The existence and decay rates of strong solutions for Navier-Stokes Equations in Bessel-potential spaces

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Abstract: In this paper, we prove some results on the existence and decay properties of high order derivatives in time and space variables for local and global solutions of the Cauchy problem for the Navier-Stokes equations in Bessel-potential spaces.

§1. Introduction

This paper studies the Cauchy problem of the incompressible Navier-Stokes equations (NSE) in the whole space $\mathbb{R}^d$ for $d \geq 2$,

$$\begin{cases} \partial_t u = \Delta u - \nabla.(u \otimes u) - \nabla p, \\ \nabla.u = 0, \\ u(0,x) = u_0, \end{cases}$$

which is a condensed writing for

$$\begin{cases} 1 \leq k \leq d, \quad \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l (u_l u_k) - \partial_k p, \\ \sum_{l=1}^d \partial_l u_l = 0, \\ 1 \leq k \leq d, \quad u_k(0,x) = u_{0k}. \end{cases}$$

The unknown quantities are the velocity $u(t,x) = (u_1(t,x), \ldots, u_d(t,x))$ of the fluid element at time t and position x and the pressure $p(t,x)$.
There is an extensive literature on the existence and decay rate of strong solutions of the Cauchy problem for NSE. Maria E. Schonbek [1] established the decay of the homogeneous H^m norms for solutions to NSE in two dimensions. She showed that if u is a solution to NSE with an arbitrary $u_0 \in H^m \cap L^1(\mathbb{R}^2)$ with $m \geq 3$ then

$$\|D^\alpha u\|_2^2 \leq C_\alpha (t+1)^{-(|\alpha|+1)} \text{ and } \|D^\alpha u\|_\infty \leq C_\alpha (t+1)^{-(|\alpha|+\frac{1}{2})} \text{ for all } t \geq 1, \alpha \leq m.$$

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Zhi-Min Chen [2] showed that if $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), (d \leq p < \infty)$ and $\|u_0\|_1 + \|u_0\|_p$ is small enough then there is a unique solution $u \in BC([0,\infty); L^1 \cap L^p)$, which satisfies decay property

$$\sup_{t>0} t^{\frac{d}{2}} \left(\|u\|_\infty + t^{\frac{1}{2}} \|Du\|_\infty + t^{\frac{1}{2}} \|D^2 u\|_\infty \right) < \infty.$$

Kato [3] studied strong solutions in the spaces $L^q(\mathbb{R}^d)$ by applying the $L^q - L^p$ estimates for the semigroup generated by the Stokes operator. He showed that there is $T > 0$ and a unique solution u, which satisfies

$$t^{\frac{1}{2}(1-\frac{d}{q})} u \in BC([0,T); L^q), \text{ for } d \leq q \leq \infty,$$
$$t^{\frac{1}{2}(2-\frac{d}{q})} \nabla u \in BC([0,T); L^q), \text{ for } d \leq q \leq \infty,$$

as $u_0 \in L^d(\mathbb{R}^d)$. He showed that $T = \infty$ if $\left\|u_0\right\|_{L^d(\mathbb{R}^d)}$ is small enough.
Cannone [4] generalized the results of Kato. He showed that if $u_0 \in L^d$ and $\left\|u_0\right\|_{\dot{B}_q^{\frac{d}{q}-1,\infty}}, (q > d)$ is small enough then there is a unique solution u, which satisfies

$$t^{\frac{1}{2}(1-\frac{d}{q})} u \in BC([0,\infty); L^q), \text{ for } q \geq d.$$

Note that the condition on the initial data of Cannon is weaker than that of Kato. In 2002, Cheng He and Ling Hsiao [6] extended the results of Kato. They estimated the decay rates of higher order derivatives in time and space variables for the strong solution to NSE with initial data in $L^d(\mathbb{R}^d)$. They showed that if $\left\|u_0\right\|_{L^d(\mathbb{R}^d)}$ is small enough then there is a unique solution u, which satisfies

$$t^{\frac{1}{2}(1+|\alpha|+2\alpha_0-\frac{d}{q})} D_x^\alpha D_t^{\alpha_0} u \in BC([0,\infty); L^q), \text{ for } q \geq d,$$
$$t^{\frac{1}{2}(2+|\alpha|-\frac{d}{q})} D_x^\alpha p \in BC([0,\infty); L^q), \text{ for } q \geq d,$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d), |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d$ and $\alpha_0 \in \mathbb{N}$. D_x^α denotes $\partial_x^{|\alpha|} = \partial^{|\alpha|}/(\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_d}^{\alpha_d})$, $\partial_t^{\alpha_0} = \partial^{\alpha_0}/\partial t^{\alpha_0}$.
In 2005, Okihiro Sawada [7] obtained the decay rate of solutions to NSE with initial data in $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$. He showed that every mild solution in the class

$$u \in BC([0,T); \dot{H}^{\frac{d}{2}-1}) \text{ and } t^{\frac{1}{2}(\frac{d}{2}-\frac{d}{p})} u \in BC([0,T); \dot{H}_q^{\frac{d}{2}-1}),$$

for some $T > 0$ and $p \in (2, \infty]$ satisfies

$$\|u(t)\|_{\dot{H}_q^{\frac{d}{2}-1+\alpha}} \leq K_1 (K_2 \tilde{\alpha})^{\tilde{\alpha}} t^{-\frac{\tilde{\alpha}}{2}} \text{ for } \alpha > 0, \tilde{\alpha} := \alpha + \frac{d}{2} - \frac{d}{q},$$

where constants K_1 and K_2 depend only on d, p, M_1, and M_2 with $M_1 = \sup\limits_{0<t<T} \|u(t)\|_{\dot{H}^{\frac{d}{2}-1}}$ and $M_2 = \sup\limits_{0<t<T} t^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}\|u(t)\|_{\dot{H}_p^{\frac{d}{2}-1}}$.

In this paper, we discuss the existence and decay properties of high order derivatives in time and space variables for local and global solutions of the Cauchy problem for the NSE with initial data in Bessel-potential spaces $\dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d), (1 < p < \infty)$. By using several tools from harmonic analysis, we obtain decay estimates for derivatives of arbitrary order. The estimate for the decay rate is optimal in the sense that it coincides with the decay rate of a solution to the heat equation. This result improves the previous ones.

The content of this paper is as follows: in Section 2, we state our main theorem after introducing some notations. In Section 3, we first establish some estimates concerning the heat semigroup with differential. We also recall some auxiliary lemmas and several estimates in the homogeneous Sobolev spaces and Besov spaces. Finally, in Section 4, we will give the proof of the main theorem.

§2. Statement of the results

Now, for $T > 0$, we say that u is a mild solution of NSE on $[0, T]$ corresponding to a divergence-free initial datum u_0 when u solves the integral equation

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau,.)\otimes u(\tau,.)\big)\mathrm{d}\tau.$$

Above we have used the following notation: for a tensor $F = (F_{ij})$ we define the vector $\nabla.F$ by $(\nabla.F)_i = \sum_{j=1}^d \partial_j F_{ij}$ and for two vectors u and v, we define their tensor product $(u\otimes v)_{ij} = u_i v_j$. The operator $\mathbb{P}$ is the Helmholtz-Leray projection onto the divergence-free fields

$$(\mathbb{P}f)_j = f_j + \sum_{1\leq k\leq d} R_j R_k f_k,$$

where R_j is the Riesz transforms defined as

$$R_j = \frac{\partial_j}{\sqrt{-\Delta}} \quad \text{i.e.} \quad \widehat{R_j g}(\xi) = \frac{i\xi_j}{|\xi|}\hat{g}(\xi).$$

The heat kernel $e^{t\Delta}$ is defined as

$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2}e^{-|.|^2/4t} * u)(x).$$

For a space of functions defined on $\mathbb{R}^d$, say $E(\mathbb{R}^d)$, we will abbreviate it as E. We denote by $L^q := L^q(\mathbb{R}^d)$ the usual Lebesgue space for $q \in [1, \infty]$ with the norm $\|.\|_q$, and we do not distinguish between the vector-valued and scalar-value spaces of functions. We define the Bessel-potential space by $\dot{H}^s_q := \dot{\Lambda}^{-s}L^q$ equipped with the norm $\big\|f\big\|_{\dot{H}^s_q} := \|\dot{\Lambda}^s f\|_q$. Here $\dot{\Lambda}^s := \mathcal{F}^{-1}|\xi|^s\mathcal{F}$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse, respectively. Now we can state our result

Theorem 1. *Let $1 < p < \infty$ be fixed, then*

(A) (Local existence) *For any initial data $u_0 \in \dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)$ with $\nabla.u_0 = 0$, there exists a positive $T = T(u_0)$ such that NSE has a unique mild solution u satisfying*

(i) $t^{\frac{1}{2}(s+1-\frac{d}{q})}\dot{\Lambda}^s u \in BC([0,T);L^q),\ \ for\ q \geq p, s \geq \dfrac{d}{p} - 1,$

(ii) $t^{\frac{1}{2}(s+1+2n-\frac{d}{q})}\dot{\Lambda}^s D^n_t u \in BC([0,T);L^q),\ \ for\ q \geq p, s \geq \dfrac{d}{p} - 1, n \in \mathbb{N},$

(iii) $t^{\frac{1}{2}(s+2-\frac{d}{q})}\dot{\Lambda}^s p \in BC([0,T);L^q)\ q \geq p, s \geq \dfrac{d}{p} - 1.$

(B) (Global existence) *For all $\tilde{q} > \max\{p, d\}$ there exists a positive constant $\sigma_{p,\tilde{q},d}$ such that if*

$$\big\|u_0\big\|_{\dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}} \leq \sigma_{p,\tilde{q},d}, \tag{1}$$

then the existence time T in point (A) *for the solution u is equal to $+\infty$. Moreover, we have*

(a) $\lim\limits_{t\to\infty} t^{\frac{1}{2}(s+1-\frac{d}{q})}\big\|u\big\|_{\dot{H}^s_q} = 0,\ \ for\ q \geq p, s \geq \dfrac{d}{p} - 1,$

(b) $\lim\limits_{t\to\infty} t^{\frac{1}{2}(s+1+2n-\frac{d}{q})}\big\|D^n_t u\big\|_{\dot{H}^s_q} = 0,\ \ for\ q \geq p, s \geq \dfrac{d}{p} - 1, n \in \mathbb{N},$

(c) $\lim\limits_{t\to\infty} t^{\frac{1}{2}(s+2-\frac{d}{q})}\big\|p\big\|_{\dot{H}^s_q} = 0\ \ for\ q \geq p, s \geq \dfrac{d}{p} - 1.$

Remark 1. Our result improves the previous ones for $L^d(\mathbb{R}^d)$ and $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$. These spaces, studied in [6] and [7], are particular cases of the Bessel spaces $\dot{H}_p^{\frac{d}{p}-1}$ with $p = d$ and $p = 2$, respectively. We have the following imbeddings

$$\dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)_{(1<p<2)} \hookrightarrow \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d) \hookrightarrow \dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)_{(p>d)}.$$

Furthermore, we obtain statements that are stronger than those of Cheng He and Ling Hsiao [6] and Okihiro Sawada [7] but under a much weaker condition on the initial data.

We show that the condition (1) on the initial data in Theorem 1 is weaker than the condition in [6]. We have $L^d(\mathbb{R}^d) \hookrightarrow \dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}(\mathbb{R}^d)$, $(\tilde{q} > d)$, but these two spaces are different. Indeed, we have $\big|x\big|^{-1} \notin L^d$ and $\big|x\big|^{-1} \in \dot{B}_{\tilde{q}}^{\frac{d}{\tilde{q}}-1,\infty}$ for all $\tilde{q} > d$.

§3. Tools from harmonic analysis

In this section we prove some auxiliary lemmas.
We first establish the $L^p - L^q$ estimate for the heat semigroup with differential.

Lemma 1. *Assume that $d \geq 1$ and $s \geq 0, t > 0$ and $1 \leq p \leq q \leq \infty$. Then for all $f \in L^p$ we have*

$$t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})+\frac{s}{2}}\dot{\Lambda}^s e^{t\Delta}f \in BC([0,\infty); L^q(\mathbb{R}^d)) \text{ and } \big\|\dot{\Lambda}^s e^{t\Delta}f\big\|_q \leq Ct^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-\frac{s}{2}}\|f\|_p.$$

Proof. See [7]. □

In order to obtain our theorem we must establish the estimates for bilinear terms. We thus need a version of Hölder type inequality in Bessel-potential spaces.

Lemma 2. *Let $1 < r, p_1, p_2, q_1, q_2 < \infty$ and $s \geq 0$ satisfying $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Then there exists a constant $C = C(d, s, p_1, p_2, q_1, q_2)$ such that for all $f \in \dot{H}^s_{p1}(\mathbb{R}^d) \cap L^{p_2}(\mathbb{R}^d)$ and for all $g \in \dot{H}^s_{q_2}(\mathbb{R}^d) \cap L^{q_1}(\mathbb{R}^d)$ we have*

$$\big\|fg\big\|_{\dot{H}^s_r} \leq C\big(\big\|f\big\|_{\dot{H}^s_{p_1}}\|g\|_{q_1} + \|f\|_{p_2}\big\|g\big\|_{\dot{H}^s_{q_2}}\big).$$

Proof. See [11]. □

Lemma 3. *Let $\gamma, \theta \in \mathbb{R}$ and $t > 0$, then*
(a) *If $\theta < 1$ then*

$$\int_0^{\frac{t}{2}}(t-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau = Ct^{1-\gamma-\theta}, \text{ where } C = \int_0^{\frac{1}{2}}(1-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau < \infty.$$

(b) *If $\gamma < 1$ then*

$$\int_{\frac{t}{2}}^{t}(t-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau = Ct^{1-\gamma-\theta}, \text{ where } C = \int_{\frac{1}{2}}^{1}(1-\tau)^{-\gamma}\tau^{-\theta}\mathrm{d}\tau < \infty.$$

The proof of this lemma is elementary and may be omitted. □

Theorem 2. (Calderon-Zygmund theorem).
The Riesz transforms $R_j = \frac{\partial_j}{\sqrt{-\Delta}}$ *defined by* $\mathcal{F}(R_j g)(\xi) = \frac{i\xi_j}{|\xi|}\hat{f}(\xi)$ *are bounded from* $\mathcal{H}^1$ *to* L^1*, from* L^∞ *to* BMO*, and* L^q *to* L^q *for* $1 < q < \infty$*.*

Lemma 4. (Sobolev inequalities).
(a) *For* $0 < \alpha < d$*, the operator* $(\frac{1}{\sqrt{-\Delta}})^\alpha$ *is bounded from the Hardy space* $\mathcal{H}^1$ *to* $L^{\frac{d}{d-\alpha}}$ *and from* $L^{d/\alpha}$ *to* $BMO = (\mathcal{H}^1)^*$*.*
(b) *For* $1 < p < \infty$ *and* $0 < \alpha < d/p$ *the operator* $(\frac{1}{\sqrt{-\Delta}})^\alpha$ *is bounded from* L^p *to* L^q *where* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$*.*

In this paper we use the definition of the homogeneous Besov space $\dot{B}_q^{s,p}$ in [8, 9]. A known property of these spaces is that the Riesz potential $\dot{\Lambda}^s = (-\Delta)^{s/2}$ is an isomorphism from $\dot{B}_q^{s_0,p}$ onto $\dot{B}_q^{s_0-s,p}$, see [10].
The following lemmas will provide a different characterization of Besov spaces $\dot{B}_q^{s,p}$ in terms of the heat semigroup and will be one of the staple ingredients of the proof of Theorem 1.

Lemma 5. *Let* $1 \leq p, q \leq \infty$ *and* $s \in \mathbb{R}$*.*
(a) *If* $s < 1$ *then the two quantities*

$$\Big(\int_0^\infty \big(t^{-\frac{s}{2}}\big\|e^{t\Delta}t^{\frac{1}{2}}\dot{\Lambda}f\big\|_q\big)^p\frac{\mathrm{d}t}{t}\Big)^{1/p} \text{ and } \|f\|_{\dot{B}_q^{s,p}} \text{ are equivalent.}$$

(b) *If* $s < 0$ *then the two quantities*

$$\Big(\int_0^\infty \big(t^{-\frac{s}{2}}\big\|e^{t\Delta}f\big\|_q\big)^p\frac{\mathrm{d}t}{t}\Big)^{1/p} \text{ and } \|f\|_{\dot{B}_q^{s,p}} \text{ are equivalent,}$$

where $\dot{B}_q^{s,p}$ *is the homogeneous Besov space.*

Proof. See ([5], Proposition 1, p. 181 and Proposition 3, p. 182), or see ([12], Theorem 5.4, p. 45). □
The following lemma is a generalization of the above lemma.

Lemma 6. *Let* $1 \leq p, q \leq \infty,\ \alpha \geq 0$*, and* $s < \alpha$*. Then the two quantities*

$$\Big(\int_0^\infty (t^{-\frac{s}{2}}\big\|e^{t\Delta}t^{\frac{\alpha}{2}}\dot{\Lambda}^\alpha f\big\|_{L^q})^p\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}} \text{ and } \|f\|_{\dot{B}_q^{s,p}} \text{ are equivalent,}$$

Proof. Note that $\dot{\Lambda}^{s_0}$ is an isomorphism from $\dot{B}_q^{s,p}$ to $\dot{B}_q^{s-s_0,p}$, then we can easily prove the lemma. □
Let us recall following result on solutions of a quadratic equation in Banach spaces (Theorem 22.4 in [12], p. 227).

Theorem 3. *Let E be a Banach space, and $B : E \times E \to E$ be a continuous bilinear map such that there exists $\eta > 0$ so that*

$$\|B(x, y)\| \leq \eta \|x\| \|y\|,$$

for all x and y in E. Then for any fixed $y \in E$ such that $\|y\| \leq \frac{1}{4\eta}$, the equation $x = y - B(x, x)$ has a unique solution $\overline{x} \in E$ satisfying $\|\overline{x}\| \leq \frac{1}{2\eta}$.

§4. Proof of Theorem 1

In this section we shall give the proof of Theorem 1. We now need three more lemmas. In order to proceed, we define an auxiliary space $K^s_{q,T}$.
Let s, q, T be such that

$$q \in (1, +\infty), s \geq \frac{d}{q} - 1, \text{ and } 0 < T \leq \infty,$$

we set

$$\alpha = \alpha(s, q) = s + 1 - \frac{d}{q}.$$

In the case $T < \infty$, we define the auxiliary space $\mathcal{K}^s_{q,T}$ which is made up by the functions $u(t, x)$ such that

$$t^{\frac{\alpha}{2}} u \in C([0, T]; \dot{H}^s_q)$$

and

$$\lim_{t \to 0} t^{\frac{\alpha}{2}} \big\|u(t, .)\big\|_{\dot{H}^s_q} = 0. \tag{2}$$

In the case $T = \infty$, we define the auxiliary space $\mathcal{K}^s_{q,\infty}$ which is made up by the functions $u(t, x)$ such that

$$t^{\frac{\alpha}{2}} u \in BC([0, \infty); \dot{H}^s_q),$$

$$\lim_{t \to 0} t^{\frac{\alpha}{2}} \big\|u(t, .)\big\|_{\dot{H}^s_q} = 0, \tag{3}$$

and

$$\lim_{t \to \infty} t^{\frac{\alpha}{2}} \big\|u(t, .)\big\|_{\dot{H}^s_q} = 0. \tag{4}$$

The auxiliary space $\mathcal{K}^s_{q,T}$ is equipped with the norm

$$\big\|u\big\|_{\mathcal{K}^s_{q,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}} \big\|u(t, .)\big\|_{\dot{H}^s_q}.$$

In the case $s = \frac{d}{q} - 1$ it is also convenient to define the space $\mathcal{K}^s_{q,T}$ as the natural space $C([0,T); \dot{H}^s_q(\mathbb{R}^d))$ with the additional condition that its elements $u(t,x)$ satisfy

$$\lim_{t\to 0} \big\| u(t,.) \big\|_{\dot{H}^s_q} = 0,$$

if $T = \infty$ then its elements $u(t,x)$ satisfy the additional condition

$$\lim_{t\to \infty} \big\| u(t,.) \big\|_{\dot{H}^s_q} = 0.$$

Remark 2. The auxiliary space $\mathcal{K}_q := \mathcal{K}^0_{q,T}$ $(q \geq d, 0 < T < \infty)$ was introduced by Weissler and systematically used by Kato [3] and Cannone [4]. In the case $T = \infty$, the space $\mathcal{K}_q$ of Kato isn't restricted by the condition (4).

Lemma 7. *If* $u_0 \in \dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d), (1 < p < \infty)$ *then*
(a) *For all q and s satisfying $q \geq p$ and $s \geq \frac{d}{p} - 1$ we have*

$$t^{\frac{1}{2}(s+1-\frac{d}{q})} e^{t\Delta} u_0 \in BC([0,\infty); \dot{H}^s_q) \tag{5}$$

and

$$\lim_{t\to\infty} t^{\frac{1}{2}(s+1-\frac{d}{q})} \big\| e^{t\Delta} u_0 \big\|_{\dot{H}^s_q} = 0. \tag{6}$$

(b) *For all q and s satisfying $q > p$ and $s \geq \frac{d}{p} - 1$ we have*

$$e^{t\Delta} u_0 \in \mathcal{K}^s_{q,\infty}.$$

Proof. (a) By applying Lemma 1 for $\dot{\Lambda}^{\frac{d}{p}-1} u_0 \in L^p$, we have

$$t^{\frac{1}{2}(s+1-\frac{d}{q})} \dot{\Lambda}^s e^{t\Delta} u_0 = t^{\frac{1}{2}(s+1-\frac{d}{q})} \dot{\Lambda}^{s-\frac{d}{p}+1} e^{t\Delta} \dot{\Lambda}^{\frac{d}{p}-1} u_0 \in BC([0,\infty); L^q). \tag{7}$$

The relation (5) is equivalent to the relation (7).
We now prove (6), it is easily prove that

$$\lim_{n\to\infty} \Big\| \mathcal{X}_n \dot{\Lambda}^{\frac{d}{p}-1} u_0 \Big\|_p = 0, \tag{8}$$

where $\mathcal{X}_n(x) = 0$ for $x \in \{x : \ |x| < n\} \cap \{x : \big|\dot{\Lambda}^{\frac{d}{p}-1} u_0(x)\big| < n\}$ and $\mathcal{X}_n(x) = 1$ otherwise. Let p^* be such that $1 < p^* < p$. Applying Lemma 1, we get

$$t^{\frac{1}{2}(s+1-\frac{d}{q})} \| e^{t\Delta} u_0 \|_{\dot{H}^s_q} = t^{\frac{1}{2}(s+1-\frac{d}{q})} \big\| \dot{\Lambda}^{s-\frac{d}{p}+1} e^{t\Delta} \dot{\Lambda}^{\frac{d}{p}-1} u_0 \big\|_q \leq$$

$$t^{\frac{1}{2}(s+1-\frac{d}{q})} \big\| \dot{\Lambda}^{s-\frac{d}{p}+1} e^{t\Delta} (\mathcal{X}_n \dot{\Lambda}^{\frac{d}{p}-1} u_0) \big\|_q + t^{\frac{1}{2}(s+1-\frac{d}{q})} \big\| \dot{\Lambda}^{s-\frac{d}{p}+1} e^{t\Delta} \big((1-\mathcal{X}_n) \dot{\Lambda}^{\frac{d}{p}-1} u_0\big) \big\|_q$$

$$\leq \big\| \mathcal{X}_n \dot{\Lambda}^{\frac{d}{p}-1} u_0 \big\|_p + t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p^*})} \big\| n(1-\mathcal{X}_n) \big\|_{p^*}. \tag{9}$$

For any $\epsilon > 0$, from (8) we have

$$\left\|\mathcal{X}_n \dot{\Lambda}^{\frac{d}{p}-1} u_0\right\|_p < \frac{\epsilon}{2} \tag{10}$$

for large enough n. Fixed one of such n, there exists $t_0 = t_0(n) > 0$ satisfying

$$t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p^*})}\left\|n(1-\mathcal{X}_n)\right\|_{p^*} < \frac{\epsilon}{2} \tag{11}$$

for all $t > t_0$, from the inequalities (9), (10), and (11) we deduce that

$$t^{\frac{1}{2}(s+1-\frac{d}{q})}\|e^{t\Delta}u_0\|_{\dot{H}^s_q} < \epsilon \text{ for all } t > t_0.$$

(b) We only need to prove

$$\lim_{t\to 0} t^{\frac{1}{2}(s+1-\frac{d}{q})}\left\|e^{t\Delta}u_0\right\|_{\dot{H}^s_q} = 0.$$

Let p^* be such that $p < p^* \leq q$. For any $\epsilon > 0$, applying Lemma 1, by an argument similar to the previous one, there exist a sufficiently large n and a sufficiently small $t_0 = t_0(n)$ such that

$$t^{\frac{1}{2}(s+1-\frac{d}{q})}\|e^{t\Delta}u_0\|_{\dot{H}^s_q} \leq \left\|\mathcal{X}_n \dot{\Lambda}^{\frac{d}{p}-1}u_0\right\|_p + t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{p^*})}\left\|n(1-\mathcal{X}_n)\right\|_{p^*} < \epsilon$$

for all $t < t_0$. □

In the following lemmas a particular attention will be devoted to the study of the bilinear operator $B(u,v)(t)$ defined by

$$B(u,v)(t) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big)\mathrm{d}\tau. \tag{12}$$

Lemma 8. *Let $r, q, \tilde{q}, s \in \mathbb{R}$ be such that*

$$1 < r, q < \infty, \tilde{q} > d, s \geq \max\{\frac{d}{q}-1, \frac{d}{r}-1, 0\}, \frac{1}{\tilde{q}} + \frac{1}{q} - \frac{1}{d} < \frac{1}{r} \leq \frac{1}{\tilde{q}} + \frac{1}{q}.$$

Then the bilinear operator $B(u,v)(t)$ is continuous from

$$\big(\mathcal{K}^0_{\tilde{q},T} \cap \mathcal{K}^s_{q,T}\big) \times \big(\mathcal{K}^0_{\tilde{q},T} \cap \mathcal{K}^s_{q,T}\big) \textit{ into } \mathcal{K}^s_{r,T},$$

and the following inequality holds

$$\left\|B(u,v)\right\|_{\mathcal{K}^s_{r,T}} \leq C\left\|u\right\|_{\mathcal{K}^0_{\tilde{q},T}\cap\mathcal{K}^s_{q,T}}\left\|v\right\|_{\mathcal{K}^0_{\tilde{q},T}\cap\mathcal{K}^s_{q,T}}, \tag{13}$$

where the space $\mathcal{K}^0_{\tilde{q},T} \cap \mathcal{K}^s_{q,T}$ is equipped with the norm

$$\left\|u\right\|_{\mathcal{K}^0_{\tilde{q},T}\cap\mathcal{K}^s_{q,T}} := \left\|u\right\|_{\mathcal{K}^0_{\tilde{q},T}} + \left\|u\right\|_{\mathcal{K}^s_{q,T}},$$

and C is a positive constant independent of T.

Proof. We split the integral given in (12) into two parts coming from the subintervals $(0, \frac{t}{2})$ and $(\frac{t}{2}, t)$

$$B(u,v)(t) = \int_0^{\frac{t}{2}} e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\mathrm{d}\tau + \int_{\frac{t}{2}}^{t} e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\mathrm{d}\tau. \qquad (14)$$

To estimate the first term on the right-hand side of the equation (14), applying Lemma 1, 3, the Holder inequality, and the Calderon-Zygmund theorem to obtain

$$\Big\|\int_0^{\frac{t}{2}} e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\mathrm{d}\tau\Big\|_{\dot{H}^s_r} \leq \int_0^{\frac{t}{2}} \Big\|\dot{\Lambda}^s e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\Big\|_r \mathrm{d}\tau =$$
$$\int_0^{\frac{t}{2}} \Big\|\dot{\Lambda}^{s+1} e^{(t-\tau)\Delta}\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\big(u\otimes v\big)\Big\|_r \mathrm{d}\tau \leq$$
$$\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{s+1}{2}+\frac{d}{2}(\frac{1}{r}-\frac{2}{\tilde{q}})}\|u(\tau)\|_{\tilde{q}}\|v(\tau)\|_{\tilde{q}}\mathrm{d}\tau \leq$$
$$\int_0^{\frac{t}{2}} (t-\tau)^{-\frac{s+1}{2}+\frac{d}{2}(\frac{1}{r}-\frac{2}{\tilde{q}})}\tau^{-(1-\frac{d}{\tilde{q}})} \sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u(\eta)\|_{\tilde{q}} \sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v(\eta)\|_{\tilde{q}}\mathrm{d}\tau \simeq$$
$$t^{-\frac{1}{2}(1+s-\frac{d}{r})} \sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u(\eta)\|_{\tilde{q}}. \sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v(\eta)\|_{\tilde{q}}. \qquad (15)$$

Note that since $\tilde{q} > d$, it follows that $\theta = (1-\frac{d}{\tilde{q}}) < 1$ is satisfied. So we can apply Lemma 3 (a).
To estimate the second term on the right-hand side of the equation (14), applying Lemmas 1, 2, 3, and the Calderon-Zygmund theorem to obtain

$$\Big\|\int_{\frac{t}{2}}^{t} e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\mathrm{d}\tau\Big\|_{\dot{H}^s_r} \leq \int_{\frac{t}{2}}^{t} \Big\|\dot{\Lambda}^s e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\Big\|_r \mathrm{d}\tau =$$
$$\int_{\frac{t}{2}}^{t} \Big\|\dot{\Lambda} e^{(t-\tau)\Delta}\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\dot{\Lambda}^s\big(u\otimes v\big)\Big\|_r \mathrm{d}\tau \leq$$
$$\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}+\frac{d}{2}(\frac{1}{r}-\frac{1}{\tilde{q}}-\frac{1}{q})}\|u(\tau)\|_{\dot{H}^s_q}\|v(\tau)\|_{\tilde{q}}\mathrm{d}\tau +$$
$$\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}+\frac{d}{2}(\frac{1}{r}-\frac{1}{\tilde{q}}-\frac{1}{q})}\|u(\tau)\|_{\tilde{q}}\|v(\tau)\|_{\dot{H}^s_q}\mathrm{d}\tau \leq$$
$$\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}+\frac{d}{2}(\frac{1}{r}-\frac{1}{\tilde{q}}-\frac{1}{q})}\tau^{-\frac{1}{2}(2+s-\frac{d}{\tilde{q}}-\frac{d}{q})} \sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|u\|_{\dot{H}^s_q} \sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v\|_{\tilde{q}}\mathrm{d}\tau +$$
$$\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}+\frac{d}{2}(\frac{1}{r}-\frac{1}{\tilde{q}}-\frac{1}{q})}\tau^{-\frac{1}{2}(2+s-\frac{d}{\tilde{q}}-\frac{d}{q})} \sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u\|_{\tilde{q}} \sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|v\|_{\dot{H}^s_q}\mathrm{d}\tau$$

$$\simeq t^{-\frac{1}{2}(1+s-\frac{d}{r})}\Big(\sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|u(\eta)\|_{\dot{H}^s_q}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v(\eta)\|_{\tilde{q}}+$$

$$\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u(\eta)\|_{\tilde{q}}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|v(\eta)\|_{\dot{H}^s_q}\Big). \tag{16}$$

Note that since $\frac{1}{\tilde{q}}+\frac{1}{q}-\frac{1}{d}<\frac{1}{r}$, it follows that $\gamma=\frac{1}{2}-\frac{d}{2}(\frac{1}{r}-\frac{1}{\tilde{q}}-\frac{1}{q})<1$ is satisfied. So we can apply Lemma 3 (b). From the inequalities (15) and (16) we have

$$t^{\frac{1}{2}(1+s-\frac{d}{r})}\big\|B(u,v)(t)\big\|_{\dot{H}^s_r}\lesssim\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u(\eta)\|_{\tilde{q}}.\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v(\eta)\|_{\tilde{q}}+$$

$$\sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|u(\eta)\|_{\dot{H}^s_q}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v(\eta)\|_{\tilde{q}}+$$

$$\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u(\eta)\|_{\tilde{q}}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|v(\eta)\|_{\dot{H}^s_q}. \tag{17}$$

The estimate (13) is deduced from the inequality (17). Let us now check the validity of the condition (2) for the bilinear term $B(u,v)(t)$. Indeed, from the inequality (17) we have

$$\lim_{t\to 0}t^{\frac{1}{2}(1+s-\frac{d}{q})}\big\|B(u,v)(t)\big\|_{\dot{H}^s_r}=0 \tag{18}$$

whenever

$$\lim_{t\to 0}t^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u(t)\|_{\tilde{q}}=\lim_{t\to 0}t^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v(t)\|_{\tilde{q}}=$$

$$\lim_{t\to 0}t^{\frac{1}{2}(1+s-\frac{d}{q})}\|u(t)\|_{\dot{H}^s_q}=\lim_{t\to 0}t^{\frac{1}{2}(1+s-\frac{d}{q})}\|v(t)\|_{\dot{H}^s_q}=0. \tag{19}$$

In the case of $T=\infty$, we check the validity of the condition (4) for the bilinear term $B(u,v)(t)$. Firstly we estimate the first term on the right-hand side of equation (14). From the estimates in the inequality (15) we get

$$t^{\frac{1}{2}(1+s-\frac{d}{r})}\Big\|\int_0^{\frac{t}{2}}e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\mathrm{d}\tau\Big\|_{\dot{H}^s_r}\lesssim$$

$$\int_0^{\frac{1}{2}}(1-\tau)^{-\frac{s+1}{2}+\frac{d}{2}(\frac{1}{r}-\frac{2}{\tilde{q}})}\tau^{-(1-\frac{d}{\tilde{q}})}\big((t\tau)^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u(t\tau)\|_{\tilde{q}}(t\tau)^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v(t\tau)\|_{\tilde{q}}\big)\mathrm{d}\tau.$$

Applying Lebesgue's convergence theorem we deduce that

$$\lim_{t\to\infty}t^{\frac{1}{2}(1+s-\frac{d}{r})}\Big\|\int_0^{\frac{t}{2}}e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\mathrm{d}\tau\Big\|_{\dot{H}^s_r}=0 \tag{20}$$

whenever

$$t^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}u\in BC([0,\infty);L^{\tilde{q}}),\ \ t^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}v\in BC([0,\infty);L^{\tilde{q}}), \tag{21}$$

and

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{d}{\tilde q})}\|u(t)\|_{\tilde q} = \lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{d}{\tilde q})}\|v(t)\|_{\tilde q} = 0. \tag{22}$$

By an argument similar to the previous one, we have

$$\lim_{t\to\infty} t^{\frac{1}{2}(1+s-\frac{d}{r})}\Big\| \int_{\frac{t}{2}}^{t} e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\mathrm{d}\tau\Big\|_{\dot H^s_r} = 0 \tag{23}$$

whenever

$$\begin{aligned} &t^{\frac{1}{2}(1+s-\frac{d}{q})}u \in BC([0,\infty);\dot H^s_q),\ t^{\frac{1}{2}(1+s-\frac{d}{q})}v \in BC([0,\infty);\dot H^s_q),\\ &t^{\frac{1}{2}(1-\frac{d}{\tilde q})}u \in BC([0,\infty);L^{\tilde q}),\ \text{and}\ \ t^{\frac{1}{2}(1-\frac{d}{\tilde q})}v \in BC([0,\infty);L^{\tilde q}),\end{aligned} \tag{24}$$

and

$$\begin{aligned} &\lim_{t\to\infty} t^{\frac{1}{2}(1+s-\frac{d}{q})}\|u(t)\|_{\dot H^s_q} = \lim_{t\to\infty} t^{\frac{1}{2}(1+s-\frac{d}{q})}\|v(t)\|_{\dot H^s_q} =\\ &\quad \lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{d}{\tilde q})}\|u(t)\|_{\tilde q} = \lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{d}{\tilde q})}\|v(t)\|_{\tilde q} = 0.\end{aligned} \tag{25}$$

It follows readily from (20) and (23) that

$$\lim_{t\to\infty} t^{\frac{1}{2}(1+s-\frac{d}{q})}\big\|B(u,v)(t)\big\|_{\dot H^s_r} = 0,$$

whenever (24) and (25) are satisfied.
Finally, the continuity at $t = 0$ zero of $t^{\frac{1}{2}(1+s-\frac{d}{r})}B(u,v)(t)$ follows from the equality (18). The continuity elsewhere follows from carefully rewriting the expression $\int_0^{t+\epsilon} - \int_0^t$ and applying the same argument. □

Remark 3. Lemma 8 is a generalization of Lemma 10 in ([5], p. 196). In particular, when $q = \tilde q, s = 0$, we get back Lemma 10 in ([5], p. 196).

Lemma 9. *Let $\tilde q, q, s \in \mathbb{R}$ be such that*

$$q > 1, \tilde q > d, s \geq \max\{\frac{d}{q} - 1, 0\},\ \text{and}\ \frac{d}{q} - 1 \leq s^* < s + 1 - \frac{d}{q}.$$

Then the bilinear operator $B(u,v)(t)$ is continuous from

$$\big(\mathcal{K}^0_{\tilde q,T}\cap\mathcal{K}^s_{q,T}\big)\times\big(\mathcal{K}^0_{\tilde q,T}\cap\mathcal{K}^s_{q,T}\big)\ \textit{into}\ \mathcal{K}^{s^*}_{q,T},$$

and the following inequality holds

$$\big\|B(u,v)\big\|_{\mathcal{K}^{s^*}_{q,T}} \leq C\big\|u\big\|_{\mathcal{K}^0_{\tilde q,T}\cap\mathcal{K}^s_{q,T}}\big\|v\big\|_{\mathcal{K}^0_{\tilde q,T}\cap\mathcal{K}^s_{q,T}}, \tag{26}$$

where C is a positive constant independent of T.

Proof. To estimate the first term on the right-hand side of the equation (14), we apply Lemmas 1, 3, the Holder inequality, and the Calderon-Zygmund theorem

$$
\begin{aligned}
\Big\|\int_0^{\frac{t}{2}} e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\mathrm{d}\tau\Big\|_{\dot{H}^{s^*}_q} \leq \int_0^{\frac{t}{2}} \Big\|\dot{\Lambda}^{s^*}e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\Big\|_q\mathrm{d}\tau = \\
\int_0^{\frac{t}{2}} \Big\|\dot{\Lambda}^{s^*+1}e^{(t-\tau)\Delta}\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\big(u\otimes v\big)\Big\|_q\mathrm{d}\tau \leq \\
\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{s^*+1}{2}+\frac{d}{2}(\frac{1}{q}-\frac{2}{\tilde{q}})}\|u(\tau)\|_{\tilde{q}}\|v(\tau)\|_{\tilde{q}}\mathrm{d}\tau \leq \\
\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{s^*+1}{2}+\frac{d}{2}(\frac{1}{q}-\frac{2}{\tilde{q}})}\tau^{-(1-\frac{d}{\tilde{q}})}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u(\eta)\|_{\tilde{q}}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v(\eta)\|_{\tilde{q}}\mathrm{d}\tau \simeq \\
t^{-\frac{1}{2}(1+s^*-\frac{d}{q})}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u(\eta)\|_{\tilde{q}}.\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v(\eta)\|_{\tilde{q}}. \qquad (27)
\end{aligned}
$$

Note that since $\tilde{q} > d$, it follows that $\theta = 1 - \frac{1}{\tilde{q}} < 1$ is satisfied. So we can apply Lemma 3 (a). To estimate the second term on the right-hand side of equation (14), we apply Lemmas 1, 2, 3, and the Calderon-Zygmund theorem

$$
\begin{aligned}
\Big\|\int_{\frac{t}{2}}^{t} e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\mathrm{d}\tau\Big\|_{\dot{H}^{s^*}_q} \leq \int_{\frac{t}{2}}^{t} \Big\|\dot{\Lambda}^{s^*}e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u\otimes v\big)\Big\|_q\mathrm{d}\tau = \\
\int_{\frac{t}{2}}^{t} \Big\|\dot{\Lambda}^{s^*-s+1}e^{(t-\tau)\Delta}\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\dot{\Lambda}^s\big(u\otimes v\big)\Big\|_q\mathrm{d}\tau \leq \\
\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{s^*-s+1}{2}-\frac{d}{2\tilde{q}}}\|u(\tau)\|_{\dot{H}^s_q}\|v(\tau)\|_{\tilde{q}}\mathrm{d}\tau + \\
\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{s^*-s+1}{2}-\frac{d}{2\tilde{q}}}\|u(\tau)\|_{\tilde{q}}\|v(\tau)\|_{\dot{H}^s_q}\mathrm{d}\tau \leq \\
\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{s^*-s+1}{2}-\frac{d}{2\tilde{q}}}\tau^{-\frac{1}{2}(2+s-\frac{d}{\tilde{q}}-\frac{d}{q})}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|u\|_{\dot{H}^s_q}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v\|_{\tilde{q}}\mathrm{d}\tau + \\
\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{s^*-s+1}{2}-\frac{d}{2\tilde{q}}}\tau^{-\frac{1}{2}(2+s-\frac{d}{\tilde{q}}-\frac{d}{q})}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u\|_{\tilde{q}}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|v\|_{\dot{H}^s_q}\mathrm{d}\tau \\
\simeq t^{-\frac{1}{2}(1+s^*-\frac{d}{q})}\Big(\sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|u\|_{\dot{H}^s_q}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v\|_{\tilde{q}} + \\
\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u\|_{\tilde{q}}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|v\|_{\dot{H}^s_q}\Big). \qquad (28)
\end{aligned}
$$

Note that since $s^* < s+1-\frac{d}{q}$, it follows that $\gamma = \frac{s^*-s+1}{2} + \frac{d}{2\tilde{q}} < 1$ is satisfied. So we can apply Lemma 3 (b).

From the inequalities (27) and (28), we have

$$t^{\frac{1}{2}(1+s^*-\frac{d}{q})}\big\|B(u,v)(t)\big\|_{\dot{H}^{s^*}_q} \lesssim \sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u(\eta)\|_{\tilde{q}}.\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v(\eta)\|_{\tilde{q}}+ \\ \sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|u(\eta)\|_{\dot{H}^s_q}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|v(\eta)\|_{\tilde{q}}+ \\ \sup_{0<\eta<t}\eta^{\frac{1}{2}(1-\frac{d}{\tilde{q}})}\|u(\eta)\|_{\tilde{q}}\sup_{0<\eta<t}\eta^{\frac{1}{2}(1+s-\frac{d}{q})}\|v(\eta)\|_{\dot{H}^s_q}. \tag{29}$$

The estimate (26) is deduced from the inequality (29). Finally, by an argument similar to the one used in Lemma 9, we easily check the validity of the conditions (2) and (4) for the bilinear term $B(u,v)(t)$ and the continuity of $t^{\frac{1}{2}(1+s^*-\frac{d}{q})}B(u,v)(t)$ at all $t \geq 0$. $\square$

Proof of Theorem 1

(A) To prove point (i), we take arbitrary $\tilde{q}$ satisfying $\tilde{q} > \max\{p,d\}$. Applying Lemma 8 we deduce that the bilinear operator B is bounded from $K^0_{\tilde{q}} \times K^{\max\{\frac{d}{p}-1,0\}}_{\tilde{q}}$ into $K^{\max\{\frac{d}{p}-1,0\}}_{\tilde{q}}$ and from $K^0_{\tilde{q}} \times K^0_{\tilde{q}}$ into $K^0_{\tilde{q}}$. Therefore, bilinear operator B is bounded from

$$(K^0_{\tilde{q}} \cap K^{\max\{\frac{d}{p}-1,0\}}_{\tilde{q}}) \times (K^0_{\tilde{q}} \cap K^{\max\{\frac{d}{p}-1,0\}}_{\tilde{q}}) \text{ into } (K^0_{\tilde{q}} \cap K^{\max\{\frac{d}{p}-1,0\}}_{\tilde{q}}).$$

Applying Theorem 3 to the bilinear operator B, we deduce that there exists a positive constant $\delta_{p,\tilde{q},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^{\frac{d}{p}-1}_p(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying

$$\big\|e^{t\Delta}u_0\big\|_{K^0_{\tilde{q},T}\cap K^{\max\{\frac{d}{p}-1,0\}}_{\tilde{q},T}} \leq \delta_{p,\tilde{q},d} \tag{30}$$

NSE has a unique mild solution u satisfying

$$u \in K^0_{\tilde{q},T} \cap K^{\max\{\frac{d}{p}-1,0\}}_{\tilde{q},T}. \tag{31}$$

Assuming that the inequality (30) is valid, we prove that

$$u \in K^s_{\tilde{q},T} \text{ for all } s \geq \frac{d}{p} - 1. \tag{32}$$

Indeed, from (31) applying Lemmas 7 (b) and 9 to obtain

$$u \in K^s_{\tilde{q},T} \text{ for all } s \in [\frac{d}{p} - 1, \max\{\frac{d}{p} - 1, 0\} + 1 - \frac{d}{\tilde{q}})$$

Note that $1-\frac{d}{\tilde q}>0$, applying again Lemmas 7 (b) and 9 to get

$$u \in K^s_{\tilde q,T} \ \text{ for all } s \in \big[\frac{d}{p}-1, \max\{\frac{d}{p}-1,0\}+2(1-\frac{d}{\tilde q})\big).$$

By induction, we obtaine

$$u \in K^s_{\tilde q} \ \text{ for all } n\in\mathbb{N} \text{ and } s \in \big[\frac{d}{p}-1, \max\{\frac{d}{p}-1,0\}+n(1-\frac{d}{\tilde q})\big),$$

therefore, the relation (32) is valid.
We prove that

$$u \in K^s_{q,T} \ \text{ for all } s \geq \frac{d}{p}-1 \text{ and } q>p. \tag{33}$$

Indeed, let s be fixed in $\frac{d}{p}-1\leq s<\infty$, applying Lemmas 7 (b) and 8, by an argument similar to the previous one, we get

$$u \in K^s_{q,T} \ \text{ for all } q>p, \tag{34}$$

therefore, the relation (34) is valid. Applying again Lemma 8 to get

$$B(u,u) \in K^s_{q,T} \ \text{ for all } s \geq \frac{d}{p}-1 \text{ and } q\geq p. \tag{35}$$

From $u=e^{t\Delta}u_0+B(u,u)$, the definition of $K^s_{q,T}$, the relation (35) and Lemma 7 (a), we deduce that (i) is valid.
(ii) We have proven that (ii) is valid for $n=0$. We will prove that (ii) is valid for $n=1$. Applying Lemma 2 and the Calderon-Zygmund theorem to obtain

$$\begin{aligned} t^{\frac{1}{2}(s+1+2-\frac{d}{q})}\big\|\dot\Lambda^s u_t\big\|_q &\lesssim t^{\frac{1}{2}(s+1+2-\frac{d}{q})}\big\|\dot\Lambda^{s+2}u\big\|_q + t^{\frac{1}{2}(s+1+2-\frac{d}{q})}\big\|\mathbb{P}\frac{\nabla}{\dot\Lambda}\dot\Lambda^{s+1}(u\otimes u)\big\|_q \lesssim \\ & t^{\frac{1}{2}(s+1+2-\frac{d}{q})}\big\|\dot\Lambda^{s+2}u\big\|_q + t^{\frac{1}{2}(s+1+2-\frac{d}{q})}\big\|\Lambda^{s+1}(u\otimes u)\big\|_q \lesssim \\ & t^{\frac{1}{2}(s+1+2-\frac{d}{q})}\big\|\dot\Lambda^{s+2}u\big\|_q + t^{\frac{1}{2}(s+2-\frac{d}{2q})}\big\|\Lambda^{s+1}u\big\|_{2q} t^{\frac{1}{2}(1-\frac{d}{2q})}\big\|u\big\|_{2q}<\infty, \end{aligned} \tag{36}$$

the last inequality in (36) is deduced by applying (ii) for $n=0$.
We prove that (ii) is valid for $n=2$. Applying again Lemma 2 and the Calderon-Zygmund theorem to obtain

$$\begin{aligned} t^{\frac{1}{2}(s+1+4-\frac{d}{q})}\big\|\dot\Lambda^s u_{tt}\big\|_q &\lesssim t^{\frac{1}{2}(s+2+1+2-\frac{d}{q})}\big\|\dot\Lambda^{s+2}u_t\big\|_q + t^{\frac{1}{2}(s+1+4-\frac{d}{q})}\big\|\dot\Lambda^{s+1}D_t(u\otimes u)\big\|_q \\ &\lesssim t^{\frac{1}{2}(s+2+1+2-\frac{d}{q})}\big\|\dot\Lambda^{s+2}u_t\big\|_q + t^{\frac{1}{2}(s+1+1+2-\frac{d}{2q})}\big\|\dot\Lambda^{s+1}u_t\big\|_{2q} t^{\frac{1}{2}(1-\frac{d}{2q})}\big\|u\big\|_{2q} + \\ & t^{\frac{1}{2}(1+2-\frac{d}{2q})}\big\|u_t\big\|_{2q} t^{\frac{1}{2}(s+1+1-\frac{d}{2q})}\big\|\dot\Lambda^{s+1}u\big\|_{2q}<\infty, \end{aligned} \tag{37}$$

the last inequality in (36) is deduced by applying (ii) for $n = 1$.
By continuing this procedure, we can prove that (ii) is valid for all $n \in \mathbb{N}$.
(iii) Applying Lemma 2 and the Calderon-Zygmund theorem we have

$$t^{\frac{1}{2}(s+2-\frac{d}{q})}\big\|\dot{\Lambda}^s p\big\|_q = t^{\frac{1}{2}(s+2-\frac{d}{q})}\big\|\frac{\nabla\otimes\nabla}{\Delta}\dot{\Lambda}^s(u\otimes u)\big\|_q \lesssim t^{\frac{1}{2}(s+2-\frac{d}{q})}\big\|\dot{\Lambda}^s(u\otimes u)\big\|_q$$
$$\lesssim t^{\frac{1}{2}(s+1-\frac{d}{2q})}\big\|\dot{\Lambda}^s u\big\|_{2q} t^{\frac{1}{2}(1-\frac{d}{2q})}\big\|u\big\|_{2q} < \infty. \quad (38)$$

The last inequality in (37) is deduced by applying (i).
Finally, we will show that the inequality (30) is valid when T is small enough. We first prove that

$$e^{\Delta}u_0 \in K^0_{\tilde{q},T} \cap K^{\max\{\frac{d}{p}-1,0\}}_{\tilde{q},T}. \quad (39)$$

Indeed, we consider two cases $p \geq d$ and $p < d$. In the case $p > d$, applying Lemma 7 (b) obtain $e^{\Delta}u_0 \in K^0_{\tilde{q},T}$. Therefore, the relation (39) is valid. In the case $p \leq d$, we invoke Lemma 7 (b) to deduce that $e^{\Delta}u_0 \in K^{\frac{d}{p}-1}_{\tilde{q},T}$, using Lemma 4 we get $u_0 \in L^d$ then applying Lemma 7 (b) to obtain $e^{\Delta}u_0 \in K^0_{\tilde{q},T}$. Therefore, the relation (39) is valid.
From the definition of $\mathcal{K}^s_{q,T}$ and the relation (39), we deduce that the left-hand side of the inequality (30) converges to 0 when T tends to 0. Therefore the inequality (30) holds for arbitrary $u_0 \in \dot{H}^{\frac{d}{p}-1}_p$ when $T(u_0)$ is small enough.
(B) Applying Lemma 6, we deduce that the three quantities

$$\big\|u_0\big\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}}, \big\|e^{t\Delta}u_0\big\|_{\mathcal{K}^0_{\tilde{q},\infty}}, \text{ and } \big\|e^{t\Delta}u_0\big\|_{\mathcal{K}^{\max\{\frac{d}{p}-1,0\}}_{\tilde{q},\infty}}$$

are equivalent, then there exists a positive constant $\sigma_{p,\tilde{q},d}$ such that if $\big\|u_0\big\|_{\dot{B}^{\frac{d}{\tilde{q}}-1,\infty}_{\tilde{q}}} \leq \sigma_{p,\tilde{q},d}$ the inequality (30) holds for $T = \infty$. Therefore, in point (A) we can take $T = +\infty$.
(a) From Lemma 7 (a), we only need to prove

$$\lim_{t\to\infty} t^{\frac{1}{2}(s+1-\frac{d}{q})}\big\|w\big\|_{\dot{H}^s_q} = 0, \text{ where } w = B(u,u). \quad (40)$$

Indeed, this is deduced from the relation (35) and the definition of $K^s_{q,\infty}$.
(b) For $n = 1$, (b) is deduced from (a) and the inequality (36). In the case $n = 2$, (b) is deduced by using the inequality (37) and applying (b) for $n = 1$. By continuing this procedure, we can prove that (b) is valid for all $n \in \mathbb{N}$.
(c) Using (a) and the inequality (38), we deduce that (c) is valid. □

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