The existence and space-time decay rates of strong solutions to Navier-Stokes Equations in weighed $L^\infty(|x|^\gamma \mathrm{dx}) \cap L^\infty(|x|^\beta \mathrm{dx})$ spaces

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Abstract: In this paper, we prove some results on the existence and space-time decay rates of global strong solutions of the Cauchy problem for the Navier-Stokes equations in weighed $L^\infty(\mathbb{R}^d, |x|^\gamma \mathrm{dx}) \cap L^\infty(\mathbb{R}^d, |x|^\beta \mathrm{dx})$ spaces.

§1. Introduction

This paper studies the Cauchy problem of the incompressible Navier-Stokes equations (NSE) in the whole space $\mathbb{R}^d$ for $d \geq 2$,

$$\begin{cases} \partial_t u = \Delta u - \nabla.(u \otimes u) - \nabla p, \\ \nabla.u = 0, \\ u(0, x) = u_0, \end{cases}$$

which is a condensed writing for

$$\begin{cases} 1 \leq k \leq d, \quad \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l (u_l u_k) - \partial_k p, \\ \sum_{l=1}^d \partial_l u_l = 0, \\ 1 \leq k \leq d, \quad u_k(0, x) = u_{0k}. \end{cases}$$

The unknown quantities are the velocity $u(t, x) = (u_1(t, x), \ldots, u_d(t, x))$ of the fluid element at time t and position x and the pressure $p(t, x)$.
There is an extensive literature on the existence and decay rate of strong solutions of the Cauchy problem for NSE. Maria E. Schonbek [1] established the decay of the homogeneous H^m norms for solutions to NSE in two dimensions. She showed that if u is a solution to NSE with an arbitrary $u_0 \in H^m \cap L^1(\mathbb{R}^2)$ with $m \geq 3$ then

$$\|D^\alpha u\|_2^2 \leq C_\alpha (t+1)^{-(|\alpha|+1)} \text{ and } \|D^\alpha u\|_\infty \leq C_\alpha (t+1)^{-(|\alpha|+\frac{1}{2})} \text{ for all } t \geq 1, \alpha \leq m.$$

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Zhi-Min Chen [2] showed that if $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), (d \leq p < \infty)$ and $\|u_0\|_1 + \|u_0\|_p$ is small enough then there is a unique solution $u \in BC([0,\infty); L^1 \cap L^p)$, which satisfies decay property

$$\sup_{t>0} t^{\frac{d}{2}} \left(\|u\|_\infty + t^{\frac{1}{2}} \|Du\|_\infty + t^{\frac{1}{2}} \|D^2 u\|_\infty \right) < \infty.$$

Kato [3] studied strong solutions in the spaces $L^q(\mathbb{R}^d)$ by applying the $L^q - L^p$ estimates for the semigroup generated by the Stokes operator. He showed that there is $T > 0$ and a unique solution u, which satisfies

$$t^{\frac{1}{2}(1-\frac{d}{q})} u \in BC([0,T); L^q), \text{ for } d \leq q \leq \infty,$$
$$t^{\frac{1}{2}(2-\frac{d}{q})} \nabla u \in BC([0,T); L^q), \text{ for } d \leq q \leq \infty,$$

as $u_0 \in L^d(\mathbb{R}^d)$. He showed that $T = \infty$ if $\left\|u_0\right\|_{L^d(\mathbb{R}^d)}$ is small enough.
In 2002, Cheng He and Ling Hsiao [4] extended the results of Kato, they estimated on decay rates of higher order derivatives about time variable and space variables for the strong solution to NSE with initial data in $L^d(\mathbb{R}^d)$. They showed that if $\left\|u_0\right\|_{L^d(\mathbb{R}^d)}$ is small enough then there is a unique solution u, which satisfies

$$t^{\frac{1}{2}(1+|\alpha|+2\alpha_0-\frac{d}{q})} D_x^\alpha D_t^{\alpha_0} u \in BC([0,\infty); L^q), \text{ for } q \geq d$$
$$t^{\frac{1}{2}(2+|\alpha|-\frac{d}{q})} D_x^\alpha p \in BC([0,\infty); L^q), \text{ for } q \geq d,$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d), |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d$ and $\alpha_0 \in \mathbb{N}$. D_x^α denotes $\partial_x^{|\alpha|} = \partial^{|\alpha|}/(\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_d}^{\alpha_d}),\ \partial_t^{\alpha_0} = \partial^{\alpha_0}/\partial t^{\alpha_0}$.
In 2005, Okihiro Sawada [5] obtained the decay rate of solution to NSE with initial data in $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$. He showed that every mild solution in the class

$$u \in BC([0,T); \dot{H}^{\frac{d}{2}-1}) \text{ and } t^{\frac{1}{2}(\frac{d}{2}-\frac{d}{p})} u \in BC([0,T); \dot{H}_q^{\frac{d}{2}-1}),$$

for some $T > 0$ and $p \in (2, \infty]$ satisfies

$$\|u(t)\|_{\dot{H}_q^{\frac{d}{2}-1+\alpha}} \leq K_1 (K_2 \tilde{\alpha})^{\tilde{\alpha}} t^{-\frac{\tilde{\alpha}}{2}} \text{ for } \alpha > 0, \tilde{\alpha} := \alpha + \frac{d}{2} - \frac{d}{q}$$

where constants K_1 and K_2 depend only on d, p, M_1, and M_2 with $M_1 = \sup\limits_{0<t<T} \|u(t)\|_{\dot{H}^{\frac{d}{2}-1}}$ and $M_2 = \sup\limits_{0<t<T} t^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})} \|u(t)\|_{\dot{H}_p^{\frac{d}{2}-1}}$.
The time-decay properties are therefore well understood. However, there are few results on the spatial decay properties. Farwing and Sohr [7] showed a class of weighted $|x|^\alpha$ weak solutions with second derivatives in space

variables and one order derivatives in time variable in $L^s([0,+\infty);L^q)$ for $1<q<3=2, 1<s<2$ and $0\le 3/q+2/s-4\le\alpha<\min\{1/2,3-3/q\}$ in the case of exterior domains. In [10], they also showed that there exists a class of weak solutions satisfying

$$\||x|^{\frac{\alpha}{2}}u\|_2^2+\int_0^t \||x|^{\frac{\alpha}{2}}\nabla u\|_2^2\mathrm{d}t\le\begin{cases} C(u_0,f,\alpha) & \text{if } 0\le\alpha<\frac{1}{2},\\ C(u_0,f,\alpha',\alpha)t^{\frac{\alpha'}{2}-1/4} & \text{if } \frac{1}{2}\le\alpha<\alpha'<1,\\ C(u_0,f)(t^{1/4}+t^{1/2}) & \text{if } \alpha=1.\end{cases}$$

While in [11], a class of weak solutions

$$(1+|x|^2)^{1/4}u\in L^\infty([0,+\infty);L^p(\mathbb{R}^3))$$

was constructed for $6/5\le p<3/2$, which satisfies

$$\||x|^{\frac{1}{2}}u\|_2^2+\int_0^t \||x|^{\frac{1}{2}}\nabla u\|_2^2\mathrm{d}t\le C(u_0,f)(t^{1/4}+t^{1/2}).$$

In 2002 Takahashi [9] studied the existence and space-time decay rates of global strong solutions of the Cauchy problem for the Navier-Stokes equations in the weighted $L^\infty(\mathbb{R}^d,(1+|x|)^\beta\mathrm{dx})$ spaces. Takahashi showed that if u_0 satisfies

$$|e^{t\Delta}u_0(x)|<\delta(1+|x|)^{-\beta}, |e^{t\Delta}u_0(x)|<\delta(1+t)^{-\frac{\beta}{2}}, \tag{1}$$

with sufficiently small δ, then NSE has a global mild solution u such that

$$|u(x,t)|\le C(1+|x|)^{-\beta}, |u(x,t)|\le(1+t)^{-\frac{\beta}{2}},$$

where β is restricted by the condition $1\le\beta\le d+1$.
Takahashi also showed that if

$$u_0(x)\le c(1+|x|)^{-\beta}\qquad \text{for some } 0<\beta\le d,$$

then

$$e^{t\Delta}u_0(x)\le c(1+|x|)^{-\beta},\quad e^{t\Delta}u_0(x)\le c(1+t)^{\frac{-\beta}{2}}.$$

In this paper, we discuss the existence and space-time decay rates of global strong solutions of the Cauchy problem for the Navier-Stokes equations in the weighted $L^\infty(\mathbb{R}^d,|x|^\gamma\mathrm{dx})\cap L^\infty(\mathbb{R}^d,|x|^\beta\mathrm{dx})$ spaces. The spaces $L^\infty(\mathbb{R}^d,|x|^\gamma\mathrm{dx})\cap L^\infty(\mathbb{R}^d,|x|^\beta\mathrm{dx})$ are more general than the spaces $L^\infty(\mathbb{R}^d,(1+|x|)^\beta\mathrm{dx})$. In particular, $L^\infty(\mathbb{R}^d,|x|^\gamma\mathrm{dx})\cap L^\infty(\mathbb{R}^d,|x|^\beta\mathrm{dx})=L^\infty(\mathbb{R}^d,(1+|x|)^\beta\mathrm{dx})$ when $\gamma=0$, and so this result improves the previous one.
The content of this paper is as follows: in Section 2, we state our main theorem after introducing some notations. In Section 3, we first prove the some

estimates concerning the heat semigroup with the Helmholtz-Leray projection and some auxiliary lemmas. Finally, in Section 4, we will give the proof of the main theorems.

§2. Statement of the results

Now, for $T > 0$, we say that u is a mild solution of NSE on $[0, T]$ corresponding to a divergence-free initial datum u_0 when u solves the integral equation

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau,.)\otimes u(\tau,.)\big)\mathrm{d}\tau.$$

Above we have used the following notation: for a tensor $F = (F_{ij})$ we define the vector $\nabla.F$ by $(\nabla.F)_i = \sum_{j=1}^d \partial_j F_{ij}$ and for two vectors u and v, we define their tensor product $(u\otimes v)_{ij} = u_i v_j$. The operator $\mathbb{P}$ is the Helmholtz-Leray projection onto the divergence-free fields

$$(\mathbb{P}f)_j = f_j + \sum_{1\le k\le d} R_j R_k f_k,$$

where R_j is the Riesz transforms defined as

$$R_j = \frac{\partial_j}{\sqrt{-\Delta}} \quad \text{i.e.} \quad \widehat{R_j g}(\xi) = \frac{i\xi_j}{|\xi|}\hat{g}(\xi).$$

The heat kernel $e^{t\Delta}$ is defined as

$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2}e^{-|.|^2/4t} * u)(x).$$

For a space of functions defined on $\mathbb{R}^d$, say $E(\mathbb{R}^d)$, we will abbreviate it as E. We define the space $L^\infty(|x|^\beta \mathrm{dx}) := L^\infty(\mathbb{R}^d, |x|^\beta \mathrm{dx})$ which is made up by the measurable functions u such that

$$\|u\|_{L^\infty(|x|^\alpha \mathrm{dx})} := \operatorname*{esssup}_{x\in\mathbb{R}^d} |x|^\alpha |u(x)| < +\infty.$$

Now we can state our result

Theorem 1. *Let $0 \le \gamma \le 1 \le \beta < d$ be fixed, then for all $\tilde{\gamma}, \alpha$, and $\tilde{\beta}$ satisfying*

$$0 \le \tilde{\gamma} \le \gamma, \tilde{\beta} \ge 0, \beta - 2 < \tilde{\beta} \le \beta, 0 < \alpha < 1, \text{ and } \beta - \tilde{\beta} - 1 < \alpha < d - \tilde{\beta},$$

there exists a positive constant $\delta_{\gamma,\tilde{\gamma},\alpha,\beta,\tilde{\beta},d}$ such that for all $u_0 \in L^\infty(|x|^\gamma \mathrm{dx}) \cap L^\infty(|x|^\beta \mathrm{dx})$ with $\mathrm{div}(u_0) = 0$ *satisfying*

$$\sup_{x\in\mathbb{R}^d, t>0} (|x|^{\tilde{\gamma}} t^{\frac{1}{2}(\gamma-\tilde{\gamma})} + |x|^\alpha t^{\frac{1}{2}(1-\alpha)} + |x|^{\tilde{\beta}} t^{\frac{1}{2}(\beta-\tilde{\beta})})|e^{t\Delta}u_0|) \le \delta_{\gamma,\tilde{\gamma},\alpha,\beta,\tilde{\beta},d}, \tag{2}$$

NSE has a global mild solution u on $(0,\infty)\times\mathbb{R}^d$ such that

$$\sup_{x\in\mathbb{R}^d, t>0}(|x|^\gamma + t^{\frac{\gamma}{2}} + |x|^\beta + t^{\frac{\beta}{2}})|u(x,t)|) < +\infty. \tag{3}$$

Remark 1. Our result improves the previous results for $L^\infty(\mathbb{R}^d,(1+|x|)^\beta \mathrm{dx})$. This space, studied in [9], is a particular case of the space $L^\infty(|x|^\gamma \mathrm{d}x)\cap L^\infty(|x|^\beta \mathrm{d}x)$ when $\gamma = 0$. Furthermore, we prove that Takahashi's result holds true under a much weaker condition on the initial data. Indeed, from Lemma 4, it is easily seen that the condition (2) of Theorem 1 is weaker than the condition (1).

Theorem 2. *Assume that $d\ge 1$, and $0\le\gamma\le 1\le\beta<d$. Then for all $f\in L^\infty(|x|^\gamma \mathrm{dx})\cap L^\infty(|x|^\beta \mathrm{dx})$ we have*

$$\sup_{x\in\mathbb{R}^d, t>0}(|x|^{\tilde\gamma}t^{\frac{1}{2}(\gamma-\tilde\gamma)} + |x|^\alpha t^{\frac{1}{2}(1-\alpha)} + |x|^{\tilde\beta}t^{\frac{1}{2}(\beta-\tilde\beta)})|e^{t\Delta}f|) \le$$
$$C(\|f\|_{L^\infty(|x|^\gamma \mathrm{dx})} + \|f\|_{L^\infty(|x|^\beta \mathrm{dx})})$$

for $0\le\tilde\gamma\le\gamma, 0\le\alpha\le 1$ and $0\le\tilde\beta\le\beta$.

Remark 2. We invoke Theorem 2 to deduce that if $u_0\in L^\infty(|x|^\gamma \mathrm{dx})\cap L^\infty(|x|^\beta \mathrm{dx})$ and $\|u_0\|_{L^\infty(|x|^\gamma \mathrm{dx})} + \|u_0\|_{L^\infty(|x|^\beta \mathrm{dx})}$ is small enough then the condition (2) of Theorem 1 is valid.

§3. Some auxiliary results

In this section we establish some auxiliary lemmas. We first prove a version of Young's inequality type for convolutions in $L^\infty(|x|^\beta \mathrm{dx})$ spaces.

Lemma 1. *Assume that $d\ge 1, 0<\alpha<d, 0<\beta<d$ and $\alpha+\beta>d$. Then for all $f\in L^\infty(|x|^\alpha \mathrm{dx})$ and for all $g\in L^\infty(|x|^\beta \mathrm{dx})$ we have*

$$\|f*g\|_{L^\infty(|x|^{\alpha+\beta-d}\mathrm{dx})} \le C\|f\|_{L^\infty(|x|^\alpha \mathrm{dx})}\|g\|_{L^\infty(|x|^\beta \mathrm{dx})}.$$

Proof. Since $f*g$ is bilinear on $L^\infty(|x|^\alpha \mathrm{dx})\times L^\infty(|x|^\beta \mathrm{dx})$, we may assume $\|f\|_{L^\infty(|x|^\alpha \mathrm{dx})} = \|g\|_{L^\infty(|x|^\beta \mathrm{dx})} = 1$. We have

$$f*g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy = \int_{|y|<\frac{|x|}{2}} + \int_{\{|y|>\frac{|x|}{2}\}\cap\{|y|<\frac{3|x|}{2}\}} + \int_{|y|>\frac{3|x|}{2}}$$
$$= I_1 + I_2 + I_3.$$

From

$$|f(x)| \le |x|^{-\alpha}, \text{ and } |g(x)| \le |x|^{-\beta},$$

we get

$$|I_1| \le \int_{|y|<\frac{|x|}{2}} \frac{\mathrm{d}y}{|x-y|^\alpha |y|^\beta} \lesssim \frac{1}{|x|^\alpha} \int_{|y|<\frac{|x|}{2}} \frac{\mathrm{d}y}{|y|^\beta} \simeq \frac{1}{|x|^{\alpha+\beta-d}}.$$

$$|I_2| \le \int_{\{|y|>\frac{|x|}{2}\}\cap\{|y|<\frac{3|x|}{2}\}} \frac{\mathrm{d}y}{|x-y|^\alpha |y|^\beta} \lesssim \frac{1}{|x|^\beta} \int_{|y|<\frac{5|x|}{2}} \frac{\mathrm{d}y}{|y|^\alpha} \simeq \frac{1}{|x|^{\alpha+\beta-d}}.$$

$$|I_3| \le \int_{|y|>\frac{3|x|}{2}} \frac{\mathrm{d}y}{|x-y|^\alpha |y|^\beta} \lesssim \int_{|y|>\frac{3|x|}{2}} \frac{\mathrm{d}y}{|y|^\alpha |y|^\beta} \simeq \frac{1}{|x|^{\alpha+\beta-d}}.$$

We thus obtain

$$|f*g(x)| \lesssim \frac{1}{|x|^{\alpha+\beta-d}}.$$

The proof Lemma 1 is complete. □

We now deduce the $L^\infty(|x|^\gamma \mathrm{d}x)-L^\infty(|x|^\beta \mathrm{d}x)$ estimate for the heat semigroup.

Lemma 2. *Assume that $d \ge 1$ and $0 \le \gamma \le \beta < d$. Then for all $f \in L^\infty(|x|^\beta \mathrm{d}x)$ we have*

$$\left\|e^{t\Delta} f\right\|_{L^\infty(|x|^\gamma \mathrm{d}x)} \le C t^{-\frac{1}{2}(\beta-\gamma)} \|f\|_{L^\infty(|x|^\beta \mathrm{d}x)}, \text{ for } t>0. \tag{4}$$

Proof. We have

$$e^{t\Delta} f(x) = \int_{\mathbb{R}^d} \frac{1}{t^{d/2}} E\left(\frac{x-y}{\sqrt{t}}\right) f(y) \mathrm{d}y, \text{ where } E(x) = (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}.$$

Recall the simate

$$t^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}} \le C|x|^{-\alpha} t^{\frac{1}{2}(d-\alpha)}, \text{ for } 0 \le \alpha \le d. \tag{5}$$

We first consider the case $0 < \gamma < \beta$. From (5) and Lemma 1, it follows that

$$|e^{t\Delta} f(x)| \lesssim \int_{\mathbb{R}^d} \frac{\|f\|_{L^\infty(|x|^\beta \mathrm{d}x)}}{t^{\frac{1}{2}(\beta-\gamma)} |x-y|^{\gamma+d-\beta} |y|^\beta} \mathrm{d}y \lesssim t^{-\frac{1}{2}(\beta-\gamma)} |x|^{-\gamma} \|f\|_{L^\infty(|x|^\beta \mathrm{d}x)}.$$

This proves (4).

We consider the case $0 = \gamma < \beta$. Applying Proposition 2.4 (*b*) in ([6], pp. 20) and note that $|x|^{-\beta} \in L^{\frac{d}{\beta},\infty}$

$$|e^{t\Delta} f(x)| \lesssim t^{-\frac{d}{2}} \left\|E\left(\frac{\cdot}{\sqrt{t}}\right)\right\|_{L^{\frac{d}{d-\beta},1}} \|f\|_{L^{\frac{d}{\beta},\infty}} \lesssim \left\|E\right\|_{L^{\frac{d}{d-\beta},1}} t^{-\frac{\beta}{2}} \|f\|_{L^\infty(|x|^\beta \mathrm{d}x)}.$$

This proves (4).
Suppose finally that $0 \leq \gamma = \beta$. From

$$\int_{\mathbb{R}^d} \frac{1}{t^{d/2}} E(\frac{x-y}{\sqrt{t}}) f(y) \mathrm{d}y = \int_{|y|<\frac{|x|}{2}} + \int_{|y|>\frac{|x|}{2}} = I_1 + I_2,$$

we get

$$|I_1| \leq \|f\|_{L^\infty(|x|^\gamma \mathrm{dx})} \int_{|y|<\frac{|x|}{2}} |x-y|^{-d} |y|^{-\gamma} \mathrm{d}y \leq$$

$$\|f\|_{L^\infty(|x|^\gamma \mathrm{dx})} \big(\frac{|x|}{2}\big)^{-d} \int_{|y|<\frac{|x|}{2}} |y|^{-\gamma} \mathrm{d}y \simeq \|f\|_{L^\infty(|x|^\gamma \mathrm{dx})} |x|^{-\gamma}.$$

$$|I_2| \leq \|f\|_{L^\infty(|x|^\gamma \mathrm{dx})} \int_{|y|>\frac{|x|}{2}} \frac{1}{t^{d/2}} E(\frac{x-y}{\sqrt{t}}) |y|^{-\gamma} \mathrm{d}y \leq$$

$$\|f\|_{L^\infty(|x|^\gamma \mathrm{dx})} \big(\frac{|x|}{2}\big)^{-\gamma} \int_{y\in\mathbb{R}^d} \frac{1}{t^{d/2}} E(\frac{y}{\sqrt{t}}) \mathrm{d}y = C \|f\|_{L^\infty(|x|^\gamma \mathrm{dx})} |x|^{-\gamma},$$

where

$$C = 2^\gamma \int_{y\in\mathbb{R}^d} E(y) \mathrm{d}y < +\infty.$$

Therefore,

$$|e^{t\Delta} f(x)| \lesssim \|f\|_{L^\infty(|x|^\gamma \mathrm{dx})} |x|^{-\gamma}.$$

The proof of Lemma 2 is complete. □

We now deduce the $L^\infty(|x|^\gamma \mathrm{dx}) - L^\infty(|x|^\beta \mathrm{dx})$ estimate for the operator $e^{t\Delta}\mathbb{P}\nabla$. As shown in [6], the kernel function F_t of $e^{t\Delta}\mathbb{P}\nabla$ satisfies the following inequalities

$$F_t(x) = t^{-\frac{d+1}{2}} F\big(\frac{x}{\sqrt{t}}\big), |F(x)| \lesssim \frac{1}{(1+|x|)^{d+1}}, \tag{6}$$

$$F_t(x) \leq C|x|^{-\alpha} t^{\frac{1}{2}(d+1-\alpha)}, \text{ for } 0 \leq \alpha \leq d+1. \tag{7}$$

By using the inequalities (6) and (7) and arguing as in the proof of Lemma 2, we can easily prove the following lemma.

Lemma 3. *Assume that $d \geq 1$ and $0 \leq \gamma \leq \beta < d$. Then for all $f \in L^\infty(|x|^\beta \mathrm{dx})$ we have*

$$\big\|e^{t\Delta}\mathbb{P}f\big\|_{L^\infty(|x|^\gamma \mathrm{dx})} \leq Ct^{-\frac{1}{2}(\beta+1-\gamma)} \|f\|_{L^\infty(|x|^\beta \mathrm{dx})}, \text{ for } t > 0.$$

Lemma 4. *Let $0 \leq \gamma < \beta \leq d$. Assume that $f \in \mathcal{S}'(\mathbb{R}^d)$ and satisfies the following inequality*

$$\operatorname*{esssup}_{x\in\mathbb{R}^d, t>0} (|x|^\gamma + |x|^\beta)|e^{t\Delta} f(x)| = C < +\infty, \tag{8}$$

then

$$f \in L^\infty(|x|^\gamma \mathrm{dx}) \cap L^\infty(|x|^\beta \mathrm{dx})$$

and

$$\operatorname*{esssup}_{x\in\mathbb{R}^d} (|x|^\gamma + |x|^\beta)|f(x)| \leq C. \tag{9}$$

Proof. We have

$$e^{t\Delta} f \in L^{\frac{d}{\beta},\infty} \cap L^{\frac{d}{\gamma},\infty} \subset L^q \text{ for } d/\beta < q < d/\gamma.$$

Therefore $e^{t\Delta} f \in L^\infty(0,\infty; L^q)$, by a compactness theorem in Banach space, there exists a sequence t_k which converges to 0 such that $e^{t_k\Delta} f$ converges weakly to f' in L^q with $f' \in L^q$. Since $e^{t\Delta}$ is a continuous semigroup on $\mathcal{S}'(\mathbb{R}^d)$, it follows that $f = f' \in L^q$ and so we have

$$\lim_{k\to\infty} \left\| e^{t_k\Delta} f - f \right\|_{L^q} = 0.$$

Therefore, there exists a subsequence t_{k_j} of the sequence t_k such that

$$\lim_{k\to\infty} e^{t_k\Delta} f(x) = f(x) \text{ for almost everywhere } x \in \mathbb{R}^d. \tag{10}$$

The inequality (9) is deduced from equalities (8) and (10). □

Remark 3. From Lemma 4, we see that the condition (1) of Takahashi on the initial data is equivalent to the condition $\|u_0\|_{L^\infty((1+|x|)^\beta \mathrm{dx})} \leq \delta$.

Lemma 5. *Let $\gamma, \theta \in \mathbb{R}$ and $t > 0$, then*
(a) *If $\theta < 1$ then*

$$\int_0^{\frac{t}{2}} (t-\tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau = C t^{1-\gamma-\theta}, \text{ where } C = \int_0^{\frac{1}{2}} (1-\tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau < \infty.$$

(b) *If $\gamma < 1$ then*

$$\int_{\frac{t}{2}}^{t} (t-\tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau = C t^{1-\gamma-\theta}, \text{ where } C = \int_{\frac{1}{2}}^{1} (1-\tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau < \infty.$$

(c) *If $\gamma < 1$ and $\theta < 1$ then*

$$\int_0^{t} (t-\tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau = C t^{1-\gamma-\theta}, \text{ where } C = \int_0^{1} (1-\tau)^{-\gamma} \tau^{-\theta} \mathrm{d}\tau < \infty.$$

The proof of this lemma is elementary and may be omitted. □

Let us recall the following result on solutions of a quadratic equation in Banach spaces (Theorem 22.4 in [6], p. 227).

Theorem 3. *Let E be a Banach space, and $B : E \times E \to E$ be a continuous bilinear map such that there exists $\eta > 0$ so that*

$$\|B(x, y)\| \leq \eta \|x\| \|y\|,$$

for all x and y in E. Then for any fixed $y \in E$ such that $\|y\| \leq \frac{1}{4\eta}$, the equation $x = y - B(x, x)$ has a unique solution $\overline{x} \in E$ satisfying $\|\overline{x}\| \leq \frac{1}{2\eta}$.

§4. Proofs of Theorems 1 and 2

In this section we will give the proofs of Theorems 1 and 2. We now need seven more lemmas. In order to proceed, we define an auxiliary space K^{β}_{α}. Let α and β be such that $0 \leq \alpha \leq \beta < d$, we define the auxiliary space K^{β}_{α} which is made up by the measurable functions $u(t, x)$ such that

$$\operatorname*{esssup}_{x \in \mathbb{R}^d, t > 0} |x|^{\beta} t^{\frac{1}{2}(\beta - \alpha)} |u(x, t)| < +\infty.$$

The auxiliary space K^{α}_{β} is equipped with the norm

$$\big\| u \big\|_{K^{\beta}_{\alpha}} := \operatorname*{esssup}_{x \in \mathbb{R}^d, t > 0} |x|^{\beta} t^{\frac{1}{2}(\beta - \alpha)} |u(x, t)|.$$

We rewrite Lemma 2 as follows

Lemma 6. *Assume that $d \geq 1$ and $0 \leq \alpha \leq \beta < d$. Then for all $f \in L^{\infty}(|x|^{\beta}\mathrm{dx})$ we have $e^{t\Delta} f \in K^{\beta}_{\alpha}$ and $\|e^{t\Delta} f\|_{K^{\beta}_{\alpha}} \leq C \|f\|_{L^{\infty}(|x|^{\beta}\mathrm{dx})}$.*

Lemma 7. *Assume that $d \geq 1$ and $0 \leq \alpha \leq \beta < d$. Then $K^{\beta}_{\alpha} \subset K^{\beta}_{\beta} \cap K^{\beta}_{0}$.*

The proof of this lemma is elementary and may be omitted. □

In the following lemmas a particular attention will be devoted to the study of the bilinear operator $B(u, v)(t)$ defined by

$$B(u, v)(t) = \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla . \big(u(\tau) \otimes v(\tau)\big) \mathrm{d}\tau. \tag{11}$$

Lemma 8. *Let $\beta, \tilde{\beta}, \hat{\beta}$ and α be such that*

$$0 \leq \beta < d, \tilde{\beta} > \beta - 2, 0 \leq \tilde{\beta} \leq \beta, 0 < \alpha < 1, \beta - \tilde{\beta} - 1 < \alpha < d - \tilde{\beta},$$
$$0 \leq \hat{\beta} \leq \alpha + \tilde{\beta}, \text{ and } \alpha + \tilde{\beta} - 1 < \hat{\beta} \leq \beta.$$

Then the bilinear operator $B(u,v)(t)$ is continuous from $K^1_\alpha \times K^\beta_{\tilde\beta}$ into $K^\beta_{\hat\beta}$ and the following inequality holds

$$\big\|B(u,v)\big\|_{K^\beta_{\hat\beta}} \le C\big\|u\big\|_{K^1_\alpha}\big\|v\big\|_{K^\beta_{\tilde\beta}}, \tag{12}$$

where C is a positive constant independent of T.

Proof. Since $B(.,.)$ is bilinear on $K^1_\alpha \times K^\beta_{\tilde\beta}$, we may assume $\|u\|_{K^1_\alpha} = \|u\|_{K^\beta_{\tilde\beta}} = 1$. From

$$|(u \otimes v)| \le |y|^{-(\alpha+\tilde\beta)} t^{-\frac{1}{2}(1-\alpha+\beta-\tilde\beta)},$$

by using Lemma 3, we have

$$\big|e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\big| \lesssim |x|^{-\hat\beta}\frac{1}{(t-s)^{\frac{1}{2}(1+\alpha+\tilde\beta-\hat\beta)}t^{\frac{1}{2}(1-\alpha+\beta-\tilde\beta)}}$$

then applying Lemma 5 (c), we get

$$|B(u,v)| \le |x|^{-\hat\beta}\int_0^t \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha+\tilde\beta-\hat\beta)}t^{\frac{1}{2}(1-\alpha+\beta-\tilde\beta)}}\mathrm{d}s = |x|^{-\hat\beta}t^{-\frac{1}{2}(\beta-\hat\beta)}.$$

This proves Lemma 8.
Note that since $\alpha > \beta-\tilde\beta-1$ and $\hat\beta > \alpha+\tilde\beta-1$, it follows that the conditions $\frac{1-\alpha+\beta-\tilde\beta}{2} < 1$ and $\frac{1+\alpha+\tilde\beta-\hat\beta}{2} < 1$ are valid. So we can apply Lemma 5 (c). □

Lemma 9. *Assume that NSE has a mild solution $u \in K^1_{\tilde\alpha}$ for some $\tilde\alpha \in (0,1)$ with initial data $u_0 \in L^\infty(|x|\mathrm{d}x)$ then $u \in K^1_\alpha$ for all $\alpha \in [0,1]$.*

Proof. From $u = e^{t\Delta}u_0 + B(u,u)$, applying Lemmas 6 and 8 with $\beta = 1$ and $\alpha = \tilde\beta = \tilde\alpha$, we get $u \in K^1_{\hat\beta}$ for all $\hat\beta \in \big(\tilde\alpha-(1-\tilde\alpha), 2\tilde\alpha\big)\cap[0,1]$. Applying again Lemmas 6 and 8 with $\beta = 1, \alpha = \tilde\alpha$, and $\tilde\beta \in \big(\tilde\alpha-(1-\tilde\alpha), 2\tilde\alpha\big)\cap[0,1]$ to get $u \in K^1_{\hat\beta}$ for all $\hat\beta \in \big(\tilde\alpha-2(1-\tilde\alpha), 3\tilde\alpha\big)\cap[0,1]$. By induction, we get $u \in K^1_{\hat\beta}$ for all $\hat\beta \in \big(\tilde\alpha-n(1-\tilde\alpha), (n+1)\tilde\alpha\big)\cap[0,1]$ with $n \in \mathbb{N}$. Since $\tilde\alpha \in (0,1)$, it follows that there exists sufficiently large n satisfying

$$\big(\tilde\alpha-n(1-\tilde\alpha), (n+1)\tilde\alpha\big) \supset [0,1].$$

This proves Lemma 9. □

Lemma 10. *Let β be a fixed number in the interval $[0,d)$. Assume that NSE has a mild solution $u \in \bigcap_{\alpha\in[0,1]} K^1_\alpha \cap K^\beta_{\tilde\beta}$ for some $\tilde\beta \in [0,\beta]\cap(\beta-2,\beta]$ with initial data $u_0 \in L^\infty(|x|^\beta\mathrm{d}x)$, then $u \in K^\beta_{\hat\beta}$ for all $\hat\beta \in [0,\beta]\cap(\tilde\beta-2,\beta]$.*

Proof. We first prove that $u \in K^{\beta}_{\hat{\beta}}$ for all $\hat{\beta} \in [0, \beta] \cap (\tilde{\beta} - 2, \tilde{\beta} + 1)$. Let α_1 and α_2 be such that

$$\max\{\beta - \tilde{\beta} - 1, \hat{\beta} - \tilde{\beta}, 0\} < \alpha_1 < 1 \text{ and } \max\{\hat{\beta} - \tilde{\beta}, \tilde{\beta} - \hat{\beta} - 1, 0\} < \alpha_2 < 1.$$

We split the integral given in (11) into two parts coming from the subintervals $(0, \frac{t}{2})$ and $(\frac{t}{2}, t)$

$$B(u, u)(t) = \int_0^{\frac{t}{2}} e^{(t-\tau)\Delta} \mathbb{P}\nabla.\big(u \otimes u\big) \mathrm{d}\tau + \int_{\frac{t}{2}}^{t} e^{(t-\tau)\Delta} \mathbb{P}\nabla.\big(u \otimes u\big) \mathrm{d}\tau = I_1 + I_2.$$

Since $u \in \bigcap\limits_{\alpha \in [0,1]} K^1_{\alpha}$, it follows that

$$|u(x, t)| \lesssim |x|^{-\alpha_1} t^{-\frac{1}{2}(1-\alpha_1)}, \tag{13}$$

$$|u(x, t)| \lesssim |x|^{-\alpha_2} t^{-\frac{1}{2}(1-\alpha_2)}, \tag{14}$$

and since $u \in K^{\beta}_{\tilde{\beta}}$, it follows that

$$|u(x, t)| \lesssim |x|^{-\tilde{\beta}} t^{-\frac{1}{2}(\beta-\tilde{\beta})}. \tag{15}$$

From the inequalities (13) and (15), and Lemma 3, we get

$$\Big| e^{(t-\tau)\Delta} \mathbb{P}\nabla.\big(u \otimes u\big) \Big| \lesssim |x|^{-\hat{\beta}} \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha_1+\tilde{\beta}-\hat{\beta})} t^{\frac{1}{2}(1-\alpha_1+\beta-\tilde{\beta})}}.$$

Then applying Lemma 5 (a), we have

$$I_1 \leq |x|^{-\hat{\beta}} \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha_1+\tilde{\beta}-\hat{\beta})} t^{\frac{1}{2}(1-\alpha_1+\beta-\tilde{\beta})}} \mathrm{d}s = |x|^{-\hat{\beta}} t^{-\frac{1}{2}(\beta-\hat{\beta})}. \tag{16}$$

From the inequalities (14) and (15), and Lemma 3, we get

$$\Big| e^{(t-\tau)\Delta} \mathbb{P}\nabla.\big(u \otimes u\big) \Big| \lesssim |x|^{-\hat{\beta}} \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha_2+\tilde{\beta}-\hat{\beta})} t^{\frac{1}{2}(1-\alpha_2+\beta-\tilde{\beta})}}.$$

Then applying Lemma 5 (b), we have

$$I_1 \leq |x|^{-\hat{\beta}} \int_{\frac{t}{2}}^{t} \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha_2+\tilde{\beta}-\hat{\beta})} t^{\frac{1}{2}(1-\alpha_2+\beta-\tilde{\beta})}} \mathrm{d}s = |x|^{-\hat{\beta}} t^{-\frac{1}{2}(\beta-\hat{\beta})}. \tag{17}$$

From the inequalities (16) and (17), we get $B(u, u) \in K^{\beta}_{\hat{\beta}}$, and from $u = e^{t\Delta} u_0 + B(u, u)$ and Lemma 6, we have $u \in K^{\beta}_{\hat{\beta}}$. Therefore, $u \in K^{\beta}_{\hat{\beta}}$ for all

$\hat\beta \in [0,\beta] \cap (\tilde\beta - 2, \tilde\beta + 1)$. This proves the result.
We now prove $u \in K^{\beta}_{\hat\beta}$ for all $\hat\beta \in (\beta - 2, \beta] \cap [0, \beta]$. Indeed, if $\tilde\beta > \beta - 1$ then $u \in K^{\beta}_{\hat\beta}$ for all $\hat\beta \in [0,\beta] \cap (\tilde\beta - 2, \beta]$ and so the lemma is proved. In the case $\tilde\beta \le \beta - 1$, the proof is analogous to the previous one, we have $u \in K^{\beta}_{\hat\beta}$ for all $\hat\beta \in [0,\beta] \cap (\tilde\beta - 2, \tilde\beta + 2) = [0,\beta] \cap (\tilde\beta - 2, \beta]$. Therefore the proof of Lemma 10 is complete. □

Lemma 11. *Assume that NSE has a mild solution* $u \in \bigcap\limits_{\alpha\in[0,1]} K^1_\alpha \cap \bigcap\limits_{\hat\beta \in [\tilde\beta, \beta]} K^{\beta}_{\hat\beta}$ *for some* $\tilde\beta \in [0, \beta]$ *with initial data* $u_0 \in L^\infty(|x|^\beta \mathrm{d}x)$. *Then* $u \in K^{\beta}_{\hat\beta}$ *for all* $\hat\beta \in [0,\beta]$.

Proof. We first prove that $u \in K^{\beta}_{\hat\beta}$ for all $\hat\beta \in [0,\beta] \cap (\tilde\beta - 1, \tilde\beta]$.
Let α_1 be such that $0 < \alpha_1 < 1$. Since $u \in K^1_{\alpha_1} \cap K^{\beta}_{\beta}$, it follows that

$$|u(x,t)| \lesssim |x|^{-\alpha_1} t^{-\frac{1}{2}(1-\alpha_1)}, \tag{18}$$

$$|u(x,t)| \lesssim |x|^{-\beta}. \tag{19}$$

From the inequalities (18) and (19), and Lemma 3, we get

$$\left|e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes u\big)\right| \lesssim |x|^{-\hat\beta}\frac{1}{(t-s)^{\frac{1}{2}(1+\alpha_1+\beta-\hat\beta)}t^{\frac{1}{2}(1-\alpha_1)}}.$$

Then applying Lemma 5 (a), we have

$$I_1 \le |x|^{-\hat\beta}\int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{1}{2}(1+\alpha_1+\beta-\hat\beta)}t^{\frac{1}{2}(1-\alpha_1)}}\mathrm{d}s = |x|^{-\hat\beta} t^{-\frac{1}{2}(\beta-\hat\beta)}. \tag{20}$$

Since $u \in K^1_0 \cap K^{\beta}_{\tilde\beta}$, it follows that

$$|u(x,t)| \lesssim t^{-\frac{1}{2}} \text{ and } |u(x,t)| \lesssim |x|^{\tilde\beta} t^{-\frac{1}{2}(\beta-\tilde\beta)}. \tag{21}$$

From the inequality (21), and Lemma 3, we get

$$\left|e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes u\big)\right| \lesssim |x|^{-\hat\beta}\frac{1}{(t-s)^{\frac{1}{2}(1+\tilde\beta-\hat\beta)}t^{\frac{1}{2}(1+\beta-\tilde\beta)}}.$$

Then applying Lemma 5 (b), we obtain

$$I_2 \le |x|^{-\hat\beta}\int_{\frac{t}{2}}^{t} \frac{1}{(t-s)^{\frac{1}{2}(1+\tilde\beta-\hat\beta)}t^{\frac{1}{2}(1+\beta-\tilde\beta)}}\mathrm{d}s = |x|^{-\hat\beta} t^{-\frac{1}{2}(\beta-\hat\beta)}. \tag{22}$$

From the inequalities (20) and (22), we get $B(u,u) \in K^{\beta}_{\hat{\beta}}$. From $u = e^{t\Delta}u_0 + B(u,u)$ and Lemma 6, we deduce $u \in K^{\beta}_{\hat{\beta}}$ for all $\hat{\beta} \in [0,\beta] \cap (\tilde{\beta}-1,\tilde{\beta}]$. Therefore, we get $u \in K^{\beta}_{\hat{\beta}}$ for all $\hat{\beta} \in [0,\beta] \cap (\tilde{\beta}-1,\beta]$.
We now prove that $u \in K^{\beta}_{\hat{\beta}}$ for all $\hat{\beta} \in [0,\beta]$. Indeed, in exactly the same way, since $u \in K^{\beta}_{\hat{\beta}}$ for all $\hat{\beta} \in [0,\beta] \cap (\tilde{\beta}-1,\beta]$, it follows that $u \in K^{\beta}_{\hat{\beta}}$ for all $\hat{\beta} \in [0,\beta] \cap (\tilde{\beta}-2,\beta]$. By induction, we get $u \in K^{\beta}_{\hat{\beta}}$ for all $\hat{\beta} \in [0,\beta] \cap (\tilde{\beta}-n,\beta]$ with $n \in \mathbb{N}$. However, there exists a sufficiently large number n satisfying $b - n < 0$ and therefore $u \in K^{\beta}_{\hat{\beta}}$ for all $\hat{\beta} \in [0,\beta]$. The proof of Lemma 11 is complete. $\square$

Lemma 12. *Let $0 \le \beta < d$ be fixed, then for all α and $\tilde{\beta}$ satisfying*

$$\tilde{\beta} \geq 0, 0 < \alpha < 1, \beta - 2 < \tilde{\beta} \leq \beta, \text{ and } \beta - \tilde{\beta} - 1 < \alpha < d - \tilde{\beta},$$

there exists a positive constant $\delta_{\alpha,\beta,\tilde{\beta},d}$ such that for all $u_0 \in L^{\infty}(|x|^1\mathrm{d}x) \cap L^{\infty}(|x|^{\beta}\mathrm{d}x)$ with $\mathrm{div}(u_0) = 0$ *satisfying*

$$\sup_{x\in\mathbb{R}^d, t>0} (|x|^{\alpha} t^{\frac{1}{2}(1-\alpha)} + |x|^{\tilde{\beta}} t^{\frac{1}{2}(\beta-\tilde{\beta})}) |e^{t\Delta}u_0|) \leq \delta_{\alpha,\beta,\tilde{\beta},d}, \tag{23}$$

NSE has a global mild solution u on $(0,\infty) \times \mathbb{R}^d$ such that

$$\sup_{x\in\mathbb{R}^d, t>0} (|x| + t^{\frac{1}{2}} + |x|^{\beta} + t^{\frac{\beta}{2}}) |u(x,t)|) < +\infty. \tag{24}$$

Proof. Applying Lemma 8 we deduce that the bilinear operator B is bounded from $K^1_{\alpha} \times K^1_{\alpha}$ into K^1_{α} and from $K^1_{\alpha} \times K^{\beta}_{\tilde{\beta}}$ into $K^{\beta}_{\tilde{\beta}}$. Therefore, the bilinear operator B is bounded from

$$(K^1_{\alpha} \cap K^{\beta}_{\tilde{\beta}}) \times (K^1_{\alpha} \cap K^{\beta}_{\tilde{\beta}}) \text{ into } (K^1_{\alpha} \cap K^{\beta}_{\tilde{\beta}}).$$

where the space $K^1_{\alpha} \cap K^{\beta}_{\tilde{\beta}}$ is equipped with the norm

$$\big\|u\big\|_{K^1_{\alpha} \cap K^{\beta}_{\tilde{\beta}}} := \max\{\big\|u\big\|_{K^1_{\alpha}}, \big\|u\big\|_{K^{\beta}_{\tilde{\beta}}}\}.$$

Applying Theorem 3 to the bilinear operator B, we deduce that there exists a positive constant $\delta_{\alpha,\beta,\tilde{\beta},d}$ such that for all $T > 0$ and for all $u_0 \in L^{\infty}(|x|^1\mathrm{d}x) \cap L^{\infty}(|x|^{\beta}\mathrm{d}x)$ with $\mathrm{div}(u_0) = 0$ satisfying

$$\big\|e^{t\Delta}u_0\big\|_{K^1_{\alpha} \cap K^{\beta}_{\tilde{\beta}}} \leq \delta_{\alpha,\beta,\tilde{\beta},d},$$

then NSE has a unique mild solution u satisfying

$$u \in K_{\alpha}^{1} \cap K_{\tilde{\beta}}^{\beta}.$$

Applying Lemmas 9, 10, and 11, we get $u \in K_{\hat{\beta}}^{\beta}$ for all $\hat{\beta} \in [0, \beta]$.
The proof of Lemma 12 is now complete. □

Proof of Theorem 1

Since $\in L^{\infty}(|x|\mathrm{d}x) \subset L^{\infty}(|x|^{\gamma}\mathrm{d}x) \cap L^{\infty}(|x|^{\beta}\mathrm{d}x)$, it follows that $u_0 \in L^{\infty}(|x|\mathrm{d}x)$. Applying Lemma 12 then there exists a positive constant $\delta_{\alpha,\beta,\tilde{\beta},d}$ such that if

$$\sup_{x\in\mathbb{R}^d, t>0} (|x|^{\alpha} t^{\frac{1}{2}(1-\alpha)} + |x|^{\tilde{\beta}} t^{\frac{1}{2}(\beta-\tilde{\beta})})|e^{t\Delta}u_0|) \leq \delta_{\alpha,\beta,\tilde{\beta},d},$$

NSE has a global mild solution u on $(0, \infty) \times \mathbb{R}^d$ such that

$$\sup_{x\in\mathbb{R}^d, t>0} (|x| + t^{\frac{1}{2}} + |x|^{\beta} + t^{\frac{\beta}{2}})|u(x,t)|) < +\infty.$$

Applying Lemma 12 for $\beta = \gamma$ then there exists a positive constant $\delta_{\alpha,\gamma,\tilde{\gamma},d}$ such that if

$$\sup_{x\in\mathbb{R}^d, t>0} (|x|^{\alpha} t^{\frac{1}{2}(1-\alpha)} + |x|^{\tilde{\gamma}} t^{\frac{1}{2}(\gamma-\tilde{\gamma})})|e^{t\Delta}u_0|) \leq \delta_{\alpha,\gamma,\tilde{\gamma},d},$$

NSE has a global mild solution u on $(0, \infty) \times \mathbb{R}^d$ such that

$$\sup_{x\in\mathbb{R}^d, t>0} (t^{\frac{1}{2}} + |x| + |x|^{\gamma} + t^{\frac{\gamma}{2}})|u(x,t)|) < +\infty.$$

Therefore, if u_0 satisfies the following inequality

$$\sup_{x\in\mathbb{R}^d, t>0} (|x|^{\tilde{\gamma}} t^{\frac{1}{2}(\gamma-\tilde{\gamma})} + |x|^{\alpha} t^{\frac{1}{2}(1-\alpha)} + |x|^{\tilde{\beta}} t^{\frac{1}{2}(\beta-\tilde{\beta})})|e^{t\Delta}u_0|) \leq \min\{\delta_{\alpha,\beta,\tilde{\beta},d}, \delta_{\alpha,\gamma,\tilde{\gamma},d}\}$$

NSE has a global mild solution u on $(0, \infty) \times \mathbb{R}^d$ such that (3).
The proof of Theorem 1 is complete. □

Proof of Theorem 2

Since $|x| \leq C(|x|^{\gamma} + |x|^{\beta})$, it follows that

$$\|f\|_{L^{\infty}}(|x|\mathrm{d}x) \leq C(\|f\|_{L^{\infty}}(|x|^{\gamma}\mathrm{d}x) + \|f\|_{L^{\infty}}(|x|^{\beta}\mathrm{d}x)).$$

From Lemma 2 we have

$$|x|^{\alpha}t^{\frac{1}{2}(1-\alpha)}|e^{t\Delta}u_0| \lesssim \|f\|_{L^\infty(|x|\mathrm{d}x)} \lesssim \|f\|_{L^\infty(|x|^{\gamma}\mathrm{d}x)} + \|f\|_{L^\infty(|x|^{\beta}\mathrm{d}x)},$$

$$|x|^{\tilde{\gamma}}t^{\frac{1}{2}(\gamma-\tilde{\gamma})}|e^{t\Delta}u_0| \lesssim \|f\|_{L^\infty(|x|^{\gamma}\mathrm{d}x)}, \text{ and } |x|^{\tilde{\beta}}t^{\frac{1}{2}(\beta-\tilde{\beta})}|e^{t\Delta}u_0| \lesssim \|f\|_{L^\infty(|x|^{\beta}\mathrm{d}x)}.$$

This proves Theorem 2. □

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