GLOBAL MIXED ŁOJASIEWICZ INEQUALITIES AND ASYMPTOTIC CRITICAL VALUES

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Abstract. In this paper, we prove a version of global Łojasiewicz inequality for C^1 semialgebraic functions and relate its existence to the set of asymptotic critical values.

1. Introduction

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a C^1 semialgebraic function and let $\mathrm{dist}(\cdot,\cdot)$ be the distance function. For each $x \in \mathbb{R}^n$, set $\mathrm{dist}(x,\emptyset) := 1$. Denote by $E(f)$ the set of $t \in \mathbb{R}$ for which there are no positive constants c, α and β such that the following global Łojasiewicz inequality holds:

$$|f(x) - t|^{\alpha} + |f(x) - t|^{\beta} \geq c\,\mathrm{dist}(x, \{f = t\}) \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{1}$$

Let

$$K_\infty(f) := \{y \in \mathbb{R} \ \mid \ \text{there exists a sequence } x^k \in \mathbb{R}^n \text{ such that } \|x^k\| \to +\infty,\ f(x^k) \to y, \ \ \|x^k\|\|\nabla f(x^k)\| \to 0\}$$

and

$$\widetilde{K}_\infty(f) := \{y \in \mathbb{R} \ \mid \ \text{there exists a sequence } x^k \in \mathbb{R}^n \text{ such that } \|x^k\| \to +\infty,\ f(x^k) \to y, \ \ \|\nabla f(x^k)\| \to 0\}.$$

Remark 1.1. (i) The set $K_\infty(f)$ is finite, but $\widetilde{K}_\infty(f)$ is not. See, for example, [5, 7, 11].

(ii) By [3, Theorems 2 and 3], it follows that $E(f) \subset \widetilde{K}_\infty(f)$.

(iii) The set $E(f)$ may be infinite; for example let $f(x,y) := \frac{x}{y^2+1}$, then $E(f) = \mathbb{R}$ and so $E(f) \neq K_\infty(f)$.

(iv) Suppose that f is a polynomial. If $n = 2$, we have $E(f) \subset \widetilde{K}_\infty(f) \subset \widetilde{K}_\infty(f_{\mathbb{C}}) = K_\infty(f_{\mathbb{C}})$ (see [5]), where $f_{\mathbb{C}}$ is the complexification of f. Then $E(f)$ is finite. If $n > 2$, it maybe happen that $E(f)$ is infinite (see, for instance, [11, Example 1.11]).

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In this paper, we propose a version of Łojasiewicz inequality by changing slightly the left side of (1) such that the new inequality still holds for all but a finite number of values t. The existence of the new inequality is also related to the set of asymptotic critical values. In fact, we will prove the following result.

Theorem 1.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a C^1 semialgebraic function. Assume that $t \notin K_\infty(f)$. Then there exist some constants $\alpha > 0$ and $c > 0$ such that*

$$|f(x) - t|^\alpha + \|x\||f(x) - t| \geq c\,\mathrm{dist}(x, V_t) \quad \textit{for all} \quad x \in \mathbb{R}^n. \tag{2}$$

2. Proof of the main result

Without loss of generality, we may suppose that $t = 0$ and from now on, we write V instead of V_0.

First of all, assume that $V = \emptyset$. In this situation, it holds that

$$\inf_{x \in \mathbb{R}^n} [f(x)]^2 > 0.$$

In fact, if it is not the case, then we can see that

$$\lim_{\tau \to +\infty} \min_{\|x\|^2 = \tau^2} [f(x)]^2 = 0.$$

Consequently, there exists an analytic curve $(R, +\infty) \to \mathbb{R}^n \times \mathbb{R}, \tau \mapsto (\varphi(\tau), \mu(\tau))$, such that

(a) $f(\varphi(\tau))\nabla f(\varphi(\tau)) = \mu(\tau)\varphi(\tau)$;
(b) $\|\varphi(\tau)\| = \tau$; and
(c) $\lim_{\tau \to +\infty} f(\varphi(\tau)) = 0$.

Note that $f(\varphi(\tau)) \neq 0$ for all $\tau \geq R$, so we can define $\lambda(\tau) := \frac{\mu(\tau)}{f(\varphi(\tau))}$. Furthermore, we can write

$$f(\varphi(\tau)) \;=\; c\tau^\nu + \text{ lower order terms in } \tau,$$

where $c \neq 0$ and $\nu < 0$. We have

$$\begin{aligned} \frac{d(f \circ \varphi)(\tau)}{d\tau} &= \left\langle \nabla f(\varphi(\tau)), \frac{d\varphi(\tau)}{d\tau} \right\rangle \\ &= \lambda(\tau) \left\langle \varphi(\tau), \frac{d\varphi(\tau)}{d\tau} \right\rangle. \end{aligned}$$

(The second equality follows from Condition (a).) Hence,

$$2\frac{d(f \circ \varphi)(\tau)}{d\tau} = \lambda(\tau)\frac{d\|\varphi(\tau)\|^2}{d\tau} = 2\lambda(\tau)\tau = 2\frac{\mu(\tau)}{f(\varphi(\tau))}\tau.$$

This, together with Conditions (a) and (b), implies that

$$\left| \frac{d(f \circ \varphi)(\tau)}{d\tau} \right| = \|\nabla f(\varphi(\tau))\|.$$

It follows that

$$\|\nabla f(\varphi(\tau))\|\|\varphi(\tau)\| = c\nu\tau^{\nu} + \text{ higher order terms in } \tau.$$

This combined with Condition (c) and $\nu < 0$ yields $0 \in K_\infty(f)$, which contradicts to our assumption. Therefore,

$$\inf_{x\in\mathbb{R}^n} [f(x)]^2 > 0.$$

Consequently, there exists $\delta > 0$ such that $|f|^{-1}(s) = \emptyset$ for $s \le \delta$, then for all $x \in \mathbb{R}^n$, we have $|f(x)| \ge \delta$. We prove that for all $\alpha > 0$, there exists $c = c(\alpha) > 0$ such that (2) holds. Indeed, for $\|x\| \le 1$, we have $|f(x)|^\alpha \ge \delta^\alpha = \delta^\alpha \mathrm{dist}(x, V)$ and for $\|x\| \ge 1$, we have $\|x\||f(x)| \ge \delta = \delta\, \mathrm{dist}(x, V)$. Hence (2) holds for $c = \min\{\delta^\alpha, \delta\}$.

Now we assume that $V \neq \emptyset$. We list the following facts :

(d) Since $0 \notin K_\infty(f)$, there exist $c_0 > 0$, $\delta > 0$, and $R > 0$ such that

$$\|x\|\|\nabla f(x)\| \ge c_0 \ \text{ for } \ \|x\| \ge R \ \text{ and } \ |f(x)| \le \delta. \tag{3}$$

Without loss of generality, we may assume that $\delta < \frac{c_0}{3}$ and $R \ge \mathrm{dist}(0, V)$.

(e) By [8, 9], there exist constants $\alpha > 0$ and $c_1 > 0$ such that

$$|f(x)|^\alpha \ge c_1\, \mathrm{dist}(x, V) \ \text{ for } \ \|x\| \le 2R. \tag{4}$$

(f) For each $x \in \mathbb{R}^n$ such that $|f(x)| \ge \delta$ and $\|x\| \ge 2R$, we have

$$\begin{aligned}\|x\||f(x)| &\ge \delta\|x\| = \frac{2\delta}{3}\frac{3}{2}\|x\| \ge \frac{2\delta}{3}\big(\|x\| + R\big)\\ &\ge \frac{2\delta}{3}\big(\|x\| + \mathrm{dist}(0, V)\big) \ge \frac{2\delta}{3}\, \mathrm{dist}(x, V).\end{aligned}$$

Now we consider the remain case where $\|x\| \ge 2R$ and $|f(x)| \le \delta$. Assume that we have proved:

$$\|x\||f(x)| \ge \frac{2c_0}{5}\, \mathrm{dist}(x, V). \tag{5}$$

Of course, this, together with (4), completes the proof of Theorem 1.1.

So we are left with proving (5). By contradiction, assume that there exists x^0 such that $\|x^0\| \ge 2R$, $|f(x^0)| \le \delta$ and

$$\|x^0\||f(x^0)| < \frac{2c_0}{5}\, \mathrm{dist}(x^0, V). \tag{6}$$

It is clear that $f(x^0) \neq 0$ so we have $0 = \min_{x\in\mathbb{R}^n} |f(x)| < |f(x^0)|$. We consider two cases:

Case 1: $\operatorname{dist}(x^0, V) \le \frac{\|x^0\|}{2}$.

By Ekeland variational principle (see [4]) with the data $\epsilon := |f(x^0)|$ and $\lambda := \frac{\operatorname{dist}(x^0,V)}{2}$, there exists y^0 such that

$$|f(y^0)| \le |f(x^0)| \le \delta, \tag{7}$$

$$\|x^0 - y^0\| \le \lambda = \frac{\operatorname{dist}(x^0, V)}{2} \le \frac{\|x^0\|}{4}, \tag{8}$$

$$|f(x)| + \frac{\epsilon}{\lambda}\|x - y^0\| \ge |f(y^0)| \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{9}$$

From (8), it follows that $\|x^0 - y^0\| \le \lambda < \operatorname{dist}(x^0, V)$, and so $y^0 \notin V$ and $f(y^0) \neq 0$. Without loss of generality, we may assume that $f(y^0) > 0$, then $f(x) > 0$ for all x close enough from y^0. Now (9) implies that y^0 is a local minimum of $f(x)+\frac{\epsilon}{\lambda}\|x-y^0\|$. Consequently $0 \in \nabla f(y^0) + \frac{\epsilon}{\lambda}\mathbb{B}^n$, where $\mathbb{B}^n$ denotes the unit closed ball in $\mathbb{R}^n$. Hence by (6), we have

$$\|\nabla f(y^0)\| \le \frac{\epsilon}{\lambda} = \frac{2|f(x^0)|}{\operatorname{dist}(x^0, V)} < \frac{4c_0}{5\|x^0\|}.$$

Hence

$$\|x^0\|\|\nabla f(y^0)\| < \frac{4c_0}{5}.$$

By (8), we have

$$\|y^0\| \le \|x^0\| + \|x^0 - y^0\| \le \|x^0\| + \lambda \le \frac{5\|x^0\|}{4}.$$

Consequently

$$\|y^0\|\|\nabla f(y^0)\| < c_0. \tag{10}$$

Note that, by (8),

$$\|y^0\| \ge \|x^0\| - \|x^0 - y^0\| \ge \|x^0\| - \lambda \ge \frac{3\|x^0\|}{4} > R$$

and $|f(y^0)| \le \delta$ by (7). So (10) contradicts to (3).

Case 2: $\operatorname{dist}(x^0, V) > \frac{\|x^0\|}{2}$.

By Ekeland variational principle (see [4]) with the data $\epsilon := |f(x^0)|$ and $\lambda := \frac{\|x^0\|}{2}$, there exists y^0 such that

$$|f(y^0)| \le |f(x^0)| \le \delta, \tag{11}$$

$$\|x^0 - y^0\| \le \lambda = \frac{\|x^0\|}{2}, \tag{12}$$

$$|f(x)| + \frac{\epsilon}{\lambda}\|x - y^0\| \ge |f(y^0)| \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{13}$$

Similarly to Case 1, we have

$$\|\nabla f(y^0)\| \le \frac{\epsilon}{\lambda} = \frac{2|f(x^0)|}{\|x^0\|},$$

which implies that

$$\|x^0\|\|\nabla f(y^0)\| \leq 2|f(x^0)| \leq 2\delta.$$

By (12), we have

$$\|y^0\| \leq \|x^0\| + \|x^0 - y^0\| \leq \|x^0\| + \lambda = \frac{3\|x^0\|}{2}.$$

Therefore

$$\|y^0\|\|\nabla f(y^0)\| \leq 3\delta < c_0. \tag{14}$$

Note that, by (12),

$$\|y^0\| \geq \|x^0\| - \|x^0 - y^0\| \geq \|x^0\| - \lambda = \frac{\|x^0\|}{2} \geq R$$

and $|f(y^0)| \leq \delta$ by (11). So (14) contradicts to (3).

3. Some remarks

(i) For the class of C^0 semialgebraic functions, by replacing the gradient norm $\|\nabla f\|$ by the nonsmooth slope $\mathfrak{m}_f$ (see e.g., [10, 12]), Theorem 2 still holds with the same proof. Note that by a Sard theorem for tame set-valued mappings with closed graphs ([6]), the set of asymptotic critical values of f is still finite.

(ii) If f is a polynomial of degree d in n variables, by [1], the exponent α can be made explicit by $\alpha := \frac{1}{\mathscr{R}(n,d)}$ where $\mathscr{R}(n,d) := d(3d-3)^{n-1}$ if $d > 1$ and $\mathscr{R}(n,d) := 1$ if $d = 1$.

(iii) The converse of Theorem 1.1 does not always hold, i.e., Inequality (2) may hold even if $t \in K_\infty(f)$ as we see in the following example. Consider the Broughton polynomial (see [2])

$$f(x,y) := x(xy-1) = x^2y - x.$$

We have three cases:

(a) $|x| \leq 1$ and $|y| \leq 1$. Then by Item (i), there exists a constant $c_1 > 0$ such that

$$|f(x,y)|^{\frac{1}{18}} \geq c_1 \operatorname{dist}\big((x,y), V\big).$$

(b) $|x| \geq 1$. We have $\|(x,y)\| \geq 1$ and $y = \frac{f(x,y)+x}{x^2}$, so

$$\operatorname{dist}\big((x,y), V\big) < \left|\frac{f(x,y)+x}{x^2} - \frac{1}{x}\right| = \left|\frac{f(x,y)}{x^2}\right| \leq \|(x,y)\||f(x,y)|.$$

(c) $|y| \geq 1$. We have $\|(x,y)\| \geq 1$ and by solving the equation $f(x,y) = x^2y - x$ with x as variable, we get $x = \frac{1 \pm \sqrt{1+4yf(x,y)}}{2y}$, so

$$\begin{aligned}
\operatorname{dist}\big((x,y),V\big) &\leq \max\Big\{\Big|\frac{1+\sqrt{1+4yf(x,y)}}{2y} - \frac{1}{y}\Big|, \Big|\frac{1-\sqrt{1+4yf(x,y)}}{2y}\Big|\Big\} \\
&= \Big|\frac{1-\sqrt{1+4yf(x,y)}}{2y}\Big| \\
&= \Big|\frac{1-(1+4yf(x,y))}{2y\big(1+\sqrt{1+4yf(x,y)}\big)}\Big| \\
&= \Big|\frac{2f(x,y)}{\big(1+\sqrt{1+4yf(x,y)}\big)}\Big| \leq 2\|(x,y)\||f(x,y)|.
\end{aligned}$$

We have finally

$$|f(x,y)|^{\frac{1}{18}} + \|(x,y)\||f(x,y)| \geq \min\{c_1, \frac{1}{2}\} \operatorname{dist}\big((x,y),V\big).$$

(iv) The statement of Theorem 1.1 does not always hold if we replace $K_\infty(f)$ by $B_\infty(f)$, where $B_\infty(f)$ is the set of bifurcation values of f. Indeed, let

$$f(X) = f(x,y,z) := z\Big(x^4 + (xy-1)^2\Big).$$

It is clear that f is a trivial fibration over $\mathbb{R}$ so $B_\infty(f) = \emptyset$. Consider the following parameterized curve $s \mapsto X(s) = \big(\frac{1}{s}, s, s\big)$, $s \gg 1$. We have $\nabla f(x,y,z) = \Big(z(4x^3 + 2y(xy-1), 2xz(xy-1), x^4 + (xy-1)^2)\Big)$, so

$$\|X(s)\|\|\nabla f(X(s))\| = \|\big(\frac{1}{s}, s, s\big)\|\|\big(\frac{4}{s^2}, 0, \frac{1}{s^4}\big)\| \sim s.\frac{1}{s^2} = \frac{1}{s} \to 0 \quad \text{as} \quad s \to \infty.$$

Moreover $f(X(s)) = \frac{1}{s^3} \to 0$, so $0 \in K_\infty(f)$. On the other side, since

$$\|X(s)\||f(X(s))| \sim s\frac{1}{s^3} = \frac{1}{s^2} \to 0 \text{ and } \operatorname{dist}(X(s),V) = s \to \infty,$$

there are no constants α, c such that

$$|f(X)|^\alpha + \|X\||f(X)| \geq c \operatorname{dist}(X,V) \quad \text{for all} \quad X \in \mathbb{R}^3.$$

(v) We can not put an exponent $\beta < 1$ on $\|x\|$ in Inequality (2) as we see as follows. Let

$$f(x,y) := \frac{x}{\sqrt{y^2+1}},$$

and $t := 2$. We have $\nabla f(x,y) = \big(\frac{1}{\sqrt{y^2+1}}, \frac{-xy}{(y^2+1)^{\frac{3}{2}}}\big)$. So $\|(x,y)\|\|\nabla f(x,y)\| \geq \frac{\sqrt{x^2+y^2}}{\sqrt{1+y^2}}$. Hence $\|(x,y)\|\|\nabla f(x,y)\| \to 0$ if and only if $(x,y) \to (0,0)$. Consequently $K_\infty(f) = \emptyset$. Consider the following parameterized curve $s \mapsto X(s) = \big(\sqrt{s^2+1}, s\big)$, $s \gg 1$. It is clear that $X(s) \in f^{-1}(1)$ for all s. Let $\mathbb{B}(X(s), \frac{s}{4})$ be the closed ball of radius $\frac{s}{4}$

centered at $X(s)$ and let $B(X(s), \frac{s}{4}) := \{(x, y) \ : \ |x - X(s)| \leq \frac{s}{4}, \ |y - Y(s)| \leq \frac{s}{4}\}$. Set $g(s) := \max_{(x,y) \in \mathbb{B}(X(s), \frac{s}{4})} f(x, y)))$. Then

$$
\begin{aligned}
g(s) \leq \max_{(x,y) \in B(X(s), \frac{s}{4})} f(x, y) &= \frac{\sqrt{s^2 + 1} + \frac{s}{4}}{\sqrt{(s - \frac{s}{4})^2 + 1}} \\
&= \frac{\sqrt{s^2 + 1} + \frac{s}{4}}{\sqrt{\frac{9}{16} s^2 + 1}} \\
&= \frac{\sqrt{1 + \frac{1}{s^2}} + \frac{1}{4}}{\frac{3}{4}\sqrt{1 + \frac{16}{9s^2}}} \to \frac{5}{3} < t.
\end{aligned}
$$

Consequently, for s big enough, the ball $\mathbb{B}(X(s), \frac{s}{4})$ does not intersect the fiber $f^{-1}(t)$. Hence $\frac{s}{4} \leq \operatorname{dist}(X(s), V_t)$. So

$$
\|X(s)\||f(X(s)) - t| = \|X(s)\| = \sqrt{s^2 + 1 + s^2} < 2s \leq 8 \operatorname{dist}(X(s), V_t).
$$

Therefore Inequality (2) does not hold any longer if we put an exponent $\beta < 1$ on $\|x\|$.

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