<\infty} t^{\frac{\alpha}{2}} \big\|e^{t\Delta}u_0\big\|_{L^{\tilde{q}}} \lesssim \big\|u_0\big\|_{\dot{H}^s_p}, \text{ for } \tilde{q} \geq p.$$

We have

$$e^{t\Delta}u_0 = e^{t\Delta}\dot{\Lambda}^{-s}\dot{\Lambda}^s u_0 = \frac{1}{t^{\frac{d-s}{2}}} K\Big(\frac{\cdot}{\sqrt{t}}\Big) * (\dot{\Lambda}^s u_0),$$

where

$$\hat{K}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-|\xi|^2}|\xi|^{-s},\ |K(x)| \lesssim \frac{1}{(1+|x|)^{d-s}}.$$

Set

$$\frac{1}{h} = 1 + \frac{1}{\tilde{q}} - \frac{1}{p}.$$

Applying Young's inequality to obtain

$$\big\|e^{t\Delta}u_0\big\|_{L^{\tilde{q}}} \lesssim t^{-\frac{\alpha}{2}} \big\|K\big\|_{L^h} \big\|\dot{\Lambda}^s u_0\big\|_{L^p} \simeq t^{-\frac{\alpha}{2}} \big\|u_0\big\|_{\dot{H}^s_p}.$$

This proves the result. We now claim that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \big\|e^{t\Delta}u_0\big\|_{L^{\tilde{q}}} = 0, \quad \text{for all } \tilde{q} > p.$$

Set $\mathcal{X}_{n,s}(x) = 0$ for $x \in \{x :\ |x| < n\} \cap \{x :\ |\dot{\Lambda}^s u_0(x)| < n\}$ and $\mathcal{X}_{n,s}(x) = 1$ otherwise. From the above proof we deduce that, for any $\epsilon > 0$, there exist a sufficiently large n and a sufficiently small $t_0 = t_0(n)$ such that

$$t^{\frac{\alpha}{2}} \big\|e^{t\Delta}u_0\big\|_{L^{\tilde{q}}} \leq$$
$$C_1\big\|K\big\|_{L^h}\big\|\mathcal{X}_{n,s}\dot{\Lambda}^s u_0\big\|_{L^p} + C_2 n t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{\tilde{q}})}\big\|K\big\|_{L^1}\big\|1-\mathcal{X}_{n,s}\big\|_{L^{\tilde{q}}} < \epsilon, \text{ for } t < t_0.$$

In the following lemmas a particular attention will be devoted to the study of the bilinear operator $B(u,v)(t)$ defined by

$$B(u,v)(t) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\mathrm{d}\tau. \tag{4.6}$$

Lemma 6 *Let p and s be such that*

$$p > \frac{d}{2} \text{ and } \frac{d}{p} - 1 \leq s < \frac{d}{2p}.$$

Then the bilinear operator B is continuous from $\mathcal{K}^{\tilde{q}}_{q,T} \times \mathcal{K}^{\tilde{q}}_{q,T}$ into $\mathcal{N}^s_{p,T}$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{d},\ q < \tilde{q} \leq 2p,$$

and the following inequality holds

$$\big\|B(u,v)\big\|_{\mathcal{N}^s_{p,T}} \leq CT^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}, \tag{4.7}$$

where C is a positive constant and independent of T.

Proof By Lemma 1, we have

$$\begin{aligned}\big\|B(u,v)(t)\big\|_{\dot{H}^s_p} &\le \int_0^t \big\|\dot{\Lambda}^s \mathrm{e}^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\big\|_{L^p}\mathrm{d}\tau\\ &= \int_0^t \Big\|\frac{1}{(t-\tau)^{\frac{d+1+s}{2}}}K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\big(u(\tau)\otimes v(\tau)\big)\Big\|_{L^p}\mathrm{d}\tau. \quad (4.8)\end{aligned}$$

Applying Young's inequality, we have

$$\begin{aligned}&\Big\|\frac{1}{(t-\tau)^{\frac{d+1+s}{2}}}K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\big(u(\tau)\otimes v(\tau)\big)\Big\|_{L^p}\\ &\lesssim \frac{1}{(t-\tau)^{\frac{d+1+s}{2}}}\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{L^r}\big\|u(\tau)\otimes v(\tau)\big\|_{L^{\frac{\tilde{q}}{2}}}, \quad (4.9)\end{aligned}$$

where

$$\frac{1}{r} = 1+\frac{1}{p}-\frac{2}{\tilde{q}}, \quad (4.10)$$

Applying Hölder's inequality, we have

$$\big\|u(\tau)\otimes v(\tau)\big\|_{L^{\frac{\tilde{q}}{2}}} \le \big\|u(\tau)\big\|_{L^{\tilde{q}}}\big\|v(\tau)\big\|_{L^{\tilde{q}}}. \quad (4.11)$$

Since the equality (4.10) and Lemma 1 it follows that

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{L^r} = (t-\tau)^{\frac{d}{2r}}\big\|K\big\|_{L^r} \simeq (t-\tau)^{\frac{d}{2}(1+\frac{1}{p}-\frac{2}{\tilde{q}})}. \quad (4.12)$$

From the inequalities (4.9), (4.11), and (4.12) we deduce that

$$\big\|\mathrm{e}^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\big\|_{\dot{H}^s_p} \lesssim (t-\tau)^{\frac{d}{2p}-\frac{d}{\tilde{q}}-\frac{s+1}{2}}\big\|u(\tau)\big\|_{L^{\tilde{q}}}\big\|v(\tau)\big\|_{L^{\tilde{q}}}. \quad (4.13)$$

By the inequalities (4.8), (4.13), and Lemma 4, we have

$$\begin{aligned}\big\|B(u,v)(t)\big\|_{\dot{H}^s_p} &\lesssim \int_0^t (t-\tau)^{\frac{d}{2p}-\frac{d}{\tilde{q}}-\frac{s+1}{2}}\big\|u(\tau)\big\|_{L^{\tilde{q}}}\big\|v(\tau)\big\|_{L^{\tilde{q}}}\mathrm{d}\tau\\ &\le \int_0^t (t-\tau)^{\frac{d}{2p}-\frac{d}{\tilde{q}}-\frac{s+1}{2}}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}\mathrm{d}\tau\\ &= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}\int_0^t (t-\tau)^{\frac{d}{2p}-\frac{d}{\tilde{q}}-\frac{s+1}{2}}\tau^{-\alpha}\mathrm{d}\tau\\ &\simeq t^{\frac{1}{2}(1+s-\frac{d}{p})}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}. \quad (4.14)\end{aligned}$$

The estimate (4.7) is deduced from the inequality (4.14).
Let us now check the validity of condition (4.1) for the bilinear term $B(u,v)(t)$. In fact, from the estimate (4.14) it follows that

$$\lim_{t\to 0}\big\|B(u,v)(t)\big\|_{\dot{H}^s_p} = 0, \quad (4.15)$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|u(t)\big\|_{L^{\tilde{q}}} = \lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|v(t)\big\|_{L^{\tilde{q}}} = 0.$$

Finally, the continuity at $t=0$ of $B(u,v)(t)$ follows from the equality (4.15). The continuity elsewhere follows from carefully rewriting the expression $\int_0^{t+\epsilon}-\int_0^t$ and applying the same argument.

Lemma 7 *Let q and $\tilde{q}$ be such that $\tilde{q} > q \geq d$. Then the bilinear operator B is continuous from $\mathcal{K}^{\tilde{q}}_{q,T} \times \mathcal{K}^{\tilde{q}}_{q,T}$ into $\mathcal{K}^{\tilde{q}}_{q,T}$ and the following inequality holds*

$$\big\|B(u,v)\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} \leq CT^{\frac{1}{2}(1-\frac{d}{q})}\big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}, \tag{4.16}$$

where C is a positive constant and independent of T.

Proof Applying the estimate (4.13) for $s = 0$ and $p = \tilde{q}$, we have

$$\big\|\mathrm{e}^{(t-\tau)\Delta}\mathbb{P}\nabla\cdot\big(u(\tau)\otimes v(\tau)\big)\big\|_{L^{\tilde{q}}} \lesssim (t-\tau)^{-\frac{d}{2\tilde{q}}-\frac{1}{2}}\big\|u(\tau)\big\|_{L^{\tilde{q}}}\big\|v(\tau)\big\|_{L^{\tilde{q}}}.$$

Applying Lemma 4, we have

$$\begin{aligned}\big\|B(u,v)(t)\big\|_{L^{\tilde{q}}} &\lesssim \int_0^t (t-\tau)^{-\frac{d}{2\tilde{q}}-\frac{1}{2}}\big\|u(\tau)\big\|_{L^{\tilde{q}}}.\big\|v(\tau)\big\|_{L^{\tilde{q}}}\mathrm{d}\tau \\ &\leq \int_0^t (t-\tau)^{-\frac{d}{2\tilde{q}}-\frac{1}{2}}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}\mathrm{d}\tau \\ &= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}\int_0^t (t-\tau)^{-\frac{d}{2\tilde{q}}-\frac{1}{2}}\tau^{-\alpha}\mathrm{d}\tau \\ &\simeq t^{-\frac{\alpha}{2}}t^{\frac{1}{2}(1-\frac{d}{q})}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}.\end{aligned}$$

Thus

$$t^{\frac{\alpha}{2}}\big\|B(u,v)(t)\big\|_{L^{\tilde{q}}} \lesssim t^{\frac{1}{2}(1-\frac{d}{q})}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|u(\eta)\big\|_{L^{\tilde{q}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\big\|v(\eta)\big\|_{L^{\tilde{q}}}. \tag{4.17}$$

The estimate (4.16) is deduced from the inequality (4.17).
Now we check the validity of condition (4.2) for the bilinear term $B(u,v)(t)$. From the estimate (4.17) it follows that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|B(u,v)(t)\big\|_{L^{\tilde{q}}} = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|u(t)\big\|_{L^{\tilde{q}}} = \lim_{t\to 0} t^{\frac{\alpha}{2}}\big\|v(t)\big\|_{L^{\tilde{q}}} = 0.$$

Finally, the continuity at $t = 0$ of $t^{\frac{\alpha}{2}}B(u,v)(t)$ follows from the equality (4.15). The continuity elsewhere follows from carefully rewriting the expression $\int_0^{t+\epsilon} - \int_0^t$ and applying the same argument.

The following lemma, the proof of which is omitted, is a generalization of Lemma 7.

Lemma 8 *Let $d \leq q \leq \tilde{q}_2 < \infty$ and $q < \tilde{q}_1 < \infty$ be such that one of the following conditions is satisfied.*

$$q < \tilde{q}_1 < 2d, q \leq \tilde{q}_2 < \frac{d\tilde{q}_1}{2d - \tilde{q}_1},$$

or

$$2d \leq \tilde{q}_1 \leq 2q, q \leq \tilde{q}_2 < \infty,$$

or

$$2q < \tilde{q}_1 < \infty, \frac{\tilde{q}_1}{2} < \tilde{q}_2 < \infty.$$

Then the bilinear operator B is continuous from $\mathcal{K}^{\tilde{q}_1}_{q,T} \times \mathcal{K}^{\tilde{q}_1}_{q,T}$ into $\mathcal{K}^{\tilde{q}_2}_{q,T}$, and we have the inequality

$$\big\|B(u,v)\big\|_{\mathcal{K}^{\tilde{q}_2}_{q,T}} \leq CT^{\frac{1}{2}(1-\frac{d}{q})}\big\|u\big\|_{\mathcal{K}^{\tilde{q}_1}_{q,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{q}_1}_{q,T}},$$

where C is a positive constant and independent of T.

Proof of Theorem 1

(a) From Lemma 7, B is continuous from $\mathcal{K}^{\tilde{q}}_{q,T} \times \mathcal{K}^{\tilde{q}}_{q,T}$ to $\mathcal{K}^{\tilde{q}}_{q,T}$ and we have the inequality

$$\big\|B(u,v)\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} \leq C_{q,\tilde{q},d}T^{\frac{1}{2}(1-\frac{d}{q})}\big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} = C_{q,\tilde{q},d}T^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|u\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}}\big\|v\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}},$$

where $C_{q,\tilde{q},d}$ is a positive constant independent of T. From Theorem 2 and the above inequality, we deduce that for any $u_0 \in \dot{H}^s_p$ satisfying

$$T^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|\mathrm{e}^{t\Delta}u_0\big\|_{\mathcal{K}^{\tilde{q}}_{q,T}} = T^{\frac{1}{2}(1+s-\frac{d}{p})}\sup_{0<t<T} t^{\frac{\alpha}{2}}\big\|\mathrm{e}^{t\Delta}u_0\big\|_{L^{\tilde{q}}} \leq \frac{1}{4C_{q,\tilde{q},d}},$$

where

$$\alpha = d\Big(\frac{1}{q} - \frac{1}{\tilde{q}}\Big) = d\Big(\frac{1}{p} - \frac{s}{d} - \frac{1}{\tilde{q}}\Big),$$

NSE has a solution u on the interval $(0,T)$ so that $u \in \mathcal{K}^{\tilde{q}}_{q,T}$.
We prove that $u \in \bigcap_{r>\max\{p,q\}} \mathcal{K}^r_{q,T}$. We consider three cases $q < \tilde{q} < 2d$ and $2d \leq \tilde{q} \leq 2q$, and $2q < \tilde{q} < \infty$ separately.
Note that if $\max\{p,q\} \geq 2d$ then there does not exist $\tilde{q}$ satisfying the condition of the first case, and if $p \geq 2q$ then there does not exist $\tilde{q}$ satisfying the condition of the second case. Therefore the number of cases that can occur depends on s and p.
First, we consider the case $q < \tilde{q} < 2d$. There are two possibilities $\tilde{q} > \frac{4d}{3}$ and $\tilde{q} \leq \frac{4d}{3}$. In the case $\tilde{q} > \frac{4d}{3}$, we apply Lemmas 5 and 8 to obtain $u \in \mathcal{K}^r_{q,T}$ for all r satisfying $\max\{p,q\} < r < \tilde{q}_1$ where $\tilde{q}_1 = \frac{d\tilde{q}}{2d-\tilde{q}} > 2d$. Thus, $u \in \mathcal{K}^{2d}_{q,T}$. Applying again Lemmas 5 and 8, we deduce that $u \in \mathcal{K}^r_{q,T}$ for all $r > \max\{p,q\}$. In the case $\tilde{q} \leq \frac{4d}{3}$, we set up the following series of numbers $\{\tilde{q}_i\}_{0\leq i\leq N}$ by induction. Set $\tilde{q}_0 = \tilde{q}$ and $\tilde{q}_1 = \frac{d\tilde{q}_0}{2d-\tilde{q}_0}$. We have $\tilde{q}_1 > \tilde{q}_0$. If $\tilde{q}_1 > \frac{4d}{3}$ then set $N = 1$ and stop here. In the case $\tilde{q}_1 \leq \frac{4d}{3}$ set $\tilde{q}_2 = \frac{d\tilde{q}_1}{2d-\tilde{q}_1}$. We have $\tilde{q}_2 > \tilde{q}_1$. If $\tilde{q}_2 > \frac{4d}{3}$ then set $N = 2$ and stop here. In the case $\tilde{q}_2 \leq \frac{4d}{3}$, set $\tilde{q}_3 = \frac{d\tilde{q}_2}{2d-\tilde{q}_2}$. We have $\tilde{q}_3 > \tilde{q}_2$, and so on, there exists $k \geq 0$ such that $\tilde{q}_k \leq \frac{4d}{3}, \tilde{q}_{k+1} = \frac{d\tilde{q}_k}{2d-\tilde{q}_k} > \frac{4d}{3}$. We set $N = k+1$ and stop here, and we have

$$\tilde{q}_0 = \tilde{q}, \tilde{q}_i = \frac{d\tilde{q}_{i-1}}{2d-\tilde{q}_{i-1}}, \tilde{q}_i > \tilde{q}_{i-1} \text{ for } i = 1,2,3,..,N,$$

$$2d \geq \tilde{q}_N > \frac{4d}{3} \geq \tilde{q}_{N-1}.$$

From $u \in \mathcal{K}^{\tilde{q}_0}_{q,T}$, applying Lemmas 5 and 8 to obtain $u \in \mathcal{K}^r_{q,T}$ for all r satisfying $\max\{p,q\} < r < \tilde{q}_1$. Then applying again Lemmas 5 and 8 to get $u \in \mathcal{K}^r_{q,T}$ for all r satisfying $\max\{p,q\} < r < \tilde{q}_2$, and so on, finishing we have $u \in \mathcal{K}^r_{q,T}$ for all r satisfying $\max\{p,q\} < r < \tilde{q}_N$. Therefore $u \in \mathcal{K}^r_{q,T}$ for all r satisfying $\frac{4d}{3} < r < \tilde{q}_N$. From the proof of the case $\tilde{q} > \frac{4d}{3}$, we have $u \in \mathcal{K}^r_{q,T}$ for all $r > \max\{p,q\}$.
We now consider the case $2d \leq \tilde{q} \leq 2q$. We show that $u \in \mathcal{K}^r_{q,T}$ for all $r > \max\{p,q\}$. This is easily deduced by applying Lemmas 5 and 8.
Finally, we consider the case $2q < \tilde{q} < \infty$. Let $i \in \mathbb{N}$ be such that

$$\frac{\tilde{q}}{2^{i-1}} \geq \max\{2q,p\} > \frac{\tilde{q}}{2^i}.$$

From $\tilde{q} > \max\{p,q\}$ and $\tilde{q} > 2q$, we have $\tilde{q} > \max\{2q,p\}$, hence $i \geq 1$. Applying Lemmas 5 and 8 to obtain $u \in \mathcal{K}^r_{q,T}$ for all $r > \frac{\tilde{q}}{2}$. Applying again Lemmas 5 and 8 to get $u \in \mathcal{K}^r_{q,T}$ for all $r > \frac{\tilde{q}}{2^2}$, and so on, finishing we have $u \in \mathcal{K}^r_{q,T}$ for all $r > \frac{\tilde{q}}{2^{i-1}}$. Applying again Lemmas 5 and 8 to obtain $u \in \mathcal{K}^r_{q,T}$ for all $r > \max\{p,q,\frac{\tilde{q}}{2^i}\}$. If $\max\{p,q\} \geq \frac{\tilde{q}}{2^i}$ then we have $u \in \mathcal{K}^r_{q,T}$ for all $r > \max\{p,q\}$. If $\max\{p,q\} < \frac{\tilde{q}}{2^i}$ then $2q > \frac{\tilde{q}}{2^i}$. Thus $u \in \mathcal{K}^r_{q,T}$ for all r satisfying $r > \frac{\tilde{q}}{2^i}$, hence $u \in \mathcal{K}^{2q}_{q,T}$. Applying Lemmas 5 and 8 to obtain $u \in \mathcal{K}^r_{q,T}$ for all $r > \max\{p,q\}$. This proves the result.
We now prove that $u \in BC([0,T);\dot{H}^s_p)$. Indeed, from $u \in \mathcal{K}^r_{q,T}$ for all $r > \max\{p,q\}$, applying Lemma 6 to obtain $B(u,u) \in \mathcal{N}^s_{p,T} \subseteq BC\big([0,T);\dot{H}^s_p\big)$. On the other hand, since $u \in \dot{H}^s_p$, it follows that $\mathrm{e}^{t\Delta}u_0 \in BC\big([0,T);\dot{H}^s_p\big)$. Therefore

$$u = \mathrm{e}^{t\Delta}u_0 - B(u,u) \in BC\big([0,T);\dot{H}^s_p\big).$$

Finally, we will show that the condition (2.1) is valid when T is small enough. Indeed, from the definition of $\mathcal{K}^{\tilde{q}}_{q,T}$ and Lemma 5, we deduce that the left-hand side of the condition (2.1) converges to 0 when T goes to 0. Therefore the condition (2.1) holds for arbitrary $u_0 \in \dot{H}^s_p(\mathbb{R}^d)$ when $T(u_0)$ is small enough.
(b) From Lemma 3, the two quantities $\big\|u_0\big\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}}$ and $\sup\limits_{0<t<\infty} t^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\big\|\mathrm{e}^{t\Delta}u_0\big\|_{L^{\tilde{q}}}$ are equivalent. Thus, there exists a positive constant $\sigma_{\tilde{q},d}$ such that the condition (2.1) holds for $T = \infty$ whenever $\big\|u_0\big\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}} \leq \sigma_{\tilde{q},d}$. □

Proof of Proposition 1

By Theorem 1, we only need to prove that $w \in \mathcal{N}^{\frac{d}{\tilde{p}}-1}_{\tilde{p},T}$ for all $\tilde{p} > \frac{1}{2}\max\{p,d\}$. Indeed, applying Lemma 6, we deduce that the bilinear operator B is continuous from $\mathcal{K}^r_{d,T} \times \mathcal{K}^r_{d,T}$ into $\mathcal{N}^{\frac{d}{\tilde{p}}-1}_{\tilde{p},T}$ for all $\tilde{p} > \frac{d}{2}$ and r satisfying $d < r \leq 2\tilde{p}$, hence from $u \in \bigcap\limits_{r>\max\{p,d\}} \mathcal{K}^r_{d,T}$ and $2\tilde{p} > \max\{p,d\}$, we have $w = -B(u,u) \in \mathcal{N}^{\frac{d}{\tilde{p}}-1}_{\tilde{p},T}$.
The proof Proposition 1 is complete. □

Proof of Proposition 2

By Lemma 3, we deduce that the two quantities $\big\|u_0\big\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}}$ and $\sup\limits_{0<t<\infty} t^{\frac{d}{2}(\frac{1}{p}-\frac{s}{d}-\frac{1}{\tilde{q}})}\big\|\mathrm{e}^{t\Delta}u_0\big\|_{L^{\tilde{q}}}$ are equivalent. Therefore

$$\sup_{0<t<T} t^{\frac{d}{2}(\frac{1}{p}-\frac{s}{d}-\frac{1}{\tilde{q}})}\big\|\mathrm{e}^{t\Delta}u_0\big\|_{L^{\tilde{q}}} \lesssim \big\|u_0\big\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{p}-\frac{d}{\tilde{q}}),\infty}}.$$

Proposition 2 is proved by applying the above inequality and Theorem 1. □

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