

Well-posedness for the Navier-Stokes equations with data in homogeneous Sobolev-Lorentz spaces

D. Q. Khai, N. M. Tri

Institute of Mathematics, VAST
18 Hoang Quoc Viet, 10307 Cau Giay, Hanoi, Vietnam

Abstract: In this paper, we study local well-posedness for the Navier-Stokes equations (NSE) with arbitrary initial data in homogeneous Sobolev-Lorentz spaces $\dot{H}_{L^{q,r}}^s(\mathbb{R}^d) := (-\Delta)^{-s/2}L^{q,r}$ for $d \geq 2, q > 1, s \geq 0, 1 \leq r \leq \infty$, and $\frac{d}{q} - 1 \leq s < \frac{d}{q}$. The obtained result improves the known ones for $q > d, r = q, s = 0$ (see [4, 7]), for $q = r = 2, \frac{d}{2} - 1 < s < \frac{d}{2}$ (see [4, 10]), and for $s = 0, d < q < +\infty, 1 \leq r \leq +\infty$ (see [31]). In the case of critical indexes ($s = \frac{d}{q} - 1$), we prove global well-posedness for NSE provided the norm of the initial value is small enough. This result is a generalization of the one in [5] and [9, 30] in which ($q = r = d, s = 0$) and ($q = d, s = 0, r = +\infty$), respectively.

§1. Introduction

We consider the Navier-Stokes equations in \mathbb{R}^d :

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\ \nabla \cdot u = 0, \\ u(0, x) = u_0, \end{cases}$$

which is a condensed writing for

$$\begin{cases} 1 \leq k \leq d, \quad \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l (u_l u_k) - \partial_k p, \\ \sum_{l=1}^d \partial_l u_l = 0, \\ 1 \leq k \leq d, \quad u_k(0, x) = u_{0k}. \end{cases}$$

The unknown quantities are the velocity $u(t, x) = (u_1(t, x), \dots, u_d(t, x))$ of the fluid element at time t and position x and the pressure $p(t, x)$.

In the 1960s, mild solutions were first constructed by Kato and Fujita ([20], [15]) that are continuous in time and take values in the Sobolev spaces $H^s(\mathbb{R}^d)$, ($s \geq \frac{d}{2} - 1$), say $u \in C([0, T]; H^s(\mathbb{R}^d))$. In 1992, a modern treatment for mild solutions in $H^s(\mathbb{R}^d)$, ($s \geq \frac{d}{2} - 1$) was given by Chemin [10]. In

1995, using the simplified version of the bilinear operator, Cannone proved the existence of mild solutions in $\dot{H}^s(\mathbb{R}^d)$, ($s \geq \frac{d}{2} - 1$), see [4]. Results on the existence of mild solutions with value in $L^q(\mathbb{R}^d)$, ($q > d$) were established in the papers of Fabes, Jones and Rivière [12] and of Giga [16]. Concerning the initial data in the space L^∞ , the existence of a mild solution was obtained by Cannone and Meyer in ([4], [7]). In 1994, Kato and Ponce [24] showed that the NSE are well-posed when the initial data belong to the homogeneous Sobolev spaces $\dot{H}_q^{\frac{d}{2}-1}(\mathbb{R}^d)$, ($d \leq q < \infty$). Recently, the authors of this article have considered NSE in mixed-norm Sobolev-Lorentz spaces and Sobolev-Fourier-Lorentz spaces, see [25] and [26] respectively. In [28] we prove that NSE are well-posed when the initial datum belongs to the Sobolev spaces $\dot{H}_p^{\frac{d}{2}-1}(\mathbb{R}^d)$ with ($1 < p \leq d$). In [27], We considered the initial value problem for the non stationary Navier-Stokes equations on torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ and showed that NSE are well-posed when the initial datum belongs to Sobolev spaces $V_\alpha := D(-\Delta)^{\alpha/2}$ with $\frac{1}{2} < \alpha < \frac{3}{2}$. In this paper, for $d \geq 2, q > 1, s \geq 0, 1 \leq r \leq \infty$, and $\frac{d}{q} - 1 \leq s < \frac{d}{q}$, we investigate mild solutions to NSE in the spaces $L^\infty([0, T]; \dot{H}_{L^q, r}^s(\mathbb{R}^d))$ when the initial data belong to the Sobolev-Lorentz spaces $\dot{H}_{L^q, r}^s(\mathbb{R}^d)$, which are more general than the spaces $\dot{H}_q^s(\mathbb{R}^d)$, ($\dot{H}_q^s(\mathbb{R}^d) = \dot{H}_{L^q, q}^s(\mathbb{R}^d)$). We obtain the existence of mild solutions with arbitrary initial value when T is small enough, and existence of mild solutions for any $T > 0$ when the norm of the initial value in the Besov spaces $\dot{B}_q^{s-d(\frac{1}{q}-\frac{1}{q}), \infty}(\mathbb{R}^d)$, ($\frac{1}{2}(\frac{1}{q} + \frac{s}{d}) < \frac{1}{q} < \min\{\frac{1}{2} + \frac{s}{2d}, \frac{1}{q}\}$) is small enough.

In the particular case ($q > d, r = q, s = 0$), we get the result which is more general than that of Cannone and Meyer ([4], [7]). Here we obtained a statement that is stronger than that of Cannone and Meyer but under a much weaker condition on the initial data.

In the particular case ($q = r = 2, \frac{d}{2} - 1 < s < \frac{d}{2}$), we get the result which is more general than those of Chemin in [10] and Cannone in [4]. Here we obtained a statement that is stronger than those of Chemin in [10] and Cannone in [4] but under a much weaker condition on the initial data.

In the case of critical indexes ($1 < q \leq d, r \geq 1, s = \frac{d}{q} - 1$), we get a result that is a generalization of a result of Cannone [5]. In particular, when $q = r = d, s = 0$, we get back the Cannone theorem (Theorem 1.1 in [5]).

The paper is organized as follows. In Section 2 we prove some inequalities for pointwise products in the Sobolev spaces and some auxiliary lemmas. In Section 3 we present the main results of the paper. In the sequence, for a space of functions defined on \mathbb{R}^d , say $E(\mathbb{R}^d)$, we will abbreviate it as E .

§2. Some auxiliary results

In this section, we recall the following results and notations.

Definition 1. (Lorentz spaces). (See [1].)

For $1 \leq p, r \leq \infty$, the Lorentz space $L^{p,r}(\mathbb{R}^d)$ is defined as follows:

A measurable function $f \in L^{p,r}(\mathbb{R}^d)$ if and only if

$$\|f\|_{L^{p,r}(\mathbb{R}^d)} := \left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^r \frac{dt}{t} \right)^{\frac{1}{r}} < \infty \text{ when } 1 \leq r < \infty,$$

$$\|f\|_{L^{p,\infty}(\mathbb{R}^d)} := \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty \text{ when } r = \infty,$$

where $f^*(t) = \inf \{ \tau : \mathcal{M}^d(\{x : |f(x)| > \tau\}) \leq t \}$, with \mathcal{M}^d being the Lebesgue measure in \mathbb{R}^d .

Before proceeding to the definition of Sobolev-Lorentz spaces, let us introduce several necessary notations. For real number s , the operator $\dot{\Lambda}^s$ is defined through Fourier translation by

$$(\dot{\Lambda}^s f)^\wedge(\xi) = |\xi|^s \hat{f}(\xi).$$

For $0 < s < d$, the operator $\dot{\Lambda}^s$ can be viewed as the inverse of the Riesz potential I_s up to a positive constant

$$I_s(f)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-s}} dy \text{ for } x \in \mathbb{R}^d.$$

For $q > 1, r \geq 1$, and $0 \leq s < \frac{d}{q}$, the operator I_s is continuous from $L^{q,r}$ to $L^{\tilde{q},r}$, where $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{s}{d}$, see ([31], Theorem 2.4 *iii*), p. 20).

Definition 2. (Sobolev-Lorentz spaces). (See [13].)

For $q > 1, r \geq 1$, and $0 \leq s < \frac{d}{q}$, the Sobolev-Lorentz space $\dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$ is defined as the space $I_s(L^{q,r}(\mathbb{R}^d))$, equipped with the norm

$$\|f\|_{\dot{H}_{L^{q,r}}^s} := \|\dot{\Lambda}^s f\|_{L^{q,r}}.$$

Lemma 1. Let $q > 1, 1 \leq r \leq \tilde{r} \leq \infty$, and $0 \leq s < \frac{d}{q}$. Then we have the following imbedding maps

(a)

$$\dot{H}_{L^{q,1}}^s \hookrightarrow \dot{H}_{L^{q,r}}^s \hookrightarrow \dot{H}_{L^{q,\tilde{r}}}^s \hookrightarrow \dot{H}_{L^{q,\infty}}^s.$$

(b) $\dot{H}_q^s = \dot{H}_{L^{q,q}}^s$ (equality of the norm).

Proof. It is easily deduced from the properties of the standard Lorentz spaces. \square

In the following lemmas, we estimate the pointwise product of two functions in $\dot{H}_q^s(\mathbb{R}^d)$, ($d \geq 2$) which is a generalization of the Holder inequality. In the case when $s = 0$ we get back the usual Holder inequality. Pointwise multiplication results for Sobolev spaces are also obtained in literature, see for example [11], [31], [23] and the references therein.

Lemma 2. *Assume that*

$$1 < p, q < d, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{d}.$$

Then the following inequality holds

$$\|uv\|_{\dot{H}_r^1} \lesssim \|u\|_{\dot{H}_p^1} \|v\|_{\dot{H}_q^1}, \quad \forall u \in \dot{H}_p^1, v \in \dot{H}_q^1,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{1}{d}$.

Proof. By applying the Leibniz formula for the derivatives of a product of two functions, we have

$$\|uv\|_{\dot{H}_r^1} \simeq \sum_{|\alpha|=1} \|\partial^\alpha(uv)\|_{L^r} \leq \sum_{|\alpha|=1} \|(\partial^\alpha u)v\|_{L^r} + \sum_{|\alpha|=1} \|u(\partial^\alpha v)\|_{L^r}.$$

By applying the Hölder and Sobolev inequalities we obtain

$$\sum_{|\alpha|=1} \|(\partial^\alpha u)v\|_{L^r} \leq \sum_{|\alpha|=1} \|\partial^\alpha u\|_{L^p} \|v\|_{L^{q_1}} \lesssim \|u\|_{\dot{H}_p^1} \|v\|_{\dot{H}_q^1},$$

where

$$\frac{1}{q_1} = \frac{1}{q} - \frac{1}{d}.$$

Similar to the above reasoning, we have

$$\sum_{|\alpha|=1} \|u(\partial^\alpha v)\|_{L^r} \lesssim \|u\|_{\dot{H}_p^1} \|v\|_{\dot{H}_q^1}.$$

This gives the desired result

$$\|uv\|_{\dot{H}_r^1} \lesssim \|u\|_{\dot{H}_p^1} \|v\|_{\dot{H}_q^1}.$$

\square

Lemma 3. *Assume that*

$$0 \leq s \leq 1, \frac{1}{p} > \frac{s}{d}, \frac{1}{q} > \frac{s}{d}, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}. \quad (1)$$

Then the following inequality holds

$$\|uv\|_{\dot{H}_r^s} \lesssim \|u\|_{\dot{H}_p^s} \|v\|_{\dot{H}_q^s}, \quad \forall u \in \dot{H}_p^s, v \in \dot{H}_q^s,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}$.

Proof. It is not difficult to show that if p, q , and s satisfy (1) then there exists numbers $p_1, p_2, q_1, q_2 \in (1, +\infty)$ (may be many of them) such that

$$\begin{aligned} \frac{1}{p} &= \frac{1-s}{p_1} + \frac{s}{p_2}, \frac{1}{q} = \frac{1-s}{q_1} + \frac{s}{q_2}, \frac{1}{p_1} + \frac{1}{q_1} < 1, \\ p_2 < d, q_2 < d, \text{ and } \frac{1}{p_2} + \frac{1}{q_2} < 1 + \frac{1}{d}. \end{aligned}$$

Setting

$$\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{q_1}, \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{d},$$

we have

$$\frac{1}{r} = \frac{1-s}{r_1} + \frac{s}{r_2}.$$

Therefore, applying Theorem 6.4.5 (page 152) of [1] (see also [19] for \dot{H}_p^s), we get

$$\dot{H}_p^s = [L^{p_1}, \dot{H}_{p_2}^1]_s, \dot{H}_q^s = [L^{q_1}, \dot{H}_{q_2}^1]_s, \dot{H}_r^s = [L^{r_1}, \dot{H}_{r_2}^1]_s.$$

Applying the Holder inequality and Lemma 2 in order to obtain

$$\begin{aligned} \|uv\|_{L^{r_1}} &\lesssim \|u\|_{L^{p_1}} \|v\|_{L^{q_1}}, \quad \forall u \in L^{p_1}, v \in L^{q_1}, \\ \|uv\|_{\dot{H}_{r_2}^1} &\lesssim \|u\|_{\dot{H}_{p_2}^1} \|v\|_{\dot{H}_{q_2}^1}, \quad \forall u \in \dot{H}_{p_2}^1, v \in \dot{H}_{q_2}^1. \end{aligned}$$

From Theorem 4.4.1 (page 96) of [1] we get

$$\|uv\|_{\dot{H}_r^s} \lesssim \|u\|_{\dot{H}_p^s} \|v\|_{\dot{H}_q^s}.$$

□

Lemma 4. *Assume that*

$$q > 1, p > 1, 0 \leq \frac{s}{d} < \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}. \quad (2)$$

Then we have the inequality

$$\|uv\|_{\dot{H}_r^s} \lesssim \|u\|_{\dot{H}_p^s} \|v\|_{\dot{H}_q^s}, \quad \forall u \in \dot{H}_p^s, v \in \dot{H}_q^s,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}$.

Proof. Denote by $[s]$ the integer part of s and by $\{s\}$ the fraction part of the argument s . Using the formula for the derivatives of a product of two functions, we have

$$\begin{aligned} \|uv\|_{\dot{H}_r^s} &= \|\dot{\Lambda}^s(uv)\|_{L^r} = \|\dot{\Lambda}^{\{s\}}(uv)\|_{\dot{H}_r^{[s]}} \simeq \\ &\sum_{|\alpha|=[s]} \|\partial^\alpha \dot{\Lambda}^{\{s\}}(uv)\|_{L^r} = \sum_{|\alpha|=[s]} \|\dot{\Lambda}^{\{s\}} \partial^\alpha(uv)\|_{L^r} \\ &= \sum_{|\alpha|=[s]} \|\partial^\alpha(uv)\|_{\dot{H}_r^{\{s\}}} \lesssim \sum_{|\gamma|+|\beta|=[s]} \|\partial^\gamma u \partial^\beta v\|_{\dot{H}_r^{\{s\}}}. \end{aligned}$$

Set

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{s - |\gamma| - \{s\}}{d}, \quad \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{s - |\beta| - \{s\}}{d}.$$

Applying Lemma 3 and the Sobolev inequality in order to obtain

$$\|\partial^\gamma u \partial^\beta v\|_{\dot{H}_r^{\{s\}}} \lesssim \|\partial^\gamma u\|_{\dot{H}_{\tilde{p}}^{\{s\}}} \|\partial^\beta v\|_{\dot{H}_{\tilde{q}}^{\{s\}}} \lesssim \|u\|_{\dot{H}_{\tilde{p}}^{|\gamma|+\{s\}}} \|v\|_{\dot{H}_{\tilde{q}}^{|\beta|+\{s\}}} \lesssim \|u\|_{\dot{H}_p^s} \|v\|_{\dot{H}_q^s}.$$

This gives the desired result

$$\|uv\|_{\dot{H}_r^s} \lesssim \|u\|_{\dot{H}_p^s} \|v\|_{\dot{H}_q^s}.$$

□

Lemma 5. *Let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$.*

(a) *If $s < 1$ then the two quantities*

$$\left(\int_0^\infty (t^{-\frac{s}{2}} \|e^{t\Delta} t^{\frac{1}{2}} \dot{\Lambda} f\|_q)^p \frac{dt}{t} \right)^{1/p} \text{ and } \|f\|_{\dot{B}_q^{s,p}}$$

(b) *If $s < 0$ then the two quantities*

$$\left(\int_0^\infty (t^{-\frac{s}{2}} \|e^{t\Delta} f\|_q)^p \frac{dt}{t} \right)^{1/p} \text{ and } \|f\|_{\dot{B}_q^{s,p}}$$

where $\dot{B}_q^{s,p}$ is the homogeneous Besov space.

Proof. See ([14], Proposition 1, p. 181 and Proposition 3, p. 182), or see ([31], Theorem 5.4, p. 45). \square

The following lemma is a generalization of the above lemma.

Lemma 6. *Let $1 \leq p, q \leq \infty$, $\alpha \geq 0$, and $s < \alpha$. Then the two quantities*

$$\left(\int_0^\infty (t^{-\frac{s}{2}} \|e^{t\Delta} t^{\frac{\alpha}{2}} \dot{\Lambda}^\alpha f\|_{L^q})^p \frac{dt}{t} \right)^{\frac{1}{p}} \text{ and } \|f\|_{\dot{B}_q^{s,p}}$$

Proof. Note that $\dot{\Lambda}^{s_0}$ is an isomorphism from $\dot{B}_q^{s,p}$ to $\dot{B}_q^{s-s_0,p}$, see [3], then we can easily prove the lemma. \square

Lemma 7. *Assume that $q > 1, 1 \leq r \leq \infty$, and $0 \leq s < \frac{d}{q}$. The following statement is true: If $u_0 \in \dot{H}_{L^q, r}^s$ then $e^{t\Delta} u_0 \in L^\infty([0, \infty); \dot{H}_{L^q, r}^s)$ and $\|e^{t\Delta} u_0\|_{L^\infty([0, \infty); \dot{H}_{L^q, r}^s)} \leq \|u_0\|_{\dot{H}_{L^q, r}^s}$.*

Proof. We have

$$\begin{aligned} \|e^{t\Delta} u_0\|_{\dot{H}_{L^q, r}^s} &= \|e^{t\Delta} \dot{\Lambda}^s u_0\|_{L^q, r} = \frac{1}{(4\pi t)^{d/2}} \left\| \int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}} \dot{\Lambda}^s u_0(\cdot - \xi) d\xi \right\|_{L^q, r} \\ &\leq \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}} \|\dot{\Lambda}^s u_0(\cdot - \xi)\|_{L^q, r} d\xi \\ &= \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}} \|u_0\|_{\dot{H}_{L^q, r}^s} d\xi = \|u_0\|_{\dot{H}_{L^q, r}^s}. \end{aligned}$$

\square

Let us recall following result on solutions of a quadratic equation in Banach spaces (Theorem 22.4 in [31], p. 227).

Theorem 1. *Let E be a Banach space, and $B : E \times E \rightarrow E$ be a continuous bilinear map such that there exists $\eta > 0$ so that*

$$\|B(x, y)\| \leq \eta \|x\| \|y\|,$$

for all x and y in E . Then for any fixed $y \in E$ such that $\|y\| \leq \frac{1}{4\eta}$, the equation $x = y - B(x, x)$ has a unique solution $\bar{x} \in E$ satisfying $\|\bar{x}\| \leq \frac{1}{2\eta}$.

§3. Main results

Now, for $T > 0$, we say that u is a mild solution of NSE on $[0, T]$ corresponding to a divergence-free initial datum u_0 when u solves the integral equation

$$u = e^{t\Delta} u_0 - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau, \cdot) \otimes u(\tau, \cdot)) d\tau.$$

Above we have used the following notation: For a tensor $F = (F_{ij})$ we define the vector $\nabla.F$ by $(\nabla.F)_i = \sum_{j=1}^d \partial_j F_{ij}$ and for two vectors u and v , we define their tensor product $(u \otimes v)_{ij} = u_i v_j$. The operator \mathbb{P} is the Helmholtz-Leray projection onto the divergence-free fields

$$(\mathbb{P}f)_j = f_j + \sum_{1 \leq k \leq d} R_j R_k f_k, \quad (3)$$

where R_j is the Riesz transforms defined as

$$R_j = \frac{\partial_j}{\sqrt{-\Delta}}, \quad \text{i. e.} \quad \widehat{R_j g}(\xi) = \frac{i\xi_j}{|\xi|} \hat{g}(\xi)$$

with $\hat{\cdot}$ denoting the Fourier transform. The heat kernel $e^{t\Delta}$ is defined as

$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2} e^{-|\cdot|^2/4t} * u)(x).$$

If X is a normed space and $u = (u_1, u_2, \dots, u_d)$, $u_i \in X$, $1 \leq i \leq d$, then we write

$$u \in X, \quad \|u\|_X = \left(\sum_{i=1}^d \|u_i\|_X^2 \right)^{1/2}.$$

We define the auxiliary space $\mathcal{K}_{q,r,T}^{s,\tilde{q}}$ which is made up by the functions $u(t, x)$ such that

$$\|u\|_{\mathcal{K}_{q,r,T}^{s,\tilde{q}}} := \sup_{0 < t < T} t^{\frac{\alpha}{2}} \|u(t, \cdot)\|_{\dot{H}_{L^{\tilde{q},r}}^s} < \infty,$$

and

$$\lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|u(t, \cdot)\|_{\dot{H}_{L^{\tilde{q},r}}^s} = 0, \quad (4)$$

where r, q, \tilde{q}, s being fixed constants satisfying

$$q, \tilde{q} \in (1, +\infty), r \geq 1, s \geq 0, \frac{s}{d} < \frac{1}{\tilde{q}} \leq \frac{1}{q} \leq \frac{s+1}{d},$$

and

$$\alpha = \alpha(q, \tilde{q}) = d \left(\frac{1}{q} - \frac{1}{\tilde{q}} \right).$$

In the case $\tilde{q} = q$, it is also convenient to define the space $\mathcal{K}_{q,r,T}^{s,\tilde{q}}$ as the natural space $L^\infty([0, T]; \dot{H}_{L^{q,r}}^s(\mathbb{R}^d))$ with the additional condition that its elements $u(t, x)$ satisfy

$$\lim_{t \rightarrow 0} \|u(t, \cdot)\|_{\dot{H}_{L^{q,r}}^s} = 0. \quad (5)$$

Remark 1. The auxiliary space $\mathcal{K}_{\tilde{q}} := \mathcal{K}_{d,\tilde{q},T}^{0,\tilde{q}}$ ($\tilde{q} \geq d$) was introduced by Weissler and systematically used by Kato [21] and Cannone [5].

Lemma 8. Let $1 \leq r \leq \tilde{r} \leq \infty$. Then we have the following imbedding maps

$$\mathcal{K}_{q,1,T}^{s,\tilde{q}} \hookrightarrow \mathcal{K}_{q,r,T}^{s,\tilde{q}} \hookrightarrow \mathcal{K}_{q,\tilde{r},T}^{s,\tilde{q}} \hookrightarrow \mathcal{K}_{q,\infty,T}^{s,\tilde{q}}.$$

Proof. It is easily deduced from Lemma 1 (a) and the definition of $\mathcal{K}_{q,r,T}^{s,\tilde{q}}$. \square

Lemma 9. If $u_0 \in \dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$ with $q > 1, r \geq 1, s \geq 0$, and $\frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}$ then for all \tilde{q} satisfying

$$\frac{s}{d} < \frac{1}{\tilde{q}} < \frac{1}{q},$$

we have

$$e^{t\Delta} u_0 \in \mathcal{K}_{q,1,\infty}^{s,\tilde{q}},$$

and the following imbedding map

$$\dot{H}_{L^{q,r}}^s(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d). \quad (6)$$

Proof. Before proving this lemma, we need to prove the following lemma.

Lemma 10. Suppose that $u_0 \in L^{q,r}(\mathbb{R}^d)$ with $1 \leq q \leq \infty$ and $1 \leq r < \infty$. Then $\lim_{n \rightarrow \infty} \|\mathcal{X}_n u_0\|_{L^{q,r}} = 0$, where $n \in \mathbb{N}, \mathcal{X}_n(x) = 0$ for $x \in \{x : |x| < n\} \cap \{x : |u_0(x)| < n\}$ and $\mathcal{X}_n(x) = 1$ otherwise.

Proof. With $\delta > 0$ being fixed, we have

$$\{x : |\mathcal{X}_n u_0(x)| > \delta\} \supseteq \{x : |\mathcal{X}_{n+1} u_0(x)| > \delta\}, \quad (7)$$

and

$$\bigcap_{n=0}^{\infty} \{x : |\mathcal{X}_n u_0(x)| > \delta\} = \{x : |u_0(x)| = +\infty\}. \quad (8)$$

We prove that

$$\mathcal{M}^d(\{x : |u_0(x)| = +\infty\}) = 0, \quad (9)$$

with \mathcal{M}^d being the Lebesgue measure in \mathbb{R}^d , assuming on the contrary

$$\mathcal{M}^d(\{x : |u_0(x)| = +\infty\}) > 0.$$

We have $u_0^*(t) := \inf \{\tau : \mathcal{M}^d(\{x : |u_0(x)| > \tau\}) \leq t\} = +\infty$ for all t such that $0 < t < \mathcal{M}^d(\{x : |u_0(x)| = +\infty\})$ and then $\|u_0\|_{L^{q,r}} = +\infty$, a contradiction.

Note that

$$\mathcal{M}^d(\{x : |\mathcal{X}_0 u_0(x)| > \delta\}) = \mathcal{M}^d(\{x : |u_0(x)| > \delta\}).$$

We prove that

$$\mathcal{M}^d(\{x : |u_0(x)| > \delta\}) < \infty, \quad (10)$$

assuming on the contrary

$$\mathcal{M}^d(\{x : |u_0(x)| > \delta\}) = \infty.$$

We have $u_0^*(t) \geq \delta$ for all $t > 0$, from the definition of the Lorentz space, we get

$$\|u_0\|_{L^{q,r}} = \left(\int_0^\infty (t^{\frac{1}{q}} u_0^*(t))^r \frac{dt}{t} \right)^{\frac{1}{r}} \geq \left(\int_0^\infty (t^{\frac{1}{q}} \delta)^r \frac{dt}{t} \right)^{\frac{1}{r}} = \delta \left(\int_0^\infty t^{\frac{r}{q}-1} dt \right)^{\frac{1}{r}} = \infty,$$

a contradiction.

From (7), (8), (9), and (10), we infer that

$$\lim_{n \rightarrow \infty} \mathcal{M}^d(\{x : |\mathcal{X}_n u_0(x)| > \delta\}) = \mathcal{M}^d(\{x : |u_0(x)| = +\infty\}) = 0. \quad (11)$$

Set

$$u_n^*(t) = \inf \{ \tau : \mathcal{M}^d(\{x : |\mathcal{X}_n u_0(x)| > \tau\}) \leq t \}.$$

We have

$$u_n^*(t) \geq u_{n+1}^*(t). \quad (12)$$

Fixed $t > 0$. For any $\epsilon > 0$, from (11) it follows that there exists a number $n_0 = n_0(t, \epsilon)$ large enough such that

$$\mathcal{M}^d(\{x : |\mathcal{X}_n u_0(x)| > \epsilon\}) \leq t, \forall n \geq n_0.$$

From this we deduce that

$$u_n^*(t) \leq \epsilon, \forall n \geq n_0,$$

therefore

$$\lim_{n \rightarrow \infty} u_n^*(t) = 0. \quad (13)$$

From (12) and (13), we apply Lebesgue's monotone convergence theorem to get

$$\lim_{n \rightarrow \infty} \|\mathcal{X}_n u_0\|_{L^{q,r}} = \lim_{n \rightarrow \infty} \left(\int_0^\infty (t^{\frac{1}{q}} u_n^*(t))^r \frac{dt}{t} \right)^{\frac{1}{r}} = 0. \quad \square$$

Now we return to prove Lemma 9. We prove that

$$\sup_{0 < t < \infty} t^{\frac{\alpha}{2}} \|e^{t\Delta} u_0\|_{\dot{H}_{L^{\tilde{q},1}}^s} \lesssim \|u_0\|_{\dot{H}_{L^{q,r}}^s}. \quad (14)$$

Set

$$\frac{1}{h} = 1 + \frac{1}{\tilde{q}} - \frac{1}{q}.$$

Applying Proposition 2.4 (c) in ([31], pp. 20) for convolution in the Lorentz spaces, we have

$$\begin{aligned} \|e^{t\Delta}u_0\|_{\dot{H}_{L^{\tilde{q},1}}^s} &= \left\| e^{t\Delta}\dot{\Lambda}^s u_0 \right\|_{L^{\tilde{q},1}} = \frac{1}{(4\pi t)^{d/2}} \left\| e^{-\frac{|\cdot|^2}{4t}} * \dot{\Lambda}^s u_0 \right\|_{L^{\tilde{q},1}} \lesssim \\ \frac{1}{t^{d/2}} \|e^{-\frac{|\cdot|^2}{4t}}\|_{L^{h,1}} \|\dot{\Lambda}^s u_0\|_{L^{q,\infty}} &= t^{-\frac{\alpha}{2}} \|e^{-\frac{|\cdot|^2}{4t}}\|_{L^{h,1}} \|u_0\|_{\dot{H}_{L^{q,\infty}}^s} \lesssim t^{-\frac{\alpha}{2}} \|u_0\|_{\dot{H}_{L^{q,r}}^s}. \end{aligned}$$

We claim now that

$$\lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|e^{t\Delta}u_0\|_{\dot{H}_{L^{\tilde{q},1}}^s} = 0.$$

From Lemma 10, we have

$$\lim_{n \rightarrow \infty} \left\| \mathcal{X}_{n,s} \dot{\Lambda}^s u_0 \right\|_{L^{q,r}} = 0, \quad (15)$$

where $\mathcal{X}_{n,s}(x) = 0$ for $x \in \{x : |x| < n\} \cap \{x : |\dot{\Lambda}^s u_0(x)| < n\}$ and $\mathcal{X}_{n,s}(x) = 1$ otherwise. We have

$$\begin{aligned} t^{\frac{\alpha}{2}} \|e^{t\Delta}u_0\|_{\dot{H}_{L^{\tilde{q},1}}^s} &\leq \frac{t^{\frac{\alpha}{2} - \frac{d}{2}}}{(4\pi)^{d/2}} \left\| e^{-\frac{|\cdot|^2}{4t}} * (\mathcal{X}_{n,s} \dot{\Lambda}^s u_0) \right\|_{L^{\tilde{q},1}} + \\ \frac{t^{\frac{\alpha}{2} - \frac{d}{2}}}{(4\pi)^{d/2}} \left\| e^{-\frac{|\cdot|^2}{4t}} * ((1 - \mathcal{X}_{n,s}) \dot{\Lambda}^s u_0) \right\|_{L^{\tilde{q},1}}. \end{aligned} \quad (16)$$

For any $\epsilon > 0$, applying Proposition 2.4 (c) in ([31], pp. 20) and note that (15), we have

$$\begin{aligned} &\frac{t^{\frac{\alpha}{2} - \frac{d}{2}}}{(4\pi)^{d/2}} \left\| e^{-\frac{|\cdot|^2}{4t}} * (\mathcal{X}_{n,s} \dot{\Lambda}^s u_0) \right\|_{L^{\tilde{q},1}} \\ &\leq C_1 \|e^{-\frac{|\cdot|^2}{4t}}\|_{L^{h,1}} \left\| \mathcal{X}_{n,s} \dot{\Lambda}^s u_0 \right\|_{L^{q,\infty}} \leq C_2 \left\| \mathcal{X}_{n,s} \dot{\Lambda}^s u_0 \right\|_{L^{q,r}} < \frac{\epsilon}{2}, \end{aligned} \quad (17)$$

for large enough n . Fixed one of such n , applying Proposition 2.4 (a) in ([31], pp. 20), we conclude that

$$\begin{aligned} &\frac{t^{\frac{\alpha}{2} - \frac{d}{2}}}{(4\pi)^{d/2}} \left\| e^{-\frac{|\cdot|^2}{4t}} * ((1 - \mathcal{X}_{n,s}) \dot{\Lambda}^s u_0) \right\|_{L^{\tilde{q},1}} \\ &\leq C_3 t^{\frac{\alpha}{2} - \frac{d}{2}} \|e^{-\frac{|\cdot|^2}{4t}}\|_{L^1} \left\| (1 - \mathcal{X}_{n,s}) \dot{\Lambda}^s u_0 \right\|_{L^{\tilde{q},1}} \\ &\leq C_4 t^{\frac{\alpha}{2}} \|e^{-\frac{|\cdot|^2}{4t}}\|_{L^1} \|n(1 - \mathcal{X}_{n,s})\|_{L^{\tilde{q},1}} = \\ &C_5 n t^{\frac{\alpha}{2}} \|(1 - \mathcal{X}_{n,s})\|_{L^{\tilde{q},1}} = C_6(n) t^{\frac{\alpha}{2}} < \frac{\epsilon}{2}, \end{aligned} \quad (18)$$

for small enough $t > 0$. From the estimates (16), (17), and (18) it follows that

$$t^{\frac{\alpha}{2}} \|e^{t\Delta} u_0\|_{\dot{H}_{L^{\tilde{q},1}}^s} \leq C_2 \left\| \mathcal{X}_{n,s} \dot{\Lambda}^s u_0 \right\|_{L^{q,r}} + C_6(n) t^{\frac{\alpha}{2}} < \epsilon.$$

Finally, the embedding (6) is derived from the inequality (14), Lemma 1, and Lemma 6.

Remark 2. In the case $s = 0$ and $q = r = d$, Lemma 11 is a generalization of Lemma 9 in ([8], p. 196).

In the following lemmas a particular attention will be devoted to study of the bilinear operator $B(u, v)(t)$ defined by

$$B(u, v)(t) = \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau) \otimes v(\tau)) d\tau.$$

Lemma 11. *Let $s, q \in \mathbb{R}$ be such that*

$$s \geq 0, q > 1, \text{ and } \frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}. \quad (19)$$

Then for all \tilde{q} satisfying

$$\frac{s}{d} < \frac{1}{\tilde{q}} < \min \left\{ \frac{1}{2} + \frac{s}{2d}, \frac{1}{q} \right\}, \quad (20)$$

the bilinear operator $B(u, v)(t)$ is continuous from $\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}} \times \mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}$ into $\mathcal{K}_{q,1,T}^{s,\tilde{q}}$ and the following inequality holds

$$\|B(u, v)\|_{\mathcal{K}_{q,1,T}^{s,\tilde{q}}} \leq C.T^{\frac{1}{2}(1+s-\frac{d}{q})} \|u\|_{\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}} \|v\|_{\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}}, \quad (21)$$

where C is a positive constant independent of T .

Proof. We have

$$\begin{aligned} \|B(u, v)(t)\|_{\dot{H}_{L^{\tilde{q},1}}^s} &\leq \int_0^t \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{\dot{H}_{L^{\tilde{q},1}}^s} d\tau = \\ &\int_0^t \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{\tilde{q},1}} d\tau. \end{aligned} \quad (22)$$

From the properties of the Fourier transform

$$\begin{aligned} &\left(e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right)_j^\wedge(\xi) = \\ &e^{-(t-\tau)|\xi|^2} \sum_{l,k=1}^d \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (i\xi_l) \left(\dot{\Lambda}^s (u_l(\tau, \cdot) v_k(\tau, \cdot)) \right)^\wedge(\xi), \end{aligned}$$

and then

$$\begin{aligned} & \left(e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s(u(\tau, \cdot) \otimes v(\tau, \cdot)) \right)_j = \\ & \frac{1}{(t-\tau)^{\frac{d+1}{2}}} \sum_{l,k=1}^d K_{l,k,j} \left(\frac{\cdot}{\sqrt{t-\tau}} \right) * \left(\dot{\Lambda}^s(u_l(\tau, \cdot) v_k(\tau, \cdot)) \right), \end{aligned} \quad (23)$$

where

$$\widehat{K_{l,k,j}}(\xi) = \frac{1}{(2\pi)^{d/2}} \cdot e^{-|\xi|^2} \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (i\xi_l).$$

Applying Proposition 11.1 ([31], p. 107) with $|\alpha| = 1$ we see that the tensor $K(x) = \{K_{l,k,j}(x)\}$ satisfies

$$|K(x)| \lesssim \frac{1}{(1+|x|)^{d+1}}. \quad (24)$$

So, we can rewrite the equality (23) in the tensor form

$$\begin{aligned} & e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s(u(\tau, \cdot) \otimes v(\tau, \cdot)) = \\ & \frac{1}{(t-\tau)^{\frac{d+1}{2}}} K \left(\frac{\cdot}{\sqrt{t-\tau}} \right) * \left(\dot{\Lambda}^s(u(\tau, \cdot) \otimes v(\tau, \cdot)) \right). \end{aligned} \quad (25)$$

Set

$$\frac{1}{r} = \frac{2}{\tilde{q}} - \frac{s}{d}, \quad \frac{1}{h} = \frac{s}{d} - \frac{1}{\tilde{q}} + 1. \quad (26)$$

From the inequalities (19) and (20), we can check that the following conditions are satisfied

$$1 < h, r < \infty \text{ and } \frac{1}{\tilde{q}} + 1 = \frac{1}{h} + \frac{1}{r}.$$

Applying Proposition 2.4 (c) in ([31], pp. 20) for convolution in the Lorentz spaces, we have

$$\begin{aligned} & \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s(u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{\tilde{q},1}} \lesssim \\ & \frac{1}{(t-\tau)^{\frac{d+1}{2}}} \left\| K \left(\frac{\cdot}{\sqrt{t-\tau}} \right) \right\|_{L^{h,1}} \left\| \dot{\Lambda}^s(u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{r,\infty}}. \end{aligned} \quad (27)$$

Applying Lemma 4 we obtain

$$\begin{aligned} & \left\| \dot{\Lambda}^s(u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{r,\infty}} \leq \left\| \dot{\Lambda}^s(u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^r} = \|u(\tau, \cdot) \otimes v(\tau, \cdot)\|_{\dot{H}_r^s} \\ & \lesssim \|u(\tau, \cdot)\|_{\dot{H}_q^s} \|v(\tau, \cdot)\|_{\dot{H}_q^s}. \end{aligned} \quad (28)$$

From the inequalities (24) and (26) we infer that

$$\left\| K\left(\frac{\cdot}{\sqrt{t-\tau}}\right) \right\|_{L^{h,1}} = (t-\tau)^{\frac{d}{2h}} \|K\|_{L^{h,1}} \simeq (t-\tau)^{\frac{s}{2} - \frac{d}{2\tilde{q}} + \frac{d}{2}}. \quad (29)$$

From the inequalities (27), (28), and (29) we deduce that

$$\begin{aligned} & \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{\tilde{q},1}} \lesssim \\ & (t-\tau)^{\frac{s}{2} - \frac{d}{2\tilde{q}} - \frac{1}{2}} \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s}. \end{aligned} \quad (30)$$

From the estimates (22) and (30), and note that from the inequalities (19) and (20), we can check that $\frac{s}{2} - \frac{d}{2\tilde{q}} - \frac{1}{2} > -1$ and $\alpha = d(\frac{1}{q} - \frac{1}{\tilde{q}}) < 1$, this gives the desired result

$$\begin{aligned} & \|B(u, v)(t)\|_{\dot{H}_{L^{\tilde{q},1}}^s} \lesssim \int_0^t (t-\tau)^{\frac{s}{2} - \frac{d}{2\tilde{q}} - \frac{1}{2}} \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} d\tau \lesssim \\ & \int_0^t (t-\tau)^{\frac{s}{2} - \frac{d}{2\tilde{q}} - \frac{1}{2}} \tau^{-\alpha} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} d\tau = \\ & \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \int_0^t (t-\tau)^{\frac{s}{2} - \frac{d}{2\tilde{q}} - \frac{1}{2}} \tau^{-\alpha} d\tau \simeq \\ & t^{-\frac{\alpha}{2}} t^{\frac{1}{2}(1+s-\frac{d}{q})} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{L^{\tilde{q},\tilde{q}}}^s} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{L^{\tilde{q},\tilde{q}}}^s}. \end{aligned} \quad (31)$$

Let us now check the validity of the condition (4) for the bilinear term $B(u, v)(t)$. Indeed, we have

$$\lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|B(u, v)(t)\|_{\dot{H}_{L^{\tilde{q},1}}^s} = 0,$$

whenever

$$\lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|u(t, \cdot)\|_{\dot{H}_{\tilde{q}}^s} = \lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|v(t, \cdot)\|_{\dot{H}_{\tilde{q}}^s} = 0.$$

The estimate (21) is now deduced from the inequality (31). \square

Remark 3. In the case $s = 0$ and $q = d$, Lemma 9 is a generalization of Lemma 10 in ([8], p. 196).

Lemma 12. *Let $s, q \in \mathbb{R}$ be such that*

$$s \geq 0, q > 1, \text{ and } \frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}. \quad (32)$$

Then for all \tilde{q} satisfying

$$\frac{1}{2} \left(\frac{1}{q} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} < \min \left\{ \frac{1}{2} + \frac{s}{2d}, \frac{1}{q} \right\}, \quad (33)$$

the bilinear operator $B(u, v)(t)$ is continuous from $\mathcal{K}_{q, \tilde{q}, T}^{s, \tilde{q}} \times \mathcal{K}_{q, \tilde{q}, T}^{s, \tilde{q}}$ into $\mathcal{K}_{q, 1, T}^{s, q}$ and the following inequality holds

$$\|B(u, v)\|_{\mathcal{K}_{q, 1, T}^{s, q}} \leq C.T^{\frac{1}{2}(1+s-\frac{d}{q})} \|u\|_{\mathcal{K}_{q, \tilde{q}, T}^{s, \tilde{q}}} \|v\|_{\mathcal{K}_{q, \tilde{q}, T}^{s, \tilde{q}}}, \quad (34)$$

where C is a positive constant independent of T .

Proof. Set

$$\frac{1}{r} = \frac{2}{\tilde{q}} - \frac{s}{d}, \quad \frac{1}{h} = 1 + \frac{1}{q} - \frac{2}{\tilde{q}} + \frac{s}{d}. \quad (35)$$

From the inequalities (32) and (33), we can check that h and r satisfy

$$1 < h, r < \infty \text{ and } \frac{1}{q} + 1 = \frac{1}{h} + \frac{1}{r}.$$

From the equality (25), applying Proposition 2.4 (c) in ([31], pp. 20), we obtain

$$\begin{aligned} & \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{q, 1}} \lesssim \\ & \frac{1}{(t-\tau)^{\frac{d+1}{2}}} \left\| K \left(\frac{\cdot}{\sqrt{t-\tau}} \right) \right\|_{L^{h, 1}} \left\| \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{r, \infty}}. \end{aligned} \quad (36)$$

Applying Lemma 4, we have

$$\begin{aligned} \left\| \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^{r, \infty}} & \leq \left\| \dot{\Lambda}^s (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{L^r} \\ & \lesssim \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s}. \end{aligned} \quad (37)$$

From the inequalities (24) and (35) it follows that

$$\left\| K \left(\frac{\cdot}{\sqrt{t-\tau}} \right) \right\|_{L^{h, 1}} = (t-\tau)^{\frac{d}{2h}} \|K\|_{L^{h, 1}} \simeq (t-\tau)^{\frac{d}{2} + \frac{d}{2q} - \frac{d}{\tilde{q}} + \frac{s}{2}}. \quad (38)$$

From the estimates (36), (37), (38) we deduce that

$$\begin{aligned} \left\| e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot (u(\tau, \cdot) \otimes v(\tau, \cdot)) \right\|_{\dot{H}_{L^{q, 1}}^s} & \lesssim (t-\tau)^{\frac{d}{2q} - \frac{d}{\tilde{q}} + \frac{s}{2} - \frac{1}{2}} \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \\ & = (t-\tau)^{\alpha + \frac{s}{2} - \frac{d}{2q} - \frac{1}{2}} \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s}. \end{aligned}$$

From the inequalities (32) and (33), we can check that $\alpha + \frac{s}{2} - \frac{d}{2q} - \frac{1}{2} > -1$ and $\alpha = d(\frac{1}{q} - \frac{1}{\tilde{q}}) < 1$, this gives the desired result

$$\begin{aligned}
\|B(u, v)(t)\|_{\dot{H}_{L^q, 1}^s} &\lesssim \int_0^t (t - \tau)^{\alpha + \frac{s}{2} - \frac{d}{2q} - \frac{1}{2}} \|u(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \|v(\tau, \cdot)\|_{\dot{H}_{\tilde{q}}^s} d\tau \lesssim \\
&\int_0^t (t - \tau)^{\alpha + \frac{s}{2} - \frac{d}{2q} - \frac{1}{2}} \tau^{-\alpha} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \cdot \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} d\tau = \\
&\sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \cdot \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{\tilde{q}}^s} \int_0^t (t - \tau)^{\alpha + \frac{s}{2} - \frac{d}{2q} - \frac{1}{2}} \tau^{-\alpha} d\tau \simeq \\
&t^{\frac{1}{2}(1+s-\frac{d}{q})} \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|u(\eta, \cdot)\|_{\dot{H}_{L^{\tilde{q}}, \tilde{q}}^s} \cdot \sup_{0 < \eta < t} \eta^{\frac{\alpha}{2}} \|v(\eta, \cdot)\|_{\dot{H}_{L^{\tilde{q}}, \tilde{q}}^s}. \quad (39)
\end{aligned}$$

Let us now check the validity of the condition (5) for the bilinear term $B(u, v)(t)$. Indeed, we have

$$\lim_{t \rightarrow 0} \|B(u, v)(t)\|_{\dot{H}_{L^q, 1}^s} = 0$$

whenever

$$\lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|u(t, \cdot)\|_{\dot{H}_{\tilde{q}}^s} = \lim_{t \rightarrow 0} t^{\frac{\alpha}{2}} \|v(t, \cdot)\|_{\dot{H}_{\tilde{q}}^s} = 0.$$

The estimate (34) is now deduced from the inequality (39). \square

Combining Theorem 1 with Lemmas 7, 9, 11, 12, we obtain the following existence result.

Theorem 2. *Let s, q , and $r \in \mathbb{R}$ be such that*

$$s \geq 0, q > 1, r \geq 1, \text{ and } \frac{s}{d} < \frac{1}{q} \leq \frac{s+1}{d}. \quad (40)$$

(a) *For all \tilde{q} satisfying*

$$\frac{1}{2} \left(\frac{1}{q} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} < \min \left\{ \frac{1}{2} + \frac{s}{2d}, \frac{1}{q} \right\}, \quad (41)$$

there exists a positive constant $\delta_{s, q, \tilde{q}, d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_{L^q, r}^s(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying

$$T^{\frac{1}{2}(1+s-\frac{d}{q})} \sup_{0 < t < T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \|e^{t\Delta} u_0\|_{\dot{H}_{\tilde{q}}^s} \leq \delta_{s, q, \tilde{q}, d}, \quad (42)$$

NSE has a unique mild solution $u \in \mathcal{K}_{q, 1, T}^{s, \tilde{q}} \cap L^\infty([0, T]; \dot{H}_{L^q, r}^s)$. In particular, for arbitrary $u_0 \in \dot{H}_{L^q, r}^s$ with $\operatorname{div}(u_0) = 0$, there exists $T(u_0)$ small enough

such that the inequality (42) holds.

(b) If $1 < q \leq d$, and $s = \frac{d}{q} - 1$ then for any \tilde{q} be such that

$$\frac{1}{q} - \frac{1}{2d} < \frac{1}{\tilde{q}} < \min\left\{\frac{1}{2} + \frac{1}{2q} - \frac{1}{2d}, \frac{1}{q}\right\},$$

there exists a positive constant $\sigma_{q,\tilde{q},d}$ such that if $\|u_0\|_{\dot{B}_{\tilde{q}}^{\frac{d}{q}-1,\infty}} \leq \sigma_{q,\tilde{q},d}$ and $T = \infty$ then the inequality (42) holds.

Proof. From Lemmas 11 and 8, the bilinear operator $B(u, v)(t)$ is continuous from $\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}} \times \mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}$ into $\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}$ and we have the inequality

$$\|B(u, v)\|_{\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}} \leq \|B(u, v)\|_{\mathcal{K}_{q,1,T}^{s,\tilde{q}}} \leq C_{s,q,\tilde{q},d} T^{\frac{1}{2}(1+s-\frac{d}{q})} \|u\|_{\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}} \|v\|_{\mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}},$$

where $C_{s,q,\tilde{q},d}$ is a positive constant independent of T . From Theorem 1 and the above inequality, we deduce following: for any $u_0 \in \dot{H}_{L^q,r}^s(\mathbb{R}^d)$ such that

$$\operatorname{div}(u_0) = 0, \quad T^{\frac{1}{2}(1+s-\frac{d}{q})} \sup_{0 < t < T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \|e^{t\Delta} u_0\|_{\dot{H}_{\tilde{q}}^s} \leq \frac{1}{4C_{s,q,\tilde{q},d}},$$

NSE has a mild solution u on the interval $(0, T)$ so that

$$u \in \mathcal{K}_{q,\tilde{q},T}^{s,\tilde{q}}. \quad (43)$$

Lemma 12 and the relation (43) imply that

$$B(u, u) \in \mathcal{K}_{q,1,T}^{s,q} \subseteq \mathcal{K}_{q,r,T}^{s,q} \subseteq L^\infty([0, T]; \dot{H}_{L^q,r}^s).$$

On the other hand, from Lemma 7, we have $e^{t\Delta} u_0 \in L^\infty([0, T]; \dot{H}_{L^q,r}^s)$.

Therefore

$$u = e^{t\Delta} u_0 - B(u, u) \in L^\infty([0, T]; \dot{H}_{L^q,r}^s).$$

From Lemma 9 and Lemma 11, we deduce that $u \in \mathcal{K}_{q,1,T}^{s,\tilde{q}}$.

From the definition of $\mathcal{K}_{q,r,T}^{s,\tilde{q}}$ and Lemma 9, we deduce that the left-hand side of the inequality (42) converges to 0 when T tends to 0. Therefore the inequality (42) holds for arbitrary $u_0 \in \dot{H}_{L^q,r}^s(\mathbb{R}^d)$ when $T(u_0)$ is small enough.

(b) From Lemma 6, the two quantities

$$\|u_0\|_{\dot{B}_{\tilde{q}}^{\frac{d}{q}-1,\infty}} \quad \text{and} \quad \sup_{0 < t < \infty} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \|e^{t\Delta} u_0\|_{\dot{H}_{\tilde{q}}^{\frac{d}{q}-1}}$$

are equivalent, then there exists a positive constant $\sigma_{q,\tilde{q},d}$ such that if $\|u_0\|_{\dot{B}_{\tilde{q}}^{\frac{d}{q}-1,\infty}} \leq \sigma_{q,\tilde{q},d}$ and $T = \infty$ then the inequality (42) holds. \square

Remark 4. In the case when the initial data belong to the critical Sobolev-Lorentz spaces $\dot{H}_{L^{q,r}}^{\frac{d}{q}-1}(\mathbb{R}^d)$, ($1 < q \leq d, r \geq 1$), from Theorem 2 (b), we get the existence of global mild solutions in the spaces $L^\infty([0, \infty); \dot{H}_{L^{q,r}}^{\frac{d}{q}-1}(\mathbb{R}^d))$ when the norm of the initial value in the Besov spaces $\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1, \infty}(\mathbb{R}^d)$ is small enough. Note that a function in $\dot{H}_{L^{q,r}}^{\frac{d}{q}-1}(\mathbb{R}^d)$ can be arbitrarily large in the $\dot{H}_{L^{q,r}}^{\frac{d}{q}-1}(\mathbb{R}^d)$ norm but small in the $\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1, \infty}(\mathbb{R}^d)$ norm. This is deduced from the following imbedding maps (see Lemma 9)

$$\dot{H}_{L^{q,r}}^{\frac{d}{q}-1}(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1, \infty}(\mathbb{R}^d), \left(\frac{1}{q} - \frac{1}{d} < \frac{1}{\tilde{q}} < \frac{1}{q} \right).$$

This result is stronger than that of Cannone. In particular, when $q = r = d, s = 0$, we get back the Cannone theorem (Theorem 1.1 in [5]).

Next, we consider the super-critical indexes $s > \frac{d}{q} - 1$.

Theorem 3. *Let*

$$s \geq 0, q > 1, r \geq 1, \text{ and } \frac{s}{d} < \frac{1}{q} < \frac{s+1}{d}.$$

Then for any \tilde{q} be such that

$$\frac{1}{2} \left(\frac{1}{q} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} < \min \left\{ \frac{1}{2} + \frac{s}{2d}, \frac{1}{q} \right\},$$

there exists a positive constant $\delta_{s,q,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying

$$T^{\frac{1}{2}(1+s-\frac{d}{q})} \|u_0\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}), \infty}} \leq \delta_{s,q,\tilde{q},d},$$

NSE has a unique mild solution $u \in \mathcal{K}_{q,1,T}^{s,\tilde{q}} \cap L^\infty([0, T]; \dot{H}_{L^{q,r}}^s)$.

Proof. Applying Lemma 6, the two quantities $\|u_0\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}), \infty}}$ and $\sup_{0 < t < \infty} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \|e^{t\Delta} u_0\|_{\dot{H}_{\tilde{q}}^s}$ are equivalent. Thus

$$\sup_{0 < t < T} t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})} \|e^{t\Delta} u_0\|_{\dot{H}_{\tilde{q}}^s} \lesssim \|u_0\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}), \infty}},$$

the theorem is proved by applying the above inequality and Theorem 2. \square

Remark 5. In the case when the initial data belong to the Sobolev-Lorentz spaces $\dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$, ($q > 1, r \geq 1, s \geq 0$, and $\frac{d}{q} - 1 < s < \frac{d}{q}$), we obtain the existence of mild solutions in the spaces $L^\infty([0, T]; \dot{H}_{L^{q,r}}^s(\mathbb{R}^d))$ for any $T > 0$ when the norm of the initial value in the Besov spaces $\dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}), \infty}(\mathbb{R}^d)$ is small enough. Note that a function in $\dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$ can be arbitrarily large in the $\dot{H}_{L^{q,r}}^s(\mathbb{R}^d)$ norm but small in $\dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}), \infty}(\mathbb{R}^d)$ norm. This is deduced from the following imbedding maps (see Lemma 9)

$$\dot{H}_{L^{q,r}}^s(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{s-(\frac{d}{q}-\frac{d}{\tilde{q}}), \infty}(\mathbb{R}^d), \quad \left(\frac{s}{d} < \frac{1}{\tilde{q}} < \frac{1}{q}\right).$$

Applying Theorem 3 for $q > d, r = q$ and $s = 0$, we get the following proposition which is stronger than the result of Cannone and Meyer ([4], [7]). In particular, we obtained a result that is stronger than that of Cannone and Meyer but under a much weaker condition on the initial data.

Proposition 1. *Let $q > d$. Then for any \tilde{q} be such that*

$$q < \tilde{q} < 2q,$$

there exists a positive constant $\delta_{q,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in L^q(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying

$$T^{\frac{1}{2}(1-\frac{d}{q})} \|u_0\|_{\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q}, \infty}} \leq \delta_{q,\tilde{q},d}, \quad (44)$$

NSE has a unique mild solution $u \in \mathcal{K}_{q,1,T}^{0,\tilde{q}} \cap L^\infty([0, T]; L^q)$.

Remark 6. If in (44) we replace the $\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q}, \infty}$ norm by the L^q norm then we get the assumption made in ([4], [7]). We show that the condition (44) is weaker than the condition in ([4], [7]). In Remark 5 we have showed that

$$L^q(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q}, \infty}(\mathbb{R}^d), \quad (\tilde{q} > q \geq d),$$

but these two spaces are different. Indeed, we have $|x|^{-\frac{d}{q}} \notin L^q(\mathbb{R}^d)$. On the other hand by using Lemma 6, we can easily prove that $|x|^{-\frac{d}{q}} \in \dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-\frac{d}{q}, \infty}(\mathbb{R}^d)$ for all $\tilde{q} > q$.

Applying Theorem 3 for $q = r = 2, \frac{d}{2} - 1 < s < \frac{d}{2}$, we get the following proposition which is stronger than the results of Chemin in [10] and Cannone in [4]. In particular, we obtained the result that is stronger than that of Chemin and Cannone but under a much weaker condition on the initial data.

Proposition 2. Let $\frac{d}{2} - 1 < s < \frac{d}{2}$. Then for any \tilde{q} be such that

$$\frac{1}{2} \left(\frac{1}{2} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} < \frac{1}{2},$$

there exists a positive constant $\delta_{s,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^s(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying

$$T^{\frac{1}{2}(1+s-\frac{d}{2})} \|u_0\|_{\dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty}} \leq \delta_{s,\tilde{q},d}, \quad (45)$$

NSE has a unique mild solution $u \in \mathcal{K}_{2,1,T}^{s,\tilde{q}} \cap L^\infty([0, T]; \dot{H}^s)$.

Remark 7. If in (45) we replace the $\dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty}$ norm by the $\dot{H}^s(\mathbb{R}^d)$ norm then we get the assumption made in ([10], [4]). We show that the condition (45) is weaker than the condition in ([10], [4]). In Remark 5 we showed that

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty}, \quad \frac{1}{2} \left(\frac{1}{2} + \frac{s}{d} \right) < \frac{1}{\tilde{q}} < \frac{1}{2},$$

but that these two spaces are different. Indeed, we have $\dot{\Lambda}^{-s}|\cdot|^{-\frac{d}{2}} \notin \dot{H}^s(\mathbb{R}^d)$, on the other hand by using Lemma 6, we easily prove that $\dot{\Lambda}^{-s}|\cdot|^{-\frac{d}{2}} \in \dot{B}_{\tilde{q}}^{s-(\frac{d}{2}-\frac{d}{\tilde{q}}),\infty}(\mathbb{R}^d)$ for all $\tilde{q} > 2$.

Acknowledgments. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2014.50.

References

- [1] J. Bergh and J. Lofstrom, *Interpolation Spaces*, Springer-Verlag, 1976, 264 pp.
- [2] J. Bourgain and N. Pavlović, *Ill-posedness of the Navier-Stokes equations in a critical space in 3D*, J. Funct. Anal., **255** (9) (2008), 2233-2247.
- [3] B. Jawerth, *Some observations on Besov and Lizorkin-Triebel space*, Math. Scand., **40** (1977), 94-104.
- [4] M. Cannone, *Ondelettes, Paraproducts et Navier-Stokes*, Diderot Editeur, Paris, 1995, 191 p.

- [5] M. Cannone, *A generalization of a theorem by Kato on Navier-Stokes equations*, Rev. Mat. Iberoamericana, **13** (3) (1997), 515-541.
- [6] M. Cannone and F. Planchon, *On the nonstationary Navier-Stokes equations with an external force*, Adv. in Diff. Eq., **4** (5) (1999), 697-730.
- [7] M. Cannone and Y. Meyer, *Littlewood-Paley decomposition and the Navier-Stokes equations*, Meth. and Appl. of Anal., **2** (1995), 307-319.
- [8] M. Cannone, *Harmonic analysis tools for solving the incompressible Navier-Stokes equations*, in: S.J. Friedlander, D. Serre (Eds.), Handbook of Mathematical Fluid Dynamics, Vol. III, Elsevier, Amsterdam, 2004, pp. 161-244
- [9] Y. Meyer, *Wavelets, paraproducts, and Navier-Stokes equations* Wavelets, paraproducts, and Navier-Stokes equations. Current developments in mathematics, 1996 (Cambridge, MA), 105-212, Int. Press, Boston, MA, 1997.
- [10] J. M. Chemin, *Remarques sur l'existence globale pour le système de Navier-Stokes incompressible*, SIAM J. Math. Anal., **23** (1992), 20-28.
- [11] Jean-Yves Chemin , *Le système de Navier-Stokes incompressible soixante dix ans après Jean Leray*, in: Actes des Journées Mathématiques à la Mémoire de Jean Leray, in: Sémin. Congr., vol. 9, Soc. Math. France, Paris, 2004, pp. 99-123.
- [12] E. Fabes, B. Jones and N. Riviere, *The initial value problem for the Navier-Stokes equations with data in L^p* , Arch. Rat. Mech. Anal., **45** (1972), 222-240.
- [13] Hajaiej Hichem, Yu Xinwei, and Zhai Zhichun, *Fractional Gagliardo-Nirenberg and Hardy inequalities under Lorentz norms*, J. Math. Anal. Appl., **396** (2012), 569-577.
- [14] S. Friedlander and D. Serre, Handbook of Mathematical Fluid Dynamics, Volume 3, Elsevier, 2004.
- [15] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem I*, Arch. Rat. Mech. Anal., **16** (1964), 269-315.
- [16] Y. Giga, *Solutions of semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differ. Eq., **62** (1986), 186-212.

- [17] Y. Giga and T. Miyakawa, *Solutions in L^r of the Navier-Stokes initial value problem*, Arch. Rat. Mech. Anal., **89** (1985), 267-281.
- [18] Lars Hörmander, *Linear partial differential operators*, Springer Verlag, Berlin Heidelberg, New York 1976.
- [19] N. Kalton, S. Mayboroda and M. Mitrea, *Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin spaces and applications to problems in partial differential equations. Interpolation theory and applications*, Contemp. Math., **445**, Amer. Math. Soc., Providence, RI, 2007, 121-177.
- [20] T. Kato and H. Fujita, *On the non-stationary Navier-Stokes system*, Rend. Sem. Mat. Univ. Padova, **32** (1962), 243-260.
- [21] T. Kato, *Strong L^p solutions of the Navier-Stokes equations in \mathbb{R}^m with applications to weak solutions*, Math. Zeit., **187** (1984), 471-480.
- [22] T. Kato, *Strong solutions of the Navier-Stokes equations in Morrey spaces*, Bol. Soc. Brasil. Math., **22** (1992), 127-155.
- [23] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math., XLI (1988), 891-907.
- [24] T. Kato and G. Ponce, *The Navier-Stokes equations with weak initial data*, Int. Math. Res. Notes, **10** (1994), 435-444.
- [25] D. Q. Khai and N. M. Tri, *Solutions in mixed-norm Sobolev-Lorentz spaces to the initial value problem for the Navier-Stokes equations*, J. Math. Anal. Appl., **417** (2014), 819-833.
- [26] D. Q. Khai and N. M. Tri, *Well-posedness for the Navier-Stokes equations with datum in Sobolev-Fourier-Lorentz spaces*. To appear in J. Math. Anal. Appl. doi:10.1016/j.jmaa.2016.01.015
- [27] D. Q. Khai and N. M. Tri, *On the Hausdorff dimension of the singular set in time for weak solutions to the nonstationary Navier-Stokes equation on torus*, Vietnam Journal of Mathematics, Volume 43, Issue **2** (2015), 283-295.
- [28] D. Q. Khai and N. M. Tri, *On the initial value problem for the Navier-Stokes equations with the initial datum in critical Sobolev and Besov spaces*. To appear in the Journal of Mathematical Sciences, The University of Tokyo. Preprint arXiv:1601.01726

- [29] Herbert Koch and Daniel Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math., **157** (1) (2001), 22-35.
- [30] H. Kozono, M. Yamazaki, *Local and global unique solvability of the Navier-Stokes exterior problem with Cauchy data in the space $L^{n,+\infty}$* Houston J. Math. **21** (1995), no. 4, 755-799
- [31] P. G. Lemarie-Rieusset, *Recent Developments in the Navier-Stokes Problem*, Chapman and Hall/CRC Research Notes in Mathematics, vol. 431, Chapman and Hall/CRC, Boca Raton, FL, 2002.
- [32] M. E. Taylor, *Analysis on Morrey spaces and applications to Navier-Stokes equations and other evolution equations*, Comm. Partial Differential Equations, **17** (1992), 1407-1456.
- [33] F. B. Weissler, *The Navier-Stokes initial value problem in L^p* , Arch. Rat. Mech. Anal., **74** (1981), 219-230.