

# LEAVITT PATH ALGEBRAS HAVING UNBOUNDED GENERATING NUMBER

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**ABSTRACT.** We present a result of P. Ara which establishes that the Unbounded Generating Number property is a Morita invariant for unital rings. Using this, we give necessary and sufficient conditions on a graph  $E$  so that the Leavitt path algebra associated to  $E$  has UGN. We conclude by identifying the graphs for which the Leavitt path algebra is (equivalently) directly finite; stably finite; Hermite; and has cancellation of projectives.

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**Key words:** Cohn path algebra; Invariant Basis Number; Leavitt path algebra; Unbounded Generating Number.

## 1. INTRODUCTION

The concept of projective modules over rings is a generalization of the idea of a vector space; and their structure theory, in some sense, may be considered as a generalization of the theorem asserting the existence and uniqueness of cardinalities of bases for vector spaces. Projective modules play an important role in different branches of mathematics, in particular, homological algebra and algebraic K-theory. In general ring theory it is often convenient to impose certain conditions on the projective modules, either to exclude pathological cases or to ensure better behavior. For rings we have the following successively more restrictive cancellation-type conditions on the projective (and, in particular, the free) modules:

- (1) Invariant Basis Number (in short: IBN),
- (2) Unbounded Generating Number (in short: UGN)
- (3) stable finiteness,
- (4) the Hermite property (in P. M. Cohn's sense), and
- (5) cancellation of projectives.

Definitions of these properties are given below. It is easily verified that each of these conditions is left-right symmetric and each implies the previous condition. Moreover, in general, all these classes are distinct.

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The conditions (1) - (3) have been well studied in both algebraic and topological settings. For basic properties of rings with these first three properties we suggest [9]. For additional examples of rings satisfying property (3), see [8] and the references given there. By finding conditions for an embedding of a (non-commutative) ring in a skew field to be possible, P. M. Cohn discovered the theory of free ideal rings, in which properties (1) - (5) above play an important role (see, e.g., [12]). We refer the reader to [10] and [11] for an investigation of rings having (4) and (5), respectively. It is fair to say that, in general, it is not at all easy to decide whether a given ring has any one of these properties.

Given a (row-finite) directed graph  $E$  and field  $K$ , Aranda Pino and the first author in [1], and independently Ara, Moreno, and Pardo in [6], introduced the *Leavitt path algebra*  $L_K(E)$ . These Leavitt path algebras generalize the Leavitt algebras  $L_K(1, n)$  of [16], and also contain many other interesting classes of algebras. In addition, Leavitt path algebras are intimately related to graph  $C^*$ -algebras (see [17]). In [5] Ara and Goodearl introduced and investigated the *Cohn path algebra*  $C_K(E)$  of  $E$  having coefficients in a field  $K$ . Recently, Kanuni and the first author [3] have shown that  $C_K(E)$  has IBN for every finite graph  $E$ . On the other hand, as of the writing of this article, it is an open question to give necessary and sufficient conditions on  $E$  so that  $L_K(E)$  has IBN. However, regarding the remaining four aforementioned properties, we are able to completely classify those graphs  $E$  for which  $L_K(E)$  has UGN (Theorem 3.16), as well as classify those graphs  $E$  for which  $L_K(E)$  satisfies (equivalently) properties (3), (4), and (5) (Theorem 4.2). We achieve similar results for  $C_K(E)$  as well.

The article is organized as follows. For the remainder of this introductory section we recall the germane background information. In Section 2 we present Ara's proof that the Unbounded Generating Number property is a Morita invariant property in the class of all unital rings (Theorem 2.8). In Section 3 we give a necessary and sufficient condition for the Leavitt path algebra of a finite source-free graph to have Unbounded Generating Number (Theorem 3.9). Then, by using Theorem 2.8 and the source elimination process, we obtain a criterion for the Leavitt path algebra of an arbitrary finite graph to have Unbounded Generating Number (Theorem 3.16). Consequently, we get a criterion for the Cohn path algebra of a finite graph to have Unbounded Generating Number (Corollary 3.17). We conclude with Section 4, in which we describe (Theorem 4.2, resp. Corollary 4.5) those graphs  $E$  for which  $L_K(E)$  (resp.,  $C_K(E)$ ) have any one of the (equivalent) aforementioned properties (3) - (5).

Throughout this note, all rings are nonzero, associative with identity and all modules are unitary. The set of nonnegative integers is denoted by  $\mathbb{N}$ , the positive integers by  $\mathbb{N}^+$ .

A (directed) graph  $E = (E^0, E^1, s, r)$  (or simply  $E = (E^0, E^1)$ ) consists of two disjoint sets  $E^0$  and  $E^1$ , called *vertices* and *edges* respectively, together with two maps  $s, r : E^1 \rightarrow E^0$ . The vertices  $s(e)$  and  $r(e)$  are referred to as the *source* and the *range* of the edge  $e$ , respectively. The graph is called *row-finite* if  $|s^{-1}(v)| < \infty$  for all  $v \in E^0$ . All graphs in this paper will be assumed to be row-finite. A graph  $E$  is *finite* if both sets  $E^0$  and  $E^1$  are finite. A vertex  $v$  for which  $s^{-1}(v)$  is empty is called a *sink*; a vertex  $v$  for which  $r^{-1}(v)$  is empty is

called a *source*; a vertex  $v$  is called *isolated* if it is both a source and a sink; and a vertex  $v$  is *regular* iff  $0 < |s^{-1}(v)| < \infty$ . A graph  $E$  is said to be *source-free* if it has no sources.

A *path*  $p = e_1 \cdots e_n$  in a graph  $E$  is a sequence of edges  $e_1, \dots, e_n$  such that  $r(e_i) = s(e_{i+1})$  for  $i = 1, \dots, n-1$ . In this case, we say that the path  $p$  starts at the vertex  $s(p) := s(e_1)$  and ends at the vertex  $r(p) := r(e_n)$ , and has *length*  $|p| := n$ . We denote by  $p^0$  the set of its vertices, that is,  $p^0 = \{s(e_i) \mid i = 1, \dots, n\} \cup \{r(e_n)\}$ . If  $p$  is a path in  $E$ , and if  $v = s(p) = r(p)$ , then  $p$  is a *closed path based at  $v$* . A closed path  $p = e_1 \cdots e_n$  based at  $v$  is a *closed simple path based at  $v$*  if  $s(e_i) \neq v$  for every  $i > 1$ . If  $p = e_1 \cdots e_n$  is a closed path and all vertices  $s(e_1), \dots, s(e_n)$  are distinct, then the subgraph  $F = (F^0, F^1)$  of  $E$  defined by  $F^0 = \{s(e_1), \dots, s(e_n)\}$ ,  $F^1 = \{e_1, \dots, e_n\}$  is called a *cycle*. A graph  $E$  is *acyclic* if it has no cycles.

For any graph  $E = (E^0, E^1)$  and vertices  $v, w \in E^0$  we write  $v \geq w$  in case  $v = w$  or there exists a path  $p$  in  $E$  with  $s(p) = v$  and  $r(p) = w$ . For  $v \in E^0$ , the set  $T(v) := \{w \in E^0 \mid v \geq w\}$  is the *tree* of  $v$ . (We will denote it by  $T_E(v)$  when it is necessary to emphasize the dependence on the graph  $E$ .)

For any finite graph  $E = (E^0, E^1)$  we denote by  $A_E$  the *incidence matrix* of  $E$ . Formally, if  $E^0 = \{v_1, \dots, v_n\}$ , then  $A_E = (a_{ij})$ , the  $n \times n$  matrix for which  $a_{ij}$  is the number of edges in  $E$  having  $s(e) = v_i$  and  $r(e) = v_j$ . Note that, if  $v_i \in E^0$  is a sink (resp., source), then  $a_{ij} = 0$  (resp.,  $a_{ji} = 0$ ) for all  $j = 1, \dots, n$ .

The notion of a Cohn path algebra has been defined and investigated by Ara and Goodearl [5] (see also [2]). Specifically, for an arbitrary graph  $E = (E^0, E^1, s, r)$  and an arbitrary field  $K$ , the *Cohn path algebra*  $C_K(E)$  of the graph  $E$  with coefficients in  $K$  is the  $K$ -algebra generated by the sets  $E^0$  and  $E^1$ , together with a set of variables  $\{e^* \mid e \in E^1\}$ , satisfying the following relations for all  $v, w \in E^0$  and  $e, f \in E^1$ :

- (1)  $vw = \delta_{v,w}w$ ;
- (2)  $s(e)e = e = er(e)$  and  $r(e)e^* = e^* = e^*s(e)$ ; and
- (3)  $e^*f = \delta_{e,f}r(e)$ .

Let  $I$  be the ideal of  $C_K(E)$  generated by all elements of the form  $v - \sum_{e \in s^{-1}(v)} ee^*$ , where  $v$  is a regular vertex. Then the  $K$ -algebra  $C_K(E)/I$  is called the *Leavitt path algebra* of  $E$  with coefficients in  $K$ , denoted  $L_K(E)$ .

Typically the Leavitt path algebra  $L_K(E)$  is defined without reference to Cohn path algebras, rather, it is defined as the  $K$ -algebra generated by the set  $\{v, e, e^* \mid v \in E^0, e \in E^1\}$  which satisfies the above conditions (1), (2), (3), and the additional condition:

- (4)  $v = \sum_{e \in s^{-1}(v)} ee^*$  for any regular vertex  $v$ .

If the graph  $E$  is finite, then both  $C_K(E)$  and  $L_K(E)$  are unital rings, each having identity  $1 = \sum_{v \in E^0} v$  (see, e.g., [1, Lemma 1.6]).

## 2. RINGS HAVING UNBOUNDED GENERATING NUMBER

The goal of this section is to show that the UGN property is a Morita invariant for unital rings.

For many fundamental rings  $R$  (e.g., fields and  $\mathbb{Z}$ ), it is well-known that any two bases for a free right  $R$ -module necessarily contain the same number of elements; rephrased, if  $R^m \cong R^n$  as right  $R$ -modules, then  $m = n$ . Such rings are said to have the Invariant Basis Number (IBN) property. On the other hand, in fundamental work done by W.G. Leavitt, it is shown (among other things) that, for any pair  $(n, N)$  of positive integers with  $N > n$ , and any field  $K$ , there exists a  $K$ -algebra  $L_K(n, N)$  for which  $R^n \cong R^N$ . Germane in this context is the observation that for the graph  $R_N$  consisting of one vertex and  $N$  loops, the Leavitt path algebra  $L_K(R_N)$  is isomorphic to  $L_K(1, N)$ . Additional examples of Leavitt path algebras which lack the IBN property abound. Appropriate in this context is the observation that rings which lack the IBN property fail to have a “cancellation of projectives”: specifically, if  $R^n \cong R^N$  with  $n < N$ , then  $R^n \oplus R^{N-n} \cong R^N \cong R^n \cong R^n \oplus \{0\}$ , but obviously  $R^{N-n} \not\cong \{0\}$ .

There are natural, well-studied “cancellation-type” properties of projective modules over general rings which are stronger than the IBN property.

**Definition 2.1.** A ring  $R$  is said to have *Unbounded Generating Number* (UGN for short) if, for each positive integer  $m$ , any set of generators for the free right  $R$ -module  $R^m$  has cardinality  $\geq m$ .  $\square$

Another terminology which has been used for the UGN property is the “rank condition” (see, e.g., [13] and [14, Section 1C]). We note the following easily verified equivalent characterizations of the UGN property.

**Remark 2.2.** The following conditions are equivalent for any ring  $R$ :

- (1)  $R$  has Unbounded Generating Number;
- (2) For any pair of positive integers  $m$  and  $n$ , and any right  $R$ -module  $K$ ,  $R^n \cong R^m \oplus K$  implies that  $n \geq m$ ;
- (3) For any  $A \in M_{m \times n}(R)$  and  $B \in M_{n \times m}(R)$ , if  $AB = I_m$ , then  $n \geq m$ .  $\square$

**Remark 2.3.** By condition (3) in Remark 2.2 we see that the UGN property is indeed a left-right symmetric condition in general. Moreover, using condition (2), it is clear that if  $R$  is UGN, then necessarily  $R$  is IBN. We will show in Example 3.18 that the converse is not true, even in the context of Leavitt path algebras.  $\square$

**Lemma 2.4** (cf. [9, Proposition 2.4]). *Let  $f : R \rightarrow S$  be a unital ring homomorphism. If  $S$  has Unbounded Generating Number, then so too does  $R$ .*

*Proof.* If  $A \in M_{m \times n}(R)$  and  $B \in M_{n \times m}(R)$  are matrices for which  $AB = I_m$ , then we get an analogous equation in matrices over  $S$  by applying the homomorphism  $f$  entrywise, so  $n \geq m$  by the UGN property on  $S$ .  $\square$

There are many classes of rings which have Unbounded Generating Number. For example, any field easily has UGN. More generally, using Lemma 2.4, any commutative ring  $R$  has UGN: pick a maximal ideal  $M$  of  $R$ , and consider the natural surjection from  $R$  to the field  $R/M$ . Additionally, any Hopfian ring  $R$  (a ring for which every right module epimorphism  $\varphi : R^n \rightarrow R^n$  is an isomorphism for each  $n \in \mathbb{N}^+$ ) is UGN; these include the Noetherian rings and self-injective rings.

The rest of this section is taken up in showing that the UGN property is a Morita invariant for unital rings. Before doing so, we recall some fundamental concepts, as well as establish some useful facts.

**Definition 2.5.** Let  $M$  be an abelian monoid (i.e.,  $M$  is a set, and  $+$  is an associative commutative binary operation on  $M$ , with a neutral element).

(1) We define a relation  $\leq$  on  $M$  by setting

$$x \leq y \text{ in case there exists } z \in M \text{ for which } x + z = y.$$

Then  $\leq$  is a preorder (reflexive and transitive).

(2) We call an element  $u \in M$  *properly infinite* if  $2u \leq u$  in  $M$ . It is easy to check that if  $u \leq v$  and  $v \leq u$  in  $M$ , and  $u$  is properly infinite, then  $v$  is also properly infinite.

(3) An element  $d \in M$  is called an *order-unit* if, for any  $x \in M$ , there exist a positive integer  $n$  such that  $x \leq nd$ .

(4) An order-unit  $d \in M$  is said to have *Unbounded Generating Number* (for short, *UGN*) if, for every pair of positive integers  $n, n'$ , if  $nd \leq n'd$  in  $M$ , then  $n \leq n'$ .  $\square$

For any ring  $R$  we denote by  $\mathcal{V}(R)$  the set of isomorphism classes (denoted by  $[P]$ ) of finitely generated projective right  $R$ -modules, and we endow  $\mathcal{V}(R)$  with the structure of an abelian monoid by imposing the operation:

$$[P] + [Q] = [P \oplus Q]$$

for any isomorphism classes  $[P]$  and  $[Q]$ . By Remark 2.2(2), we see that the ring  $R$  has UGN if and only if the order-unit  $[R]$  of  $\mathcal{V}(R)$  has UGN.

**Lemma 2.6.** *Let  $M$  be an abelian monoid and  $\mu \in M$ . Then  $\mu$  does not have Unbounded Generating Number if and only if  $n\mu$  is properly infinite for some positive integer  $n$ .*

*Proof.* Assume that  $\mu$  does not have UGN, that is, there exist two positive integers  $m, n$  such that  $m > n$  and  $m\mu + x = n\mu$  for some  $x \in M$ . Set  $k := m - n > 0$ . We then have that

$$m\mu = n\mu + k\mu = (m\mu + x) + k\mu = m\mu + (k\mu + x),$$

which by substituting gives  $m\mu = (m\mu + (k\mu + x)) + (k\mu + x) = m\mu + 2k\mu + 2x$ , which by an easy induction gives  $m\mu = (m + tk)\mu + tx$  for all  $t \in \mathbb{N}$ . Then adding  $x$  to both sides yields  $n\mu = m\mu + x = (m + tk)\mu + (t + 1)x$  for all  $t \in \mathbb{N}^+$ . In particular,  $(m + tk)\mu \leq n\mu$  for all  $t \in \mathbb{N}^+$ . So, without loss of generality, we may assume that  $m \geq 2n$ . But then

$$2n\mu + (m - 2n)\mu + x = m\mu + x = n\mu,$$

that is,  $2n\mu \leq n\mu$ . Therefore,  $n\mu$  is properly infinite.

The converse is obvious.  $\square$

**Proposition 2.7.** *Let  $M$  be an abelian monoid. Let  $d_1$  and  $d_2$  be order-units in  $M$ . Then  $d_1$  has Unbounded Generating Number if and only if so does  $d_2$ .*

*Proof.* Assume that  $d_1$  does not have UGN; we show that the same holds for  $d_2$  as well. By Lemma 2.6, there exists a positive integer  $n$  such that  $nd_1$  is properly infinite, i.e.,  $2nd_1 \leq nd_1$ . Since  $d_2$  is an order-unit in  $M$ , there exists a positive integer  $\ell$  such that

$u := nd_1 \leq \ell d_2 =: v$ . Furthermore, as  $d_1$  is an order-unit in  $M$ , there exists a positive integer  $k$  such that  $v \leq kd_1$ .

We show now that  $v \leq u$ . If  $k \leq 2n$ , then we have that  $v \leq kd_1 \leq 2nd_1 \leq nd_1 = u$ . Otherwise, let  $t$  be the minimum positive integer for which  $0 < k - tn \leq 2n$ . But then

$$kd_1 = 2nd_1 + (k - 2n)d_1 \leq nd_1 + (k - 2n)d_1 = (k - n)d_1,$$

which similarly gives

$$(k - n)d_1 = 2nd_1 + (k - 3n)d_1 \leq nd_1 + (k - 3n)d_1 = (k - 2n)d_1,$$

which then by induction and the transitivity of  $\leq$  gives  $kd_1 \leq (k - tn)d_1$ . But then we have  $v \leq kd_1 \leq (k - tn)d_1 \leq 2nd_1 \leq nd_1 = u$ .

So we have  $u \leq v$  and  $v \leq u$ . From these observations and the assumption that  $u$  is properly infinite, we conclude by the observation made in Definition 2.5(2) that  $v = \ell d_2$  is also properly infinite. Therefore,  $d_2$  does not have UGN, by Lemma 2.6.  $\square$

It is known that the IBN property is not a Morita invariant property for rings (see, e.g., [14, Exercise 11, page 502]; such examples where both of the rings are Leavitt path algebras can be constructed as well). In contrast, we now present the main result of this section.

**Theorem 2.8.** *Let  $R$  and  $S$  be Morita equivalent unital rings. Then  $R$  and  $S$  have Unbounded Generating Number simultaneously.*

*Proof.* Let  $\Phi : \text{Mod} - R \rightarrow \text{Mod} - S$  be the presumed equivalence of categories. Then the restriction  $\varphi = \Phi|_{\mathcal{V}(R)} : \mathcal{V}(R) \rightarrow \mathcal{V}(S)$  is a monoid isomorphism. Since a monoid isomorphism clearly takes order-units to order-units, and the UGN property of an element in a monoid is a monoid-isomorphism invariant, we see that if  $[R]$  has UGN in  $\mathcal{V}(R)$ , then  $\varphi([R])$  is an order-unit in  $\mathcal{V}(S)$  having UGN. But then by Proposition 2.7 we get that the order-unit  $[S]$  of  $\mathcal{V}(S)$  has UGN as well.  $\square$

**Remark 2.9.** Using the *separative* property of  $\mathcal{V}(L_K(E))$  established in [6] (for any finite graph  $E$ ), we had been able to fairly easily verify that the UGN property is a Morita invariant within the class of unital Leavitt path algebras. (This property was of sufficient strength to allow us to use it to achieve the original proof of our main result, Theorem 3.16.) Subsequently, when informed about this property of Leavitt path algebras, P. Ara realized that such Morita invariance indeed holds for *all* unital rings. We thank him for allowing us to use his proof of this more general property in our article; it has been presented here as Lemma 2.6, Proposition 2.7, and Theorem 2.8. We note that Theorem 2.8 answers [13, Problem 5.2].<sup>4</sup>  $\square$

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<sup>4</sup>It is interesting to note also that the following question appears as an Exercise in Section 0.1 of Cohn's book [12]:

9\*. *Which of IBN, UGN, and weak finiteness (if any) are Morita invariants?*

We know of no place in the literature where a solution to the UGN portion of the question appears. The asterisk \* indicates that Cohn viewed this as a ‘‘harder’’ question; however, it was not considered an ‘‘open’’ question (which would have instead merited a  $\circ$  designation).

### 3. LEAVITT PATH ALGEBRAS HAVING UNBOUNDED GENERATING NUMBER

In this section we establish the main result of the article, to wit, we give necessary and sufficient conditions for the Leavitt path algebra  $L_K(E)$  of a finite graph  $E$  with coefficients in a field  $K$  to have Unbounded Generating Number.

Following [6], for any directed graph  $E = (E^0, E^1, s, r)$  we define the monoid  $M_E$  as follows.

**Definition 3.1.** We denote by  $Y_E$  (or simply by  $Y$ , if the graph  $E$  is clear) the free abelian monoid (written additively) with generators  $E^0$ . Relations are defined on  $Y_E$  by setting

$$v = \sum_{e \in s^{-1}(v)} r(e) \quad (\text{M})$$

for every regular vertex  $v \in E^0$ . Let  $\sim_E$  be the congruence relation on  $Y_E$  generated by these relations. Then  $M_E$  is defined to be the monoid  $Y_E/\sim_E$ . The elements of  $M_E$  are usually denoted by  $[x]$ , for  $x \in Y_E$ .  $\square$

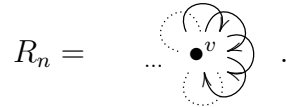
In the literature the generators of  $Y$  are sometimes denoted  $\{a_v \mid v \in E^0\}$  (rather than by  $E^0$  itself) to indicate that  $M_E$  is not being viewed as any sort of quotient of elements of  $L_K(E)$ ; we have chosen to use the less cumbersome of the two notations. Alternatively,  $Y$  may be viewed as  $\mathbb{N}^{|E^0|}$ , where  $E^0 = \{v_1, v_2, \dots, v_n\}$ , and  $v_i$  is associated with the  $i^{\text{th}}$  standard basis vector in  $\mathbb{N}^{|E^0|}$  for each  $1 \leq i \leq n$ .

**Examples 3.2.** We identify the monoid  $M_E$  for some important classes of graphs.

(1) For each  $n \in \mathbb{N}^+$ , let  $A_n = \bullet^{v_1} \longrightarrow \bullet^{v_2} \longrightarrow \dots \bullet^{v_{n-1}} \longrightarrow \bullet^{v_n}$ . Then in  $M_{A_n}$  we have  $[v_1] = [v_2] = \dots = [v_n]$ , and  $M_{A_n} = \{j[v_n] \mid j \in \mathbb{N}\} \cong \mathbb{N}$ .

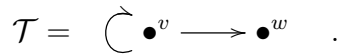
(2) For each  $n \in \mathbb{N}^+$  let  $C_n$  be the “single cycle graph of  $n$  vertices”, with vertices labelled  $v_1, v_2, \dots, v_n$ . Then in  $M_{C_n}$  we have, as in the previous example,  $[v_1] = \dots = [v_n]$ , and  $M_{C_n} = \{j[v_n] \mid j \in \mathbb{N}\} \cong \mathbb{N}$ .

(3) For each integer  $n \geq 2$  let



( $R_n$  is the “rose with  $n$  petals” graph; it is central to the theory of Leavitt path algebras, as  $L_K(R_n) \cong L_K(1, n)$ , the aforementioned Leavitt algebra of order  $n$ .) Then  $M_{R_n} = \{0, 1[v], 2[v], \dots, (n-1)[v]\}$ , where  $n[v] = [v]$ .

(4) The Toeplitz graph is the graph



Then  $M_{\mathcal{T}} = \{n[v] + n'[w] \mid n, n' \in \mathbb{N}, \text{ and } [v] = [v + w]\}$ .  $\square$

In [6] Ara, Moreno and Pardo establish the following fundamental result.

**Theorem 3.3** ([6, Theorem 3.5]). *Let  $E = (E^0, E^1)$  be a row-finite graph and  $K$  any field. Then the map  $[v] \mapsto [vL_K(E)]$  yields an isomorphism of abelian monoids  $M_E \cong \mathcal{V}(L_K(E))$ . In particular, under this isomorphism, we have  $[\sum_{v \in E^0} v] \mapsto [L_K(E)]$ .*

Applying Theorem 3.3 and Remark 2.2(2), we immediately get the following corollary, which provides us with a criterion to check the UGN property of  $L_K(E)$  in terms of the monoid  $M_E$ .

**Corollary 3.4.** *Let  $E = (E^0, E^1)$  be a finite graph and  $K$  any field. Then the following are equivalent:*

- (1)  $L_K(E)$  has Unbounded Generating Number.
- (2) For any pair of positive integers  $m$  and  $n$ , and any  $[x] \in M_E$ ,

$$\text{if } m[\sum_{v \in E^0} v] + [x] = n[\sum_{v \in E^0} v] \text{ in } M_E, \text{ then } m \leq n.$$

As usual,  $( )^t$  notation denotes the standard transpose of a matrix. Also, for matrices  $A = (a_{ij})$  and  $B = (b_{ij}) \in M_{m \times n}(\mathbb{Z})$ ,  $A \leq B$  means that  $a_{ij} \leq b_{ij}$  for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ .

**Definition 3.5.** Let  $c$  be a cycle in the graph  $E$ . We call  $c$  a *source cycle* in case  $|r^{-1}(v)| = 1$  for all  $v \in c^0$ .  $\square$

We will utilize heavily the following graph-theoretic result.

**Lemma 3.6.** *Let  $E$  be a finite source-free graph for which no cycle is a source cycle. Then there exists a vertex  $v \in E^0$  for which there are two distinct cycles based at  $v$ , and for which  $|r^{-1}(v)| \geq 2$ .*

*Proof.* Let  $v_1 \in E^0$  be an arbitrary vertex. Then, as  $v_1$  is not a source, there exists  $e_1 \in E^1$  such that  $r(e_1) = v_1$ . Set  $v_2 := s(e_1)$ . If  $v_2 = v_1$ , then we get a cycle  $c = e_1$ . Otherwise, as  $v_2$  is not a source, there exists  $e_2 \in E^1$  such that  $r(e_2) = v_2$ . Let  $v_3 := s(e_2)$ , and we continue to repeat this process. Since  $E$  is finite, there exists a smallest integer  $n$  such that  $r(e_n) = v_n$  and  $s(e_n) = v_i$  for some  $i$  ( $1 \leq i \leq n$ ), and  $c := e_n \cdots e_{i+1} e_i$  is a cycle in  $E$ .

Using the “no source cycle” hypothesis on  $c$ , there then exists  $j \in \{i, \dots, n\}$  such that  $|r^{-1}(v_j)| \geq 2$ . Let  $f_1 \in r^{-1}(v_j)$  such that  $f_1 \neq e_j$ . If  $s(f_1) = v_j$ , then  $v_j$  is the base of two distinct cycles  $c$  and  $c' := f_1$ , as desired. Otherwise, since  $w_1 := s(f_1)$  is not a source, there exists  $f_2 \in E^1$  such that  $r(f_2) = w_1$ . Since  $E$  is finite, we must eventually arrive at one of these two cases:

Case 1. There exists an integer  $m$  such that  $r(f_m) = w_{m-1}$  and  $s(f_m) = v_k$  for some  $k$  ( $1 \leq k \leq n$ ). Choose the smallest such  $m$ . Then, we have that

$$c' = \begin{cases} e_{j-1} \cdots e_k f_m \cdots f_1 & \text{if } k < j, \\ f_m \cdots f_1 & \text{if } k = j, \\ e_{j-1} \cdots e_i e_n e_{n-1} \cdots e_k f_m \cdots f_1 & \text{if } k > j \end{cases}$$

is a cycle based at  $v_j$ , which is different from  $c$ , for which  $|r^{-1}(v_j)| \geq 2$ , as desired.

Case 2. There exists an integer  $m$  such that  $r(f_m) = w_{m-1}$  and  $s(f_m) = w_\ell$  for some  $\ell$  ( $1 \leq \ell \leq m-1$ ). Choose the smallest such  $m$ . Then  $c_1 := f_m \cdots f_\ell$  is a cycle in  $E$ . In this case, we repeat the process described above, starting with the cycle  $c_1$ .



In this way we produce a sequence of cycles  $c, c_1, \dots, c_t$ . If for some  $c_i$  we are in Case 1, then we are done. We note that if we start the process with some  $c_i$ , and one of the vertices appearing in the process for  $c_i$  is a vertex which has previously appeared in the process corresponding to one of the cycles  $c, c_1, \dots, c_{i-1}$ , then we may find a vertex of the desired type by constructing two cycles in a manner similar to that done in Case 1. Therefore, as  $E$  is finite, we must eventually reach Case 1, thus completing the proof.  $\square$

The existence of a vertex of the type described in Lemma 3.6 will play a key role in the following result.

**Lemma 3.7.** *Let  $E = (E^0, E^1, r, s)$  be a finite source-free graph in which no cycle is a source cycle. Let  $n := |E^0|$ . Then for each positive integer  $a$  there exists a row vector  $\vec{m}_a = [m_1 \dots m_n] \in \mathbb{N}^n$  such that  $m_i \geq a$  for all  $i = 1, \dots, n$ , and*

$$(A_E^t - I_n)\vec{m}_a^t \geq [a \dots a]^t.$$

*Proof.* By Lemma 3.6 there exists a vertex  $w_1 \in E^0$  such that  $w_1$  is a base of distinct cycles, and  $|r^{-1}(w_1)| \geq 2$ . If  $E^0 \setminus T_E(w_1) \neq \emptyset$ , then we consider the subgraph  $F = (F^0, F^1, r|_{F^1}, s|_{F^1})$ , where  $F^0 := E^0 \setminus T(w_1)$  and  $F^1 := r^{-1}(F^0)$ . Note that we always have  $r_F^{-1}(v) = r_E^{-1}(v)$  for any vertex  $v \in F^0$ . This implies that  $F$  is a source-free graph in which no cycle is a source cycle.

So we may apply Lemma 3.6 to  $F$ , to conclude the existence of a vertex  $w_2 \in F^0$  such that  $w_2$  is a base of distinct cycles in  $F$  and  $|r_F^{-1}(w_2)| \geq 2$ . If  $F^0 \setminus T_F(w_2) \neq \emptyset$ , we continue to repeat the process. Since  $E$  is finite, this process ends after finitely many (say,  $k$ ) steps. We consider the set of vertices

$$\{w_1, w_2, \dots, w_k\}.$$

Anticipating an induction argument, we note that the number of steps required to complete the same process starting with either of the two graphs  $E^0 \setminus T(w_1)$  or  $T(w_1)$  is less than  $k$ .

We use induction on  $k$  to establish the result. If  $k = 1$  we have that  $E^0 = T_E(w_1)$ . By renumbering vertices in  $E^0$ , without loss of generality, we may assume that

$$E^0 = \{v_1, v_2, \dots, v_n\}, v_1 := w_1, \text{ and } |r^{-1}(v_1)| \geq 2.$$

For each positive integer  $a$ , we choose the row vector  $\vec{m}_a := [m_1 \dots m_n]$  according to the following algorithm.

- Define  $m_1 := 3na$ .
- For any  $i \in \{2, \dots, n\}$ , since  $E^0 = T_E(v_1)$  we have  $v_i \in T_E(v_1)$ , and hence, there exists a path  $p$  (which can be chosen of minimal length) such that  $s(p) = v_1$  and  $r(p) = v_i$ . Then we define

$$m_i := m_1 - |p|a = (3n - |p|)a,$$

where we denote by  $|p|$  the length of the path  $p$ .

We note that since  $|E^0| = n$ , we always have that  $1 \leq |p| \leq n$ , so that  $m_i \geq 2na$ . Also, for any  $j \in \{2, \dots, n\}$ ,  $v_j \in r(s^{-1}(v_i))$  for some vertex  $v_i \neq v_j$ , so there exists an  $i \in \{1, \dots, n\}$  such that  $m_j = m_i - a$ .

We will prove that the vector  $\vec{m}_a$  satisfies the statement, in other words, that  $(A_E^t - I_n)\vec{m}_a^t \geq [a \dots a]^t$ . Equivalently, we show

$$a_{1j}m_1 + \dots + (a_{jj} - 1)m_j + \dots + a_{nj}m_n \geq a \text{ for all } j = 1, \dots, n.$$

For  $j \in \{2, \dots, n\}$ : as  $v_j \in T(v_1)$ , there exists a vertex  $v_i$  such that  $v_j \in r(s^{-1}(v_i))$  and  $v_i \neq v_j$ . As noted above, we then may find an element  $i \in \{1, \dots, n\}$  such that  $a_{ij} \geq 1$  and  $m_j = m_i - a$ . This implies that

$$\begin{aligned} a_{1j}m_1 + \dots + (a_{jj} - 1)m_j + \dots + a_{nj}m_n &\geq a_{ij}m_i + (a_{jj} - 1)m_j \\ &\geq m_i - m_j = a. \end{aligned}$$

For  $j = 1$ : as  $|r^{-1}(v_1)| \geq 2$ , there exist two distinct elements  $k, h \in \{1, \dots, n\}$  such that  $a_{k1} \geq 1$  and  $a_{h1} \geq 1$ . If  $k \geq 2$  and  $h \geq 2$ , we have that

$$\begin{aligned} (a_{11} - 1)m_1 + \dots + a_{n1}m_n &\geq -m_1 + m_k + m_h \\ &\geq -3na + 2na + 2na = a. \end{aligned}$$

Otherwise, without loss of generality, we may assume that  $h \geq 2$ . We then have that  $a_{11} - 1 \geq 0$  and

$$(a_{11} - 1)m_1 + \dots + a_{h1}m_h + \dots + a_{n1}m_n \geq m_h \geq 2a.$$

These two cases establish the claim. Now we proceed inductively. For  $k > 1$ , let  $F = (F^0, F^1, r|_{F^1}, s|_{F^1})$  and  $G = (G^0, G^1, r|_{G^1}, s|_{G^1})$  be the subgraphs of  $E$  defined by:

$$F^0 := E^0 \setminus T(w_1) \text{ and } F^1 := r^{-1}(F^0)$$

and

$$G^0 := T(w_1) \text{ and } G^1 := \{f \in E^1 \mid s(f), r(f) \in G^0\}.$$

Clearly,  $F$  and  $G$  satisfy the same conditions as the graph  $E$ . Then, by the induction hypothesis, for each positive integer  $a$ , there exist row vectors  $\vec{x}_a = [m_1 \dots m_f] \in \mathbb{N}^f$  ( $m_i \geq a$ ) and  $\vec{y}_a = [m'_1 \dots m'_g] \in \mathbb{N}^g$  ( $m'_j \geq a$ ) such that

$$(A_F^t - I_f)\vec{x}_a^t \geq [a \dots a]^t$$

and

$$(A_G^t - I_g)\vec{y}_a^t \geq [a \dots a]^t,$$

where  $f = |F^0|$ ,  $g = |G^0|$ , and  $A_F$  and  $A_G$  are the incidence matrices of  $F$  and  $G$ , respectively.

We write the matrix  $(A_E^t - I)$  of the form:

$$A_E^t - I = \begin{pmatrix} A_F^t - I_f & A_{21} \\ A_{12} & A_G^t - I_g \end{pmatrix}$$

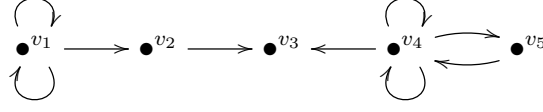
where  $A_{12}$  and  $A_{21}$  are the appropriately sized rectangular submatrices of  $A_E^t - I_n$ , each having only nonnegative integer entries (since none of these entries is on the main diagonal of  $A_E^t - I_n$ ). Let  $\vec{m}_a := [m_1 \dots m_f \ m'_1 \dots m'_g] \in \mathbb{N}^n$ . We then get

$$(A_E^t - I_n)\vec{m}_a^t = \begin{pmatrix} A_F^t - I_f & A_{21} \\ A_{12} & A_G^t - I_g \end{pmatrix} \begin{pmatrix} \vec{x}_a^t \\ \vec{y}_a^t \end{pmatrix} \geq [a \dots a]^t,$$

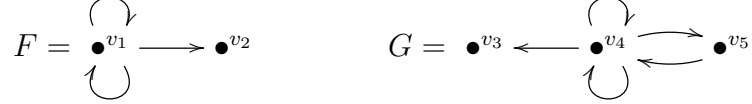
which ends the proof.  $\square$

For clarification, we illustrate the ideas which arise in the proof of Lemma 3.7 by presenting the following example.

**Example 3.8.** Let  $E$  be the graph



Note that  $v_4$  is the base of two distinct cycles, and  $|r^{-1}(v_4)| \geq 2$ . We designate  $w_1 = v_4$ . Let  $G$  denote the subgraph  $T(w_1) = T(v_4)$ , and let  $F$  denote  $E \setminus G = E \setminus T(w_1)$ . Then,  $F$  and  $G$  are the following graphs:



Let  $a$  be an arbitrary positive integer. As shown in the proof of Lemma 3.7, we choose vectors  $\vec{x}_a = [m_1 \ m_2] \in \mathbb{N}^2$  and  $\vec{y}_a = [m_3 \ m_4 \ m_5] \in \mathbb{N}^3$  as follows:

$$m_1 = 3a|F^0| = 6a \quad \text{and} \quad m_2 = m_1 - a = 5a,$$

$$m_4 = 3a|G^0| = 9a \quad \text{and} \quad m_3 = m_5 = m_4 - a = 8a.$$

We then have that

$$(A_F^t - I_2)\vec{x}_a^t = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 6a \\ 5a \end{pmatrix} = \begin{pmatrix} 6a \\ a \end{pmatrix} \geq \begin{pmatrix} a \\ a \end{pmatrix}$$

and

$$(A_G^t - I_3)\vec{y}_a^t = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 8a \\ 9a \\ 8a \end{pmatrix} = \begin{pmatrix} a \\ 17a \\ a \end{pmatrix} \geq \begin{pmatrix} a \\ a \\ a \end{pmatrix}.$$

Furthermore,

$$A_E^t - I_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} A_F^t - I_2 & A_{21} \\ A_{12} & A_G^t - I_3 \end{pmatrix}$$

where  $A_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$  and  $A_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ .

Let  $\vec{m}_a := [6a \ 5a \ 8a \ 9a \ 8a] \in \mathbb{N}^5$ . We then get

$$(A_E^t - I_5)\vec{m}_a^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 6a \\ 5a \\ 8a \\ 9a \\ 8a \end{pmatrix} = \begin{pmatrix} 6a \\ a \\ 6a \\ 17a \\ a \end{pmatrix} \geq \begin{pmatrix} a \\ a \\ a \\ a \\ a \end{pmatrix},$$

which concludes the example. □

We are now in position to give a necessary and sufficient condition for the Leavitt path algebra of a finite source-free graph to have Unbounded Generating Number. We recall an important property of the monoid  $M_E$ . Let  $E$  be a finite graph having  $|E^0| = h$ , and regular (i.e., non-sink) vertices  $\{v_i \mid 1 \leq i \leq z\}$ . For  $x = n_1v_1 + \cdots + n_hv_h \in Y_E$  (the free abelian monoid on generating set  $E^0$ ), and  $1 \leq i \leq z$ , let  $M_i(x)$  denote the element of  $Y_E$  which results by applying to  $x$  the relation  $(M)$  (given in Definition 3.1) corresponding to vertex  $v_i$ . For any sequence  $\sigma$  taken from  $\{1, 2, \dots, z\}$ , and any  $x \in Y$ , let  $\Lambda_\sigma(x) \in Y$  be the element which results by applying relation  $(M)$  in the order specified by  $\sigma$ .

**The Confluence Lemma.** ([6, Lemma 4.3]) For each pair  $x, y \in Y_E$ ,  $[x] = [y]$  in  $M_E$  if and only if there are sequences  $\sigma, \sigma'$  taken from  $\{1, 2, \dots, z\}$  such that  $\Lambda_\sigma(x) = \Lambda_{\sigma'}(y)$  in  $Y_E$ .

Here is the key precursor to our main result.

**Theorem 3.9.** *Let  $E = (E^0, E^1, r, s)$  be a finite source-free graph and  $K$  any field. Then  $L_K(E)$  has Unbounded Generating Number if and only if  $E$  contains a source cycle.*

*Proof.* We denote  $E^0$  by  $\{v_1, v_2, \dots, v_h\}$ , in such a way that the non-sink vertices of  $E$  appear as  $v_1, \dots, v_z$ .

( $\Leftarrow$ ) Assume that  $E$  contains a source cycle  $c$ ; we prove that  $L_K(E)$  has Unbounded Generating Number. We use Corollary 3.4 to do so. Namely, let  $m$  and  $n$  be positive integers such that

$$m[\sum_{i=1}^h v_i] + [x] = n[\sum_{i=1}^h v_i] \text{ in } M_E$$

for some  $[x] \in M_E$ . We must show that  $m \leq n$ . We write  $x \in Y_E$  as

$$x = \sum_{i=1}^h n_i v_i,$$

where  $n_i$  ( $i = 1, \dots, h$ ) are nonnegative integers. By the Confluence Lemma and the hypothesis  $m[\sum_{i=1}^h v_i] + [x] = n[\sum_{i=1}^h v_i]$ , there are two sequences  $\sigma$  and  $\sigma'$  taken from  $\{1, \dots, z\}$  for which

$$\Lambda_\sigma\left(\sum_{i=1}^h (m + n_i)v_i\right) = \gamma = \Lambda_{\sigma'}\left(n \sum_{i=1}^h v_i\right)$$

for some  $\gamma \in Y$ . But each time a substitution of the form  $M_j$  ( $1 \leq j \leq z$ ) is made to an element of  $Y$ , the effect on that element is to:

- (i) subtract 1 from the coefficient on  $v_j$ ;
- (ii) add  $a_{ji}$  to the coefficient on  $v_i$  (for  $1 \leq i \leq h$ ).

For each  $1 \leq j \leq z$ , denote the number of times that  $M_j$  is invoked in  $\Lambda_\sigma$  (resp.,  $\Lambda_{\sigma'}$ ) by  $k_j$  (resp.,  $k'_j$ ). Recalling the previously observed effect of  $M_j$  on an element of  $Y$ , we see that

$$\begin{aligned}
\gamma &= \Lambda_\sigma(\sum_{i=1}^h (m + n_i)v_i) \\
&= ((m + n_1 - k_1) + a_{11}k_1 + a_{21}k_2 + \dots + a_{z1}k_z)v_1 \\
&\quad + ((m + n_2 - k_2) + a_{12}k_1 + a_{22}k_2 + \dots + a_{z2}k_z)v_2 \\
&\quad + \dots \\
&\quad + ((m + n_z - k_z) + a_{1z}k_1 + a_{2z}k_2 + \dots + a_{zz}k_z)v_z \\
&\quad + ((m + n_{z+1}) + a_{1(z+1)}k_1 + a_{2(z+1)}k_2 + \dots + a_{z(z+1)}k_z)v_{z+1} \\
&\quad + \dots \\
&\quad + ((m + n_h) + a_{1h}k_1 + a_{2h}k_2 + \dots + a_{zh}k_z)v_h.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\gamma &= \Lambda_{\sigma'}(n \sum_{i=1}^h v_i) \\
&= ((n - k'_1) + a_{11}k'_1 + a_{21}k'_2 + \dots + a_{z1}k'_z)v_1 \\
&\quad + ((n - k'_2) + a_{12}k'_1 + a_{22}k'_2 + \dots + a_{z2}k'_z)v_2 \\
&\quad + \dots \\
&\quad + ((n - k'_z) + a_{1z}k'_1 + a_{2z}k'_2 + \dots + a_{zz}k'_z)v_z \\
&\quad + (n + a_{1(z+1)}k'_1 + a_{2(z+1)}k'_2 + \dots + a_{z(z+1)}k'_z)v_{z+1} \\
&\quad + \dots \\
&\quad + (n + a_{1h}k'_1 + a_{2h}k'_2 + \dots + a_{zh}k'_z)v_h.
\end{aligned}$$

For each  $1 \leq i \leq z$ , define  $m_i := k'_i - k_i$ . Then from the above observations, equating coefficients on the free generators  $\{v_i \mid 1 \leq i \leq h\}$  of  $Y_E$ , we get the following system of equations:

$$\left\{ \begin{array}{l} m - n + n_1 = (a_{11} - 1)m_1 + a_{21}m_2 + \dots + a_{z1}m_z \\ m - n + n_2 = a_{12}m_1 + (a_{22} - 1)m_2 + \dots + a_{z2}m_z \\ \vdots \\ m - n + n_z = a_{1z}m_1 + a_{2z}m_2 + \dots + (a_{zz} - 1)m_z \\ m - n + n_{z+1} = a_{1(z+1)}m_1 + a_{2(z+1)}m_2 + \dots + a_{z(z+1)}m_z \\ \vdots \\ m - n + n_h = a_{1h}m_1 + a_{2h}m_2 + \dots + a_{zh}m_z \end{array} \right. \quad (1)$$

By hypothesis  $c$  is a source cycle in  $E$ , i.e.,  $|r^{-1}(v)| = 1$  for all  $v \in c^0$ . By renumbering vertices if necessary, we may assume without loss of generality that  $c^0 = \{v_1, \dots, v_p\}$ . (Note that, as each vertex in  $c^0$  emits at least one edge, we have that each of  $\{v_1, \dots, v_p\}$  is a regular vertex.) The condition  $|r^{-1}(v)| = 1$  then yields:

- $a_{i,i+1} = 1$  for  $1 \leq i \leq p - 1$ ;
- $a_{p,1} = 1$ ;
- $a_{j,i+1} = 0$  for  $1 \leq i \leq p - 1$  and  $j \neq i$  ( $1 \leq j \leq h$ ); and
- $a_{j,1} = 0$  if  $j \neq p$  ( $1 \leq j \leq h$ ).

If  $p = 1$  (i.e., if  $c$  is a loop), then  $a_{11} = 1$ , and first equation in the system of equations (1) becomes

$$m - n + n_1 = (1 - 1)m_1 + 0m_2 + \cdots + 0m_z = 0,$$

so  $m - n = -n_1 \leq 0$ , i.e.,  $m \leq n$ .

If  $p \geq 2$ , then using the noted information about the  $a_{i,j}$ , the  $p$  first equations of the system of equations (1) can be written as:

$$\begin{cases} m - n + n_1 = & -m_1 & & & & +m_p \\ m - n + n_2 = & m_1 & -m_2 & & & \\ m - n + n_3 = & & m_2 & -m_3 & & \\ & \vdots & & & & \\ m - n + n_p = & & & & m_{p-1} & -m_p \end{cases}.$$

Then adding both sides yields that  $p(m - n) + (n_1 + \dots + n_p) = 0$ , so that  $p(m - n) = -(n_1 + \dots + n_p) \leq 0$ , which gives  $m \leq n$ .

Therefore,  $L_K(E)$  has Unbounded Generating Number.

( $\implies$ ) Assume conversely that  $E$  does not contain any source cycles. We will prove that  $L_K(E)$  does not have Unbounded Generating Number. So let  $m$  and  $n$  be two positive integers such that  $m > n$ . We will establish the existence of an element  $x = \sum_{i=1}^h n_i v_i \in Y_E$  such that

$$m\left[\sum_{i=1}^h v_i\right] + [x] = n\left[\sum_{i=1}^h v_i\right]$$

in  $M_E$ . Equivalently, arguing as in the previous half of the proof, we show that we can find nonnegative integers  $n_i$ ,  $k_j$  and  $k'_j$  ( $i = 1, \dots, h$  and  $j = 1, \dots, z$ ) such that

$$\begin{cases} m - n + n_1 = (a_{11} - 1)m_1 + a_{21}m_2 + \dots + a_{z1}m_z \\ m - n + n_2 = a_{12}m_1 + (a_{22} - 1)m_2 + \dots + a_{z2}m_z \\ \vdots \\ m - n + n_z = a_{1z}m_1 + a_{2z}m_2 + \dots + (a_{zz} - 1)m_z \\ m - n + n_{z+1} = a_{1(z+1)}m_1 + a_{2(z+1)}m_2 + \dots + a_{z(z+1)}m_z \\ \vdots \\ m - n + n_h = a_{1h}m_1 + a_{2h}m_2 + \dots + a_{zh}m_z \end{cases} \quad (2)$$

where  $m_j := k'_j - k_j$  for all  $j = 1, \dots, z$ .

We apply Lemma 3.7 to find such elements. Namely, let  $F$  be the subgraph of  $E$  defined by:

$$F^0 := \{v_1, \dots, v_z\} \text{ and } F^1 := r^{-1}(F^0).$$

In other words,  $F$  is the graph produced from  $E$  by removing the sinks. Specifically, we have that  $A_F^t$  is the  $z \times z$  matrix

$$A_F^t = \begin{pmatrix} a_{11} & \dots & a_{z1} \\ a_{12} & \dots & a_{z2} \\ \vdots & \dots & \vdots \\ a_{1z} & \dots & a_{zz} \end{pmatrix}$$

Also, we note that the first  $z$  equations of the system of equations (2) in the proof of Theorem 3.9 is induced by the matrix  $A_F^t - I_z$ . Easily we see that  $F$  contains neither sources nor source cycles (because  $E$  contains neither). By Lemma 3.7 (applied to the graph  $F$  and positive integer  $a = m - n$ ), there is a row vector  $\vec{m} = [m_1 \dots m_z] \in \mathbb{N}^z$  such that  $m_j \geq m - n$  for all  $j = 1, \dots, z$ , and

$$(A_F^t - I_z)\vec{m}^t \geq [m - n \dots m - n]^t.$$

That is, we have

$$a_{1j}m_1 + \dots + (a_{jj} - 1)m_j + \dots + a_{zj}m_z \geq m - n$$

for all  $j = 1, \dots, z$ . For each  $j = 1, \dots, z$ , let

$$n_j := a_{1j}m_1 + \dots + (a_{jj} - 1)m_j + \dots + a_{zj}m_z - (m - n).$$

For each  $j = z + 1, \dots, h$ , as  $v_j$  is not a source, there exists  $i \in \{1, \dots, z\}$  such that  $a_{ij} \geq 1$ , and hence, for such  $j$ ,

$$a_{1j}m_1 + a_{2j}m_2 + \dots + a_{zj}m_z \geq a_{ij}m_i \geq m_i \geq m - n.$$

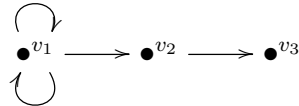
We then choose the non-negative integers  $n_j$  ( $j = z + 1, \dots, h$ ) as follows:

$$n_j := a_{1j}m_1 + a_{2j}m_2 + \dots + a_{zj}m_z - (m - n).$$

Finally, positive integers  $k_j$  and  $k'_j$  ( $j = 1, \dots, z$ ) are chosen arbitrarily such that  $m_j = k'_j - k_j$  for all  $j = 1, \dots, z$ . Then, a tedious but straightforward computation yields that this choice of integers indeed satisfies the system of equations (2) above, thus completing the proof of the theorem.  $\square$

**Remark 3.10.** We have in fact shown in the proof of Theorem 3.9 that the UGN property fails for the order-unit  $[\sum_{i=1}^h v_i]$  of  $M_E$  for *every* pair of positive integers  $m > n$ ; of course, it was required only to show that it fails for *some* such pair. Using the previously-mentioned *separativity* of  $\mathcal{V}(L_K(E))$  (and so of  $M_E$ ), one can easily show that failure of UGN for one pair is equivalent to failure of UGN for every pair.  $\square$

**Example 3.11.** We present a specific example of the construction presented in the proof of Theorem 3.9 which shows that source-free graphs having no source cycles do not have UGN. Let  $K$  be a field and let  $E$  be the graph



Clearly,  $E$  is a source-free graph in which no cycle is a source cycle, and

$$A_E^t = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We will show that  $L_K(E)$  does not have UGN. So let  $m$  and  $n$  be two positive integers such that  $m > n$ . We will establish the existence of an element  $x = \sum_{i=1}^3 n_i v_i \in Y_E$  such that

$$m\left[\sum_{i=1}^3 v_i\right] + [x] = n\left[\sum_{i=1}^3 v_i\right].$$

Equivalently, we show that we can find nonnegative integers  $n_i, k_j$  and  $k'_j$  ( $1 \leq i \leq 3$  and  $1 \leq j \leq 2$ ) such that

$$\begin{cases} m - n + n_1 = m_1 \\ m - n + n_2 = m_1 - m_2 \\ m - n + n_3 = m_2 \end{cases} \quad (3)$$

where  $m_j := k'_j - k_j$  for  $j = 1, 2$ . Let  $F$  be the graph produced from  $E$  by deleting  $v_3$ . Note that

$$A_F^t = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$A_E^t = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} A_F^t & \vdots & 0 \\ \dots & & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Also, the two first equations of the above system can be written as

$$(A_F^t - I_2) \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} m - n + n_1 \\ m - n + n_2 \end{pmatrix}.$$

As in the proof of Theorem 3.9, we define  $m_1$  and  $m_2$  as follows:

$$m_1 = 3|F^0|(m - n) = 6(m - n), \quad \text{and} \quad m_2 = m_1 - (m - n) = 5(m - n).$$

Subsequently, we define

$$\begin{aligned} n_1 &= m_1 - (m - n) = 5(m - n), \\ n_2 &= m_1 - m_2 - (m - n) = 0, \quad \text{and} \\ n_3 &= m_2 - (m - n) = 4(m - n). \end{aligned}$$

Then the construction described in the proof of Theorem 3.9 yields that the element

$$[x] = [5(m - n)v_1 + 4(m - n)v_3]$$

of  $M_E$  satisfies

$$m\left[\sum_{i=1}^3 v_i\right] + [x] = n\left[\sum_{i=1}^3 v_i\right]$$

in  $M_E$ . It is instructive to verify the validity of this equation directly; we achieve this by verifying the equivalent version

$$[(6m - 5n)v_1 + mv_2 + (5m - 4n)v_3] = [nv_1 + nv_2 + nv_3]$$

in  $M_E$ . In  $M_E$  we have:

$$(i) \ [v_1] = [2v_1 + v_2], \quad \text{and} \quad (ii) \ [v_2] = [v_3].$$



Recall that  $m_1 = 6(m - n)$  and  $m_2 = 5(m - n)$ . We must choose positive integers  $k_i$  and  $k'_i$  such that  $m_i = k'_i - k_i$  ( $i = 1, 2$ ). We choose these as follows:  $k_1 = 1 = k_2$ ,  $k'_1 = 6(m - n) + 1$ , and  $k'_2 = 5(m - n) + 1$ . The left side can be transformed as follows:

$$\begin{aligned}
& [(6m - 5n)v_1 + mv_2 + (5m - 4n)v_3] \\
&= [(6m - 5n - 1)v_1 + v_1 + mv_2 + (5m - 4n)v_3] \\
&= [(6m - 5n - 1)v_1 + (2v_1 + v_2) + mv_2 + (5m - 4n)v_3] && \text{by (i)} \\
&= [(6m - 5n + 1)v_1 + mv_2 + v_2 + (5m - 4n)v_3] \\
&= [(6m - 5n + 1)v_1 + mv_2 + v_3 + (5m - 4n)v_3] && \text{by (ii)} \\
&= [(6m - 5n + 1)v_1 + mv_2 + (5m - 4n + 1)v_3].
\end{aligned}$$

On the other hand, an application of (i) yields

$$[nv_1 + nv_2] = [(n - 1)v_1 + v_1 + nv_2] = [(n - 1)v_1 + (2v_1 + v_2) + nv_2] = [(n + 1)v_1 + (n + 1)v_2].$$

By an easy induction this gives

$$[nv_1 + nv_2] = [(n + u)v_1 + (n + u)v_2]$$

for every  $u \in \mathbb{N}$ ; in particular, applying (i)  $u = k'_1 = 6(m - n) + 1$  times gives the first step in the following transformation of the right side:

$$\begin{aligned}
& [nv_1 + nv_2 + nv_3] \\
&= [(6m - 5n + 1)v_1 + (6m - 5n + 1)v_2 + nv_3] \\
&= [(6m - 5n - 1)v_1 + mv_2 + (5m - 5n + 1)v_2 + nv_3] \\
&= [(6m - 5n - 1)v_1 + mv_2 + (5m - 5n + 1)v_3 + nv_3] && \text{by (ii), } k'_2 \text{ times} \\
&= [(6m - 5n + 1)v_1 + mv_2 + (5m - 4n + 1)v_3].
\end{aligned}$$

This completes the verification that the two quantities are indeed equal in  $M_E$ .  $\square$

In our main result (Theorem 3.16), we show how to eliminate the “no sources” hypothesis in Theorem 3.9.

**Definition 3.12** (e.g., [7, Notation 2.4]). Let  $E = (E^0, E^1, r, s)$  be a graph, and let  $v \in E^0$  be a source. We form the *source elimination* graph  $E_{\setminus v}$  of  $E$  as follows:

$$(E_{\setminus v})^0 = E^0 \setminus \{v\}; \quad (E_{\setminus v})^1 = E^1 \setminus s^{-1}(v); \quad s_{E_{\setminus v}} = s|_{(E_{\setminus v})^1}; \quad \text{and } r_{E_{\setminus v}} = r|_{(E_{\setminus v})^1}.$$

In other words,  $E_{\setminus v}$  denotes the graph gotten from  $E$  by deleting  $v$  and all of edges in  $E$  emitting from  $v$ .  $\square$

Let  $E$  be a finite graph. If  $E$  is acyclic, then repeated application of the source elimination process to  $E$  yields the empty graph. On the other hand, if  $E$  contains a cycle, then repeated application of the source elimination process will yield a source-free graph  $E_{sf}$  which necessarily contains a cycle.

Consider the sequence of graphs which arises in some step-by-step process of source eliminations

$$E := E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_i \rightarrow \cdots \rightarrow E_\ell := E_{sf} .$$

To avoid defining a graph to be the empty set, we define  $E_{sf}$  to be the graph  $E_{triv}$  (consisting of one vertex and no edges) in case  $E_{\ell-1} = E_{triv}$ .

Although there in general are many different orders in which a step-by-step source elimination process can be carried out, the resulting source-free subgraph  $E_{sf}$  is always the same. For a cycle  $c$  in  $E$ , denote by  $T_E(c)$  the set of vertices

$$T_E(c) := \{w \in E^0 \mid v \geq w \text{ for some } v \in c^0\}.$$

**Lemma 3.13.** *Let  $E$  be a finite graph.*

- (1)  $E_{sf} = E_{triv}$  if and only if  $E$  is acyclic.
- (2) Suppose  $E$  contains cycles. Then

$$E_{sf}^0 = \bigcup_c T_E(c) \quad (\text{where } c \text{ runs over all cycles in } E),$$

and  $E_{sf}^1 = E^1|_{E_{sf}^0}$ .

*Proof.* We first note that, if  $v \in E^0$  is a source, then it is easy to verify that

- (1)  $c$  is a cycle in  $E$  if and only if  $c$  is a cycle in  $E_{\setminus v}$ , and
- (2) if  $c$  is a cycle in  $E$ , then  $T_E(c) = T_{E_{\setminus v}}(c)$ .

Now consider the sequence of graphs which arises in some step-by-step process of source eliminations

$$E := E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_i \rightarrow \cdots \rightarrow E_\ell =: E_{sf}.$$

Using the two observations, we immediately get that  $E_{sf} = E_{triv}$  if and only if  $E$  is acyclic; and that  $T_E(c) \subseteq E_{sf}^0$  for each cycle  $c$  in  $E$ . Now assume that  $v \notin E_{sf}^0$ . Let  $i$  be minimal in  $\{1, \dots, \ell\}$  such that  $v \notin E_i$ , and hence,  $v$  is a source in  $E_{i-1}$ . This implies that  $v \notin T_{E_{i-1}}(c)$  for each cycle  $c$  in  $E_{i-1}$ . Applying the second observation, we get that  $v \notin T_E(c)$  for each cycle  $c$  in  $E$ .  $\square$

The key result which will allow the extension of Theorem 3.9 is the following observation of Ara and Rangaswamy.

**Lemma 3.14** ([7, Lemma 4.3]). *Let  $E$  be a finite graph and  $K$  any field. If  $v$  is a source which is not isolated, then  $L_K(E)$  is Morita equivalent to  $L_K(E_{\setminus v})$ .*

**Lemma 3.15.** *Let  $E$  be a finite graph containing an isolated vertex, and  $K$  any field. Then  $L_K(E)$  has Unbounded Generating Number.*

*Proof.* Let  $v$  denote the presumed isolated vertex. We then get immediately that  $L_K(E) \cong K \oplus L_K(E_{\setminus v})$ , and hence there is a natural surjection from  $L_K(E)$  onto  $K$ . Obviously, the field  $K$  has UGN, so  $L_K(E)$  has UGN by Lemma 2.4.  $\square$

Using Theorem 3.9 and Lemmas 3.14 and 3.15, we are finally in position to establish the main result of this article.

**Theorem 3.16.** *Let  $E$  be a finite graph and  $K$  any field. Let*

$$E = E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_i \rightarrow \cdots \rightarrow E_\ell = E_{sf}$$

*be a sequence of graphs which arises in some step-by-step process of source eliminations. Then  $L_K(E)$  has Unbounded Generating Number if and only if either  $E_i$  contains an isolated vertex (for some  $0 \leq i \leq \ell$ ), or  $E_{sf}$  contains a source cycle.*

*Proof.* Assume first that  $E_i$  contains an isolated vertex for some  $i$ . Let  $j$  denote the minimal such  $i$ . Then, at each step of the source elimination process

$$E = E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_j ,$$

the source which is being eliminated is not an isolated vertex. By Lemma 3.14 we then have that the algebras  $L_K(E)$ ,  $L_K(E_1)$ , ...,  $L_K(E_j)$  are Morita equivalent one to the other. But  $L_K(E_j)$  has UGN by Lemma 3.15, and hence  $L_K(E)$  has UGN by Theorem 2.8.

On the other hand, suppose that no  $E_i$  contains an isolated vertex. Then Lemma 3.14 applies at each step of the source elimination process, so that  $L_K(E)$  is Morita equivalent to  $L_K(E_{sf})$ . So, by Theorem 2.8,  $L_K(E)$  has UGN if and only if  $L_K(E_{sf})$  has UGN. As  $E_{sf}$  is source-free we may apply Theorem 3.9, so that  $L_K(E)$  has UGN if and only if  $E_{sf}$  contains a source cycle, thus establishing the result.  $\square$

We emphasize that the statement of Theorem 3.16 depends not only on the subgraph  $E_{sf}$ , but on the sequence of source-eliminations as well. As an easy example, consider the two graphs  $F$  and  $G$ :

$$F = \bullet \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \bullet \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \quad \text{and} \quad G = \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \bullet \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} .$$

(So  $F$  is the disjoint union of  $E_{triv}$  with  $G$ .) Then obviously  $F_{sf} = G_{sf} = G$ . But  $L_K(F)$  has UGN (it has a direct summand isomorphic to  $K$ ), while  $L_K(G)$  does not (as  $G$  is a source-free graph containing no source cycles.)

We finish this section with a few remarks about Cohn path algebras. We present here a specific case of a more general result described in [2, Section 1.5]. Namely, let  $E = (E^0, E^1, s, r)$  be an arbitrary graph and  $\Phi$  the set of regular vertices of  $E$ . Let  $\Phi' = \{v' \mid v \in \Phi\}$  be a disjoint copy of  $\Phi$ . For  $v \in \Phi$  and for each edge  $e$  in  $E^1$  such that  $r_E(e) = v$ , we consider a new symbol  $e'$ . We define the graph  $F(E)$ , as follows:

$$F(E)^0 := E^0 \sqcup \Phi' \quad \text{and} \quad F(E)^1 := E^1 \sqcup \{e' \mid r_E(e) \in \Phi\},$$

and for each  $e \in E^1$ ,  $s_{F(E)}(e) = s_E(e)$ ,  $s_{F(E)}(e') = s_E(e)$ ,  $r_{F(E)}(e) = r_E(e)$ , and  $r_{F(E)}(e') = r_E(e)'$ . For instance, if

$$E = \begin{array}{c} \begin{array}{c} \curvearrowright \\ \bullet v \\ \curvearrowleft \\ f \end{array} \end{array}, \quad \text{then} \quad F(E) = \begin{array}{c} \begin{array}{ccc} \begin{array}{c} \curvearrowright \\ \bullet v \\ \curvearrowleft \\ f \end{array} & \begin{array}{c} \xrightarrow{e'} \\ \xrightarrow{f'} \end{array} & \begin{array}{c} \bullet v' \end{array} \end{array} .$$

Ara, Siles Molina and the first author have shown that for any graph  $E$  and any field  $K$ , there is an isomorphism of  $K$ -algebras

$$C_K(E) \cong L_K(F(E)). \quad (\text{cf. [2, Theorem 1.5.17]})$$

(So, perhaps counterintuitively, every Cohn path algebra is in fact isomorphic to a Leavitt path algebra.) From this observation and Theorem 3.16, we immediately get a criterion to determine which Cohn path algebras of finite graphs have Unbounded Generating Number.

**Corollary 3.17.** *Let  $E$  be a finite graph and  $K$  any field. Then the Cohn path algebra  $C_K(E)$  has Unbounded Generating Number if and only if  $E$  satisfies at least one of the following conditions:*

- (1)  $E$  contains a source; or
- (2)  $E$  contains a source cycle.

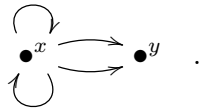
*Proof.* ( $\implies$ ) Assume that  $E$  contains neither sources nor source cycles. By the construction of the graph  $F(E)$ , it is easy to see that  $F(E)$  also contains neither sources nor source cycles. Then, by Theorem 3.16,  $C_K(E) \cong L_K(F(E))$  does not have UGN.

( $\impliedby$ ) If  $E$  contains a source cycle  $c$ , then by considering the explicit construction given above, it is clear that the graph  $F(E)$  must contain such a cycle, and hence  $F(E)_{sf}$  contains such a cycle too. So by Theorem 3.16,  $L_K(F(E))$  has UGN, and hence so does  $C_K(E)$ .

On the other hand, suppose  $E$  contains a source vertex  $v$ . If  $v$  is a sink (i.e., if  $v$  is isolated) in  $E$  then  $v$  is an isolated vertex in  $F(E)$ . Otherwise, since  $v$  is a source in  $E$ , the corresponding vertex  $v'$  is an isolated vertex in  $F(E)$ . Hence, in this case,  $F(E)$  always contains an isolated vertex. Then, by Lemma 3.15,  $L_K(F(E))$  has UGN, hence so does  $C_K(E)$ .  $\square$

In [3, Theorem 9], Kanuni and the first author showed that the Cohn path algebra of any finite graph has the IBN property. Using this result and Theorem 3.16, we easily give an example of a Leavitt path algebra which has the IBN property, but does not have the UGN property.

**Example 3.18.** Let  $K$  be a field, and let  $G$  be the graph



Then, by Theorem 3.16,  $L_K(G)$  does not have UGN. But as shown previously,  $G = F(E)$  where  $E$  is the graph



So  $L_K(G) \cong C_K(E)$ , and so has IBN by [3, Theorem 9].

It is perhaps instructive to explicitly consider  $\mathcal{V}(L_K(G)) \cong M_G$  in this case. Specifically,  $M_G$  is the free abelian monoid on  $\{x, y\}$  with the one relation  $x = 2x + 2y$ . The standard order-unit of  $M_G$  is  $[x + y]$ . It is not hard to show that any equation of the form  $n[x + y] = m[x + y]$  in  $M_G$  necessarily gives  $m = n$ . (The one relation, applied to an element of  $Y_G$  of the form  $t = nx + ny$ , will either yield  $t$  itself, or an element  $t' = ix + jy$  for which  $i \neq j$ .) On the other hand, the relation  $x = 2x + 2y$  gives  $[x + y] = [2x + 3y] = 2[x + y] + [y]$  in  $M_G$ , so that  $[x + y]$  does not have the UGN property in  $M_G$ .  $\square$

#### 4. LEAVITT PATH ALGEBRAS HAVING CANCELLATION OF PROJECTIVES

In this, the article's short final section, we identify the graphs  $E$  for which the Leavitt path algebra  $L_K(E)$  satisfies conditions (3) through (5) mentioned in the Introduction. We briefly review some terminology.

Let  $E$  be a graph, and  $p = e_1 \cdots e_n$  a path in  $E$ . Then an edge  $f \in E^1$  is an *exit* for  $p$  if  $s(f) = s(e_i)$  but  $f \neq e_i$  for some  $1 \leq i \leq n$ .  $E$  is said to be a *no-exit graph* if no cycle in  $E$  has an exit.

A ring  $R$  is called *directly finite* if, for any  $a, b \in R$ ,  $ab = 1$  implies  $ba = 1$ .

$R$  is said to be *stably finite* if for any  $n \in \mathbb{N}^+$ ,  $R^n \cong R^n \oplus K$  (as right  $R$ -modules) implies  $K = 0$ .

$R$  is called a *Hermite ring* if for all  $m, n \in \mathbb{N}^+$  and any right  $R$ -module  $K$ ,  $R^n \cong R^m \oplus K$  (as right  $R$ -modules) implies that  $n \geq m$  and  $K \cong R^{n-m}$ .

Finally,  $R$  is said to have *cancellation of projectives* if for any finitely generated projective right  $R$ -modules  $P$  and  $P'$ ,  $P \oplus R \cong P' \oplus R$  (as right  $R$ -modules) implies that  $P \cong P'$ .

Short, straightforward computations immediately establish that, for any unital ring  $R$ ,

$$\text{cancellation of projectives} \Rightarrow \text{Hermite} \Rightarrow \text{stably finite} \Rightarrow \text{directly finite}.$$

For general rings, there are examples which show that none of these implications can be reversed. Germane here is the observation that it is easy to establish that

$$\text{stably finite} \Rightarrow \text{Unbounded Generating Number};$$

however, examples exist which show that directly finite does not in general imply Unbounded Generating Number (nor does UGN imply directly finite).

An abelian monoid is called *cancellative* in case, for every  $m, m', m'' \in M$ , if  $m' + m = m'' + m$ , then  $m' = m''$ . Obviously the monoid  $\mathbb{N}$  is cancellative; almost as obviously, so too is  $\mathbb{N}^t$  for any positive integer  $t$ .

**Remark 4.1.** A ring  $R$  has cancellation of projectives if and only if the monoid  $\mathcal{V}(R)$  is cancellative. This is not hard to see. Indeed, assume that  $R$  has cancellation of projectives as defined above, and suppose  $[P] + [Q] = [P'] + [Q]$  in  $\mathcal{V}(R)$ , i.e.,  $P \oplus Q \cong P' \oplus Q$  as right  $R$ -modules. Since  $[R]$  is an order-unit in  $\mathcal{V}(R)$ , there exist  $n \in \mathbb{N}^+$  and a right  $R$ -module  $K$  such that  $Q \oplus K \cong R^n$ . But then  $P \oplus R^n \cong P \oplus Q \oplus K \cong P' \oplus Q \oplus K \cong P' \oplus R^n$ , so  $P \cong P'$  as  $R$  has cancellation of projectives, i.e.,  $[P] = [P']$ . The other implication is immediate.  $\square$

Here is the relationship between these properties in the context of Leavitt path algebras.

**Theorem 4.2.** *Let  $E$  be a finite graph and  $K$  any field. Then the following are equivalent:*

- (1)  $L_K(E)$  has cancellation of projectives;
- (2)  $L_K(E)$  is Hermite;
- (3)  $L_K(E)$  is stably finite;
- (4)  $L_K(E)$  is directly finite;
- (5)  $E$  is a no-exit graph.

*Proof.* By the discussion above, we need only show that (4) implies (5), and (5) implies (1). That (4) and (5) are equivalent is established in [19, Theorem 4.12]. We give here a very

brief outline of one direction. Suppose  $E$  contains a cycle  $c$  with an exit  $f$ , and let  $v = s(f)$ . We may view  $c$  as being based at  $v$ . Then  $c^*c = v$ . Let  $x := \sum_{w \in E^0, w \neq v} w \in L_K(E)$ . Let  $a := c + x$  and  $b := c^* + x$ . It is easily verified that  $ba = 1$  in  $L_K(E)$ . But  $ab = cc^* + x$ ; since  $c^*f = 0$  we get  $abf = 0$ , so in particular  $ab \neq 1$ , so  $L_K(E)$  is not directly finite.

So now suppose that (5) holds; we show that  $L_K(E)$  has cancellation of projectives, i.e., we show that  $\mathcal{V}(L_K(E))$  is a cancellative monoid. Let  $\{c_1, \dots, c_\ell\}$  and  $\{v_1, \dots, v_k\}$  be the sets of cycles and sinks in  $E$ , respectively. (Because  $E$  is a no-exit graph, the cycles in  $E$  are necessarily disjoint.) Then, by [4, Theorems 3.8 and 3.10] we get

$$L_K(E) \cong \left( \bigoplus_{i=1}^{\ell} M_{m_i}(K[x, x^{-1}]) \right) \oplus \left( \bigoplus_{j=1}^k M_{n_j}(K) \right),$$

where for each  $1 \leq i \leq \ell$ ,  $m_i$  is the number of paths ending in a fixed (although arbitrary) vertex of the cycle  $c_i$  which do not contain the cycle itself, and for each  $1 \leq j \leq k$ ,  $n_j$  is the number of paths ending in the sink  $v_j$ .

Every finitely generated projective  $K[x, x^{-1}]$ -module  $P$  is free (see, e.g., [15, Corollary 4.10, page 189]), and  $K[x, x^{-1}]$  has IBN, so we immediately get that  $\mathcal{V}(K[x, x^{-1}]) \cong \mathbb{N}$ . But then, as  $\mathcal{V}$  is a Morita invariant and preserves ring direct sums, the displayed ring isomorphism yields that  $\mathcal{V}(L_K(E))$  is isomorphic to the cancellative monoid  $\mathbb{N}^\ell \oplus \mathbb{N}^k \cong \mathbb{N}^{\ell+k}$ .  $\square$

By [4, Theorem 3.10], we may add the statement “ $L_K(E)$  is Noetherian” to Theorem 4.2. Although in general the Noetherian condition on a ring  $R$  is enough to yield that  $R$  is stably finite, it is not sufficient in general to yield that  $R$  is Hermite (neither, then, that  $R$  has cancellation of projectives). In particular, we must utilize the explicit structure of Noetherian Leavitt path algebras (as presented in the displayed isomorphism in the proof of Theorem 4.2) in order to conclude the cancellation of projectives property.

**Example 4.3.** Let  $K$  be a field, and consider the Toeplitz graph

$$\mathcal{T} = \begin{array}{c} \curvearrowright \\ \bullet \longrightarrow \bullet \end{array}$$

described in Examples 3.2. By Theorem 3.16,  $L_K(E)$  has UGN. However,  $L_K(E)$  does not have simultaneously the directly finite, stably finite, Hermite and cancellation of projectives properties, by Theorem 4.2.  $\square$

**Remark 4.4.** In summary, we recall the hierarchy of five cancellation properties of rings presented in the Introduction. We have established that, within the class of Leavitt path algebras, the IBN property is strictly weaker than the UGN property; the UGN property is strictly weaker than the stably finite property; and the stably finite, Hermite, and cancellation of projective properties are equivalent. Moreover, the graphs  $E$  for which  $L_K(E)$  has the UGN property, and the graphs  $F$  for which  $L_K(F)$  has any one of the final three properties, have been explicitly described.

It remains an open question to give graph-theoretic conditions on  $E$  which describe precisely the Leavitt path algebras  $L_K(E)$  having the IBN property.  $\square$

We finish this paper by giving a description of the Cohn path algebras of finite graphs that have any one of the above properties.

**Corollary 4.5.** *Let  $E$  be a finite graph and  $K$  any field. Then the following are equivalent:*

- (1)  $C_K(E)$  has cancellation of projectives;
- (2)  $C_K(E)$  is Hermite;
- (3)  $C_K(E)$  is stably finite;
- (4)  $C_K(E)$  is directly finite;
- (5)  $E$  is acyclic.

*Proof.* We have that  $C_K(E) \cong L_K(F(E))$ , where  $F(E)$  is the graph constructed from  $E$  given near the end of Section 3. Using that description, it is easy to see that  $F(E)$  is a no-exit graph if and only if  $E$  is acyclic. The result then follows immediately from Theorem 4.2.  $\square$

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