On integral separation of bounded linear random differential equations

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Abstract

Our aim in this paper is to investigate the openness and denseness of the set of integrally separated systems in the space of bounded linear random differential equations equipped with the L^∞-metric. We show that in the general case, the set of integrally separated systems is open and dense. An exception is the case when the base space is isomorphic to the ergodic rotation flow of the unit circle, in which the set of integrally separated systems is open but not generic.

1 Introduction

A continuous-time linear deterministic differential equation is of the form

$$\dot{x} = Ax, \qquad \text{where } A \in \mathbb{R}^{d\times d}. \tag{1}$$

The Lyapunov spectrum of (1) is the set of real part of eigenvalues of A. It is well-known that the eigenvalues depend continuously on A (see Kato [18]). Hence, the linear deterministic differential equations with simple Lyapunov spectrum, i.e. the spectrum has exact d distinct exponents, form an open (but not dense) set in the space of all systems. In the case that (1) has simple Lyapunov spectrum, the state space $\mathbb{R}^d$ can be decomposed into a direct sum of one dimensional subspaces

$$\mathbb{R}^d = E_1 \oplus E_2 \oplus \cdots \oplus E_d$$

with the property that the asymptotical behavior of solutions of (1) starting from subspaces E_i are uniformly separated in the sense that there exist positive numbers K, ε such that for $i < j$ with $i, j \in \{1, \ldots, d\}$ and $t \geq 0$

$$\frac{\|e^{tA}v_i\|}{\|v_i\|} \geq Ke^{\varepsilon t}\frac{\|e^{tA}v_j\|}{\|v_j\|}, \quad \text{where } v_i \in E_i \setminus \{0\}, v_j \in E_j \setminus \{0\}.$$

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A natural extension of the above analyzed uniform separation of asymptotical dynamics of autonomous systems to nonautonomous systems leads to a notion of integral separation. In [20], Millionshchikov showed that in the space of bounded piecewise-constant linear differential equations equipped with C^0-topology, the set of systems being integrally separated is dense and open. Later, the notion of integral separation and exponential dichotomy for shifted equations give rise to an adequate spectral theory for nonautonomous linear differential equations, see Sacker & Sell [22], Palmer [21], Siegmund [23] and Kloeden & Rasmussen [19].

The objects of interest in this paper are bounded linear random differential equations, i.e. a special class of nonautonomous linear differential equations in which the base space consists of randomness and is modelled by an ergodic flow on a probability space. Our main aim is to answer the question how often bounded linear random differential equations exhibit an integral separation. This question is motivated from some well-developed research areas:

Firstly, we mention some research achievements on nonautonomous differential equations whose base spaces are modelled as the Kronecker flow associated with an irrationally dependent frequency on a torus. It was showed in [13, 14, 15] that there is a Baire residual set of frequencies with the property that for a fixed frequency in this set, the set of bounded traceless linear systems having an exponential dichotomy is C^0-dense. An improvement of this result was obtained in Avila, Bochi & Damanik [6] which showed that for an arbitrary irrational independent frequency the set of bounded traceless linear systems having an exponential dichotomy is C^0-dense, see also Fabbri, Johnson & Zampogni [16].

Secondly, in the discrete-time case, i.e. product of random matrices, it was proved in Cong [11] that the set of integrally separated linear cocycle is open and dense in the space of bounded linear cocycles equipped with L^∞-metric. It is also pointed out in Cong & Doan [12] that this result cannot be extended to the space of linear cocycles satisfying integrability condition of the multiplicative ergodic theorem. In contrast to L^∞-metric, Arbieto & Bochi [3] showed that L^p-generic linear cocycles has one-point spectrum. As a consequence, the set of integrally separated linear cocycles is L^p-meagre.

Concerning with continuous-time linear random differential equations, there have been a few contributions to the genericity of Lyapunov spectrum and uniform hyperbolicity. We mention here the work in Bessa [7] in which he showed that for a C^0-generic subset of all the 2-dimensional conservative systems, either Lyapunov exponents are zero or there is a dominated splitting (an equivalent notion of integral separation). An extension of this result to arbitrary dimensional spaces was established in Bessa [8]. Recently, it has been established in Bessa & Vilarinho [9] that L^p-generic linear random differential equations has one-point spectrum.

In this paper, our goal is to extend the results on genericity of integral separation from discrete-time to continuous time in L^∞-metric. Note that coming from discrete-time setting to continuous-time setting, the main technical problem arose is the null-set property. For example, when we want to use the Millionshchikov perturbation method (see Bochi & Viana [10] and Cong [11]), we need to perturb a system along a specific orbit. For discrete-time systems, null-set property is not an issue because any countable union of null-sets will be again a null-set. However, for continuous-time systems, along any orbit the union of null-set can be no longer a null set. To get rid of this difficulty, we use Ambrose's representation theorem of ergodic flow and deal with the induced random dynamical systems. Depending on the structure of the base space in the representation theorem, we completely characterize the openness and denseness of the set of integrally separated systems.

The paper is organized as follows: In section 2, we introduce the space of linear bounded random differential equations and the spectral theory of these equations. Section 3 is devoted to our main result in this paper about denseness and openness of integrally separated linear bounded random differential equations. The main ingredients in the proof of the main result are the representation of ergodic flow and the result about denseness of integrally separated linear cocycles. These materials are collected in the Appendix.

2 Linear bounded random differential equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Lebesgue probability space and $(\theta_t)_{t\in\mathbb{R}}$ be an ergodic flow from Ω into itself preserving the probability $\mathbb{P}$ (see Appendix for more details of the definition of ergodic flow). Suppose further that $(\theta_t)_{t\in\mathbb{R}}$ has no fixed point. Let $\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ denote the space of bounded measurable matrix-valued maps $A : \Omega \to \mathbb{R}^{d\times d}$ satisfying that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega)\| < \infty.$$

We endow $\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ with the L^∞-metric ρ_∞ defined by

$$\rho_\infty(A, B) := \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega) - B(\omega)\|.$$

It is well known that the metric space $(\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}), \rho_\infty)$ is complete, see e.g., Arnold & Cong [5]. For each $A \in \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$, we consider the corresponding linear random differential equation of the following form (see Arnold [4, Subsection 2.2])

$$\dot{x} = A(\theta_t\omega)x \qquad \text{for } \omega \in \Omega, t \in \mathbb{R}. \tag{2}$$

Let $\Phi_A(t, \omega)x$ denote the solution of (2) satisfying $x(0) = x$, i.e. $\Phi_A(t, \omega)$ satisfies the following integral equation

$$\Phi_A(t, \omega) = \mathrm{id} + \int_0^t A(\theta_s\omega)\Phi_A(s, \omega)\, ds \quad \text{for all } t \in \mathbb{R}, \tag{3}$$

where id denotes the identity matrix of size d. Thus, the linear mapping $\Phi_A : \mathbb{R} \times \Omega \to \mathbb{R}^{d\times d}$ is a continuous random dynamical system, i.e. Φ_A is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}^{d\times d}))$ measurable and the following properties hold:

(i) $\Phi_A(0, \omega) = \mathrm{id}$,

(ii) $\Phi_A(t + s, \omega) = \Phi_A(t, \theta_s\omega)\Phi_A(s, \omega)$ for all $t, s \in \mathbb{R}$, $\omega \in \Omega$,

(iii) For each $\omega \in \Omega$, the mapping $t \mapsto \Phi_A(t, \omega)$ is continuous,

see e.g., Arnold [4, Subsection 2.2]. In order to provide some estimates on $\|\Phi_A(t, \omega)\|$, we need the following technical lemma. This lemma can be considered as a L^∞-version of Anrold [4, Lemma 2.2.5].

Lemma 1. Let $A \in \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ and define $K := \operatorname{ess\,sup}_{\omega\in\Omega} \|A(\omega)\|$. Then, for any $\varepsilon > 0$ there exists a measurable set $\widehat{\Omega} \subset \Omega$ (depending on ε) being of full measure and invariant under the flow $(\theta_t)_{t\in\mathbb{R}}$, i.e. $\theta_t\widehat{\Omega} \subset \widehat{\Omega}$ for all $t \in \mathbb{R}$, such that for $\alpha, \beta \in \mathbb{R}$ with $\alpha \le \beta$

$$\int_\alpha^\beta \|A(\theta_s\omega)\|\, ds \le (K + \varepsilon)(\beta - \alpha) \quad \text{for all } \omega \in \widehat{\Omega}. \tag{4}$$

Proof. Choose and fix an arbitrary $\varepsilon > 0$. Let $\ell \in \mathbb{Q}_{>0}$ be an arbitrary positive rational number. Define the following set

$$\mathcal{M}_\ell := \left\{ \omega \in \Omega : \int_a^b \|A(\theta_s\omega)\| \, ds \le (K+\varepsilon)\ell \quad \text{for all } b-a=\ell \right\}. \tag{5}$$

By continuity of Lebesgue integral, we have

$$\mathcal{M}_\ell := \bigcap_{a\in\mathbb{Q}} \left\{ \omega \in \Omega : \int_a^{a+\ell} \|A(\theta_s\omega)\| \, ds \le (K+\varepsilon)\ell \right\},$$

which implies that the set $\mathcal{M}_\ell$ is measurable. From the following observation

$$\int_a^b \|A(\theta_s \circ \theta_t\omega)\| \, ds = \int_{a+t}^{b+t} \|A(\theta_s\omega)\| \, ds \qquad \text{for all } t \in \mathbb{R},$$

we derive that

$$\theta_t \mathcal{M}_\ell \subset \mathcal{M}_\ell \qquad \text{for all } t \in \mathbb{R}.$$

Thus, by ergodicity of the flow $(\theta_t)_{t\in\mathbb{R}}$, either $\mathbb{P}(\mathcal{M}_\ell) = 0$ or $\mathbb{P}(\mathcal{M}_\ell) = 1$ holds. We now prove that $\mathbb{P}(\mathcal{M}_\ell) = 1$ by supposing the contrary that $\mathbb{P}(\mathcal{M}_\ell) = 0$. Let $\delta := \frac{\varepsilon\ell}{2K} > 0$ and for each $k \in \mathbb{Z}$ we define

$$\mathcal{N}_{k,\ell} := \left\{ \omega \in \Omega : \int_a^{a+\ell} \|A(\theta_s\omega)\| \, ds \ge (K+\varepsilon)\ell \text{ for some } a \in [k\delta, (k+1)\delta] \right\}.$$

Using analogous arguments as in the proof of measurability of $\mathcal{M}_\ell$ above, the set $\mathcal{N}_{k,\ell}$ is also measurable. By definition of $\mathcal{M}_\ell$ as in (5), we have $\mathcal{M}_\ell^c = \bigcup_{k\in\mathbb{Z}} \mathcal{N}_{k,\ell}$, where $\mathcal{M}_\ell^c$ is the complement of $\mathcal{M}_\ell$. Thus, $\mathbb{P}(\mathcal{M}_\ell^c) = 1$ implies that there exists $k \in \mathbb{Z}$ such that $\mathbb{P}(\mathcal{N}_{k,\ell}) > 0$. Note that $\mathcal{N}_{k,\ell}$ is a subset of the following set

$$\mathcal{N} := \left\{ \omega \in \Omega : \int_{k\delta}^{(k+1)\delta+\ell} \|A(\theta_s\omega)\| \, ds \ge (K+\varepsilon)\ell \right\}. \tag{6}$$

By measurability of the maps $(s,\omega) \mapsto \theta_s\omega$ and $\omega \mapsto A(\omega)$, we have that the map $(s,\omega) \mapsto \|A(\theta_s\omega)\|$ is also measurable. Thus, using Fubini's Theorem, we obtain that

$$\int_{k\delta}^{(k+1)\delta+\ell} \int_{\mathcal{N}} \|A(\theta_s\omega)\| \, d\mathbb{P}(\omega) \, ds = \int_{\mathcal{N}} \int_{k\delta}^{(k+1)\delta+\ell} \|A(\theta_s\omega)\| \, ds \, d\mathbb{P}(\omega),$$

which together with (6) implies that

$$\int_{k\delta}^{(k+1)\delta+\ell} \int_{\mathcal{N}} \|A(\theta_s\omega)\| \, d\mathbb{P}(\omega) \, ds \geq (K+\varepsilon)\ell \, \mathbb{P}(\mathcal{N}). \tag{7}$$

From the fact that $K = \operatorname{ess\,sup}_{\omega\in\Omega} \|A(\omega)\|$ we derive that

$$\int_{\mathcal{N}} \|A(\theta_s\omega)\| \, d\mathbb{P}(\omega) \leq K \, \mathbb{P}(\mathcal{N}) \quad \text{for all } s \in [k\delta, (k+1)\delta + \ell],$$

which gives

$$\int_{k\delta}^{(k+1)\delta+\ell} \int_{\mathcal{N}} \|A(\theta_s\omega)\| \, d\mathbb{P}(\omega) \, ds \leq (\delta+\ell)K \, \mathbb{P}(\mathcal{N}).$$

Since $\delta = \frac{\varepsilon\ell}{2K}$ it follows that $(\delta+\ell)K < (K+\varepsilon)\ell$. Thus, the preceding inequality provides a contradiction to (7). Therefore, $\mathbb{P}(\mathcal{M}_\ell) = 1$ for all $\ell \in \mathbb{Q}_{\geq 0}$. Hence, the measurable set $\widehat{\Omega} := \bigcap_{\ell \in \mathbb{Q}_{>0}} \mathcal{M}_\ell$ is of full measure and also invariant under all θ_t for $t \in \mathbb{R}$. To conclude the proof, we prove that $\widehat{\Omega}$ satisfies (4). For this purpose, let $\omega \in \widehat{\Omega}$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Let $(q_n)_{n\in\mathbb{N}}$ be a sequence of positive rational numbers such that $\lim_{n\to\infty} q_n = \beta - \alpha$. Since $\omega \in \widehat{\Omega}$ it follows that $\omega \in \mathcal{M}_{q_n}$ for all $n \in \mathbb{N}$. Therefore,

$$\int_{\alpha}^{\alpha+q_n} \|A(\theta_s\omega)\| \, ds \leq (K+\varepsilon)q_n.$$

Letting $n \to \infty$ in the above inequality, we obtain that

$$\int_{\alpha}^{\beta} \|A(\theta_s\omega)\| \, ds \leq (K+\varepsilon)(\beta-\alpha),$$

which completes the proof. □

According to Lemma 1, replacing Ω by a set of full measure being invariant under $(\theta_t)_{t\in\mathbb{R}}$, we can assume from now on that

$$\int_{\alpha}^{\beta} \|A(\theta_s\omega)\| \, ds \leq \hat{K}(\beta-\alpha) \quad \text{for all } \omega \in \Omega, \alpha, \beta \in \mathbb{R}, \tag{8}$$

where $\hat{K} > \operatorname{ess\,sup}_{\omega\in\Omega} \|A(\omega)\|$ is a positive constant. In the following corollary, we establish some estimates on the norm of the solution $\Phi_A(t,\omega)$.

Corollary 2. For all $\omega \in \Omega$ and $t \in \mathbb{R}$, we have

$$\|\Phi_A(t,\omega)\| \le \exp\left(|t|\hat{K}\right),$$

where $\hat{K}$ is a positive constant satisfying (8).

Proof. Choose and fix $\omega \in \Omega$. Let $\delta > 0$ be arbitrary such that $\hat{K}\delta < 1$. From (3), we obtain that

$$\|\Phi_A(\tau,\omega)\| \le 1 + \left|\int_0^\tau \|A(\theta_s\omega)\|\|\Phi_A(s,\omega)\|\, ds\right| \quad \text{for } \tau \in [0,\delta],$$

which together with (8) implies that

$$\max_{0\le s\le\delta} \|\Phi_A(s,\omega)\| \le 1 + \hat{K}\delta \max_{0\le s\le\delta} \|\Phi_A(s,\omega)\|.$$

Note that the maximum $\max_{0\le s\le\delta} \|\Phi_A(s,\omega)\|$ exists due to continuity of the map $s \mapsto \Phi_A(s,\omega)$. Since $\hat{K}\delta < 1$ it follows that

$$\max_{0\le s\le\delta} \|\Phi_A(s,\omega)\| \le \frac{1}{1-\hat{K}\delta}. \tag{9}$$

Now let $t > 0$ be arbitrary. We choose $n \in \mathbb{N}$ such that $\hat{K}\frac{t}{n} < 1$. By cocycle property we have

$$\Phi_A(t,\omega) = \prod_{k=0}^{n-1} \Phi_A(\frac{t}{n}, \theta_{k\frac{t}{n}}\omega).$$

Hence, using (9), we obtain

$$\|\Phi_A(t,\omega)\| \le \left(\frac{1}{1-\hat{K}\frac{t}{n}}\right)^n.$$

Letting $n \to \infty$ yields that $\|\Phi_A(t,\omega)\| \le \exp(\hat{K}t)$. Similarly, for the case $t < 0$ we also obtain that $\|\Phi_A(t,\omega)\| \le \exp(-\hat{K}t)$. The proof is complete. □

Using Corollary 2, the generated RDS $\Phi_A(t,\omega)$ satisfies the integrability condition of the multiplicative ergodic theorem, i.e.

$$\log^+ \|\alpha^+(\cdot)\|, \log^+ \|\alpha^-(\cdot)\| \in \mathcal{L}^1(\Omega,\mathcal{F},\mathbb{P})^1,$$

[1] A measurable function $f \in L^1(\Omega,\mathcal{F},\mathbb{P})$ if $\int_\Omega f(\omega)\, d\mathbb{P}(\omega)$ is finite.

where $\log^+ x := \max(0, \log x)$ and $\alpha^+(\omega) := \sup_{0\le t\le 1} \|\Phi_A(t,\omega)\|, \alpha^-(\omega) := \sup_{-1\le t\le 0} \|\Phi_A(t,\omega)\|$. So, we arrive at the following result on the asymptotical behavior of the solution $\Phi_A(t,\omega)v$ for an arbitrary $v \in \mathbb{R}^d \setminus \{0\}$ as $t \to \pm\infty$.

Theorem 3 (Multiplicative Ergodic Theorem for Linear Random Differential Equations)**.** Let Φ_A denote the linear random dynamical system generated by (2). Then, there exist p, where $1 \le p \le d$, non-random Lyapunov exponents $\lambda_p < \lambda_{p-1} < \cdots < \lambda_1$ and an invariant measurable decomposition

$$\mathbb{R}^d = W_1(\omega) \oplus W_2(\omega) \oplus \cdots \oplus W_p(\omega)$$

with the property that for $k = 1, \ldots, p$ the linear measurable subspace $W_k(\omega)$ is dynamically characterized by

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \|\Phi_A(t,\omega)v\| = \lambda_k \quad \text{iff} \quad v \in W_k(\omega) \setminus \{0\}.$$

Proof. See Arnold [4, Theorem 3.4.11]. □

3 Integral separation of bounded linear random differential equations

We start this section by extending the notion of integral separation of linear nonautonomous differential equations, see e.g., Adrianova [1, Definition 5.3.2], to linear random differential equations.

Definition 4 (Integral Separation)**.** System (2) is called *integrally separated* if there exist a set of full measure $\widehat{\Omega}$ being invariant under $(\theta_t)_{t\in\mathbb{R}}$, positive constants L, α and an invariant measurable decomposition

$$\mathbb{R}^d = W_1(\omega) \oplus W_2(\omega) \oplus \cdots \oplus W_d(\omega)$$

such that for $i = 1, \ldots, d$ the following assertions hold:

(i) $\dim W_i(\omega) = 1$ for all $\omega \in \widehat{\Omega}$,

(ii) For all $\omega \in \widehat{\Omega}$, $u \in \bigoplus_{j=1}^{i} W_j(\omega) \setminus \{0\}$ and $v \in \bigoplus_{j=i+1}^{d} W_j(\omega) \setminus \{0\}$ we have

$$\frac{\|\Phi_A(t,\omega)u\|}{\|u\|} \ge Le^{t\alpha} \frac{\|\Phi_A(t,\omega)v\|}{\|v\|}, \qquad t \in \mathbb{R}_{\ge 0}.$$

Let $\mathcal{R} \subset \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ be the set of bounded measurable matrices A such that the associated linear random differential equation $\dot{x} = A(\theta_t\omega)x$ is integrally separated. In this section, we focus on the question: How big is the set $\mathcal{R}$ in the complete metric space $(\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}), \rho_\infty)$? The answer of this question depends on the structure of the base space $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. Before going to formulate this answer, we need the following preparation:

Firstly, in the whole section we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue probability space and $(\theta_t)_{t\in\mathbb{R}}$ is an ergodic flow from Ω into itself without fixed point preserving the probability $\mathbb{P}$. Hence, according to Theorem 9 in the Appendix, there exists a Lebesgue probability space $(X, \mathcal{A}, m)$ and an ergodic transformation $T : X \to X$ preserving the probability m and a measurable function $f : X \to \mathbb{R}_{\geq 0}$ which is bounded from 0 and ∞ such that the flow $(S_t)_{t\in\mathbb{R}}$ built under a function (X, T, m, f) is isomorphic to the flow $(\theta_t)_{t\in\mathbb{R}}$, i.e. there exists a measure preserving bijective transformation $H : \Omega \to B$, where $B = \{(x, t) : 0 \leq t < f(x)\}$, such that

$$H \circ \theta_t(\omega) = S_t \circ H(\omega) \qquad \text{for all } \omega \in \Omega. \tag{10}$$

Since f is bounded from 0 and ∞, it follows that

$$\overline{c} := \operatorname*{ess\,sup}_{x\in X} f(x) \in (0, \infty), \quad \underline{c} := \operatorname*{ess\,inf}_{x\in X} f(x) \in (0, \infty). \tag{11}$$

Finally, since $(X, \mathcal{F}, m)$ is a Lebesgue probability space, $(X, \mathcal{F}, m)$ is of one of the following cases:

Case A: $(X, \mathcal{F}, m)$ is isomorphic to $([0, 1], \lambda)$, where λ is the standard Lebesgue probability on $[0, 1]$.

Case B: The set X is isomorphic to a probability space $[0, s] \cup \{x_1, x_2, \dots, x_k\}$, where $s = 1 - \sum_{i=1}^k p_i$ with p_i is the probability of $\{x_i\}$ and k can be equal to ∞.

Remark 5. Suppose that **Case B** holds. Since T is an invertible ergodic transformation on X, it follows that X has a finite atoms and from this we derive that X consists of only atoms. Hence, X is isomorphic to the set $\{x_1, \dots, x_k\}$ with the probability of x_i is $\frac{1}{k}$. Consequently, the base space $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is isomorphic to $(\mathcal{S}^1, \mathcal{B}(\mathcal{S}^1), \lambda, (R_t)_{t\in\mathbb{R}})$, where $\mathcal{S}^1 := \{e^{2\pi i z} : z \in [0, 1]\}$ is the unit circle, λ is the standard Lebesgue probability on $\mathcal{S}^1$ and $R_t : \mathcal{S}^1 \to \mathcal{S}^1$ is the rotation map defined by $R_t(e^{2\pi i z}) = e^{2\pi i(z+t)}$.

3.1 Case A

Theorem 6. Suppose that **Case A** holds. Then, we have:

(i) The set $\mathcal{R}$ is dense in $(\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}), \rho_\infty)$,

(ii) The set $\mathcal{R}$ is open in $(\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}), \rho_\infty)$.

As a consequence, the set of bounded linear random differential equations being integrally separated is generic.

Proof. (i) Let $A \in \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ and $\varepsilon > 0$ be arbitrary. Our aim is to construct $A_\varepsilon \in \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ with $\rho_\infty(A, A_\varepsilon) < \varepsilon$ such that the linear differential equation $\dot{x} = A_\varepsilon(\theta_t\omega)x$ is integrally separated. For this purpose, we define a measurable matrix-valued function $\mathcal{M} : X \to \mathbb{R}^{d\times d}$ by

$$\mathcal{M}(x) := \Phi_A(f(x), H^{-1}(x,0)) \qquad \text{for all } x \in X. \tag{12}$$

By Corollary 2, we have

$$\operatorname*{ess\,sup}_{x\in X} \|\mathcal{M}(x)\|, \operatorname*{ess\,sup}_{x\in X} \|\mathcal{M}(x)^{-1}\| \le \exp\left(\bar{c}\hat{K}\right), \tag{13}$$

where $\bar{c}$ is defined as in (11). Thus, $\mathcal{M}$ and $\mathcal{M}^{-1}$ are essentially bounded. Let δ is a positive number satisfying that

$$\delta < \frac{1}{\exp\left(\bar{c}\hat{K}\right)} \text{ and } \delta \frac{\exp(\bar{c}\hat{K})}{1 - \delta\exp(\bar{c}\hat{K})}\left(\frac{1}{\underline{c}} + 2\operatorname*{ess\,sup}_{\omega\in\Omega}\|A(\omega)\|\right) < \varepsilon. \tag{14}$$

In view of Theorem 10 in the Appendix, there exists a linear random mapping $\widehat{\mathcal{M}} : X \to \mathbb{R}^{d\times d}$ such that

$$\operatorname*{ess\,sup}_{x\in X} \|\mathcal{M}(x) - \widehat{\mathcal{M}}(x)\| + \operatorname*{ess\,sup}_{x\in X} \|\mathcal{M}(x)^{-1} - \widehat{\mathcal{M}}(x)^{-1}\| \le \delta, \tag{15}$$

and the generated linear cocyle $\Phi_{\widehat{\mathcal{M}}} : \mathbb{Z} \times X \to \mathbb{R}^{d\times d}$ over the metric dynamical system $(X, \mathcal{A}, m, T)$ is integrally separated. For any $x \in X$, we set

$$R(x,t) := \mathrm{id} - \frac{t}{f(x)}(\mathrm{id} - \widehat{\mathcal{M}}(x)\mathcal{M}(x)^{-1}), \quad t \in [0, f(x)].$$

Combining (13) and (15) yields that for m-a.e. $x \in X$

$$\|\widehat{\mathcal{M}}(x)\mathcal{M}(x)^{-1} - \mathrm{id}\| \le \delta \exp\left(\overline{c}\hat{K}\right),$$

which implies that for $t \in [0, f(x)]$

$$\|R(x,t) - \mathrm{id}\| \le \delta \exp\left(\overline{c}\hat{K}\right), \qquad \left\|\frac{\partial R}{\partial t}(x,t)\right\| \le \frac{\delta}{\underline{c}} \exp\left(\overline{c}\hat{K}\right). \tag{16}$$

Note that $\delta \exp\left(\overline{c}\hat{K}\right) < 1$ and thus $R(t,x)$ is invertible and we have

$$\|R(x,t)^{-1}\| \le \frac{1}{1 - \delta \exp\left(\overline{c}\hat{K}\right)}, \qquad \|R(x,t)^{-1} - \mathrm{id}\| \le \frac{\delta \exp\left(\overline{c}\hat{K}\right)}{1 - \delta \exp\left(\overline{c}\hat{K}\right)}. \tag{17}$$

We divide the remaining of the proof of this assertion into two steps:

Step 1: In this step, we construct $\widehat{A} : \Omega \to \mathbb{R}^{d\times d}$ such that $\rho_\infty(A, \widehat{A}) < \varepsilon$ and

$$\widehat{\mathcal{M}}(x) = \Phi_{\widehat{A}}(f(x), H^{-1}(x,0)) \qquad \text{for all } x \in X. \tag{18}$$

For this purpose, we define for all $(x,t) \in B$

$$\widehat{A}(H^{-1}(x,t)) := R(x,t)A(H^{-1}(x,t))R(x,t)^{-1} + \frac{\partial R}{\partial t}(x,t)R(x,t)^{-1}.$$

Thus,

$$\begin{aligned}\|\widehat{A}(H^{-1}(x,t)) - A(H^{-1}(x,t))\| \;\le\; & \|R(x,t) - \mathrm{id}\|\|A(H^{-1}(x,t))R(x,t)^{-1}\| \\ & + \|A(H^{-1}(x,t))\|\|R(x,t)^{-1} - \mathrm{id}\| + \left\|\frac{\partial R}{\partial t}(x,t)R(x,t)^{-1}\right\|.\end{aligned}$$

Using (16) and (17), we obtain that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega) - \widehat{A}(\omega)\| \le \delta \frac{\exp(\overline{c}\hat{K})}{1 - \delta\exp(\overline{c}\hat{K})} \left(\frac{1}{\underline{c}} + 2 \operatorname*{ess\,sup}_{\omega\in\Omega} \|A(\omega)\|\right),$$

which together with (14) gives that $\rho_\infty(A, \widehat{A}) < \varepsilon$. To conclude the proof of this step, let $x \in X$ be arbitrary. By definition of $\widehat{A}$ and (10), for any $t \in [0, f(x))$ the equation

$$\dot{\xi} = \widehat{A}(\theta_t \circ H^{-1}(x,0))\xi = \widehat{A}(H^{-1}(x,t))\xi, \quad \xi(0) = \xi_0 \in \mathbb{R}^d$$

has the solution

$$\xi(t) = R(x,t)\Phi_A(t, H^{-1}(x,0))\xi_0.$$

Consequently,

$$\Phi_{\widehat{A}}(f(x), H^{-1}(x,0)) = R(x, f(x))\Phi_A(f(x), H^{-1}(x,0)).$$

By definition of R, equality (18) holds.

Step 2: In this step, we show that the linear random dynamical system $\Phi_{\widehat{A}}(t,\omega)$ is integrally separated. From (10) and (12), we derive that for any $x \in X$ and $n \in \mathbb{N}$

$$\begin{aligned}
\Phi_{\widehat{\mathcal{M}}}(n,x) &= \prod_{k=0}^{n-1} \widehat{\mathcal{M}}(T^k x) \\
&= \prod_{k=0}^{n-1} \Phi_{\widehat{A}}(f(T^{k-1}x), H^{-1}(T^{k-1}x, 0)) \\
&= \Phi_{\widehat{A}}(\sum_{k=0}^{n-1} f(T^k x), H^{-1}(x,0)).
\end{aligned}$$

Hence, from the fact that $\Phi_{\widehat{\mathcal{M}}}$ is integrally separated and f is strictly separated from 0 and ∞ we can easily deduce that the linear random dynamical system $\Phi_{\widehat{A}}(t,\omega)$ is also integrally separated. The proof of (i) is complete.

(ii) Let $A \in \mathcal{R}$ be arbitrary. Define a measurable matrix-valued function $\mathcal{M} : X \to \mathbb{R}^{d\times d}$ by

$$\mathcal{M}(x) := \Phi_A(f(x), H^{-1}(x,0)) \qquad \text{for all } x \in X.$$

Since the linear random dynamical system Φ_A is integrally separated and f is strictly separated from 0 and ∞ it follows that the discrete time linear random dynamical system $\Phi_{\mathcal{M}}$ generated by $\mathcal{M}$ over $(X, \mathcal{A}, m, T)$ is also integrally separated. Note that for discrete time linear random dynamical systems, integral separation is a robust property, see e.g., Cong & Doan [12]. Thus, there exists $\varepsilon < 1$ such that for all $\widehat{\mathcal{M}}$, satisfying that

$$\operatorname*{ess\,sup}_{x\in X} \|\mathcal{M}(x) - \widehat{\mathcal{M}}(x)\| + \operatorname*{ess\,sup}_{x\in X} \|\mathcal{M}(x)^{-1} - \widehat{\mathcal{M}}(x)^{-1}\| \le \varepsilon, \tag{19}$$

the generated discrete time linear random dynamical system $\Phi_{\widehat{\mathcal{M}}}$ is also integrally separated. Let $\widehat{A} \in \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ be arbitrary such that $\rho_\infty(A, \widehat{A}) < \widehat{\varepsilon}$, where $\widehat{\varepsilon} < 1$ is chosen later. Analog to Lemma 1, there exists $\tilde{K} > 0$ (independent of $\widehat{\varepsilon}$) and a set of full measure $\widehat{\Omega}$ being invariant under $(\theta_t)_{t\in\mathbb{R}}$ such that for all $\omega \in \widehat{\Omega}, \beta > \alpha$ we have

$$\int_\alpha^\beta \|A(\theta_s\omega)\|\, ds, \int_\alpha^\beta \|\widehat{A}(\theta_s\omega)\|\, ds \le \tilde{K}(\beta - \alpha)$$

and

$$\int_\alpha^\beta \|A(\theta_s\omega) - \widehat{A}(\theta_s\omega)\|\, ds \le 2\widehat{\varepsilon}(\beta - \alpha).$$

Note that

$$\Phi_A(t,\omega) - \Phi_{\widehat{A}}(t,\omega) = \int_0^t A(\theta_s\omega)\Phi_A(s,\omega) - \widehat{A}(\theta_s\omega)\Phi_{\widehat{A}}(s,\omega)\, ds,$$

which implies that

$$\begin{aligned} \|\Phi_A(t,\omega) - \Phi_{\widehat{A}}(t,\omega)\| &\le \int_0^t \|A(\theta_s\omega) - \widehat{A}(\theta_s\omega)\|\|\Phi_A(s,\omega)\|\, ds \\ &\quad + \int_0^t \|\widehat{A}(\theta_s\omega)\|\|\Phi_A(s,\omega) - \Phi_{\widehat{A}}(s,\omega)\|\, ds. \end{aligned}$$

Hence, by Corollary 2 we have

$$\max_{0\le s\le |t|} \|\Phi_A(s,\omega) - \Phi_{\widehat{A}}(s,\omega)\| \le 2\widehat{\varepsilon} e^{\tilde{K}|t|} + \tilde{K}|t| \max_{0\le s\le |t|} \|\Phi_A(s,\omega) - \Phi_{\widehat{A}}(s,\omega)\|.$$

Letting $|t| = \frac{1}{2\tilde{K}}$ yields that

$$\max_{0\le |s|\le \frac{1}{2\tilde{K}}} \|\Phi_A(s,\omega) - \Phi_{\widehat{A}}(s,\omega)\| \le 4\widehat{\varepsilon}\sqrt{e} \qquad \text{for all } \omega \in \widehat{\Omega}.$$

Hence, using the cocycle property we obtain that for m-a.e. $x \in X$

$$\|\Phi_A(f(x), H^{-1}(x,0)) - \Phi_{\widehat{A}}(f(x), H^{-1}(x,0))\| \le 4^n \widehat{\varepsilon} \exp(3/2 + 3\bar{c}\tilde{K})$$

and

$$\|\Phi_A(-f(x), H^{-1}(x,0)) - \Phi_{\widehat{A}}(-f(x), H^{-1}(x,0))\| \le 4^n \widehat{\varepsilon} \exp(3/2 + 3\bar{c}\tilde{K}),$$

where $\bar{c}$ is given as in (11) and n is the smallest integer such that $\frac{n}{2\tilde{K}} \ge \bar{c}$. Choose $\widehat{\varepsilon} := \frac{\varepsilon}{8^n \exp(3/2+3\bar{c}\tilde{K})}$. Then for all $\widehat{A} \in \mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d})$ with $\rho_\infty(\widehat{A}, A) < \widehat{\varepsilon}$ the linear discrete-time random dynamical system generated by $\Phi_{\widehat{A}}(f(x), H^{-1}(x,0))$ is integrally separated. Analog to proof of Step 2 of (i), the linear random dynamical system $\Phi_{\widehat{A}}(t,\omega)$ is also integrally separated. The proof is complete. $\square$

3.2 Case B

Theorem 7. Suppose that **Case B** holds. Then, the set $\mathcal{R}$ is open but not generic in $(\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}), \rho_\infty)$.

Proof. By virtue of Remark 5, the base space $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ can be identified with $(\mathcal{S}^1, \mathcal{B}(\mathcal{S}^1), \lambda, (R_t)_{t\in\mathbb{R}})$. Let $A \in (\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}), \rho_\infty)$ be arbitrary. By periodicity of the flow $(R_t)_{t\in\mathbb{R}}$, i.e. $R_{t+1} \equiv R_t$, we have

$$\Phi_A(t, \theta_s\omega) = \Phi_A(t, \theta_{s+1}\omega) \qquad \text{for all } s \in \mathbb{R}, \omega \in \mathcal{S}^1.$$

Let $\omega_0 = e^{2\pi i 0}$ and define $M := \Phi_A(1, \omega_0)$. Thus, for any $\omega \in \mathcal{S}^1$ there exists a unique $s \in [0, 1)$ such that $\omega = \theta_s\omega_0$ and therefore we have for all $t \geq 0$

$$\Phi_A(t, \omega) = \Phi_A(t + s - n, \omega_0)M^n\Phi_A(s, \omega_0)^{-1},$$

where n is the largest integer smaller or equal to $t+s$. Therefore, the generated linear random dynamical system Φ_A is integrally separated if and only if the set of the logarithms of the moduli of eigenvalues of the matrix M has exactly d elements. Due to the fact that eigenvalues depend continuously on the matrix, the set $\mathcal{R}$ is open. Also note that the multiplicity 2 of pairs of conjugate complex eigenvalues is also stable under small perturbation. Therefore, there exists an open set in $(\mathcal{L}^\infty(\Omega, \mathbb{R}^{d\times d}), \rho_\infty)$ which has no intersection with $\mathcal{R}$. The proof is complete. □

4 Appendix

4.1 Representation of ergodic flow

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Lebesgue probability space and $(\theta_t)_{t\in\mathbb{R}}$ be an ergodic flow preserving the probability measure $\mathbb{P}$, i.e. the map $\Theta : \mathbb{R} \times \Omega \to \Omega, (t, \omega) \mapsto \theta_t\omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$ measurable and the following properties hold:

(i) $\theta_0 = \text{id}$ and $\theta_{t+s} = \theta_t \circ \theta_s$ for $t, s \in \mathbb{R}$,

(ii) For all $t \in \mathbb{R}$ and $U \in \mathcal{F}$, we have $\mathbb{P}(\theta_t U) = \mathbb{P}(U)$,

(iii) Let U be a measurable set satisfying that $\theta_t^{-1}U = U$ for all $t \in \mathbb{R}$. Then, $\mathbb{P}(U) = 0$ or $\mathbb{P}(U) = 1$.

It is proved in Ambrose [2] that any ergodic flow is isomorphic to a flow built under a function. To state this result, we recall the following notion.

Definition 8 (Flow built under a function)**.** Let $(X, \mathcal{A}, m)$ be a probability space and $T : X \to X$ be an invertible measurable transformation preserving the probability m. Let $f : X \to \mathbb{R}_{\geq 0}$ be a measurable function with $\int_X f(x)\, dm(x) < \infty$. The flow build under the function (X, T, m, f) is defined with the following ingredients:

(i) *The base space* (B, σ, μ): The set B is defined by $B := \{(x, s) \in X \times \mathbb{R}_{\geq 0} : x \in X, 0 \leq s < f(x)\}$. Then, B is a measurable set of the measurable space $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}))$. Let σ and μ denote the restriction of the sigma algebra $\mathcal{A} \otimes \mathcal{B}(\mathbb{R})$, the measure $m \times \lambda$ on the measurable set B, respectively.

(ii) *The flow* $(S_t)_{t\in\mathbb{R}}$: For each $t \in \mathbb{R}_{\geq 0}$, the map $S_t : B \to B$ is defined by

$$S_t(x, s) := (T^{k-1}x, t + s - \sum_{j=1}^{k-1} f(T^j x)),$$

where k is the smallest positive integer satisfying

$$\sum_{j=1}^{k-1} f(T^j x) \leq t + s < \sum_{j=1}^{k} f(T^j x),$$

and for each $t < 0$, $S_t := (S_{-t})^{-1}$.

Theorem 9 (Representation of Ergodic Flow)**.** Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and $(\theta_t)_{t\in\mathbb{R}}$ be an ergodic flow from Ω into itself without fixed point preserving the probability $\mathbb{P}$. Then, there is a flow built under a function (X, T, m, f), where $(X, \mathcal{A}, m)$ is a Lebesgue space, T is an invertible ergodic transformation on X preserving the probability m and f is bounded strictly away 0 and ∞, which is isomorphic to $(\theta_t)_{t\in\mathbb{R}}$.

Proof. From Ambrose [2, Theorem 2] (see also Cornfeld *et al.* [17, Chapter 11]), any ergodic flow is isomorphic to a flow built under a function (X, T, m, f). Furthermore, since $(\theta_t)_{t\in\mathbb{R}}$ is ergodic, it follows that T is also ergodic. The proof is complete. □

4.2 A generic bounded linear cocycle is integrally separated

Let $(X, \mathcal{A}, m)$ be a non-atomic Lebesgue probability space and $T : X \to X$ an ergodic transformation preserving the probability m. Let $\mathcal{M} : X \to Gl(d, \mathbb{R})$ be a measurable matrix-valued function. Then, $\mathcal{M}$ generates a discrete-time linear random dynamical system $\Phi_{\mathcal{M}} : \mathbb{Z} \times X \to Gl(d, \mathbb{R})$ by

$$\Phi_{\mathcal{M}}(n, x) = \begin{cases} \mathcal{M}(T^n x) \circ \cdots \circ \mathcal{M}(x), & \text{if } n > 0; \\ \mathrm{id}, & \text{if } n = 0; \\ \mathcal{M}(T^n x)^{-1} \circ \cdots \circ \mathcal{M}(T^{-1}x)^{-1}, & \text{if } n < 0. \end{cases}$$

We recall the following result on the denseness of integrally separated discrete-time linear random dynamical systems on the set of all bounded discrete-time linear random dynamical systems.

Theorem 10 (Denseness of Integrally Separated Discrete-time Linear RDS)**.** Let $\mathcal{M} : X \to Gl(d, \mathbb{R})$ be measurable matrix-valued function such that this function and its inverse are bounded. Then, for any $\varepsilon > 0$ there exists a measurable matrix $\mathcal{M}_\varepsilon : X \to Gl(d, \mathbb{R})$ such that

$$\operatorname*{ess\,sup}_{x \in X} \|\mathcal{M}(x) - \mathcal{M}_\varepsilon(x)\| + \operatorname*{ess\,sup}_{x \in X} \|\mathcal{M}(x)^{-1} - \mathcal{M}_\varepsilon(x)^{-1}\| \leq \varepsilon$$

and the generated discrete-time linear random dynamical system $\Phi_{\mathcal{M}_\varepsilon}$ is integrally seprated, i.e. there exist positive constants L, α and an invariant decomposition

$$\mathbb{R}^d = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_d(x)$$

such that for $i = 1, \ldots, d$ the following assertions hold:

(i) $\dim E_i(x) = 1$ for m-a.e. $x \in X$,

(ii) For m-a.e. $x \in X$, $u \in \bigoplus_{j=1}^{i} E_j(x) \setminus \{0\}$ and $v \in \bigoplus_{j=i+1}^{d} E_j(x) \setminus \{0\}$ we have

$$\|\Phi_{\mathcal{M}_\varepsilon}(n, x)u\| \geq L e^{n\alpha} \|\Phi_{\mathcal{M}_\varepsilon}(n, x)v\|, \qquad n \in \mathbb{N}.$$

Proof. See Cong [11, Theorem 3.6]. □

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