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An Algorithm for a Class of Split Feasibility Problems: Application to a Model in Electricity Production

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Abstract We propose a projection algorithm for solving split feasibility problems involving paramonotone equilibria and convex optimization. The proposed algorithm can be considered as a combination of the projection ones for equilibrium and convex optimization problems. We apply the algorithm for finding an equilibrium point with minimal environmental cost for a model in electricity production. Numerical results for the model are reported.

Keywords Split feasibility · equilibria · convex optimization · practical model

1 Introduction and the Problem Statement

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle ., . \rangle$ and its reduced norm $\|.\|$, and K be a nonempty closed convex subset of $\mathcal{H}$. Let $f : K \times K \to \mathbb{R}$ be a bi-function such that $f(x, x) = 0$ for all $x \in K$. We consider the following equilibrium problem, shortly EP,

$$\text{Find } x \in K \text{ such that } f(x, y) \geq 0 \quad \forall y \in K. \tag{EP}$$

The solution set of this problem is denoted by $Sol(EP)$.

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This inequality was first used by Nikaido and Isoda in [22] for noncooperative game. After the publication of the paper by Blum and Oettli [4], Problem (EP) has attracted many attention and a large number of articles on this problem have been published (see. e.g. the interesting survey paper [3] and the references therein).

An interesting feature of Problem (EP) is that, although having a very simple formulation, it gives a unified formulation for some important problems such as optimization problems, saddle point, variational inequalities, Kakutani fixed point and Nash equilibria, in the sense that it includes these problems as particular cases (see for instance [3] and the references cited there).

Some solution methods are proposed for Problem (EP), among them the projection method is commonly used. However for monotone equilibrium problems, the basic projection method may fail to converge. In order to overcome this disadvantage, the extragradient method (double projection) first proposed by Korpelevich [15] for saddle point problem was extended to pseudomonotone EP. Recently an inexact subgradient-projection method has been developed for solving Problem (EP) with paramonotone equilibrium bifunction [25] in finite dimensional Euclidean space. This method uses only one projection at each iteration ensuring convergence. Briefly, at each iteration k, having x^k the next iterate x^{k+1} is defined as an approximate projection of $x_k - \alpha_k g_k$ onto K where g_k is an approximate diagonal subgradient of the convex function $f(x^k, .)$ at x^k. With suitable choice of the stepsize α_k, the sequence of iterates converges to a solution of (EP).

The split feasibility problem in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [8] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [5]. Recently, it has been found that this problem can also be used to model the intensity-modulated radiation therapy [8], [9], and many other fields.

Mathematically, a split feasibility problem in real Hilbert spaces can be state as follows: Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1, Q \subseteq H_2$ be two nonempty convex sets, $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem is defined as

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q. \qquad (SF)$$

Recently, this problem where C and/or Q are the solution-sets of variational inequalities and/or fixed point-sets have been considered in some research papers and several solution algorithms using the metric projection combining with the proximal mapping have been developed see e.g. [16], [17], [18], [19], [20], [26].

In this paper we extend the method in [25] to the split feasibility problem (SF) with C being the solution-set of a paramonotone equilibrium problem in H_1 and Q the solution-set of a mathematical convex program in H_2. Mathematically, the problem can be formulated as

$$\text{Find } x^* \in K : f(x^*, y) \geq 0 \ \forall y \in K \text{ and } g(Ax^*) \leq g(u) \ \forall u \in H_2, \qquad (SEO)$$

where g is a properly lower semicontinuous convex function on H_2. The proposed algorithm is a combination of the projection one in [25] for the equilibrium problem with the Man-Krasnoselskii iterative scheme for the proximal operator defined by the convex optimization problem, which ensures the strong convergence. To illustrate the problem we present an equilibrium model with minimal environmental fee which arises in electricity production. Some computational results are reported to show behaviour of the proposed algorithm.

The paper is organized as follows. The next section are preliminaries. In the third section we present the algorithm and its convergence. We close the paper with a practical model in electricity market and some computational results for the model.

2 Preliminaries

The following lemmas will be used for validity and convergence of the algorithm.

Lemma 1 *For $x, y, z \in \mathcal{H}$ and $0 \leq a \leq 1$, we have*

$$\|ax + (1-a)y - z\|^2 \leq a\|x - z\|^2 + (1-a)\|y - z\|^2. \tag{1}$$

Proof One has

$$\begin{aligned} & \|ax + (1-a)y - z\|^2 \\ &= a^2\|x - z\|^2 + (1-a)^2\|y - z\|^2 + 2a(1-a)\langle x - z, y - z\rangle \\ &= a\|x - z\|^2 + (1-a)\|y - z\|^2 - a(1-a)\left[\|x - z\|^2 + \|y - z\|^2 - 2\langle x - z, y - z\rangle\right] \\ &\leq a\|x - z\|^2 + (1-a)\|y - z\|^2. \end{aligned}$$

Lemma 2 *([2] p. 61). Let C be a nonempty closed convex subset in a Hilbert space $\mathcal{H}$ and $P_C(x)$ be the metric projection of x onto C. Then*

(i) $\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2 \quad \forall x, y \in \mathcal{H}$.
(ii) $\langle x - P_C(x), P_C(x) - y\rangle \geq 0 \quad \forall x \in \mathcal{H}, y \in C$.

Lemma 3 *(see, e.g. [25]) Let $\{v_k\}$ and $\{\delta_k\}$ be nonnegative sequences of real numbers satisfying $v_{k+1} \leq v_k + \delta_k$ withe $\sum_{k=1}^{\infty} \delta_k < +\infty$. Then the sequence $\{v_k\}$ is convergent.*

Lemma 4 *(see, e.g. [1]) Let $\mathcal{H}$ be a real Hilbert space, $\{a_k\}$ be a sequence of real numbers such that $0 < a < a_k < b < 1$ for all $k = 1, 2, \ldots,$ and let $\{v_k\}, \{w_k\}$ be sequences in $\mathcal{H}$ such that*

$$\limsup_{k \to +\infty} \|v_k\| \leq c, \limsup_{k \to +\infty} \|w_k\| \leq c,$$

and

$$\lim_{k \to +\infty} \|a_k v_k + (1 - a_k) w_k\| = c, \quad \text{for some } c > 0.$$

Then, $\lim_{k \to +\infty} \|v_k - w_k\| = 0$.

3 The Algorithm and its Convergence

We need the following assumptions for the algorithm and its convergence will be presented below

Assumptions

(A1) For each $x \in K$, $f(x,x) = 0$ and $f(x,.)$ is lower semicontinuous convex on K.

(A2) $\partial_2^\epsilon f(x,x)$ is nonempty for every $\epsilon > 0$ and $x \in K$ and is bounded on any bounded subset of C.

(A3) f is pseudomonotone on K with respect to every solution of (EP), that is $f(x,x^*) \le 0$ for every $x \in K$, $x^* \in Sol(EP)$, and satisfies the following condition, called para-monotonicity property

$$x^* \in Sol(EP), y \in K, \quad f(x^*,y) = f(y,x^*) = 0 \Rightarrow y \in Sol(EP).$$

(A4) For every $x \in K$, $f(.,x)$ is weakly upper semicontinuous on K.

Comments for these assumptions, especially for (A2) and (A3) can be found in [25].

We recall that the proximal mapping of the convex function g with $\lambda > 0$, denoted by $prox_{\lambda g}$, is defined as the unique solution of the strongly convex programming problem

$$prox_{\lambda g}(u) := \text{argmin}\{g(v) + \frac{1}{\lambda}\|v-u\|^2 : v \in H_2\}. \qquad P(u)$$

For $\lambda > 0$, we set $h(x) := \frac{1}{2}\|(I - prox_{\lambda g})Ax\|^2$. By using the necessary and sufficient optimality condition for convex programming, we can see that $h(x) = 0$ if and only Ax solves P(u) with $u = Ax$. Note that (see [24] page 52), even g may not be differentiable, h is always differentiable and $\nabla h(x) = A^*(I - prox_{\lambda g})Ax$. Hence $h(x) = 0$ if and only if $\nabla h(x) = 0$.

3.1 Algorithm

Algorithm 3.1

Take a positive parameters δ, ξ and real sequences $\{a_k\}, \{\delta_k\}$, $\{\beta_k\}$, $\{\epsilon_k\}$, $\{\rho_k\}$ satisfying the conditions:

$$0 < a < a_k < b < 1, 0 < \xi \le \rho_k \le 4 - \xi, \quad \forall k \in \mathbb{N} \tag{2}$$

$$\delta_k > \delta > 0, \beta_k > 0, \epsilon_k \ge 0, \quad \forall k \in \mathbb{N} \tag{3}$$

$$\lim_{k \to +\infty} a_k = \frac{1}{2} \tag{4}$$

$$\sum_{k=1}^{\infty} \frac{\beta_k}{\delta_k} = +\infty, \sum_{k=1}^{\infty} \beta_k^2 < +\infty, \tag{5}$$

$$\sum_{k=1}^{\infty} \frac{\beta_k \epsilon_k}{\delta_k} < +\infty \tag{6}$$

Step 0: Choose $x_0 \in K$. Set $k = 1$.
Step k: Let $x_k \in K$.
Take $g_k \in \partial_2^{\epsilon_k} f(x_k, x_k)$. Define

$$\alpha_k = \frac{\beta_k}{\gamma_k} \text{ where } \gamma_k = \max\{\delta_k, \|g_k\|\}.$$

Compute $y_k = P_K(x_k - \alpha_k g_k)$, *i.e.*

$$\langle y_k - x_k + \alpha_k g_k, x - y_k \rangle \geq 0 \quad \forall x \in K.$$

Take

$$\mu_k := \begin{cases} 0 \text{ if } \nabla h(y_k) = 0, \\ \rho_k \frac{h(y_k)}{\|\nabla h(y_k)\|^2} \text{ if } \nabla h(y_k) \neq 0 \end{cases} \tag{7}$$

and compute

$$z_k = P_K(y_k - \mu_k A^*(I - prox_{\lambda g})(Ay_k)).$$

Let

$$x_{k+1} = a_k x_k + (1 - a_k) z_k.$$

Remark 1 Note that when $g \equiv 0$, the problem (SEO) becomes the problem (EP). In this case the algorithm is reduced to the projection Mann-Krasnoselskii scheme for (EP).

Remark 2 If we choose $\epsilon_k = 0$, then $x_k = y_k$ and $h(x_k) = 0$ imply that x_k is a solution. Motivated by this fact we call x_k an ϵ-solution if $\epsilon_k \leq \epsilon$ and $\|x_k - y_k\| \leq \epsilon$, $|h(x_k)| \leq \epsilon$.

Theorem 1 *Suppose that Problem (SEO) admits a solution. Then under Assumptions (A1)-(A4) the sequence (x_k) generated by Algorithm 3.1 strongly converges to a solution of (SEO).*

We need the following lemmas to the proof of the convergence of the proposed algorithm.

Lemma 5 *([19]) Let S be the set of solutions of the Problem (SEO) and $z \in S$. If $\nabla h(y_k) \neq 0$ then it holds that*

$$\|z_k - z\|^2 \leq \|y_k - z\|^2 - \rho_k(4 - \rho_k) \frac{h^2(y_k)}{\|\nabla h(y_k)\|^2}. \tag{8}$$

Lemma 6 *([25]) For every k, the following inequalities hold*

(i) $\alpha_k \|g_k\| \leq \beta_k$;
(ii) $\|y_k - x_k\| \leq \beta_k$.

Lemma 7 *Let $z \in S$. Then, for every k such that $\nabla h(y_k) \neq 0$, we have*

$$\|x_{k+1}-z\|^2 \leq \|x_k-z\|^2-(1-a_k)\rho_k(4-\rho_k)\frac{h^2(y_k)}{\|\nabla h(y_k)\|^2}+2(1-a_k)\alpha_k f(x_k,z)+A_k, \tag{9}$$

and for every k such that $\nabla h(y_k) = 0$, we have

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 + 2(1 - a_k)\alpha_k f(x_k, z) + A_k, \tag{10}$$

where $A_k = 2(1 - a_k)(\alpha_k \epsilon_k + \beta_k^2)$.

Proof By definition of x_{k+1}, in virtue of Lemma 1, we have

$$\begin{aligned} \|x_{k+1} - z\|^2 &= \|a_k x_k + (1 - a_k)z_k - z\|^2 \\ &\leq a_k\|x_k - z\|^2 + (1 - a_k)\|z_k - z\|^2. \end{aligned} \tag{11}$$

We consider two cases:
Case 1: If $\nabla h(y_k) \neq 0$, then thanks to Lemma 5, we have

$$\|x_{k+1} - z\|^2 \leq a_k\|x_k - z\|^2 + (1 - a_k)\left[\|y_k - z\|^2 - \rho_k(4 - \rho_k)\frac{h^2(y_k)}{\|\nabla h(y_k)\|^2}\right]. \tag{12}$$

Moreover,

$$\begin{aligned} \|y_k - z\|^2 &= \|z - x_k + x_k - y_k\|^2 \\ &= \|x_k - z\|^2 - \|x_k - y_k\|^2 + 2\langle x_k - y_k, z - y_k\rangle \\ &\leq \|x_k - z\|^2 + 2\langle x_k - y_k, z - y_k\rangle. \end{aligned}$$

In Algorithm 3.1, since y_k is chosen such that

$$\langle y_k - x_k + \alpha_k g_k, x - y_k\rangle \geq 0 \quad \forall x \in K,$$

by taking $x = z$, we obtain

$$\begin{aligned} &\langle y_k - x_k + \alpha_k g_k, z - y_k\rangle \geq 0 \\ \Leftrightarrow &\langle \alpha_k g_k, z - y_k\rangle \geq \langle x_k - y_k, z - y_k\rangle. \end{aligned}$$

Hence,

$$\begin{aligned} \|y_k - z\|^2 &\leq \|x_k - z\|^2 + 2\langle \alpha_k g_k, z - y_k\rangle \\ &= \|x_k - z\|^2 + 2\langle \alpha_k g_k, z - x_k\rangle + 2\langle \alpha_k g_k, x_k - y_k\rangle. \end{aligned} \tag{13}$$

It follows from $g_k \in \partial_2^{\epsilon_k} f(x_k, x_k)$ that

$$\begin{aligned} &f(x_k, z) - f(x_k, x_k) \geq \langle g_k, z - x_k\rangle - \epsilon_k \\ \Leftrightarrow &f(x_k, z) + \epsilon_k \geq \langle g_k, z - x_k\rangle. \end{aligned} \tag{14}$$

On the other hand, from Lemma 6 it holds that

$$\langle \alpha_k g_k, x_k - y_k\rangle \leq \alpha_k\|g_k\|\|x_k - y_k\| \leq \beta_k^2.$$

From (13), (14) and $\alpha_k > 0$ follows

$$\|y_k - z\|^2 \leq \|x_k - z\|^2 + 2\alpha_k f(x_k, z) + 2\alpha_k \epsilon_k + 2\beta_k^2. \tag{15}$$

Combining this inequality with (12), we obtain

$$\|x_{k+1}-z\|^2 \leq \|x_k-z\|^2+2(1-a_k)\alpha_k f(x_k,z)-(1-a_k)\rho_k(4-\rho_k)\frac{h^2(y_k)}{\|\nabla h(y_k)\|^2}+A_k,$$

where $A_k = 2(1 - a_k)(\alpha_k \epsilon_k + \beta_k^2)$.
Case 2: If $\nabla h(y_k) = 0$ then, by definition of x_{k+1}, we can write

$$\|x_{k+1} - z\|^2 \leq a_k\|x_k - z\|^2 + (1 - a_k)\|y_k - z\|^2.$$

Now, by the same argument as in Case 1, we have

$$\|y_k - z\|^2 \leq \|x_k - z\|^2 + 2\alpha_k f(x_k, z) + 2\alpha_k \epsilon_k + 2\beta_k^2.$$

Then,

$$\|x_{k+1} - z\|^2 \leq \|x_k - z||^2 + 2(1 - a_k)\alpha_k f(x_k, z) + A_k,$$

where $A_k = 2(1 - a_k)(\alpha_k \epsilon_k + \beta_k^2)$.

Proof. (of the theorem). *Claim 1*: $\{\|x_k - z\|^2\}$ is convergent for all $z \in S$.
Indeed, let $z \in S$. Since $z \in Sol(EP)$ and f is pseudomonotone on K with respect to every solution of (EP), we have

$$f(x_k, z) \leq 0.$$

If $\nabla h(y_k) \neq 0$, then, since

$$\rho_k(4 - \rho_k)\frac{h^2(y_k)}{\|\nabla h(y_k)\|^2} \geq 0,$$

it follows from Lemma 7 that

$$\|x_{k+1} - z\|^2 \leq \|x_k - z||^2 + A_k, \tag{16}$$

where $A_k = 2(1 - a_k)(\alpha_k \epsilon_k + \beta_k^2)$.
Since $\alpha_k = \frac{\beta_k}{\gamma_k}$ with $\gamma_k = \max\{\delta_k, \|g_k\|\}$,

$$\sum_{k=1}^{+\infty} \alpha_k \epsilon_k = \sum_{k=1}^{+\infty} \frac{\beta_k}{\gamma_k}\epsilon_k \leq \sum_{k=1}^{+\infty} \frac{\beta_k}{\delta_k}\epsilon_k < +\infty.$$

Note that $\sum_{k=1}^{+\infty} \beta_k^2 < +\infty$ and $0 < a < a_k < b < 1$, we have

$$\sum_{k=1}^{+\infty} A_k < 2(1 - a)\sum_{k=1}^{+\infty}(\alpha_k \epsilon_k + \beta_k^2) < +\infty.$$

Now using Lemma 6, we see that $\{\|x_k - z\|^2\}$ is convergent for all $z \in S$. Hence, $\{x_k\}$ is bounded. Then, again by Lemma 6, we can see that $\{y_k\}$ is bounded too.

Claim 2: $\limsup_{k\to+\infty} f(x_k, z) = 0$ for every $z \in S$.

By Lemma 7, for every k, we have

$$-2(1 - a_k)\alpha_k f(x_k, z) \leq \|x_k - z\|^2 - \|x_{k+1} - z\|^2 + A_k. \tag{17}$$

Summing up we obtain

$$\sum_{k=1}^{\infty} -2(1 - a_k)\alpha_k f(x_k, z) < +\infty. \tag{18}$$

On the other hand, using Assumption (A2) and the fact that $\{x_k\}$ is bounded, we see that $\{\|g_k\|\}$ is bounded. Then, there exists $L > \delta$ such that $\|g_k\| \leq L$ for every k. Therefore,

$$\frac{\gamma_k}{\delta_k} = \max\{1, \frac{\|g_k\|}{\delta_k}\} \leq \frac{L}{\delta}.$$

Hence,

$$\alpha_k = \frac{\beta_k}{\gamma_k} \geq \frac{\delta}{L}\frac{\beta_k}{\delta_k}.$$

Since z is a solution, by pseudomonotonicity of f, we have $-f(x_k, z) \geq 0$ which together with $0 < a < a_k < b < 1$ implies

$$\sum_{k=1}^{\infty}(1 - b)\frac{\beta_k}{\delta_k}[-f(x_k, z)] < +\infty.$$

But from $\sum_{k=1}^{\infty} \frac{\beta_k}{\delta_k} = +\infty$, it holds that

$$\limsup_{k\to+\infty} f(x_k, z) = 0, \ \forall z \in S.$$

Claim 3: For any $z \in S$, suppose that $\{x_{k_j}\}$ is the subsequence of $\{x_k\}$ such that

$$\limsup_{k\to+\infty} f(x_k, z) = \lim_{j\to+\infty} f(x_{k_j}, z), \tag{19}$$

and that x^* is a weakly cluster point of $\{x_{k_j}\}$. Then x^* belongs to $Sol(EP)$.

Without loss of generality, we can assume that x_{k_j} weakly converges to x^* as $j \to \infty$. Since $f(., z)$ is upper semicontinuous, by Claim 2, we have

$$f(x^*, z) \geq \limsup_{j\to+\infty} f(x_{k_j}, z) = 0.$$

Since $z \in S$ and f is pseudomonotone, we have $f(x^*, z) \leq 0$. Thus $f(x^*, z) = 0$. Again by pseudomotonicity $f(z, x^*) \leq 0$. Hence $f(x^*, z) = f(z, x^*) = 0$. Then by paramononotonicity (Assumption A3), we can conclude that x^* is also a solution of (EP).

Claim 4: Every weak cluster point $\overline{x}$ of the sequence $\{x_k\}$ satisfies $\overline{x} \in K$ and $A\overline{x} \in argming$.

Let $\overline{x}$ be a weak cluster point of $\{x_k\}$ and $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ weakly converging to $\overline{x}$. Then $\overline{x} \in K$. On the other hand, we know that $\|y_k - x_k\| \leq \beta_k$ and $\sum_{k=1}^{\infty} \beta_k^2 < +\infty$. Hence,

$$\lim_{k\to\infty} \|y_k - x_k\| = 0.$$

Thus, $\{y_{k_j}\}$ weakly converges to $\overline{x}$.

From Lemma 7, if $\nabla h(y_k) \neq 0$ then

$$(1-a_k)\rho_k(4-\rho_k)\frac{h^2(y_k)}{\|\nabla h(y_k)\|^2} \leq \|x_k - z\|^2 - \|x_{k+1} - z\|^2 + A_k,$$

and if $\nabla h(y_k) = 0$ then

$$0 \leq |x_k - z||^2 - \|x_{k+1} - z\|^2 + A_k.$$

Let $N_1 := \{k : \nabla h(y_k) \neq 0\}$ and summing up altogether we can write

$$\sum_{k\in N_1} (1-a_k)\rho_k(4-\rho_k)\frac{h^2(y_k)}{\|\nabla h(y_k)\|^2} \leq \|x_0 - z\|^2 + \sum_{k=1}^{\infty} A_k < +\infty.$$

Combining this fact with the assumption $\xi \leq \rho_k \leq 4-\xi$ (for some $\xi > 0$) and $0 < a < a_k < b < 1$, we can conclude that

$$\sum_{k\in N_1} \frac{h^2(y_k)}{\|\nabla h(y_k)\|^2} < +\infty. \tag{20}$$

Moreover, since ∇h is Lipschitz continuous with constant $\|A\|^2$, we see that $\|\nabla h(y_k)\|^2$ is bounded. Note that $h(y_k) = 0$ for $k \notin N_1$. Consequently,

$$\lim_{k\to+\infty} h(y_k) = 0. \tag{21}$$

By the lower-semicontinuity of h,

$$0 \leq h(\overline{x}) \leq \liminf_{j\to+\infty} h(y_{k_j}) = \lim_{k\to+\infty} h(y_k) = 0, \tag{22}$$

which implies that $A\overline{x}$ is a fixed point of the proximal mapping of g. Thus, $A\overline{x}$ is a minimizer of g.

Claim 5: $\lim_{k\to+\infty} x_k = \lim_{k\to+\infty} y_k = \lim_{k\to+\infty} P(x_k) = x^*$, where x^* is a weakly cluster point of the sequence satisfying (19).

From Claim 3 and Claim 4, we can deduce that x^* belongs to S. By Claim 1 we can assume that

$$\lim_{k\to+\infty} \|x_k - x^*\| = c < +\infty.$$

By Lemma 6, we have

$$\begin{aligned}\|z_k - x^*\| &\leq \|y_k - x^*\| \\ &\leq \|x_k - x^*\| + \|y_k - x_k\| \\ &\leq \|x_k - x^*\| + \beta_k,\end{aligned}$$

which implies that

$$\limsup_{k\to+\infty} \|z_k - x^*\| \le \limsup_{k\to+\infty}(\|x_k - x^*\| + \beta_k) = c.$$

On the other hand,

$$\lim_{k\to+\infty} \|a_k(x_k - x^*) + (1-a_k)(z_k - x^*)\| = \lim_{k\to+\infty} \|x_{k+1} - x^*\| = c.$$

By applying Lemma 4 with $v_k := x_k - x^*, w_k := z_k - x^*$, we obtain

$$\lim_{k\to+\infty} \|z_k - x_k\| = 0. \tag{23}$$

Combining this with the fact that x^* is a weak cluster point of the sequence $\{x_k\}$, we see that x^* is also a weak cluster point of the sequence $\{z_k\}$. Suppose $\{z_{k_j}\}$ weakly converge to x^*. Note that

$$\begin{aligned}\|x_{k_j+1} - P_S(x_{k_j+1})\|^2 &\le \|x_{k_j+1} - P_S(x_{k_j})\|^2 \\ &\le a_k\|x_{k_j} - P_S(x_{k_j})\|^2 + (1-a_k)\|z_{k_j} - P_S(x_{k_j})\|^2.\end{aligned}$$

On the other hand,

$$\|z_{k_j} - P_S(x_{k_j})\|^2 = \|z_{k_j} - x_{k_j}\|^2 - \|x_{k_j} - P_S(x_{k_j})\|^2 - 2\langle z_{k_j} - P_S(x_{k_j}), P_S(x_{k_j}) - x_{k_j}\rangle.$$

Hence,

$$\begin{aligned}&\|x_{k_j+1} - P_S(x_{k_j+1})\|^2 \\ &\le (2a_{k_j} - 1)\|x_{k_j} - P_S(x_{k_j})\|^2 + (1-a_{k_j})\|z_{k_j} - x_{k_j}\|^2 - 2(1-a_{k_j})\langle z_{k_j} - P_S(x_{k_j}), P_S(x_{k_j}) - x_{k_j}\rangle. \\ &\le (2a_{k_j} - 1)\|x_{k_j} - P_S(x_{k_j})\|^2 + (1-a_{k_j})\|z_{k_j} - x_{k_j}\|^2 - 2(1-a_{k_j})\langle z_{k_j} - x^*, P_S(x_{k_j}) - x_{k_j}\rangle \\ &\quad -2(1-a_{k_j})\langle x^* - P_S(x_{k_j}), P_S(x_{k_j}) - x_{k_j}\rangle.\end{aligned}$$

Since $x^* \in S$,

$$\langle x^* - P_S(x_{k_j}), P_S(x_{k_j}) - x_{k_j}\rangle \ge 0.$$

The sequence $\{x_{k_j}\}$ is bounded, $\{x_{k_j} - P_S(x_{k_j})\}$ is bounded too. Using this with the facts that $\lim_{j\to+\infty} \|z_{k_j} - x_{k_j}\| = 0$ and $\lim_{j\to+\infty} a_{k_j} = \frac{1}{2}$. we can deduce that

$$\lim_{j\to\infty} \|x_{k_j+1} - P_S(x_{k_j+1})\| = 0. \tag{24}$$

Now, we show that $\{P_S(x_{k_j})\}$ is a Cauchy sequence. Indeed, for all $m > j$, we have

$$\begin{aligned}&\|P_S(x_{k_m}) - P_S(x_{k_j})\|^2 \\ &= 2\|x_{k_m} - P_S(x_{k_m})\|^2 + 2\|x_{k_m} - P_S(x_{k_j})\|^2 - 4\|x_{k_m} - \frac{1}{2}(P_S(x_{k_{m+1}}) + P_S(x_{k_{j+1}}))\|^2 \\ &\le 2\|x_{k_m} - P_S(x_{k_m})\|^2 + 2\|x_{k_m} - P_S(x_{k_j})\|^2 - 4\|x_{k_m} - P_S(x_{k_m})\|^2 \\ &= 2\|x_{k_m} - P_S(x_{k_j})\|^2 - 2\|x_{k_m} - P_S(x_{k_m})\|^2.\end{aligned} \tag{25}$$

Then, applying Lemma 3.6 with $z = P_S(x_{k_j})$ successively we obtain

$$\begin{aligned} \|x_{k_m} - P_S(x_{k_j})\|^2 &\le \|x_{k_m-1} - P_S(x_{k_j})\|^2 + A_{k_m-1} \\ &\le \dots \\ &\le \|x_{k_j} - P_S(x_{k_j})\|^2 + \sum_{i=k_j}^{k_m-1} A_i. \end{aligned} \tag{26}$$

From (15) and (25), it follows that

$$\|P_S(x_{k_m}) - P_S(x_{k_j})\|^2 \le 2\|x_{k_j} - P_S(x_{k_j})\|^2 + 2\sum_{i=k_j}^{k_m-1} A_i - 2\|x_{k_m} - P_S(x_{k_m})\|^2.$$

From (24) and the fact that $\lim_{j\to\infty} \sum_{i=k_j}^{k_m-1} A_i = 0$, we can conclude that $\{P_S(x_{k_j})\}$ is a Cauchy sequence. Hence, $\{P_S(x_{k_j})\}$ strongly converges to some point $x \in S$. Since $\lim_{j\to\infty} \|x_{k_j+1} - P_S(x_{k_j+1})\| = 0$, we see that $\{x_{k_j}\}$ also strongly converges to x. Finally, using Claim 1, we can conclude that

$$\lim_{k\to+\infty} x_k = \lim_{k\to+\infty} y_k = \lim_{k\to+\infty} P(x_k) = x^*$$

□

3.2 A Practical Model and Computational Results

In this section, we consider an equilibrium-optimization model which can be regarded as an extension of a Nash-Cournot oligopolistic equilibrium model in electricity markets. The latter model has been investigated in some research papers (see e.g. [13,23]). In this equilibrium model, it is assumed that there are n companies, each company i may possess I_i generating units. Let x denote the the vector whose entry x_j stands for the the power generating by unit j. Following [13] we suppose that the price $p_i(s)$ is a decreasing affine function of s with $s = \sum_{j=1}^N x_j$ where N is the number of all generating units, that is $p_i(s) = \alpha - \beta_i s$. Then the profit made by company i is given by $f_i(x) = p_i(s)(\sum_{j\in I_i} x_j) - \sum_{j\in I_i} c_j(x_j)$, where $c_j(x_j)$ is the cost for generating x_j by generating unit j. Suppose that K_i is the strategy set of company i, that is the condition $\sum_{j\in I_i} x_j \in K_i$ must be satisfied for evey i. Then the strategy set of the model is $K := K_1 \times K_2 ... \times K_n$.

Actually, each company seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other companies are parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept.

We recall that a point $x^* \in K = K_1 \times K_2 \times \cdots \times K_n$ is an equilibrium point of the model if

$$f_i(x^*) \ge f_i(x^*[x_i]) \ \forall x_i \in K_i, \ \forall i = 1, 2, \dots, n,$$

where the vector $x^*[x_i]$ stands for the vector obtained from x^* by replacing x_i^* with x_i. By taking

$$f(x,y) := \psi(x,y) - \psi(x,x)$$

with

$$\psi(x,y) := -\sum_{i=1}^{n} f_i(x[y_i]), \tag{27}$$

the problem of finding a Nash equilibrium point of the model can be formulated as

$$x^* \in K : f(x^*,x) \geq 0 \ \forall x \in K. \qquad (EP)$$

We extend this equilibrium model by additionally assuming that to produce electricity the generating units use some materials. Let $a_{l,j}$ denote the quantity of material l ($l = 1,...,m$) for producing one unit electricity by the generating unit j ($j = 1,...,N$). Let A be the matrix whose entries are $a_{l,j}$. Then the entry l of the vector Ax is the quantity of material l for producing x. Using materials for production may cause pollution to environment for which companies have to pay environmental fee. Suppose that $g(Ax)$ is the total environmental fee for producing x. The task now is to find a production x^* such that it is a Nash equilibrium point with minimum environmental fee. This problem can be formulated as a split feasibility problem of the form

$$\text{Find } x^* \in K : f(x^*,x) \geq 0 \ \forall x \in K, \ g(Ax^*) \leq g(Ax) \ \forall x \in K. \qquad (SEP)$$

As usual, we suppose that, for every j, the cost c_j for production and the environmental fee g are increasingly convex functions. The convexity assumption here means that both the cost and fee for producing a unit production increases as the quantity of the production gets larger.

Under this convexity assumption, it is not hard to see (see also [23]) that Problem (EP) with f is given by (27) can be formulated as

$$\text{Find } x^* \in K : \langle \tilde{B_1}x^* - \bar{a}, x - x^* \rangle + \varphi(x) - \varphi(x^*) \geq 0 \ \forall x \in K, \tag{28}$$

where

$$\bar{a} := (\alpha, \alpha, ..., \alpha)^T$$

$$B_1 := \begin{pmatrix} \beta_1 & 0 & 0 & ... & 0 \\ 0 & \beta_2 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & \beta_n \end{pmatrix}, \ \tilde{B_1} := \begin{pmatrix} 0 & \beta_1 & \beta_1 & ... & \beta_1 \\ \beta_2 & 0 & \beta_2 & ... & \beta_2 \\ ... & ... & ... & ... & ... \\ \beta_n & \beta_n & \beta_n & ... & 0 \end{pmatrix},$$

$$\varphi(x) := x^T B_1 x + \sum_{j=1}^{N} c_j(x_j).$$

Note that when c_j is differentiable convex for every j, then Problem (28) is equivalent to the variational inequality

$$\text{Find } x^* \in K : \langle \tilde{B}_1 x^* - \bar{a} + \nabla\varphi(x^*), x - x^* \rangle \geq 0, \ \forall x \in K. \tag{29}$$

We tested the proposed algorithm with the cost function given by

$$c_j(x_j) = \frac{1}{2} p_j x_j^2 + q_j x_j, \ p_j > 0.$$

For this cost function, by using Proposition 3.2 in [14], one can check that the cost operator in the problem (29) is paramonotone.

We took the sequences of the parameters as

$$\beta_k = \frac{7}{2(k+1)}, \epsilon_k = 0, \delta_k = 3, \gamma_k = \max\{3, ||g_k||\} \ \ \forall k$$

and solved the model with different sizes, ten problems for each size.

The algorithm was coded in Matlab 7.8 on a 8Gb Ram core i7. The main subproblems were solved with the MATLAB Optimization Toolbox by using QUADPROG function for the positive semidefinite quadratic function $g(u) := 1/2u^T Du + d^T u$. The computational results are shown in Table 1. The horizontal and veritcal axes show the average iteration k, average CPU-times, and $error1 := ||x - y||$, $error2 := h(x)$, respectively. The parameters β_j, p_j, q_j, for all $j = 1, \ldots, n$, were generated randomly in the interval (0,1], [1,3], [1,3] respectively, while the entries of the matricies A, D and vector d were generated in the interval [-2,30].

Table 1 Algorithm 3.1

size N	Prob.	Iter	CPU-times(s)	Error 1	Error 2
6	10	1654	51.13	$9.9996.10^{-5}$	$1.1.10^{-6}$
10	10	19793	622.09	$9.7011.10^{-5}$	$7.8.10^{-4}$
20	10	25690	1282.10	$9.9801.10^{-5}$	0.0565
30	10	32001	1059.12	$9.9504.10^{-5}$	0.3283
50	10	67344	3213.47	$9.8034.10^{-5}$	2.9610
100	10	72469	3729.56	$9.8624.10^{-5}$	50.6554

Conclusion. We have proposed an algorithm for solving the split feasibility problems involving equilibria and optimization in Hilbert spaces. Strong convergence of the algorithm have been proved. A Nash-Cournot equilibrium model with minimal environmental cost in electricity production has been solved by the proposed algorithm. Some computational results have been reported to show efficiency of the algorithm.

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