arXiv:1605.01879v1 [math.CV] 6 May 2016

WEAK SOLUTION OF PARABOLIC COMPLEX MONGE-AMPÈRE EQUATION II

DO HOANG SON

Abstract. We study the equation $\dot{u} = \log\det(u_{\alpha\bar{\beta}}) - Au + f(z,t)$ in $\Omega \times (0,T)$, where $A \geq 0, T > 0$ and Ω is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, with the boundary condition $u = \varphi$ and the initial condition $u = u_0$. In this paper, we consider the case where φ is smooth and u_0 is an arbitrary plurisubharmonic function in a neighbourhood of $\bar{\Omega}$ satisfying $u_0|_{\partial\Omega} = \varphi(.,0)$.

Contents

Date: May 9, 2016
The author is supported by Vietnam Academy of Science and Technology, under the program "Building a research team and research trends in complex analysis", decision VAST.CTG.01/16-17.

Introduction

Let Ω be a bounded smooth strictly pseudoconvex domain of $\mathbb{C}^n$, i.e., there exists a smooth strictly plurisubharmonic function ρ defined on a bounded neighbourhood of $\bar{\Omega}$ such that

$$\Omega = \{\rho < 0\}.$$

Let $A \geq 0, T > 0$. We consider the equation

$$\begin{cases} \dot{u} = \log\det(u_{\alpha\bar{\beta}}) - Au + f(z,t) & \text{on } \Omega \times (0,T), \\ u = \varphi & \text{on } \partial\Omega \times [0,T), \\ u = u_0 & \text{on } \bar{\Omega} \times \{0\}, \end{cases} \tag{1}$$

where $\dot{u} = \frac{\partial u}{\partial t}$, $u_{\alpha\bar{\beta}} = \frac{\partial^2 u}{\partial z_\alpha \partial \bar{z}_\beta}$, u_0 is a plurisubharmonic function in a neighbourhood of $\bar{\Omega}$ and φ, f are smooth in $\bar{\Omega} \times [0,T]$.

If u_0 is a smooth strictly plurisubharmonic function in $\bar{\Omega}$ and some compatibility conditions are satisfied on $\partial\Omega \times \{0\}$, then (1) admits a unique solution $u \in C^{2;1}(\bar{\Omega} \times [0,T))$, and then u is smooth outside $\partial\Omega \times \{0\}$ ([HL10], see also Section 3.1). In more general cases, we define the weak solution

Definition 0.1. *The function $u \in USC(\bar{\Omega} \times [0,T))$ (upper semicontinuous function) is called a* weak solution *of* (1) *if there exist $u_m \in C^\infty(\bar{\Omega} \times [0,T))$ satisfying*

$$\begin{cases} u_m(.,t) \in SPSH(\Omega), \\ \dot{u}_m = \log\det(u_m)_{\alpha\bar{\beta}} - Au_m + f(z,t) & \textit{on } \Omega \times (0,T), \\ u_m \searrow \varphi & \textit{on } \partial\Omega \times [0,T), \\ u_m \searrow u_0 & \textit{on } \bar{\Omega} \times \{0\}, \\ u = \lim\limits_{m\to\infty} u_m, \end{cases} \tag{2}$$

where $SPSH(\Omega) = \{$strictly plurisubharmonic functions on $\Omega\}$.

The equation (1) always admits a unique weak solution, and the weak solution has been described in case u_0 has zero Lelong numbers or $u \geq N\sum \log|z - a_j| + O(1)$ ([Do15a, Do15b], see also Section 3.3). In this article, we will consider the general case where u_0 has positive Lelong numbers.

The corresponding problem in compact Kähler manifolds was considered and solved by Di Nezza anh Lu [DL14]. The maximal solution (as weak solution in case of domains) is smooth outside $D_t \times \{t\}$, where D_t is an analytic set. In case of domains on $\mathbb{C}^n$, by condition $u_0 = \varphi(.,0)$ on $\partial\Omega$ (and φ is smooth), $(u_0 - \sup_\Omega u_0 - 1)$ can be approximated as in Theorem 2.1 by functions contained in Cegrell's class $\mathcal{E}(\Omega)$ and then for any $\epsilon > 0$, the set $\{z \in \Omega : \nu(u_0, z) \geq \epsilon\}$ contains a finite number of points [Ceg04]. Hence we can describe more precisely the singular set of the weak solution: $\Omega \times \{t\}$ contains only finitely many singular points of the weak solution, for any $0 < t < T$. In the domain $\Omega \times [0,T)$, the set of singular points of the weak solution is $\cup\{a_j\} \times (0, \epsilon_j)$, where $\nu(u_0, a_j) > 0$ and ϵ_j is a positive number which is bounded by constants depending on A and $\nu(u_0, a_j)$.

For the convenience, we denote by ϵ_A the functions

$$\begin{cases}\epsilon_A(x) = \frac{x}{2n} \text{ if } A = 0,\\ \epsilon_A(x) = \frac{1}{A}(\log(Ax + 2n) - \log(2n)) \text{ if } A > 0.\end{cases} \tag{3}$$

where $x > 0$ and $0 < t < T$.

Our main result is following

Theorem 0.2. *Let $A \geq 0, T > 0$ and Ω be a bounded smooth strictly pseudoconvex domain of $\mathbb{C}^n$. Let φ, f be smooth functions in $\bar{\Omega} \times [0,T]$ and u_0 be a plurisubharmonic function in a neighbourhood of $\bar{\Omega}$ such that $u_0(z) = \varphi(z, 0)$ for any $z \in \partial\Omega$. Then the weak solution u of (1) satisfies*

(i) *If $m > \frac{1}{T}$ and $\{z \in \Omega : \nu_{u_0}(z) \geq \frac{2}{m}\} = \{a_{m,1}, ..., a_{m,N(m)}\}$ then there exist nonegetive numbers $\epsilon_{m,1}, ..., \epsilon_{m,N(m)} \in (0, T]$ such that*

(a) *$\epsilon_A(\nu(u_0, a_{m,j})) \leq \epsilon_{m,j} \leq \epsilon_A(n\nu(u_0, a_{m,j}))$ for any $j = 1, ..., N(m)$.*

(b) *$\nu(u(.,t), a_{m,j}) > 0$ for any $j = 1, ..., N(m)$ and $t < \epsilon_{m,j}$. Moreover, if $\epsilon_{m,j} < T$ then $\nu(u(.,t), a_{m,j}) \searrow 0$ as $t \nearrow \epsilon_{m,j}$.*

(c) *$u \in C^\infty(Q_m)$, where $Q_m = \bar{\Omega} \times (\frac{1}{m}, T) \setminus \cup_{j \leq N(m)} \{a_{m,j}\} \times (\frac{1}{m}, \epsilon_{m,j}]$.*

(d) *$\dot{u} = \log\det(u_{\alpha\bar{\beta}}) - Au + f(z,t)$ on Q_m.*

(ii) *$u = \varphi$ on $\partial\Omega \times [0, T)$ and $\lim\limits_{t\to 0} u(z,t) = u_0(z)$ for any $z \in \Omega$. In particular, $u(z,t) \xrightarrow{L^1} u_0(z)$ as $t \to 0$.*

Let us first recall some preliminaries.

1. Coherent analytic sheaves on pseudoconvex domains

In this section, we recall some properties of coherent analytic sheaves on pseudoconvex domains in $\mathbb{C}^n$. The readers can find more details in [Hor90] (chap VI and chap VII). Corollary 1.10 will be used to prove Proposition 2.2.

If Ω is a domain in $\mathbb{C}^n$, we let $\mathcal{O}(\Omega)$ denote the set of holomorphic functions on Ω. If $z \in \mathbb{C}^n$, we let $\mathcal{O}_z(\mathbb{C}^n)$ or $\mathcal{O}_z$ for short denote the set of equivalence classes of functions f which are holomorphic in some neighbourhood of z, under the equivalence relation $f \sim g$ if $f = g$ in some neighbourhood of z. If f is holomorphic in a neighbourhood of z, we write f_z for the residue class of f in $\mathcal{O}_z$, which is called the germ of f at z.

It follows from Theorem 6.3.3 and Theorem 6.3.5 in [Hor90] that:

Theorem 1.1. *Let $a \in \mathbb{C}^n$. Then $\mathcal{O}_a$ is a Noetherian ring. If $\mathcal{I}$ is a ideal of $\mathcal{O}_a$, and $\mathcal{I} \ni f_j \xrightarrow{j\to\infty} f \in \mathcal{O}_a$ in sense of simple convergence, then $f \in \mathcal{I}$. Here $f_j \longrightarrow f$ means that the coefficient of $(z-a)^\alpha$ in f_j converges to the coefficient of $(z-a)^\alpha$ in f for every α.*

By the same argument, we also have:

Theorem 1.2. *Let Ω be a neighbourhood of $0 \in \mathbb{C}^n$. Then for any $\{f_j\}_{j=1}^{\infty} \subset \mathcal{O}(\Omega)$, there exist $M > 0$ and $0 < r < d(0, \partial\Omega)$ such that the ideal $\mathcal{I}$ of $\mathcal{O}(\Delta_r^n)$ which is generated by $\{f_j\}_{j=1}^{\infty}$ is also generated by $\{f_j\}_{j=1}^{M}$. Moreover, $\mathcal{I}$ is closed in the topology of uniform convergence on compact subsets of Δ_r^n.*

Definition 1.3. *Let X and $\mathcal{F}$ be two topological spaces and π be a mapping $\mathcal{F} \to X$ such that*

(i) *π maps $\mathcal{F}$ onto X.*
(ii) *π is a local homeomorphism, that is, every point in $\mathcal{F}$ has an open neighbourhood which is mapped homeomorphically by π on an open set in X.*

Then $\mathcal{F}$ is called a sheaf on X and π is called the projection on X. If U is a subset of X, a section of $\mathcal{F}$ over U is a continuous map $\varphi : U \to \mathcal{F}$ such that $\pi\varphi = Id$ on U. The set of all sections of $\mathcal{F}$ over U is denoted by $\Gamma(U, \mathcal{F})$. If $x \in X$, then $\mathcal{F}_x = \pi^{-1}\{x\}$ is called the stalk of $\mathcal{F}$ at x.

Example 1.4. *Let Ω be a open subset of $\mathbb{C}^n$. The sheaf $\mathcal{O}_\Omega$ (or $\mathcal{O}$ for short) of germs of holomorphic functions (analytic functions) on $\Omega \subset \mathbb{C}^n$ is the topological space defined by*

(i) $\mathcal{O} = \cup_{z\in\Omega}\mathcal{O}_z$.

(ii) *The topology in $\mathcal{O}$ is the strongest topology such that for every open subset $U \subset \Omega$ and for every $f \in \mathcal{O}(U)$, the map $U \ni z \mapsto f_z \in \mathcal{O}$ is continuous.*

The projection on Ω is the map $\pi : \mathcal{O} \to \Omega$ defined by $\pi(\mathcal{O}_z) = \{z\}$. The set of sections of $\mathcal{O}$ over U is $\Gamma(U, \mathcal{O}) = \mathcal{O}(U)$ and the stalk of $\mathcal{O}$ at z is the set of germs at z of holomorphic functions in some neighbourhood of z.

Definition 1.5. *Let Ω be an open subset of $\mathbb{C}^n$. A sheaf $\mathcal{F}$ on Ω is called an analytic sheaf if it is a sheaf of $\mathcal{O}$-modules, i.e., $\mathcal{F}_z$ is an $\mathcal{O}_z$ module for every $z \in \Omega$ and the product of a section of $\mathcal{O}$ and a section of $\mathcal{F}$ is a section of $\mathcal{F}$.*

Definition 1.6. *An analytic sheaf $\mathcal{F}$ on Ω is called coherent if*

(i) *$\mathcal{F}$ is locally finitely generated, i.e., for every $z \in \Omega$ there exists a neighbourhood $U \subset \Omega$ and a finite number of sections $f_1, ..., f_q \in \Gamma(U, \mathcal{F})$ so that $\mathcal{F}_z$ is generated by $(f_1)_z, ..., (f_q)_z$ as $\mathcal{O}_z$ module for every $z \in \Omega$.*
(ii) *If U is an open subset of Ω and $f_1, ..., f_q \in \Gamma(U, \mathcal{F})$, then the sheaf of relations $\mathcal{R}(f_1, ..., f_q)$ is locally finitely generated.*

Theorem 1.7. *Every locally finitely generated subsheaf of $\mathcal{O}^p$ is coherent.*

Corollary 1.8. *Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{O}(\Omega)$. Assume that $\mathcal{F}$ is the analytic sheaf generated by $\{f_j\}_{j=1}^{\infty}$ over Ω, i.e., the germs $(f_1)_z, ..., (f_q)_z ...$ generate $\mathcal{F}_z$ for every $z \in \Omega$. Then $\mathcal{F}$ is coherent.*

Theorem 1.9. *Let Ω be a Stein manifold, K an Ω-holomorphically convex compact subset of Ω, and $\mathcal{F}$ a coherent analytic sheaf on a neighbourhood of K. Then*

(i) *There exist finitely many sections $f_1, ..., f_q$ of $\mathcal{F}$ over a neighbourhood of K which generate $\mathcal{F}$ there.*

(ii) *If $f_1, .., f_q$ are sections of $\mathcal{F}$ over a neighbourhood of K which generate $\mathcal{F}$ there and if f is an arbitraly section of $\mathcal{F}$ over a neighbourhood of K, then one can find $c_1, ..., c_q$ analytic in a neighbourhood of K so that $f = \sum_{j=1}^{q} c_j f_j$ there.*

Corollary 1.10. *Let W be a pseudoconvex domains in $\mathbb{C}^n$ and Ω be a open subset of W, $\bar{\Omega} \subset W$. Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{O}(W)$. Assume that $\mathcal{I}$ is the ideal of $\mathcal{O}(\Omega)$ generated by $f_1|_{\Omega}, ..., f_j|_{\Omega}...$ Then there exist finitely many functions $f_1, ..., f_q$ such that $\mathcal{I}$ is generated by $f_1|_{\Omega}, ..., f_q|_{\Omega}$.*

2. Demailly's approximation

We recall Demailly's approximation theorem, which allows to approximate plurisubharmonic functions by multiples of logarithms of holomorphic functions. The readers can find the proof of this theorem in [Dem92] (or [Dem14]).

Theorem 2.1. *Let φ be a plurisubharmonic function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathcal{H}_{\Omega}(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_{\Omega} |f|^2 e^{-2m\varphi} dV_{2n} < +\infty$ and let $\varphi_m = \frac{1}{2m} \log \sum |g_{m,l}|^2$ where $(g_{m,l})$ is an orthonormal basis of $\mathcal{H}_{\Omega}(m\varphi)$. Then there are constants $C_1, C_2 > 0$ independent of m such that*

(a) $\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}$ *for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to \infty$.*

(b) $\nu(\varphi, z) - \frac{n}{m} \leq \nu(\varphi_m, z) \leq \nu(\varphi, z)$ *for every $z \in \Omega$.*

Using Theorem 2.1, we will prove the following proposition, which will be use to prove the main theorem.

Proposition 2.2. *Under the assumptions of Theorem 2.1, for any open subset $U \Subset \Omega$, the following properties hold*

(a) $\int_U e^{2m(\varphi_m - \varphi)} < \infty$.

(b) *If $z \in U$ and $\nu(\varphi, z) < \frac{1}{m}$ then φ_m is smooth in a neighbourhood of z.*

(c) *If there exist only finitely many points $a_1, ..., a_l$ in $\{z \in \bar{U} : \nu(\varphi, z) \geq \frac{1}{m}\}$, then there exist $C, N > 0$ such that*

$$\varphi_m \geq N \sum_{j=1}^{l} \log |z - a_j| - C \text{ on } U.$$

Proof. (a) There is a corresponding result in the case of compact Kähler manifolds, which was proved in [Dem10, p.164], [Dem14, p.10]. The same arguments can be applied for the case of domains in $\mathbb{C}^n$.

Set $f_j : \Omega \times \Omega \to \mathbb{C}$, $f_j(z, w) = g_{m,j}(z)\overline{g_{m,j}(\bar{w})}$. We have, for any $z, w \in V$ and $l > 0$,

$$\sum_{j \leq l} |f_j(z, w)| \leq (\sum_{j \leq l} |g_{m,j}(z)|^2)^{1/2} (\sum_{j \leq l} |g_{m,j}(\bar{w})|^2)^{1/2} \leq e^{m(\varphi_m(z) + \varphi(\bar{w}))}.$$

Then the series $\sum\limits_{j=1}^{\infty} f_j(z,w)$ converges uniformly on compact subsets of $\Omega\times\Omega$.

Let $U\Subset V\Subset\Omega$. We denote by $\mathcal{J}$ the ideal of $\mathcal{O}(V\times V)$ generated by $\{f_j\}_{j=1}^{\infty}$. It follows from Theorem 1.2 that $f=\sum\limits_{j=1}^{\infty} f_j\in\mathcal{J}$.

Moreover, by Corollary 1.10 we can choose $M\gg 1$ such that $\mathcal{J}$ is generated by $f_1,...,f_M$. Hence,

$$f=\sum_{j=1}^{M} a_j f_j,$$

where $a_j\in\mathcal{O}(V\times V)$. Then there exists $C>0$ such that

$$e^{2m\varphi_m}=f(z,\bar{z})\leq\sum_{j=1}^{M}|a_j(z,\bar{z})||g_{m,j}(z)|^2\leq C\sum_{j=1}^{M}|g_{m,j}(z)|^2,$$

for any $z\in U$. Thus

$$\int_U e^{2m(\varphi_m-\varphi)}\leq CM<\infty.$$

(b) If $\in U$ and $\nu(\varphi,z)<\frac{1}{m}$, then $2m\varphi$ is intergrable in a neighborhood of z (see [Sko72]). Hence, there exists $g\in\mathcal{H}_\Omega(m\varphi)$ such that $g(z)\neq 0$. Hence $\varphi_m(z)\neq-\infty$. Thus φ_m is smooth in a neighbourhood of z.

(c) This is a corollary of Lemma 2.3. □

Lemma 2.3. *Let $g_1,...,g_M\in\mathcal{O}(\Delta^n)$ such that*

$$\{g_1=...=g_M=0\}=\{0\}.$$

Then there exist $C,N>0$ such that, on $\Delta^n_{1/2}$,

$$|g_1|^2+...+|g_M|^2\geq C(|z_1|^2+...+|z_n|^2)^N.$$

Proof. It follows from Hilbert's Nullstellensatz theorem (see [Huy05, p.19]) that there exist $N>0$, $1>r>0$ and holomorphic functions $f_{jk}\in\mathcal{O}(\Delta^n_r)$ satisfying, on Δ^n_r,

$$z_k^N=\sum_{j=1}^{m} g_j.f_{ij},$$

for $k=1,...,n$. Then there exists $C_1>0$ such that, on $\Delta^n_{r/2}$,

$$|g_1|^2+...+|g_M|^2\geq C_1(|z_1|^2+...+|z_n|^2)^N.$$

In the other hand, $\inf\limits_{\Delta^n_{1/2}\setminus\Delta^n_{r/2}}(|g_1|^2+...+|g_M|^2)>0$. Then there exists $C_2>0$ such that, on $\Delta^n_{1/2}\setminus\Delta^n_{r/2}$,

$$|g_1|^2+...+|g_M|^2\geq C_2(|z_1|^2+...+|z_n|^2)^N.$$

Denote $C=\max\{C_1,C_2\}$, we have, on $\Delta^n_{1/2}$,

$$|g_1|^2+...+|g_M|^2\geq C(|z_1|^2+...+|z_n|^2)^N.$$

□

Lemma 2.3 is also an immediate corollary of Lojasiewicz inequality:

Theorem 2.4. *Let $f : U \to \mathbb{R}$ be a real-analytic function on an open set U in $\mathbb{R}^n$, and let Z be the zero locus of f. Assume that Z is not empty. Then, for any compact set K in U, there are positive constants C and α such that, for every $x \in K$*

$$dist(x, Z)^\alpha \leq C|f(x)|.$$

We refer the reader to [Loj59, Mal66, JKS92] for more details.

Now we recall some previous results on Parabolic complex Monge-Ampère equation.

3. Parabolic complex Monge-Ampère equation

3.1. Hou-Li theorem.

The Hou-Li theorem [HL10] states that equation (1) has a unique smooth solution when the conditions are good enough. We state the precise problem to be studied:

$$\begin{cases} \dot{u} = \log\det(u_{\alpha\bar{\beta}}) + f(t, z, u) & \text{on } \Omega \times (0, T), \\ u = \varphi & \text{on } \partial\Omega \times [0, T), \\ u = u_0 & \text{on } \bar{\Omega} \times \{0\}. \end{cases} \tag{4}$$

We first need the notion of a subsolution to (4).

Definition 3.1. *A function $\underline{u} \in C^\infty(\bar{\Omega} \times [0, T))$ is called a* subsolution *of the equation* (4) *if and only if*

$$\begin{cases} \underline{u}(., t) \textit{is a strictly plurisubharmonic function,} \\ \dot{\underline{u}} \leq \log\det(\underline{u})_{\alpha\bar{\beta}} + f(t, z, \underline{u}), \\ \underline{u}|_{\partial\Omega\times(0,T)} = \varphi|_{\partial\Omega\times(0,T)}, \\ \underline{u}(., 0) \leq u_0. \end{cases} \tag{5}$$

Theorem 3.2. *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary. Let $T \in (0, \infty]$. Assume that*

- *φ is a smooth function in $\bar{\Omega} \times [0, T)$.*
- *f is a smooth function in $[0, T) \times \bar{\Omega} \times \mathbb{R}$ non increasing in the lastest variable.*
- *u_0 is a smooth strictly plurisubharmonic funtion in a neighbourhood of Ω.*
- *$u_0(z) = \varphi(z, 0)$, $\forall z \in \partial\Omega$.*
- *The compatibility condition is satisfied, i.e.*

$$\dot{\varphi} = \log\det(u_0)_{\alpha\bar{\beta}} + f(t, z, u_0), \ \ \forall (z, t) \in \partial\Omega \times \{0\}.$$

- *There exists a subsolution to the equation* (4)*.*

Then there exists a unique solution $u \in C^\infty(\Omega\times(0, T))\cap C^{2;1}(\bar{\Omega}\times[0, T))$ of the equation (4)*.*

If Ω is a bounded smooth strictly pseudoconvex domain of $\mathbb{C}^n$ then a subsolution always exists on $\bar{\Omega} \times [0, T')$ (for example, $\underline{u} = M\rho + \varphi$, where $M \gg 1$ and ρ is a smooth strictly plurisubhamonic function such that $\rho|_{\partial\Omega} = 0$), for any $0 < T' < T$, and so Theorem 3.2 does not need the additional assumption of existence of a subsolution.

Using regularity theories (see, for example, [GT83, CC95, Lieb96, Do15a]), we can conclude that the solution u of (4) is smooth outside $\partial\Omega \times \{0\}$ if the assumption of

Theorem 3.2 holds. If Ω is a bounded smooth strictly pseudoconvex domain of $\mathbb{C}^n$, φ is smooth in $\bar{\Omega}\times[0,T]$ and f is smooth in $[0,T]\times\bar{\Omega}\times\mathbb{R}$, then u is the solution of (4) in $\bar{\Omega}\times[0,T+\delta)$, where $0<\delta\ll 1$. Hence u can be approximated in $\bar{\Omega}\times[0,T]$ by the smooth functions $u(z,t+\frac{1}{m})$. Thus, for approximations, u is as good as smooth in $\bar{\Omega}\times[0,T]$.

3.2. Maximum principle.

The following maximum principle is a basic tool to establish upper and lower bounds in the sequel (see [BG13] and [IS13] for the proof).

Theorem 3.3. *Let Ω be a bounded domain of C^n and $T>0$. Let $\{\omega_t\}_{0<t<T}$ be a continuous family of continuous positive definite Hermitian forms on Ω. Denote by Δ_t the Laplacian with respect to ω_t:*

$$\Delta_t f=\frac{n\omega_t^{n-1}\wedge dd^c f}{\omega_t^n},\ \forall f\in C^\infty(\Omega).$$

Suppose that $H\in C^\infty(\Omega\times(0,T))\cap C(\bar{\Omega}\times[0,T))$ and satisfies

$$(\tfrac{\partial}{\partial t}-\Delta_t)H\leq 0\ \ or\ \ \dot{H}_t\leq\log\frac{(\omega_t+dd^cH_t)^n}{\omega_t^n}.$$

Then $\sup\limits_{\bar{\Omega}\times[0,T)}H=\sup\limits_{\partial_P(\Omega\times[0,T))}H$. Here we denote $\partial_P(\Omega\times(0,T))=(\partial\Omega\times(0,T))\cup\big(\bar{\Omega}\times\{0\}\big)$.

Corollary 3.4. *(Comparison principle) Let Ω be a bounded domain of $\mathbb{C}^n$ and $A\geq 0, T>0$. Let $u,v\in C^\infty(\Omega\times(0,T))\cap C(\bar{\Omega}\times[0,T))$ satisfy:*

- *$u(.,t)$ and $v(.,t)$ are strictly plurisubharmonic functions for any $t\in[0,T)$,*
- *$\dot{u}\leq\log\det(u_{\alpha\bar{\beta}})-Au+f(z,t)$,*
- *$\dot{v}\geq\log\det(v_{\alpha\bar{\beta}})-Av+f(z,t)$,*

where $f\in C^\infty(\bar{\Omega}\times[0,T))$.
Then $\sup\limits_{\Omega\times(0,T)}(u-v)\leq max\{0,\sup\limits_{\partial_P(\Omega\times(0,T))}(u-v)\}$.

Corollary 3.5. *Let Ω be a bounded domain of $\mathbb{C}^n$ and $T>0$. We denote by L the operator on $C^\infty(\Omega\times(0,T))$ given by*

$$L(f)=\frac{\partial f}{\partial t}-\sum a_{\alpha\bar{\beta}}\frac{\partial^2 f}{\partial z_\alpha\partial\bar{z}_\beta}-b.f,$$

where $a_{\alpha\bar{\beta}},b\in C(\Omega\times(0,T))$, $(a_{\alpha\bar{\beta}}(z,t))$ are positive definite Hermitian matrices and $b(z,t)<0$.
Assume that $\phi\in C^\infty(\Omega\times(0,T))\cap C(\bar{\Omega}\times[0,T))$ satisfies

$$L(\phi)\leq 0.$$

Then $\phi\leq max(0,\sup\limits_{\partial_P(\Omega\times(0,T))}\phi)$.

3.3. **Weak solution of Parabolic Monge-Ampère equation.**

We recall some properties of weak solution of (1), which were proved in [Do15a] and [Do15b].

Proposition 3.6. *Let $A \geq 0, T > 0$ and let Ω be a bounded smooth strictly pseudoconvex domain of $\mathbb{C}^n$. Assume that u_0 is a plurisubharmonic function in a neighbourhood of $\bar{\Omega}$ and φ, f are smooth in $\bar{\Omega} \times [0,T]$. Then there exists a unique weak solution u of* (1)*. Moreover,*

(i) $u|_{\partial\Omega \times [0,T)} = \varphi$; $u(z,t) \xrightarrow{L^1} u_0$ *as* $t \searrow 0$.

(ii) *If Lelong numbers $\nu(u,a) = 0$ for every $a \in \Omega$ then $u \in C^\infty(\bar{\Omega} \times (0,T))$ and u satisfies* (1) *in $\bar{\Omega} \times (0,T)$ in the classical sense.*

(ii) *If there exist $l \in \mathbb{N}, a_j \in \Omega, N_j \geq n_j \geq 0$ such that*

$$\sum_{j=1}^{l} n_j \log|z - a_j| + C_0 \geq u_0 \geq \sum_{j=1}^{l} N_j \log|z - a_j| - C_0,$$

then u satisfies

(a) $u \in C^\infty(Q)$, *where* $Q = (\bar{\Omega} \times (0,T)) \setminus (\cup(\{a_j\} \times (0, \epsilon_A(N_j)]))$.

(b) $u = -\infty$ *on* $\cup(\{a_j\} \times [0, \min\{T, \epsilon_A(n_j)\}))$.

(c) $\dot{u} = \log\det u_{\alpha\bar{\beta}} - Au + f(z,t)$ *in* Q.

Proposition 3.7. *Assume that there exists $u_m \in C^\infty(\bar{\Omega} \times [0,T))$ satisfying*

$$\begin{cases} u_m(.,t) \in SPSH(\Omega) & \forall t \in [0,T), \\ u_m(z,t) + 2^{-m} \geq u_{m+1}(z,t) & \forall (z,t) \in \bar{\Omega} \times [0,T), \\ \dot{u}_m = \log\det(u_m)_{\alpha\bar{\beta}} - Au_m + f(z,t) & \forall (z,t) \in \Omega \times (0,T), \\ u_m(z,t) \longrightarrow \varphi(z,t) & \forall (z,t) \in \partial\Omega \times [0,T), \\ u_m(z,0) \longrightarrow u_0(z) & \forall z \in \bar{\Omega}. \end{cases} \tag{6}$$

Then $u = \lim u_m$ is a weak solution of (1)*.*

Proposition 3.8. *If there is $a \in \Omega$ such that $\nu_{u_0}(a) > 0$, then the weak solution u of* (1) *satisfies $u(a,t) = -\infty$ for $t \in [0, \epsilon_A(\nu_{u_0}(a)))$.*

4. Proof of Theorem 0.2

4.1. **Some technique lemmas.** We present some lemmas, which will be used to prove the main theorem. The following two lemmas were proved in [Do15b].

Lemma 4.1. *Suppose that $\psi, g \in C^\infty(\bar{\Omega} \times [0,T])$. Assume that $v \in C^\infty(\bar{\Omega} \times [0,T))$ satisfies*

$$\begin{cases} v(.,t) \in SPSH(\bar{\Omega}), & \\ \dot{v} = \log\det(v_{\alpha\bar{\beta}}) - Av + g(z,t) & \text{on } \Omega \times (0,T), \\ v = \psi & \text{on } \partial\Omega \times [0,T). \end{cases} \tag{7}$$

Then

$$v(z,t) - v(z,0) \geq -C(t),$$

for any $(z,t)\in\bar\Omega\times[0,T)$. *Here* $C(t)$ *is defined by*

$$C(t)=\inf_{1>\epsilon>0}((-n\log\epsilon+A\sup|\psi|+\sup|g|)t-\epsilon\inf\rho)+\sup_{t'\in[0,t]}\sup_{z\in\partial\Omega}|\psi(z,t')-\psi(z,0)|,$$

where $\rho\in C^\infty(\bar\Omega)$ *such that* $dd^c\rho\geq dd^c|z|^2$ *and* $\rho|_{\partial\Omega}=0$.

Lemma 4.2. *Assume that* $u\in C^\infty(\bar\Omega\times[0,T))$ *satisfies*

$$\begin{cases}\dot u=\log\det(u_{\alpha\bar\beta})+f(z,t) & on\ \ \Omega\times(0,T),\\ u=\varphi\quad on\ \ \partial\Omega\times[0,T).\end{cases}\tag{8}$$

Then

$$\frac{u(z,t)-\sup u_0}{t}-B\leq\dot u(z,t)\leq\frac{u(z,t)-u_0(z)}{t}+B,\ \ \forall(z,t)\in\bar\Omega\times[0,T),$$

where $B=2\sup|\dot\varphi|+T\sup|\dot f|+n$ *and* $u_0=u(.,0)$.

We will also need the following elementary observation

Lemma 4.3. *Let* $g:\mathbb{R}^+\to\mathbb{R}^+$ *be a decreasing right-continuous function. Assume that there exist* $\alpha,B>1$ *such that* g *satisfies*

$$tg(t+s)\leq B(g(s))^\alpha\ \ \forall t,s>0.$$

Then $g(s)>0$ *for all* $s\geq s_\infty$, *where*

$$s_\infty=\frac{2Bg(0)^{\alpha-1}}{1-2^{-\alpha+1}}.$$

We refer the reader to [EGZ09] for the proof of Lemma 4.3. Using this lemma, we prove Lemma 4.4.

Lemma 4.4. *Let* Ω *be a bounded open subset of* $\mathbb{C}^n$. *Let* u,u_0,ψ *be plurisubharmonic functions in* Ω *such that* $u\in C^\infty(\bar\Omega)$ *and* ψ *is bounded near* $\partial\Omega$. *Assume also that*

$$(dd^cu)^n\leq Be^{a_1(u-a_2u_0)}dV,$$
$$\int_\Omega e^{b(\psi-u_0)}dV\leq B,$$

where $a_1,a_2,b,B>0,b>a_1a_2$. *Then there exists* $C>0$ *depending only on* a_1,a_2,b,B,n,Ω *and* $\liminf_{z\to\partial\Omega}(u-a_2\psi)$ *such that*

$$u\geq a_2\psi-C\ \ on\ \Omega.$$

There is a corresponding result in the case of compact Kähler manifolds, which was proved in [DL14]. The same arguments can be applied for the case of domains in $\mathbb{C}^n$. For the reader's convenience, we recall the arguments here.

Proof. Without loss of generality, we can assume that $\liminf_{z\to\partial\Omega}(u-a_2\psi)=0$. For $t\in\mathbb{R}$, we denote

$$U_t=\{z\in\Omega:u(z)<a_2\psi(z)-t\}.$$

Let $M>\sup|u|/a_2$ and $\psi_M=\max\{\psi,-M\}$. Then, for any $t,s>0$ and $v\in PSH(\Omega)$ such that $-1\leq v\leq 0$, we have

$$
\begin{aligned}
t^n\int\limits_{U_{t+s}}(dd^cv)^n=\int\limits_{\{u<a_2\psi_M-t-s\}}(tdd^cv)^n &\leq \int\limits_{\{u<a_2\psi_M+tv-s\}}(dd^c(a_2\psi_M+tv-s))^n\\
&\leq \int\limits_{\{u<a_2\psi_M+tv-s\}}(dd^cu)^n\\
&\leq \int\limits_{U_s} Be^{a_1(u-a_2u_0)}dV\\
&\leq \int\limits_{U_s} Be^{-s}e^{a_1a_2(\psi-u_0)}dV\\
&\leq B\int\limits_{U_s} e^{a_1a_2(\psi-u_0)}dV\\
&\leq B\left(\int\limits_{U_s} e^{b(\psi-u_0)}\right)^{\frac{a_1a_2}{b}}\left(\int\limits_{U_s} dV\right)^{1-\frac{a_1a_2}{b}}\\
&\leq B^{1+\frac{a_1a_2}{b}}\lambda(U_s)^{1-\frac{a_1a_2}{b}},
\end{aligned}
$$

where λ is Lebesgue measure. Hence, we have

$$t^nCap(U_{t+s},\Omega)\leq B^{1+\frac{a_1a_2}{b}}\lambda(U_s)^{1-\frac{a_1a_2}{b}}. \tag{9}$$

Moreover, it follows from [AT84, Zer01] that for any $p>0$, there exists $C_p>0$ depending only on p,n and Ω such that

$$\lambda(U_t)\leq C_pCap(U_t,\Omega)^p. \tag{10}$$

Combining (9) and (10), we obtain

$$t^n\lambda(U_{t+s})^{1/p}\leq B^{1+\frac{a_1a_2}{b}}C_p^{1/p}\lambda(U_s)^{1-\frac{a_1a_2}{b}}. \tag{11}$$

Let $p=\frac{2b}{b-a_1a_2}$. Applying Lemma 4.3 for $g(t)=\lambda(U_t)^{1/(pn)}$ and $\alpha=\frac{p(b-a_1a_2)}{b}$, there exists $s_\infty>0$ depending only on a_1,a_2,b,B,n,Ω such that $\lambda(U_{s_\infty})=0$. By the plurisubharmonicity of u and ψ, we conclude that $u\geq a_2\psi-s_\infty$. □

4.2. **Proof of Theorem 0.2.** We consider the case $A=0$ and then we use it to prove the result in the case $A>0$.

Step 1: Construct an approximation.

Let us first contruct a sequence of solutions as in Proposition 3.7 such that

(i) $\varphi_k=\varphi$ on $\partial\Omega\times(\epsilon,T)$ for any $\epsilon>0$ and $k\gg 1$;

(ii) $\sup\limits_{k\geq 0}\sup\limits_{z\in\partial\Omega}|\varphi_k(z,t)-\varphi_k(z,0)|\to 0$ as $t\searrow 0$.

Using the convolution of $u_0+\frac{|z|^2}{k}$ with mollifiers, we can take $u_{0,k}\in C^\infty(\bar\Omega)\cap SPSH(\bar\Omega)$ such that

$$u_{0,k}\searrow u_0. \tag{12}$$

Note that $u_0|_{\partial\Omega}$ is continuous. Then

$$\delta_k=\sup_{z\in\partial\Omega}(u_{0,k}(z)-u_0(z))\overset{k\to\infty}{\longrightarrow}0. \tag{13}$$

We define $g_k\in C^\infty(\bar\Omega)$ and $\varphi_k\in C^\infty(\bar\Omega\times[0,T))$ by

$$g_k=\log\det(u_{0,k})_{\alpha\bar\beta}+f(z,0),$$

$$\varphi_k = \zeta(\frac{t}{\epsilon_k})(tg_k + u_{0,k}) + (1 - \zeta(\frac{t}{\epsilon_k}))\varphi,$$

where ζ is a smooth function on $\mathbb{R}$ such that ζ is decreasing, $\zeta|_{(-\infty,1]} = 1$ and $\zeta|_{[2,\infty)} = 0$. $\epsilon_k > 0$ are chosen such that the sequences $\{\epsilon_k\}$, $\{\epsilon_k \sup |g_k|\}$ are decreasing to 0.

$u_{0,k}$ and φ_k satisfy the compatibility condition. By Theorem 3.2, there exists $u_k \in C^\infty(\bar{\Omega} \times [0,T))$ satisfying

$$\begin{cases} \dot{u}_k = \log\det(u_k)_{\alpha\bar{\beta}} + f(z,t) & \text{on } \Omega \times (0,T), \\ u_k = \varphi_k & \text{on } \partial\Omega \times [0,T), \\ u_k = u_{0,k} & \text{on } \bar{\Omega} \times \{0\}. \end{cases} \tag{14}$$

It is easy to verify that u_k satisfies the conditions in Proposition 3.7. Then $u(z,t) = \lim\limits_{k\to\infty} u_k(z,t)$ is the weak solution of (1). Moreover, $\varphi_k = \varphi$ on $\partial\Omega \times (\epsilon, T)$ for any $\epsilon > 0$ and $k \gg 1$.

Step 2: Smoothness of weak solution in $(\bar{\Omega} \setminus \{\nu(u_0,z) \geq \frac{2}{m}\}) \times (\frac{1}{m}, T)$.

Let $m > \frac{1}{T}$ and $\epsilon < \frac{1}{m}$. Applying Lemma 4.1 and Lemma 4.2 for $u_k(z, t+\epsilon)$, $k \gg 1$, we have, for any $(z,t) \in \bar{\Omega} \times (\epsilon, T)$,

$$\begin{aligned} (dd^c u_k)^n = e^{\dot{u}_k - f(z,t)} dV &\leq C_1 e^{\frac{u_k - u_k(z,\epsilon)}{t-\epsilon}} dV \\ &\leq C_2 e^{\frac{u_k - u_{0,k}}{t-\epsilon}} dV \\ &\leq C_2 e^{\frac{u_k - u_0}{t-\epsilon}} dV, \end{aligned}$$

where $C_1, C_2 > 0$ are independent of k.

Assume that W is an open neighbourhood of $\bar{\Omega}$ such that W is bounded pseudoconvex and $u_0 \in PSH(W)$. Let $l = \frac{1+\epsilon}{2(1/m-\epsilon)}$ and let $\{g_{l,j}\}$ be an orthonormal basis of $\mathcal{H}_W(lu_0)$. Denote $v_l = \frac{1}{2l} \log \sum\limits_{j=1}^{\infty} |g_{l,j}|^2$. Applying Lemma 4.4 for $a_1 = \frac{1}{1/m-\epsilon}$, $a_2 = 1$ and $b = 2l$, we have

$$u_k(z, \frac{1}{m}) \geq v_l(z) - C_3, \tag{15}$$

where $C_3 > 0$ is independent of k. Then

$$u(z, \frac{1}{m}) \geq v_l(z) - C_3. \tag{16}$$

Hence, applying Proposition 2.2, we have

$$u(z, \frac{1}{m}) \geq N \sum_{\nu(u_0,a) \geq \frac{1}{l}} \log|z-a| + O(1), \tag{17}$$

where $N > 0$. It follows from Proposition 3.6 that u is smooth and satisfies (1) in the classical sense in $(\bar{\Omega} \setminus \{z : \nu(u,z) \geq \frac{1}{l}\}) \times (\frac{1}{m}, T)$.

When $\epsilon \searrow 0$, we conclude that u is smooth and satisfies (1) in the classical sense in $(\bar{\Omega} \setminus \{z : \nu(u,z) \geq \frac{2}{m}\}) \times (\frac{1}{m}, T)$ and $u|_{\partial\Omega \times [0,T)} = \varphi|_{\partial\Omega \times [0,T)}$.

Step 3: The set of singular points of u.

Assume that $\{z : \nu(u_0, z) \geq \frac{2}{m}\} = \{a_{m,1}, ..., a_{m,N(m)}\}$. For any $j = 1, ..., N(m)$, we denote

$$\epsilon_{m,j} = \sup\{T > t \geq 0 : \nu(u(., t'), a_{m,j}) > 0, \forall 0 \leq t \leq t'\}. \tag{18}$$

We need to show that

- u is smooth and satisfies (1) in the classical sense in $Q_m := \bar{\Omega} \times (\frac{1}{m}, T) \setminus \cup_{j \leq N(m)}\{a_{m,j}\} \times (\frac{1}{m}, \epsilon_{m,j}]$.
- $\frac{\nu(u_0, a_{m,j})}{2n} \leq \epsilon_{m,j} \leq \frac{\nu(u_0, a_{m,j})}{2}$.
- $\nu(u(., t), a_{m,j}) > 0$ for any $j = 1, ..., N(m)$ and $t < \epsilon_{m,j}$.
- If $\epsilon_{m,j} < T$ then $\nu(u(., t), a_{m,j}) \searrow 0$ as $t \nearrow \epsilon_{m,j}$.

By Step 2, for any $T > \epsilon > 0$, there is $r > 0$ such that u is smooth in $(B(a_{m,j}, r) \setminus \{a_{m,j}\}) \times (\epsilon, T)$ for any $j = 1, ..., N(m)$. If there is $\epsilon \in (0, T)$ satisfying $\nu(u(., \epsilon), a_{m,j}) = 0$ then it follows from Proposition 3.6 that $u \in C^\infty(B(a_{m,j}, r) \times (\epsilon, T))$. By the definition of $\epsilon_{m,j}$, we conclude that u is smooth in Q_m. Clearly, u satisfies (1) in the classical sense in Q_m.

By Propostion 3.8 and by the smoothness of u in Q_m, we have $\epsilon_{m,j} \geq \frac{\nu(u_0, a_{m,j})}{2n}$. If $\nu(u_0, a_{m,j}) < 2T$, we have, as in the step 2,

$$u(a_{m,j}, c) \geq v_{\frac{1+\epsilon}{2(c-\epsilon)}} + O(1) =: v_l + O(1), \tag{19}$$

for any $0 < c < T$ and $c > \epsilon > 0$.

If $2c > \nu(u_0, a_{m,j})$ then for any $\epsilon \ll 1$ we have

$$\nu(u_0, a_{m,j}) < \frac{1}{l}.$$

It follows from Proposition 2.2 that v_l is smooth in a neighbourhood of $a_{m,j}$. By (19), we have

$$c \geq \epsilon_{m,j}.$$

Letting $c \to \frac{\nu(u_0, a_{m,j})}{2}$, we obtain $\epsilon_{m,j} \leq \frac{\nu(u_0, a_{m,j})}{2}$. Thus $\epsilon_{m,j}$ satisfies

$$\frac{\nu(u_0, a_{m,j})}{2n} \leq \epsilon_{m,j} \leq \frac{\nu(u_0, a_{m,j})}{2}.$$

By definition of $\epsilon_{m,j}$, we have $\nu(u(., t), a_{m,j}) > 0$ for any $j = 1, ..., N(m)$ and $t < \epsilon_{m,j}$. Applying Lemma 4.1 for u_k and letting $k \to \infty$, we conclude that $\nu(u(., t), a_{m,j})$ is non increasing in t. If $\epsilon_{m,j} < T$ then by Proposition 3.8 and by the smoothness of u in Q_m, we have, for any $0 < \epsilon < \epsilon_{m,j}$,

$$\frac{\nu(u(., \epsilon), a_{m,j})}{3n} + \epsilon \leq \epsilon_{m,j}.$$

Hence, $\nu(u(., t), a_{m,j}) \searrow 0$ as $t \nearrow \epsilon_{m,j}$.

Step 4: Continuity at zero.

Applying Lemma 4.1, we have

$$\liminf_{t \to 0} u(z, t) \geq u_0(z), \tag{20}$$

for any $z \in \bar{\Omega}$.

Note that u is the limit of a decreasing sequence of smooth functions, then $u \in USC(\bar{\Omega} \times [0,T))$. We have

$$\limsup_{t\to 0} u(z,t) \leq u_0(z), \tag{21}$$

for any $z \in \bar{\Omega}$.

Combining (20) and (21), we obtain

$$\lim_{t\to 0} u(z,t) = u_0(z).$$

Hence, by the dominated convergence theorem, $u(.,t) \to u_0$ in L^1, as $t \to 0$.

Step 5: The case $A > 0$.

Assume that u is the weak solution of (1) with $A > 0$. We set

$$v(z,t) = (At+1)u(z, \frac{\log(At+1)}{A}).$$

Then we can verify that v is the weak solution of

$$\begin{cases} \dot{v} = \log\det(v_{\alpha\bar{\beta}}) - n\log(At+1) + f(z, \frac{\log(At+1)}{A}) & \text{on } \Omega \times (0, \frac{e^{AT}-1}{A}), \\ v(z,0) = u_0(z) & \text{on } \Omega, \\ v(z,t) = \varphi(z, \frac{\log(At+1)}{A}) & \text{on } \partial\Omega \times [0,T). \end{cases} \tag{22}$$

Using the case $A = 0$, we conclude the similar result for the case $A > 0$.

References

[AT84] H. ALEXANDER, B. TAYLOR:*Comparison of two capacities in $\mathbb{C}^n$.* Math. Z. **186**(1984), 407–417.

[BG13] S. BOUCKSOM, V. GUEDJ: Regularizing properties of the Kähler-Ricci flow. *An introduction to the Kähler-Ricci flow,*189–237, Lecture Notes in Math., 2086, Springer, Cham, 2013.

[Ceg04] U. CEGRELL: *The general definition of the complex Monge-Ampre operator.* Annales de l'institut Fourier, Volume 54, Issue 1 (2004) 159–179.

[CC95] L. CAFFARELLI, X. CABRE:*Fully nonlinear elliptic equations,* Colloquium publications 43, American Mathematical Society, Providence, RI, 1995.

[Dem92] J-P. DEMAILLY: *Regularization of closed positive currents and Intersection Theory,* J. Alg. Geom. **1** (1992) 361–409.

[Dem10] J-P.DEMAILLY: *Analytic Methods in Algebraic Geometry,* Higher Education Press, Surveys of Modern Mathematics, Vol. 1, (2010).

[Dem14] J-P.DEMAILLY: *On the cohomology of pseudoeffective line bundles.* [arXiv:math.CV/14015432]

[Do15a] H-S. DO: *Degenerate complex Monge-Ampère flows on strictly pseudoconvex domains.* [arXiv:math.CV/150107167]

[Do15b] H-S.DO: *Weak solution of Parabolic complex Monge-Ampère equation.* [arXiv:math.CV/150601506] to appear Indiana U. Math. J.

[DL14] E. DI NEZZA, H-C. LU: *Uniqueness and short time regularity of the weak Kähler- Ricci flow.* [arXiv:math.CV/14117958]

[EGZ09] P.EYSSIDIEUX, V.GUEDJ, A.ZERIAHI: *Singular Kähler-Einstein metrics.* J. Amer. Math. Soc. **22** (2009), no. 3, 607–639.

[GT83] D. GILBARG, N. TRUDINGER:*Elliptic partial differential equations of second order. Second edition.* Grundlehren der Mathematischen Wissenschaften **224.**Springer-Verlag, Berlin, 1983. xiii+513 pp.

[Hor90] L.HÖRMANDER: *An introduction to complex analysis in several variables.* North-Holland, 1990.

[Huy05] D. HUYBRECHTS: *Complex geometry.* Universitext. Springer-Verlag, Berlin, 2005.

[HL10] Z. HOU, Q. LI: *Energy functionals and complex Monge-Ampère equations.* J. Inst. Math. Jussieu **9** (2010) no.3, 463–476.

[IS13] C. IMBERT, L. SILVERSTRE: An introduction to fully nonlinear parabolic equations. *An introduction to the Kähler-Ricci flow,*7-88, Lecture Notes in Math., 2086, Springer, Cham, 2013.

[JKS92] S. JI, J. KOLLÁR, B. SHIFFMAN: *A global Lojasiewicz inequality for algebraic varieties.* Trans. Amer. Math. Soc. **329** (1992), no.2, 813–818.

[Lieb96] G. M. LIEBERMAN: *Second order parabolic differential equations* (World Scientific, River Edge, 1996).

[Loj59] S. LOJASIEWICZ: *Sur le problème de la division.* Studia Math. **18** (1959), 87–136.

[Mal66] B. MALGRANGE:*Ideals of differentiable functions.* Oxford Univ. Press, 1966.

[Sko72] H. SKODA: *Sous-ensembles analytiques d'ordre fini ou infini dans* $\mathbb{C}^n$. Bull. Soc. Math. France **100** (1972), 353–408.

[Zer01] A. ZERIAHI: *Volume and capacity of sublevel sets of a Lelong class of plurisubharmonic functions.* Indiana Univ. Math. J. **50** (2001), no.1, 671–703.

Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi, Vietnam

E-mail address: hoangson.do.vn@gmail.com