SLOPES OF A VECTOR-VALUED MAP AND APPLICATIONS

TRƯƠNG XUÂN ĐỨC HÀ ♯

ABSTRACT. In this paper, we give other representations of the slopes recently introduced in [E.M. Bednarczuk, A.Y., Kruger, Error bounds for vector-valued functions on metric spaces. Vietnam J. Math. 40 (2012), no. 2-3, 165-180] and introduce the concepts of subdifferential slopes for a vector-valued map. The new representations allow us to show that the mentioned slopes enjoy most properties of the strong slope of a scalar function. We also explicitly compute or estimate the slopes and the subdifferential slopes in some special convex, strictly differentiable and linear cases. We further study in the slope framework several problems such as error bounds of lower level sets and a Hoffman-type error bound for a system of linear inequalities in the infinite-dimensional space setting, characterization of Pareto minima, weak sharp Pareto minima, and calmness of a vector-valued map.

1. INTRODUCTION

The concept of strong slope has been introduced by De Giorgi, Marino, and Tosques in [10]. Strong slope has proved to be an useful tool in the study of error bounds, an important object in optimality conditions, subdifferential calculus, stability and sensitivity issues, convergence of numerical methods, etc.

As remarked in [1, 2], the strong slope is the adequate notion in order to deal with error bounds (either global or local), yielding both accurate and general results, see also [11, 16]. On the other hand, the strong slope has been shown to be appropriately compared with subdifferential operators and to be estimated with the help of well-developed subdifferential calculus.

Recently, Bednarczuk and Kruger have made the first attempts to extend the concept of strong slope to a vector-valued map defined on a metric space and taking values in a normed linear space. They proposed two approaches to defining strong slope: The first

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one is based on a scalarizing function with an additional parameter [4] and the other one defines slopes directly in terms of the original map with the help of some distance function [5]. Some applications were given to characterization of error bounds for lower level sets of vector-valued maps. However, in these very first works on this subject, not much attention has been paid to properties and calculus of these slopes. Note that the strong slope has also been extended to the case with set-valued maps by Ngai and Thera [25].

In this paper, we use the distance function and the Hiriart-Urruty signed distance function to give other representations of the slopes defined in [5] and to introduce the concepts of subdifferential slopes for a vector-valued map. The new representations allow us to show that these slopes enjoy most properties of the strong slope of a scalar-valued function established in the survey papers [1, 2]. Moreover, we explicitly compute or estimate the slopes and the subdifferential slopes in some special convex, strictly differentiable and linear cases. We further study in the slope framework several problems related to error bounds, vector optimization and calmness of a vector-valued map.

The paper is organized as follows. In Section 2, we study the Hiriart-Urruty signed distance function and scalarizing functions associated to a vector-valued map f. Section 3 is devoted to the concepts of slopes and subdifferential slopes of f, their definitions, characterizations and calculus. In Section 4, we define the concepts of global error bound for a lower level set of f and establish estimations of these bounds in terms of strong slopes. As a special case, we calculate a Hoffman-type error bound for a system of linear inequalities. Section 5 is devoted to applications of the slopes to vector optimization. We formulate a version of the Ekeland variational principle and the existence of a (global) weak sharp strong Pareto minimum for a vector-valued map. We also obtain, under the convexity assumption, the characterization of a Pareto minimum through the stationarity of the slopes. In the last section, we introduce concepts of calmness of a vector-valued map and prove some necessary/sufficient conditions for such properties.

Notations: In this paper, X is either a metric or a normed space depending on the context, Y is a Banach space with the dual space Y^* and the dual pair $\langle .,. \rangle$. For a nonempty set U in a normed space, $\text{int}U$, $\text{cl}U$, $\text{co}U$ and $\text{clco}U$ denote the interior, the closure, the convex hull and the closed convex hull of U, resp., and $\mathbb{B}$ stands for the closed unit ball in a normed space. Throughout the paper, $K \subset Y$ is always a closed pointed convex cone and $K^* := \{y^* \in Y^* \mid \langle y^*, k \rangle \geq 0, \forall k \in K\}$. For any $y_1, y_2 \in Y$, we write $y_1 \leq y_2$ if $y_2 - y_1 \in K$, $y_1 < y_2$ if $y_2 - y_1 \in K \setminus \{0\}$ and $y_1 \ll y_2$ if $y_2 - y_1 \in \text{int}K$. To envisage functions defined on subsets of X we add an element ∞ not belonging to the space Y, obtaining thus the space $Y \cup \{+\infty\}$. We suppose that $y \leq \infty$ for all $y \in Y$. In the Euclidean space $\mathbb{R}^n$, we always take K being the nonnegative orthant $\mathbb{R}^n_+$. We will use the symbols $\mathbf{e} := (1, ..., 1) \in \mathbb{R}^n$ and $\mathbf{e_i} := (0, ..., 0, 1, 0, ..., 0)$ (1 at the ith-coordinate and zeros at other places).

Unless otherwise specified, $f : X \to Y \cup \{+\infty\}$ is a vector-valued map. For any $\gamma \in Y$, the lower level set of f at γ is defined by $[f \le \gamma] := \{x \in X \mid f(x) \le \gamma\}$.

For any $t \in \mathbb{R}$, $[t]_+ := \max\{t, 0\}$.

2. Scalarizing functions

This section is devoted to the signed distance function Δ_{-K} associated to the cone $-K$ and some scalarizing functions that will be used for defining slopes for the map f.

Recall that the function Δ_{-K} is defined by

$$\Delta_{-K}(y) := d_{-K}(y) - d_{Y\setminus(-K)}(y) = \begin{cases} -d_{Y\setminus(-K)}(y) & \text{if } y \in -K \\ d_{-K}(y) & \text{otherwise.} \end{cases}$$

Example 2.1. *Let $Y = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$. Then for any $y = (y_i) \in \mathbb{R}_n$ we have*

$$\begin{aligned} d_{\mathbb{R}_n\setminus(-\mathbb{R}^n_+)}(y) = &\ \min_i |y_i| && \text{if } y \in -\mathbb{R}^n_+ \\ d_{-\mathbb{R}^n_+}(y) = &\ \sqrt{\textstyle\sum_{i=1}^n [y_i]_+^2} && \text{if } y \notin -\mathbb{R}^n_+ \end{aligned}$$

and therefore,

$$\Delta_{-\mathbb{R}^n_+}(y) = \begin{cases} -\min_i |y_i| & \text{if } y_i \le 0,\ \forall i \\ \sqrt{\sum_{i=1}^n [y_i]_+^2} & \text{otherwise.} \end{cases}$$

We refer the reader to the Hiriart-Urruty paper [14] for the definition of the signed distance function associated to an arbitrary nonempty set and to [29] for properties, examples and applications of this function in vector optimization.

Some useful properties of Δ_{-K} are collected in the following proposition.

Proposition 2.1 ([29])**.** *The function Δ_{-K} has the following properties:*

(i) *It is convex, Lipschitz of rank 1 on Y.*
(ii) *It satisfies the triangle inequality: $\Delta_{-K}(y_1 + y_2) \le \Delta_{-K}(y_1) + \Delta_{-K}(y_2)$ for any $y_1, y_2 \in Y$.*
(iii) *It is K-monotone: $\Delta_{-K}(y_1) \le \Delta_{-K}(y_2)$ for any $y_1, y_2 \in Y,\ y_1 \le y_2$..*

The following result will be used to calculate subdifferential slopes.

Proposition 2.2. *The following assertions are true.*

(i) $\operatorname{clco}\{y^* \in K^*, \|y^*\| = 1\} \subseteq \partial\Delta_{-K}(0) \subseteq K^* \cap \mathbb{B}^*$.
(ii) $\partial\Delta_{-\mathbb{R}^+_n}(0) = \operatorname{co}\{v \in \mathbb{R}^n_+ \mid \|v\| = 1\} = \{v \in \mathbb{R}^n_+ \mid \|v\| \le 1 \text{ and } \langle v, \mathbf{e}\rangle \ge 1\}$.

Proof. (i) Since the second inclusion follows from [14, Propositions 2 and 5], it remains to prove the first one. As the set $\partial\Delta_{-K}(0)$ is closed and convex, it suffices to show that any $y^* \in K^*$ with $\|y^*\| = 1$ satisfies

$$\langle y^*, y\rangle \leq \Delta_{-K}(y) - \Delta_{-K}(0) = \Delta_{-K}(y), \quad \forall y \in Y.$$

Let $y \in Y$ be taken. We will consider three possible cases.

Case 1: $y \in \mathrm{bd}(-K)$. Since $y^* \in K^*$ and $y \in -K$, we get $\langle y^*, y\rangle \leq 0 = \Delta_{-K}(y)$.

Case 2: $y \in \mathrm{int}(-K)$. Set $\rho := d(y; Y \setminus -K) = d(y; Y \setminus -\mathrm{int}K)$. Let $\epsilon \in]0, \min\{1, \rho\}[$. Note that for any open set $V \subset Y$ and $v \in V$ we have

$$d(v; Y \setminus V) = \sup\{\rho > 0 \mid v + \rho\mathrm{int}\mathbb{B} \subset V\}. \tag{1}$$

This equality yields that $y + (\rho - \epsilon)\mathbb{B} \subset -\mathrm{int}K$. As $\|y^*\| = 1$, we can find $e \in \mathbb{B}$ such that $\langle y^*, e\rangle \geq 1 - \epsilon/2$. Since $y^* \in K^*$ and $y + (\rho - \epsilon)e \in -\mathrm{int}K$, we get $\langle y^*, y + (\rho - \epsilon)e\rangle \leq 0$. Hence,

$$\langle y^*, y\rangle \leq -\langle y^*, (\rho - \epsilon)e\rangle \leq -(\rho - \epsilon)(1 - \epsilon/2).$$

As $\epsilon \in]0, \min\{1, \rho\}[$ and ϵ is arbitrary, we obtain $\langle y^*, y\rangle \leq -\rho = -d(y; Y \setminus -K) = \Delta_{-K}(y)$.

Case 3: $y \in Y \setminus (-K)$. Set $\rho := d(y; -K)$. Since the set $Y \setminus (-K)$ is open, Equality (1) yields that for any $\epsilon > 0$ we can find $e \in \mathbb{B}$ with $\|e\| < 1$ such that $y - (\rho + \epsilon)e \in -K$. Then $\langle y^*, y - (\rho + \epsilon)e\rangle \leq 0$. Hence

$$\langle y^*, y\rangle \leq \langle y^*, (\rho + \epsilon)e\rangle \leq (\rho + \epsilon)\|y^*\|\|e\| \leq \rho + \epsilon.$$

Since the positive scalar ϵ is arbitrary, we get $\langle y^*, y\rangle \leq \rho = d(y; -K) = \Delta_{-K}(y)$.

(ii) The first equality has been established in [9, Theorem 4.2] and we need only to check the second one. Since $\langle v, \mathbf{e}\rangle = \sum_{i=1}^n v_i \geq \sqrt{\sum_{i=1}^n v_i^2} = 1$ for any $v = (v_i) \in \mathbb{R}^n_+$ with $\|v\| = 1$ and the set $\{v \in \mathbb{R}^n_+ \mid \|v\| \leq 1 \text{ and } \langle v, \mathbf{e}\rangle \geq 1\}$ is closed and convex, we get

$$\mathrm{co}\{v \in \mathbb{R}^n_+ \mid \|v\| = 1\} \subset \{v \in \mathbb{R}^n_+ \mid \|v\| \leq 1 \text{ and } \langle v, \mathbf{e}\rangle \geq 1\}.$$

To prove the inverse inclusion, let $\bar{v} = (\bar{v}_i) \in \mathbb{R}^n_+$ with $\|\bar{v}\| < 1$. If $\langle \bar{v}, \mathbf{e}\rangle = 1$ then $\sum_{i=1}^n \bar{v}_i = 1$ and $\bar{v} = \sum_{i=1}^n \bar{v}_i\mathbf{e_i}$, which means that $\bar{v} \in \mathrm{clco}\{v \in \mathbb{R}^n_+ \mid \|v\| = 1\}$. Hence,

$$\{v \in \mathbb{R}^n_+ \mid \langle v, \mathbf{e}\rangle = 1\} \subset \mathrm{co}\{v \in \mathbb{R}^n_+ \mid \|v\| = 1\}$$

If $\langle \bar{v}, \mathbf{e}\rangle > 1$, then

$$\bar{v} = t\frac{\bar{v}}{\langle \bar{v}, \mathbf{e}\rangle} + (1 - t)\frac{\bar{v}}{\|\bar{v}\|} \text{ with } t = \frac{1/\|\bar{v}\| - 1}{1/\|\bar{v}\| - 1/\langle \bar{v}, \mathbf{e}\rangle} \in]0, 1[$$

and since $\frac{\bar{v}}{\langle \bar{v}, \mathbf{e}\rangle} \in \{v \in \mathbb{R}^n_+ \mid \langle v, \mathbf{e}\rangle = 1\} \subset \mathrm{co}\{v \in \mathbb{R}^n_+ \mid \|v\| = 1\}$ and $\frac{\bar{v}}{\|\bar{v}\|} \in \{v \in \mathbb{R}^n_+ \mid \|v\| = 1\}$, we deduce that $\bar{v} \in \mathrm{co}\{v \in \mathbb{R}^n_+ \mid \|v\| = 1\}$. Thus, the desired inverse inclusion holds. □

Let $\Delta_{-K}f$ and $\Delta_{-K}(-f)$ be functions defined by $\Delta_{-K}f(x) := \Delta_{-K}(f(x))$ and $\Delta_{-K}(-f)(x) := \Delta_{-K}(-f(x))$. These scalarizing functions play an important role in our study and we are interested in their properties. Let us recall the concepts of lower semicontinuity and convexity of a vector-valued map.

Definition 2.1. *The map f is said to be*

(i) *K-lower semicontinuous (in brief, K-lsc) at x if for any $\epsilon > 0$ there exists $\delta > 0$ such that $f(x') \in f(x) + \epsilon\mathbb{B}_Y + K$ for any $x' \in \text{dom} f$ with $d(x', x) \leq \delta$.*

(ii) *K-level closed if $[f \leq \gamma]$ is closed for any $\gamma \in Y$.*

(iii) *K-convex if $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$ for any $x_1, x_2 \in \text{dom} f$ and $\lambda \in [0,1]$ (X is assumed to be a linear space).*

Some basic properties of the scalarizing functions are collected in the following proposition.

Proposition 2.3. *The following assertions are true.*

(i) *If f is K-lsc, then f is K-level closed.*

(ii) *If f is K-convex, then $\Delta_{-K}f$ is convex.*

(iii) *If f is K-lsc at $x \in$dom$f$, then $\Delta_{-K}f$ and $-\Delta_{-K}(-f)$ are lsc at x.*

Proof. (i) Let be given a vector $\gamma \in Y$ and points $x_i \in [f \leq \gamma]$ $(i = 1, 2, ...)$ such that $x_i \to x$. Suppose that $x \notin [f \leq \gamma]$, i.e. $f(x) \not\leq \gamma$. Then $f(x) \in (\gamma - K)^c := Y \setminus (\gamma - K)$ and since K is closed, there exists a scalar $\epsilon > 0$ such that $f(x) + \epsilon\mathbb{B}_Y \subset (\gamma - K)^c$. If there exist $e \in \mathbb{B}_Y$ and $k \in K$ such that $f(x) + \epsilon e + k \in \gamma - K$, then we get $f(x) + \epsilon e \in \gamma - K - k \subset \gamma - K$, a contradiction to $f(x) + \epsilon\mathbb{B}_Y \subset (\gamma - K)^c$. Therefore, $f(x) + \epsilon\mathbb{B}_Y + K \subset (\gamma - K)^c$. Further, as f is K-lsc we get $f(x_i) \in f(x) + \epsilon\mathbb{B}_Y + K \subset (\gamma - K)^c$ for i sufficiently large, which is a contradiction to $x_i \in [f \leq \gamma]$. Thus, the inclusion $x \in [f \leq \gamma]$ holds.

(ii) Let $x_1, x_1 \in$dom$f$ and $t \in [0,1]$. The K-convexity of f yields $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$ and the K-monotonicity and the triangle inequality property of Δ_{-K} imply

$$\begin{aligned} \Delta_{-K}f(tx_1 + (1-t)x_2) &\leq \Delta_{-K}(tf(x_1) + (1-t)f(x_2)) \\ \leq \Delta_{-K}(tf(x_1)) + \Delta_{-K}((1-t)f(x_2)) &= t\Delta_{-K}f(x_1) + (1-t)\Delta_{-K}f(x_2). \end{aligned}$$

Thus, the function $\Delta_{-K}f$ is convex.

(iii) Since f is K-lsc at x, for any $\epsilon > 0$ there exists $\delta > 0$ such that $f(x') \in f(x) + \epsilon\mathbb{B}_Y + K$ for any $x' \in$dom$f$ with $d(x', x) \leq \delta$. For such an x', we have $f(x') = f(x) + \epsilon e + k$ with some $e \in \mathbb{B}_Y$ and $k \in K$. It follows that

$$\Delta_{-K}f(x') = \Delta_{-K}(f(x) + \epsilon e + k) \geq \Delta_{-K}f(x) - \Delta_{-K}(-\epsilon e) \geq \Delta_{-K}f(x) - \epsilon\|e\| = \Delta_{-K}f(x) - \epsilon$$

and

$$\begin{aligned} -\Delta_{-K}(-f)(x') &= -\Delta_{-K}(-f(x)-\epsilon e-k) \geq -\Delta_{-K}(-f(x)) - \Delta_{-K}(\epsilon e) \\ &\geq -\Delta_{-K}(-f)(x) - \epsilon\|e\| = -\Delta_{-K}(-f)(x) - \epsilon. \end{aligned}$$

This means that the functions $\Delta_{-K}f$ and $-\Delta_{-K}(-f)$ are lsc at x. □

3. Slopes and subdifferential slopes of a vector-valued map

In this section, we discuss the concepts of upper and lower slopes for the vector-valued map f defined in [5] and introduce their subdifferential counterparts. We study properties of these slopes and calculate them in the general case, the convex case and the linear case.

3.1. **Lower and upper slopes.** The notion of strong slope introduced by De Giorgi, Marino, and Tosques in [10] as follows.

Definition 3.1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function. For $x \in \mathrm{dom} f$, set:*

$$|\nabla f|(x) := \begin{cases} 0 & \text{if } x \text{ is a local minimum of } f, \\ \limsup_{x' \to x} \frac{f(x)-f(x')}{d(x;x')} & \text{otherwise} \end{cases}$$

and for $x \notin \mathrm{dom} f$, set $|\nabla f|(x) := +\infty$. The nonnegative extended real number $|\nabla f|(x)$ is called the strong slope of f at x.

Let us recall the notions of lower and upper slopes for the vector-valued map f given in [5]. When the cone K has a nonempty interior, let $\mathring{K} := \{k \in \mathrm{int}K \mid d(k; \mathrm{bd}K) = 1\}$. One can see that $\mathring{K}$ generates $\mathrm{int}K$, i.e., $\mathrm{int}K = \cup_{t>0}(t\mathring{K})$ and if $Y = \mathbb{R}$, then $\mathring{K} = \{1\}$.

Definition 3.2. *The lower and upper slopes of f at $x \in \mathrm{dom} f$ are defined as*

$$^{-}|\nabla f|(x) := \limsup_{x' \to x} \sup\{r > 0 \mid \frac{f(x)-f(x')}{d(x;x')} \in K + r\mathring{K}\}$$

and

$$^{+}|\nabla f|(x) := \limsup_{x' \to x} \inf\{r > 0 \mid \frac{f(x')-f(x)}{d(x;x')} \in K + r\mathbb{B}\}.$$

The conventions $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$ are in force in the above definitions. From now on we assume that **the cone K has a nonempty interior** whenever the lower slope is involved.

A relation between the upper slope and the lower slope is formulated in the following proposition.

Proposition 3.1 ([5])**.** *We have*

$$^{+}|\nabla f|_u(x) \geq {}^{-}|\nabla f|(x).$$

Note that no Pareto efficiency concept has been involved in the definitions of lower and upper slopes but it turns out that these slopes equal zero at Pareto minimum. Let us recall some concepts of efficiency from vector optimization [18].

Definition 3.3. *We say that* $x \in \mathrm{dom} f$ *is a local Pareto minimum/weak Pareto minimum/strong Pareto minimum of* f *if there is a neighborhood* U *of* x *such that for all* $x' \in U \cap \mathrm{dom} f$ *one has either* $f(x') = f(x)$ *or* $f(x') \not\leq f(x)$, $f(x') \not\ll f(x)$ *and* $f(x) \leq f(x')$, *resp. When* $U = X$, *we replace "local" by "global" in the above definitions.*

It is immediate from Definition 3.3 the following fact.

Proposition 3.2. *The following assertions hold true.*

(i) *If* x *is a local Pareto minima of* f, *then* ${}^{-}|\nabla(f)|(x) = 0$.
(ii) *If* x *is a local strong Pareto minima of* f, *then* ${}^{+}|\nabla(f)|(x) = 0$.

The following characterizations of the lower and upper slopes in terms of the distance function and the signed distance function play important role in our study of their properties and calculus.

Theorem 3.1. *Let* $x \in \mathrm{dom} f$. *Then*

$$^{-}|\nabla f|(x) = \begin{cases} 0 & \text{if } x \text{ is a local Pareto minimum of } f \\ \limsup_{x' \to x, f(x') \leq f(x)} \frac{d_{Y \setminus K}(f(x) - f(x'))}{d(x;x')} & \text{otherwise} \end{cases}$$

and

$$^{+}|\nabla f|(x) = \begin{cases} 0 & \text{if } x \text{ is a local strong Pareto minimum of } f \\ \limsup_{x' \to x, f(x) \not\leq f(x')} \frac{d_{-K}(f(x) - f(x'))}{d(x;x')} & \text{otherwise} \end{cases}$$

Theorem 3.2. *Let* $x \in \mathrm{dom} f$. *Then*

$$^{-}|\nabla f|(x) = \begin{cases} 0 & \text{if } x \text{ is a local Pareto minimum of } f, \\ \limsup_{x' \to x} -\frac{\Delta_{-K}(f(x') - f(x))}{d(x;x')} & \text{otherwise} \end{cases}$$

and

$$^{+}|\nabla f|(x) = \begin{cases} 0 & \text{if } x \text{ is a local strong Pareto minimum of } f, \\ \limsup_{x' \to x} \frac{\Delta_{-K}(f(x) - f(x'))}{d(x;x')} & \text{otherwise} \end{cases}$$

To prove Theorems 3.1 and 3.2, we need some auxiliary results.

Proposition 3.3. *The following assertions hold true.*

(i) *For any* $k \in K$, *one has* $d(k; Y \setminus K) = \sup\{r \geq 0 \mid k \in K + r\mathring{K}\}$.
(ii) *For any* $u \in Y \setminus K$, *one has* $d(u; K) = \sup\{r \geq 0 \mid u \in K + r\mathbb{B}\}$.

Proof. Denote by $\mathcal{R}$ one of the quantities in the right-hand side of the equalities to be checked.

(i) Consider first the case $k \in \mathrm{bd}K$. Clearly, $d(k; Y \setminus K) = 0$. Since $K + r\mathring{K} \subset \mathrm{int}K$ whenever $r > 0$, we get $\mathcal{R} = 0$ and the desired equality follows. Next, assume that $k \in \mathrm{int}K$. Then $k = t\mathring{k}$ for some $\mathring{k} \in \mathring{K}$. The $d(k; Y \setminus K) = td(\mathring{k}; Y \setminus K) = t$. Since $k = t\mathring{k} \in K + t\mathring{K}$, we have $d(k; Y \setminus K) = t \leq \mathcal{R}$. It remains to prove that $d(k; Y \setminus K) \geq \mathcal{R}$. Let $r \in]0, \mathcal{R}[$ be an arbitrary scalar. By the definition of $\mathcal{R}$, we have $k \in K + r\mathring{K}$, i.e., $k = l + r\mathring{k}$ for some $l \in K$ and $\mathring{k} \in \mathring{K}$. Let $y \in Y \setminus K$ be an arbitrary vector. Then $y - \mathring{k} \in Y \setminus K$ because otherwise we would have $y \in K$. Clearly, we also have $u := y/r - \mathring{k}/r \in Y \setminus K$. Further, observing that

$$k - y = l + r\mathring{k} - y = r\mathring{k} - r(y/r - \mathring{k}/r) = r(\mathring{k} - u)$$

we get

$$\|k - y\| = r\|\mathring{k} - u\| \geq rd(\mathring{k}; Y \setminus K) = rd(\mathring{k}; \mathrm{bd}K) = r.$$

Since $y \in Y \setminus K$ is an arbitrary vector, it follows that $d(k; Y \setminus K) \geq r$ and since $r \in]0, \mathcal{R}[$ is an arbitrary scalar, it follows that $d(k; Y \setminus K) \geq \mathcal{R}$, as it was to be shown.

(ii) Let $r > \mathcal{R}$ be an arbitrary scalar. Then $u \in K + r\mathbb{B}$,i.e., $u = k + rb$ for some $k \in K$ and $b \in \mathbb{B}$. Then $d(u; K) \leq \|u - k\| = r\|b\| = r$ and therefore, $d(u; K) \leq \mathcal{R}$. It remains to prove the inverse inequality. Let $\epsilon \in]0, \mathcal{R}[$ be an arbitrary scalar. By the definition of $\mathcal{R}$, we have $u \notin K + (\mathcal{R} - \epsilon)\mathbb{B}$. Then for any $k \in K$ one has $u - k \notin (\mathcal{R} - \epsilon)\mathbb{B}$ or $\|u - k\| > \mathcal{R} - \epsilon)$, which means that $d(u; K) > \mathcal{R} - \epsilon$. Since $\epsilon \in]0, \mathcal{R}[$ is an arbitrary scalar, we obtain $d(u; K) \geq \mathcal{R}$, as it was to be shown. □

Proposition 3.4. *Let $x \in \mathrm{dom}f$. Denote*

$$\eta_1 := \limsup_{x' \to x} -\frac{\Delta_{-K}(f(x') - f(x))}{d(x; x')} \text{ and } \eta_2 := \limsup_{x' \to x} \frac{\Delta_{-K}(f(x) - f(x'))}{d(x; x')}.$$

(i) *Assume that x is not a local Pareto minimum. Then*

$$\eta_1 = \limsup_{x' \to x, f(x') \leq f(x)} -\frac{\Delta_{-K}(f(x') - f(x))}{d(x; x')} = \limsup_{x' \to x, f(x') \leq f(x)} \frac{d_{Y \setminus K}(f(x) - f(x'))}{d(x; x')}$$

(ii) *Assume that x is not a local strong Pareto minimum. Then*

$$\eta_2 = \limsup_{x' \to x, f(x) \not\leq f(x')} \frac{\Delta_{-K}(f(x) - f(x'))}{d(x; x')} = \limsup_{x' \to x, f(x) \not\leq f(x')} \frac{d_{-K}(f(x) - f(x'))}{d(x; x')}.$$

Proof. Observe that

$$-\Delta_{-K}(f(x') - f(x)) = \begin{cases} d_{Y \setminus K}(f(x) - f(x')) \geq 0 & \text{if } f(x') \leq f(x) \\ -d_{-K}(f(x') - f(x)) < 0 & \text{otherwise} \end{cases}$$

and

$$\Delta_{-K}(f(x)-f(x')) = \begin{cases} d_{-K}(f(x)-f(x')) > 0 & \text{if } f(x) \not\leq f(x') \\ -d_{Y\setminus(-K)}(f(x)-f(x')) \leq 0 & \text{otherwise.} \end{cases}$$

Then only those points x' near x can contribute to the value of η_1 for which $f(x') \leq f(x)$ (such x''s exist because x is not a local Pareto minimum of f) and only those points x' near x can contribute to the value of η_2 for which $f(x) \not\leq f(x')$ (such x''s exist because x is not a local strong Pareto minimum of f). Thus, the desired equalities follow. □

Let us return to the proof of Theorems 3.1 and 3.2.

Proof. Theorems 3.1 and 3.2 follows from Propositions 3.2, 3.3 and 3.4. □

From now on, we will use the *representations of the lower and upper slopes given in Theorems 3.1 and 3.2.*

Below we provide some examples of lower and upper slopes and refer the interested reader to the paper [5] for more examples.

Example 3.1. *Let* $Y = \mathbb{R}^2$ *and* $K = \mathbb{R}^2_+$.

(i) In contrast to the scalar case, a nontrivial linear vector-valued map may have a lower strong slope being equal zero everywhere. For instance, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $f(x_1, x_2) = (x_1+x_2, -x_1-x_2)$. *It is easy to see that any points* $(x_1, x_2) \in \mathbb{R}^2$ *is a global Pareto minimum of* f *and hence,* ${}^-|\nabla f|(x_1, x_2) \equiv 0$.

(ii) Let $f : \mathbb{R} \to \mathbb{R}^2$ *be a map defined by* $f(x) = (x^2, x^4)$. *Then*

$${}^-|\nabla f|(x) = \begin{cases} 2|x| & \text{if } |x| > \sqrt{2}/2 \\ 4|x|^3 & \text{if } |x| \leq \sqrt{2}/2 \end{cases} \quad \text{and } {}^+|\nabla f|(x) = 2|x|\sqrt{1+4x^4}.$$

Note that ${}^-|\nabla f|(0) =^+ |\nabla f|(0) = 0$ *and* $\bar{x} = 0$ *is a global strong Pareto minimum of* f.

(iii) Let $f : \mathbb{R} \to \mathbb{R}^2$ *be a map defined by*

$$f(x) = \begin{cases} (x^2, x^4) & \text{if } x \geq 0 \\ (x, -x^2) & \text{if } x < 0 \end{cases}$$

Then

$${}^-|\nabla f|(x) = \begin{cases} 2x & \text{if } x \geq \sqrt{2}/2 \\ 4x^3 & \text{if } 0 \leq x < \sqrt{2}/2 \\ -2x & \text{if } -1/2 \leq x < 0 \\ 1 & \text{if } x < -1/2 \end{cases} \quad \text{and } {}^+|\nabla f|(x) = \begin{cases} 2x\sqrt{1+4x^4} & \text{if } x \geq 0 \\ \sqrt{1+4x^2} & \text{if } x < 0. \end{cases}$$

Remark that ${}^-|\nabla f|(0) =^+ |\nabla f|(0) = 0$ *but* $\bar{x} = 0$ *is not a local weak Pareto minimum of* f.

(iv) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ *be a map defined by* $f(x, y) = (|x|, |y|)$. *Then* ${}^-|\nabla f|(x, y) = \sqrt{2}/2$ *and* ${}^+|\nabla f|(x, y) = 1$ *for any* $(x, y) \neq (0, 0)$ *and* ${}^-|\nabla f|(0, 0) =^+ |\nabla f|(0, 0) = 0$.

3.2. **Subdifferential slopes.** In the remainder of this section, unless otherwise specified, we assume that X *is a Banach space*. Consider an abstract subdifferential operator ∂, which associates to any lsc function $h : X \to \mathbb{R} \cup \{+\infty\}$ and any point $x \in X$ a subset $\partial h(x) \subset X^*$ in such a way that $\partial h(x) = \emptyset$ if $x \notin \mathrm{dom} h$ and

(i) If h is convex, then ∂h coincides with the subdifferential of the convex analysis.

(ii) (exact sum rule) If $g : X \to \mathbb{R}$ is convex and continuous near $\bar{x} \in \mathrm{dom} h$ and $\bar{x}$ is a local minimizer of $h + g$, then $0 \in \partial h(\bar{x}) + \partial g(\bar{x})$.

Definition 3.4. *Let $f : X \to Y \cup \{+\infty\}$ be a K-lsc function. We define the lower subdifferential slope of f at $\bar{x} \in \mathrm{dom} f$ by*

$$^{-}|\partial f|(\bar{x}) = \inf\{\|x^*\| \mid x^* \in \partial \Delta_{-K}(f(\cdot) - f(\bar{x}))(\bar{x}))\}$$

and the upper subdifferential slope of f at x by

$$^{+}|\partial f|(\bar{x}) = \inf\{\|x^*\| \mid x^* \in \partial[-\Delta_{-K}(f(\bar{x}) - f(\cdot))](\bar{x}))\}$$

in case the sets in the right-hand side of the above equalities are nonempty and set $^{-}|\partial f|(\bar{x}) := +\infty$ and $^{+}|\partial f|(\bar{x}) := +\infty$ otherwise.

Clearly, $^{-}|\partial f|(\bar{x}) = d(0; \partial \Delta_{-K}(f(\cdot) - f(\bar{x}))(\bar{x}))$ and $^{+}|\partial f|(\bar{x}) = d(0; \partial[-\Delta_{-K}(f(\bar{x}) - f(\cdot))](\bar{x}))$.

Relations among slopes and subdifferential slopes are formulated in the following proposition.

Proposition 3.5. *Assume that f is K-lsc. For any $\bar{x} \in \mathrm{dom} f$, we have*

$$^{-}|\nabla f|(\bar{x}) \geq ^{-} |\partial f|(\bar{x}) \text{ and } ^{+}|\nabla f|(\bar{x}) \geq ^{+} |\partial f|(\bar{x}).$$

Proof. By the definition of the subdifferential slopes, it suffices to prove that

$$^{-}|\nabla f|(\bar{x}) \geq \inf\{\|x^*\| \mid x^* \in \partial \Delta_{-K}(f(\cdot) - f(\bar{x}))(\bar{x}))\}$$

and

$$^{+}|\nabla f|(\bar{x}) \geq \inf\{\|x^*\| \mid x^* \in \partial[-\Delta_{-K}(f(\bar{x}) - f(\cdot))](\bar{x}))\}.$$

We may assume that $^{-}|\nabla f|(\bar{x}) < +\infty$ and $^{+}|\nabla f|(\bar{x}) < +\infty$. Let $\sigma_1 > \; ^{-}|\nabla f|(\bar{x})$ and $\sigma_2 > \; ^{+}|\nabla f|(\bar{x})$. Let $\rho > 0$ such that for all $x \in \mathbb{B}(\bar{x}, \rho)$ we have

$$-\Delta_{-K}(f(x) - f(\bar{x})) \leq \sigma_1 \|x - \bar{x}\|$$

and

$$\Delta_{-K}(f(\bar{x}) - f(x)) \leq \sigma_2 \|x - \bar{x}\|.$$

Then $\bar{x}$ is a local minimizer of the functions $\Delta_{-K}(f(\cdot) - f(\bar{x})) + \sigma_1 \|\cdot - \bar{x}\|$ and $-\Delta_{-K}(f(\bar{x}) - f(\cdot)) + \sigma_2 u \|\cdot - \bar{x}\|$. Applying Proposition 2.3, one can check that the above functions are lsc.

Since ∂ satisfies (ii), we find $x_1^* \in \partial\Delta_{-K}(f(\cdot) - f(\bar{x}))(\bar{x})$, $x_2^* \in \partial[-\Delta_{-K}(f(\bar{x}) - f(\cdot))](\bar{x})$, $z_i^* \in \partial\sigma_i\| \cdot -\bar{x}\|(\bar{x})$, $i \in \{1, 2\}$ such that $0 = x_i^* + z_i^*$. Since $\|z_i^*\| \le \sigma_i$, we get $\|x_i^*\| \le \sigma_i$. As $\sigma_1 > {}^-|\nabla f|(\bar{x})$ and $\sigma_2 > {}^+|\nabla f|(\bar{x})$ are arbitrary, the desired inequalities follow. □

From now on, by ∂ we mean either the *Clarke subdifferential* or the *Mordukhovich limiting subdifferential*, for the definitions see [8, p. 27 and 61] and [22, p.82], resp., or the subdifferential of convex analysis (under an appropriate convexity assumption). Whenever the Mordukhovich limiting subdiffential is involved, we assume that X is an Asplund space, which means that X is a Banach space where every convex continuous function is generically Fréchet differentiable. These subdifferentials satisfy (i)-(ii). Note that instead of the exact sum rule, some authors require the subdifferential operator satisfy a fuzzy sum rule see [2] but we will not consider such a case here.

Let us calculate the subdifferential slopes of a *strictly differentiable* map. Recall that f admits a strict derivative at $\bar{x}$, a continuous linear map from X to Y denoted by $f'(\bar{x})$, provided that for each $u \in X$, the following holds

$$\lim_{x \to \bar{x}, t \to 0^+} \frac{f(x + tu) - f(x)}{t} = f'(\bar{x})(u),$$

and the convergence is uniform for u in compact sets ([8, p. 30]). Denote by $f'(\bar{x})^*$ the adjoint operator of $f'(\bar{x})$. It is easy to check that if f is strictly differentiable at $\bar{x}$, then it is K-lsc at this point.

Proposition 3.6. *Suppose that $f : X \to Y$ is strictly differentiable at $\bar{x} \in X$.*

(i) *We have*

$$^-|\partial f|(\bar{x}) = \inf\{\|f'(\bar{x})^*(v^*)\| \mid v^* \in \partial\Delta_{-K}(0)\}$$

and in case ∂ is the Clarke subdifferential, we also have

$$^+|\partial f|(\bar{x}) = {}^- |\partial f|(\bar{x}).$$

(ii) *Assume that $Y = \mathbb{R}^n$ and $f = (f_1, ..., f_n)$ with $f_i : X \to \mathbb{R}$ $(i = \overline{1, n})$. We have*

$$^-|\partial f|(\bar{x}) = \min\{\|\sum_{i=1}^n v_i f_i'(\bar{x})^*\| \mid v = (v_i) \in \mathbb{R}^n_+, \sum_{i=1}^n v_i = 1\}.$$

In particular, if $X = \mathbb{R}$ and all the functions f_i $(i = \overline{1, n})$ are strictly increasing (decreasing), then we have

$$^-|\partial f|(\bar{x}) = \min_i |f_i'(\bar{x})|.$$

Proof. (i) The first equality follows from a chain rule ([8, Theorem 2.3.10, p. 45] and [23, Corollary 6.3, p. 1259])

$$\partial\Delta_{-K}(f(\cdot) - f(\bar{x}))(\bar{x}) = \{f'(\bar{x})^*(v^*) \mid v^* \in \partial\Delta_{-K}(0)\}.$$

Further, assume that ∂ is the Clarke subdifferential. Recall that the equality $\partial(-g)(x) = -\partial g(x)$ holds for the Clarke subdifferential of any function $g : X \to \mathbb{R} \cup \{+\infty\}$. We have

$$\begin{aligned}\partial(-\Delta_{-K}(f(\bar{x}) - f(\cdot))(\bar{x}) &= -\partial\Delta_{-K}(f(\bar{x}) - f(\cdot))(\bar{x}) = \{f'(\bar{x})^*(v^*) \mid v^* \in \partial\Delta_{-K}(0)\} \\ &= \partial\Delta_{-K}(f(\cdot) - f(\bar{x}))(\bar{x})\end{aligned}$$

and the assertion follows.

(ii) Proposition 2.2 and the assertion (i) yield

$$^{-}|\partial f|(\bar{x}) = \min\{\| \sum_{i=1}^{n} v_i f_i'(\bar{x})^* \| \mid v = (v_i) \in \mathbb{R}^n_+, \|v\| \le 1, \langle v, \mathbf{e}\rangle \ge 1\}. \tag{2}$$

Let $\bar{v} = (\bar{v}_i) \in \{v \in \mathbb{R}^n_+ \mid \|v\| \le 1, \langle v, \mathbf{e}\rangle \ge 1\}$ such that

$$\| \sum_{i=1}^{n} \bar{v}_i f_i'(\bar{x})^* \| = \min\{\| \sum_{i=1}^{n} v_i f_i'(\bar{x})^* \| \mid v \in \mathbb{R}^n_+, \|v\| \le 1, \langle v, \mathbf{e}\rangle \ge 1\}. \tag{3}$$

If $\langle \bar{v}, \mathbf{e}\rangle > 1$, then $\tilde{v} := \bar{v}/\langle \bar{v}, \mathbf{e}\rangle \in \{v \in \mathbb{R}^n_+, \|v\| \le 1, \langle v, \mathbf{e}\rangle = 1\}$ and

$$\| \sum_{i=1}^{n} \bar{v}_i f_i'(\bar{x})^* \| > \| \sum_{i=1}^{n} \bar{v}_i f_i'(\bar{x})^* / \langle \bar{v}, \mathbf{e}\rangle \| = \| \sum_{i=1}^{n} \tilde{v}_i f_i'(\bar{x})^* \|,$$

which is a contradiction to equalities (3). Thus, $\langle \bar{v}, \mathbf{e}\rangle = \sum_{i=1}^{n} v_i = 1$. Finally, observe that any $v \in \mathbb{R}^n_+$ satisfying $\langle v, \mathbf{e}\rangle = 1$ also satisfies $\|v\| \le 1$ and the desired equality follows from equality (2).

Finally, suppose that all the functions $f_i : \mathbb{R} \to \mathbb{R}$ ($i = \overline{1, n}$) are strictly increasing (decreasing). Then $f'(\bar{x}) = (f_1'(\bar{x}), ..., f_n'(\bar{x})) \in \text{int}\mathbb{R}^n_+ \cup -\text{int}\mathbb{R}^n_+$ and we have

$$^{-}|\partial f|(\bar{x}) = \min\{| \sum_{i=1}^{n} f_i'(\bar{x}) v_i| \mid v = (v_i) \in \mathbb{R}^n_+, \langle v, \mathbf{e}\rangle = 1\}.$$

Note that $\langle \mathbf{e}_i, \mathbf{e}\rangle = 1$ and for $v = \mathbf{e}_i$ one has $| \sum_{i=1}^{n} f_i'(\bar{x}) v_i| = |f_i'(\bar{x})|$ ($i = \overline{1, n}$). Hence,

$$^{-}|\partial f|(\bar{x}) \le \min_i |f_i'(\bar{x})|. \tag{4}$$

On the other hand, we can find $\bar{v} = (\bar{v}_i) \in \{v = (v_i) \in \mathbb{R}^n_+ \mid \langle v, \mathbf{e}\rangle = 1\}$ such that $^{-}|\partial f|(\bar{x}) = | \sum_{i=1}^{n} f_i'(\bar{x}) \bar{v}_i|$.Then

$$^{-}|\partial f|(\bar{x}) = | \sum_{i=1}^{n} f_i'(\bar{x}) \bar{v}_i| \ge \min_i |f_i'(\bar{x})| \sum_{i=1}^{n} \bar{v}_i = \min_i |f_i'(\bar{x})|,$$

which together with inequality (4) gives the desired equality. □

3.3. **The convex case.** The slopes and subdifferential slopes admit simpler representations when the map f is K-convex. First, we show that the "limsup" in the definitions of slopes given in Theorem 3.1 can be replaced by "sup" when f is K-convex. Recall [19] that the cone K is *normal* if there exists a constant $\mathcal{N} > 0$ such that

$$0 \le k_1 \le k_2 \text{ imply } \|k_1\| \le \mathcal{N}\|k_2\|,$$

e.g. the nonnegative orthants in the spaces $\mathbb{R}^n$, $C_{[0,1]}$, $L^p_{[0,1]}$ and l_p $(1 < p < +\infty)$.

Theorem 3.3. *Suppose that f is K-convex and that $\bar{x} \in \mathrm{dom} f$ is not a Pareto minimum of f. Then the strong slopes of f can be expressed as*

$$^{-}|\nabla f|(\bar{x}) = \sup_{f(x) \le f(\bar{x})} \frac{d_{Y\setminus K}(f(\bar{x}) - f(x))}{\|\bar{x} - x\|}$$

and

$$^{+}|\nabla f|(\bar{x}) = \sup_{f(\bar{x}) \not\le f(x)} \frac{d_{-K}(f(\bar{x}) - f(x))}{\|\bar{x} - x\|}.$$

If in addition, K is a normal cone with constant $\mathcal{N} = 1$ and there exists $x \in \mathrm{dom} f$, $x \ne \bar{x}$ such that $f(x) \le f(\bar{x})$, then

$$^{+}|\nabla f|(\bar{x}) \ge \sup_{f(x) \le f(\bar{x})} \frac{\|f(\bar{x}) - f(x)\|}{\|\bar{x} - x\|}.$$

Proof. Firstly, we show that

$$^{-}|\nabla f|(\bar{x}) \ge \sup_{f(x) \le f(\bar{x})} -\frac{\Delta_{-K}(f(x) - f(\bar{x}))}{\|\bar{x} - x\|} \tag{5}$$

and

$$^{+}|\nabla f|(\bar{x}) \ge \sup_{f(\bar{x}) \not\le f(x)} \frac{\Delta_{-K}(f(\bar{x}) - f(x))}{\|\bar{x} - x\|} \tag{6}$$

We will consider two cases together. Denote by ξ the quantity in the right-hand side of either of Inequalities (5) and (6). Without loss of generality, we may assume that $\xi > 0$. For each $i = 1, 2, ...$, let $\kappa_i > 0$ and $z_i \in \mathrm{dom} f$ be such that $\kappa_i \to 0$,

$$0 < \xi - \kappa_i \le -\frac{\Delta_{-K}(f(z_i) - f(\bar{x}))}{\|\bar{x} - z_i\|} \text{ and } f(z_i) \le f(\bar{x})$$

in the first case, and

$$0 < \xi - \kappa_i \le \frac{\Delta_{-K}(f(\bar{x}) - f(z_i))}{\|\bar{x} - z_i\|} \text{ and } f(\bar{x}) \not\le f(z_i)$$

in the second case. For $i = 1, 2, ...$ let t_i be scalars defined by

$$t_i = \begin{cases} \frac{1}{i+1} & \text{if the sequence } \{z_j\}_{j=1}^{\infty} \text{ is bounded} \\ \frac{1}{i(d(z_i;\bar{x})+1)} & \text{otherwise} \end{cases}$$

Then we have $t_i \in]0,1[$. Set $x_i = \bar{x} + t_i(z_i - \bar{x}) = (1-t_i)\bar{x} + t_i z_i$. The K-convexity of f implies that $f(x_i) \leq (1-t_i)f(\bar{x}) + t_i f(z_i)$, which gives $f(x_i) - f(\bar{x}) \leq t_i(f(z_i) - f(\bar{x}))$ and $f(\bar{x}) - f(x_i) \geq t_i(f(\bar{x}) - f(z_i))$. It follows from the K-monotonicity of the function Δ_{-K} that

$$\Delta_{-K}(f(x_i) - f(\bar{x})) \leq t_i \Delta_{-K}(f(z_i) - f(\bar{x}))$$

and

$$\Delta_{-K}(f(\bar{x}) - f(x_i)) \geq t_i \Delta_{-K}(f(\bar{x}) - f(z_i)).$$

Since $\|x_i - \bar{x}\| = t_i \|z_i - \bar{x}\|$, we deduce that

$$-\frac{\Delta_{-K}(f(x_i) - f(\bar{x}))}{\|x_i - \bar{x}\|} \geq -\frac{\Delta_{-K}(f(z_i) - f(\bar{x}))}{\|z_i - \bar{x}\|} \geq \xi - \kappa_i$$

in the first case and

$$\frac{\Delta_{-K}(f(\bar{x}) - f(x_i))}{\|x_i - \bar{x}\|} \geq \frac{\Delta_{-K}(f(\bar{x}) - f(z_i))}{\|z_i - \bar{x}\|} \geq \xi - \kappa_i$$

in the second case. On the other hand, the definition of t_i gives $\|x_i - \bar{x}\| = t_i \|z_i - \bar{x}\| \to 0$. Therefore, Theorem 3.2 gives

$$^{-}|\nabla f|(\bar{x}) \geq \limsup_{i \to +\infty} -\frac{\Delta_{-K}(f(x_i) - f(\bar{x}))}{\|x_i - \bar{x}\|} \geq \lim_{i \to \infty}(\xi - \kappa_i) = \xi$$

in the first case and

$$^{+}|\nabla f|(\bar{x}) \geq \limsup_{i \to +\infty} \frac{\Delta_{-K}(f(\bar{x}) - f(x_i))}{\|x_i - \bar{x}\|} \geq \lim_{i \to \infty}(\xi - \kappa_i) = \xi$$

in the second case because $\kappa_i \to 0$. Thus, inequalities (5) and (6) follow. Proposition 3.4 and Theorem 3.2 yield that the equality holds in both these inequalities. The desire equalities then follows from the equalities $-\Delta_{-K}(f(x) - f(\bar{x})) = d_{Y \setminus K}(f(\bar{x}) - f(x))$ when $f(x) \leq f(\bar{x})$ and $\Delta_{-K}(f(\bar{x}) - f(x)) = d_{-K}(f(\bar{x}) - f(x))$ when $f(\bar{x}) \not\leq f(x)$.

Finally, assume that K is a normal cone with constant $\mathcal{N} = 1$. We claim that $d_{-K}(y) = \|y\|$ holds for all $y \in K$. Indeed, it is clear that $d_{-K}(y) \leq \|y\|$ because $0 \in -K$. On the other hand, since $\|y + k\| \geq \|y\|$ for all $k \in K$, we have $d_{-K}(y) \geq \|y\|$. Therefore, $d_{-K}(f(\bar{x}) - f(x)) = \|f(x) - f(\bar{x})\|$ whenever $f(x) \leq f(\bar{x})$. The desired inequality follows from this and the equality obtained above. □

Further, we show that under the convexity assumption, the lower subdifferential slope coincides with the lower slope and they can also be calculated by means of the directional derivative. Note that Proposition 2.3 yields that the composition map $\Delta_{-K}(f(\cdot) - f(\bar{x}))$ is convex and so the subdifferential involved in the definition of the lower subdifferential slope is the subdifferential of convex analysis. Recall that f admits a directional derivative at $\bar{x}$, a map from X to Y denoted by $f'(\bar{x};\cdot)$, provided that for each $ud \in X$, the following holds

$$\lim_{t\to 0^+} \frac{f(x+td)-f(x)}{t} = f'(\bar{x};u).$$

Theorem 3.4. *Suppose that f is K-convex and that $\bar{x} \in \mathrm{dom} f$ is not a Pareto minimum of f. Then*

$$^-|\nabla f|(\bar{x}) = \ ^-|\partial f|(\bar{x}). \tag{7}$$

If $\bar{x} \in \mathrm{int}\ \mathrm{dom} f$ and the directional derivative $f'(\bar{x};\cdot)$ of f at $\bar{x}$ exists, then

$$^-|\nabla f|(\bar{x}) = \sup_{f(z)\leq f(\bar{x})} -\frac{\Delta_{-K}(f'(\bar{x};z-\bar{x}))}{\|z-\bar{x}\|}. \tag{8}$$

Proof. Let us prove equality (7). Let $x^* \in \partial\Delta_{-K}(f(\cdot)-f(\bar{x}))(\bar{x})$. For any $z \in X$ and $t > 0$, we have

$$\begin{array}{ll} \langle x^*, t(z-\bar{x})\rangle & \leq \Delta_{-K}(f(\bar{x}+t(z-\bar{x}))-f(\bar{x})) - \Delta_{-K}(f(\bar{x})-f(\bar{x})) \\ & = \Delta_{-K}(f(\bar{x}+t(z-\bar{x}))-f(\bar{x})) \end{array}. \tag{9}$$

In particular, for $t = 1$ we have $\langle x^*, z-\bar{x}\rangle \leq \Delta_{-K}(f(z)-f(\bar{x}))$ and hence,

$$-\frac{\Delta_{-K}(f(z)-f(\bar{x}))}{\|z-\bar{x}\|} \leq -\langle x^*, \frac{z-\bar{x}}{\|z-\bar{x}\|}\rangle \leq \|x^*\|.$$

This means that $^-|\nabla f|(\bar{x}) \leq \|x^*\|$. As $x^* \in \partial\Delta_{-K}(f(\cdot)-f(\bar{x}))(\bar{x})$ is arbitrarily chosen, we obtain $^-|\nabla f|(\bar{x}) \leq \ ^-|\partial f|(\bar{x})$. Proposition 3.5 now implies equality (7).

Next, we prove equality (8). Equality (7) yields that it suffices to show that

$$^-|\nabla f|(\bar{x}) \leq \sup_{f(z)\leq f(\bar{x})} -\frac{\Delta_{-K}(f'(\bar{x};z-\bar{x}))}{\|z-\bar{x}\|} \leq \inf\{\|x^*\| \mid x^* \in \partial\Delta_{-K}(f(\cdot)-f(\bar{x}))(\bar{x})\}. \tag{10}$$

Let $z \in X$ be an arbitrary vector and $t \in]0,1]$ such that $\bar{x}+t(z-\bar{x}) \in \ \mathrm{dom} f$. Since f is K-convex, we have

$$f(\bar{x}+t(z-\bar{x})) \leq tf(z)+(1-t)f(\bar{x}).$$

Hence

$$\frac{f(\bar{x}+t(z-\bar{x}))-f(\bar{x})}{t\|z-\bar{x}\|} \leq \frac{f(z)-f(\bar{x})}{\|z-\bar{x}\|}.$$

Letting $t \to 0_+$ we get

$$\frac{f'(\bar{x};z-\bar{x})}{\|z-\bar{x}\|} \leq \frac{f(z)-f(\bar{x})}{\|z-\bar{x}\|}.$$

It follows from the monotonicity of the function Δ_{-K} that

$$\Delta_{-K}(\frac{f'(\bar{x};z-\bar{x})}{\|z-\bar{x}\|}) \leq \Delta_{-K}(\frac{f(z)-f(\bar{x})}{\|z-\bar{x}\|}).$$

Then we obtain

$$\sup_{f(z)\le f(\bar{x})} -\frac{\Delta_{-K}(f(z)-f(\bar{x}))}{\|z-\bar{x}\|} \le \sup_{f(z)\le f(\bar{x})} \Delta_{-K}(\frac{f'(\bar{x};z-\bar{x})}{\|z-\bar{x}\|}).$$

Since $^{-}|\nabla f|(\bar{x}) = \sup_{f(z)\le f(\bar{x})} -\frac{\Delta_{-K}(f(z)-f(\bar{x}))}{\|z-\bar{x}\|}$ by Theorem 3.3, the first inequality of (10) follows.

Next, let $x^* \in \partial\Delta_{-K}(f(\cdot)-f(\bar{x}))(\bar{x})$. For any $z \in X$ and $t > 0$ sufficiently small, it follows from (9) that

$$\langle x^*, \frac{z-\bar{x}}{\|z-\bar{x}\|}\rangle \le \lim_{t\to 0_+} \frac{\Delta_{-K}(f(\bar{x}+t(z-\bar{x}))-f(\bar{x}))}{t\|z-\bar{x}\|} = \frac{\Delta_{-K}(f'(\bar{x};z-\bar{x}))}{\|z-\bar{x}\|}.$$

Then

$$-\frac{\Delta_{-K}(f'(\bar{x};z-\bar{x}))}{\|z-\bar{x}\|} \le -\langle x^*, \frac{z-\bar{x}}{\|z-\bar{x}\|}\rangle \le \|x^*\|.$$

It is clear that

$$\sup_{f(z)\le f(\bar{x})} -\frac{\Delta_{-K}(f'(\bar{x};z-\bar{x}))}{\|z-\bar{x}\|} \le \|x^*\|.$$

Since $x^* \in \partial\Delta_{-K}(f(\cdot)-f(\bar{x}))(\bar{x})$ is arbitrarily chosen, the second inequality of (10) follows. □

Remark 3.1. (i) *Theorems 3.3 and 3.4 are vector-valued versions of* [3, Proposition 3.1].

(ii) *We do not know whether a result for the upper slope and upper subdifferential slope similar to the one of Theorem 3.4 holds true or not. Note that in this case, the composition function $-\Delta_{-K}(f(\bar{x})-f(\cdot))$ may not be convex.*

We provide an estimation of the lower differential slope for the case f is additionally *strictly differentiable.*

Proposition 3.7. *Let $X = Y = \mathbb{R}^n$, $f = (f_1, ..., f_n)$ be a $\mathbb{R}^n_+$-convex strictly differentiable map and $\bar{x} \in \mathbb{R}^n$. Assume that the vectors $f_i'(\bar{x})$, $i = \overline{1,n}$, are linearly independent. We have*

$$^{-}|\partial f|(\bar{x}) \ge \frac{1}{\sqrt{\langle Q^{-1}\mathbf{e},\mathbf{e}\rangle}}. \tag{11}$$

Moreover, the equality

$$^{-}|\partial f|(\bar{x}) = \frac{1}{\sqrt{\langle Q^{-1}\mathbf{e},\mathbf{e}\rangle}}. \tag{12}$$

holds if and only if $Q^{-1}\mathbf{e} \in \mathbb{R}^n_+$. Here, $Q := (q_{ij})_{n\times n}$ with $q_{ij} := \langle f_i'(\bar{x}), f_j'(\bar{x})\rangle$ for $i,j = \overline{1,n}$ and Q^{-1} is the inverse matrix of Q.

Proof. Observe that since the vectors $f_i'(\bar{x})$, $i = \overline{1,n}$ are linearly independent, the matrix Q is invertible and Q^{-1} exists. Further, for any $u = (u_i) \in \mathbb{R}^n$, we have

$$\|\sum_{i=1}^{n} u_i f_i'(\bar{x})\|^2 = \sum_{i=1,j=1}^{n} u_i u_j \langle f_i'(\bar{x}), f_j'(\bar{x})\rangle = \langle Qu, u\rangle.$$

Therefore, $\langle Qu, u\rangle \geq 0$ for all $u \in \mathbb{R}^n$. Again, since the vectors $f_i'(\bar{x})$, $i = \overline{1,n}$ are linearly independent, we have $\langle Qu, u\rangle > 0$ when $u \neq 0$. Thus, Q is positive definite. Further, note that Proposition 3.6 (ii) gives

$$^{-}|\partial f|(\bar{x}) = \min\{\|\sum_{i=1}^{n} v_i f_i'(\bar{x})\| \mid v = (v_i) \in \mathbb{R}^n_+, \langle v, \mathbf{e}\rangle = 1\} = \min_{v \in \mathbb{R}^n_+, \langle v, \mathbf{e}\rangle = 1} \sqrt{\langle Qv, v\rangle}. \tag{13}$$

Consider the convex quadratic problem (QP)

$$\min_{v \in \mathbb{R}^n, \langle v, \mathbf{e}\rangle = 1} 1/2\langle Qv, v\rangle.$$

This problem has a unique solution. The Lagrange function associated to this problem has the form $L(v, \lambda) = 1/2\langle Qv, v\rangle - \lambda(\langle v, \mathbf{e}\rangle - 1)$, where $\lambda \in \mathbb{R}$. Assume that $\bar{v} \in \mathbb{R}^n$ is a solution of (QP). Applying the Lagrange multiplier rule (KKT condition) we find $\bar{\lambda}$ such that

$$\begin{cases} Q\bar{v} - \bar{\lambda}\mathbf{e} &= 0 \\ \langle \bar{v}, \mathbf{e}\rangle &= 1. \end{cases}$$

Then the following implications hold

$$Q\bar{v} = \bar{\lambda}\mathbf{e} \Rightarrow \bar{v} = \bar{\lambda}Q^{-1}\mathbf{e} \Rightarrow 1 = \langle \mathbf{e}, \bar{v}\rangle = \bar{\lambda}\langle Q^{-1}\mathbf{e}, \mathbf{e}\rangle \Rightarrow \bar{\lambda} = \frac{1}{\langle Q^{-1}\mathbf{e}, \mathbf{e}\rangle}.$$

We also have

$$\bar{v} = \frac{Q^{-1}\mathbf{e}}{\langle Q^{-1}\mathbf{e}, \mathbf{e}\rangle} \text{ and } \langle Q\bar{v}, \bar{v}\rangle = \bar{\lambda}\langle \mathbf{e}, \bar{v}\rangle = \bar{\lambda} = \frac{1}{\langle Q^{-1}\mathbf{e}, \mathbf{e}\rangle}.$$

The optimal value of (QP) is $\frac{1}{2\langle Q^{-1}\mathbf{e}, \mathbf{e}\rangle}$. Therefore, equality (13) yields inequality (11).

Next, let us consider equality (12). If $Q^{-1}\mathbf{e} \in \mathbb{R}^n_+$, then $\bar{v}$ being in $\mathbb{R}^n_+$ also is a solution to the problem (P)

$$\min_{v \in \mathbb{R}^n_+, \langle v, \mathbf{e}\rangle = 1} 1/2\langle Qv, v\rangle$$

the optimal value of which equals to $\frac{1}{2\langle Q^{-1}\mathbf{e}, \mathbf{e}\rangle}$ and equality (13) implies equality (12). Conversely, if equality (12) holds and $\bar{v} \in \mathbb{R}^n_+$ is a solution to the problem (P), then $\bar{v}$ also is a solution to the problem (QP) because the optimal values of both the problems (QP) and (P) coincide, and we get $Q^{-1}\mathbf{e} = \bar{v} \in \mathbb{R}^n_+$. □

Remark 3.2. *It may happen that* $Q^{-1}\mathbf{e} \notin \mathbb{R}^n_+$ *and inequality (11) is strict. Consider for instance the following case. Let* $X = Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $f(x_1, x_2) = (x_1, 2x_1 + x_2)$. *For any* $\bar{x} \in \mathbb{R}^2$ *we have* $f_1'(\bar{x}) = (1, 0)$, $f_2'(\bar{x}) = (2, 1)$,

$$Q = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \text{ and } Q^{-1} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}.$$

One can check that $Q^{-1}\mathbf{e} = (3, -1) \notin \mathbb{R}^2_+$ *and* $\langle Q^{-1}\mathbf{e}, \mathbf{e}\rangle = 2$. *Thus, it holds that*

$$\frac{1}{\sqrt{\langle Q^{-1}\mathbf{e}, \mathbf{e}\rangle}} = \sqrt{2}/2,$$

while Proposition 3.6 (ii) and a simple calculation give

$$^{-}|\partial f|(\bar{x}) = \min\{\|\sum_{i=1}^{2} v_i f_i'(\bar{x})\| \mid (v_1, v_2) \in \mathbb{R}^2_+, v_1 + v_2 = 1\} = 1.$$

3.4. **The linear case with $Y = \mathbb{R}^n$.** Remark that in this linear case, by Theorem 3.4 we already have $^{-}|\nabla f|(\bar{x}) = {}^{-}|\partial f|(\bar{x})$ and if ∂ is the Clarke subdifferential, then Proposition 3.6 gives $^{+}|\partial f|(\bar{x})| = {}^{-}|\partial f|(\bar{x})$. We will provide formulas for the upper slope and the lower subdifferential slope.

Proposition 3.8. *Assume that* $Y = \mathbb{R}^n$ *and* $f : X \to \mathbb{R}^n$ *is a linear map given by* $f(x) = (\langle a_1^*, x\rangle, ..., \langle a_n^*, x\rangle)$, *where* $a_i^* \in X^*$ $(i = \overline{1, n})$. *Then for any* $\bar{x} \in X$ *we have*

$$\max_i \|a_i^*\| \le {}^{+}|\nabla f|(\bar{x}) \le \sqrt{n} \max_i \|a_i^*\| \tag{14}$$

and

$$^{-}|\partial f|(\bar{x}) = \min\{\|\sum_{i=1}^{n} v_i a_i^*\| \mid (v_1, ..., v_n) \in \mathbb{R}^n_+, \sum_{i=1}^{n} v_i = 1\} = d(0; \mathrm{co}\{a_i^* \mid i = \overline{1, n}\}).$$

Proof. Let us prove the first inequality of 14. Recall that Theorem 3.3 gives

$$^{+}|\nabla f|(\bar{x}) = \sup_{f(\bar{x}) \not\leq f(x)} \frac{d_{-\mathbb{R}^n_+}(f(\bar{x}) - f(x))}{\|x - \bar{x}\|}.$$

Let be given a point $x \in X$ such that $f(\bar{x}) \not\leq f(x)$ or $f(\bar{x}) - f(x) \notin -\mathbb{R}^n_+$. Since

$$d_{-\mathbb{R}^n_+}(f(\bar{x}) - f(x)) = \sqrt{\sum_{i=1}^{n} [\langle a_i^*, \bar{x} - x\rangle]_+^2}$$

(see Example 2.1), we get

$$\sqrt{\sum_{i=1}^{n} [\langle a_i^*, \bar{x} - x\rangle]_+^2} \le \sqrt{\sum_{i=1}^{n} \|a_i^*\|^2 \|\bar{x} - x\|^2} \le \sqrt{n} \max_i \|a_i^*\| \|\bar{x} - x\| \tag{15}$$

which implies

$$^{+}|\nabla f|(\bar{x}) \leq \sqrt{n}\max_i \|a_i^*\|.$$

Next, let us prove the second inequality of (14). Assume that $\|a_j^*\| = \max_i \|a_i^*\|$. By a consequence of the Hahn-Banach theorem, we can find $v \in X$ such that $\|v\| \leq 1$ and $\langle a_j^*, v\rangle = \|a_j\|$. Set $\tilde{x} := \bar{x} - v$. Since $\langle a_j^*, v\rangle = \|a_j\| > 0$, we have $f(\bar{x}) - f(\tilde{x}) = f(v) = (\langle a_1^*, v\rangle, ..., \langle a_j^*, v\rangle, ..., \langle a_n^*, v\rangle) \notin -\mathbb{R}^n_+$. It follows from

$$\|f(\bar{x}) - f(\tilde{x})\| = \|f(v)\| = \sqrt{\sum_{i=1}^n [\langle a_i^*, v\rangle]_+^2} \geq \langle a_j^*, v\rangle = \|a_j^*\|$$

and $\|v\| \leq 1$ that

$$\frac{\|f(\tilde{x}) - f(\bar{x})\|}{\|\tilde{x} - \bar{x}\|} \geq \frac{\|a_j^*\|}{\|v\|} \geq \|a_j^*\|.$$

Therefore $^{+}|\nabla f|(\bar{x}) \geq \|a_j^*\| = \max_i |a_i^*|$, as it was to be shown.

Let us consider now the lower subdifferential slope $^{-}|\nabla f|(\bar{x})$. Observe that $f'(\bar{x}) = (a_1^*, ..., a_n^*)$. Applying Proposition 3.6(ii) and recalling that the function $\|f'(\bar{x})(\cdot)\|$ attains its minimum on the compact set $V := \{v = (v_1, ..., v_n) \in \mathbb{R}^n_+,\ \sum_{i=1}^n v_i = 1\}$, we obtain

$$^{-}|\partial f|(\bar{x}) = \inf\{\|f'(\bar{x})^*(v)\| \mid v \in V\} = \min\{\|\sum_{i=1}^n v_i a_i^*\|\ v \in V\} = d(0; \mathrm{co}\{a_i^* \mid i = \overline{1,n}\}),$$

as it was to be shown. □

We provide an example showing that the first inequality in (14) can be strict.

Example 3.2. *Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^3$, $K = \mathbb{R}^3_+$ and $f(x_1, x_2) = (x_1, 2x_1, 2x_2)$. In this case, we have $a_1^* = (1,0)$, $a_2^* = (2,0)$, $a_3^* = (0,2)$ and $\max\{\|a_i^*\| \mid i = \overline{1,3}\} = 2$. Let $\bar{x} \in \mathbb{R}^2$ be an arbitrary vector. Let $v = (1,0)$ and set $\tilde{x} := \bar{x} - v$. Observe that $f(v) = (1,2,0) \notin -\mathbb{R}^3_+$, $\|f(v)\| = \sqrt{5}$ and $\|v\| = 1$. Then $f(\bar{x}) - f(\tilde{x}) = f(v) = (1,2,0) \notin -\mathbb{R}^n_+$, $\|f(\bar{x}) - f(\tilde{x})\| = \|f(v)\| = \sqrt{5}$ and therefore,*

$$^{+}|\nabla f|(\bar{x}) = \sup_{f(\bar{x}) \not\leq f(x')} \frac{\|f(x') - f(\bar{x})\|}{\|x' - \bar{x}\|} \geq \frac{\|f(\tilde{x}) - f(\bar{x})\|}{\|\tilde{x} - \bar{x}\|} = \frac{\|f(v)\|}{\|v\|} = \frac{\sqrt{5}}{1} > 2 = \max_i \|a_j^*\|.$$

In this case, both inequalities in (14) are strict.

In the following special case, the equality holds in the first inequality of (14) and we have an exact formula for the upper slope.

Corollary 3.1. *Let $X = Y = \mathbb{R}^n$ and f be a linear map $f(x_1, ..., x_n) = (a_1x_1, ..., a_nx_n)$ with $a_i \in \mathbb{R}$ $(i = \overline{1,n})$. Then for any $\bar{x} \in \mathbb{R}^n$ we have*

$$^{+}|\nabla f|(\bar{x}) = \max_i |a_i|$$

and if $a_i \neq 0$ *for all* $i = \overline{1,n}$*, then*

$$^{-}|\partial f|(\bar{x}) = \frac{1}{\sqrt{\sum_{i=1}^{n} \frac{1}{a_i^2}}}.$$

Proof. We show that the first equality follows from Proposition 3.8. Note that in this case we have $a_i^* = (0, ..., a_i, ..., 0)$ (a_i stays at the ith place and zeros stay at the remaining places). The inequalities of (15) become

$$\sqrt{\sum_{i=1}^{n} [\langle a_i^*, \bar{x} - x\rangle]_+^2} = \sqrt{\sum_{i=1}^{n} [a_i(\bar{x}_i - x_i)]_+^2} \leq \max_i |a_i| \sqrt{\sum_{i=1}^{n} (\bar{x}_i - x_i)^2} = \max_i |a_i| \|\bar{x} - x\|.$$

Then we get

$$^{+}|\nabla f|(\bar{x}) \leq \max_i \|a_i\|,$$

which together with (14) yield $^{+}|\nabla f|(\bar{x}) = \max_i |a_i|$.

The second equality follows from Proposition 3.7. Indeed, observe that the matrix Q satisfies $q_{ii} = a_i^2$ for $i = \overline{1,n}$ and $q_{ij} = 0$ for $i, j = \overline{1,n}$, $i \neq j$ and in such a case we have $Q^{-1}\mathbf{e} = (1/a_1^2, ..., 1/a_n^2) \in \mathbb{R}_+^n$. □

Remark 3.3. *If* $0 \in \operatorname{co}\{a_i^* \mid i = \overline{1,n}\}$ *(for instance, if* $a_i^* = 0$ *for some* $i \in \{1, ..., n\}$*), then* $^{-}|\nabla f|(\bar{x}) = 0$*. It also follows that* $^{-}|\nabla f|(\bar{x}) > 0$ *provided* a_i^**,* $i = \overline{1,n}$*, are linearly independent. In contrast to Proposition 3.7, to calculate the lower subdifferential slope of a linear map in Proposition 3.8 we do not need the assumption that the vectors* a_i^* *(*$i = \overline{1,n}$*) are linear independent.*

4. Slopes and global error bounds of lower level sets of a vector-valued map

In this section, the strong slopes will be used to study the nonemptiness and global error bounds of a lower level set of the map f. In particular, we provide a version of Hoffman's error bound for an inequality system.

4.1. **Error bounds of a lower level set.** Firstly, we show that lower strong slope can provide us information about the non-emptiness of a lower level set.

Proposition 4.1. *Assume that the metric space* X *is complete, the map* f *is* K*-level closed and* $\alpha \in Y$*. Then*

$$0 < \inf_{x \notin [f \leq \alpha]} {}^{-}|\nabla f|(x) \text{ implies } [f \leq \alpha] \neq \emptyset.$$

Proof. Define a function $g_\alpha : x \to \Delta_{-K}(f(x) - \alpha)$. The implications

$$x \in [f \leq \alpha] \Leftrightarrow f(x) \leq \alpha \Leftrightarrow f(x) - \alpha \in -K \Leftrightarrow g_\alpha(x) = \Delta_{-K}(f(x) - \alpha) \leq 0$$

imply

$$[f \leq \alpha] = [g_\alpha \leq 0].$$

Then

$$\inf_{x \notin [f \leq \alpha]} {}^{-}|\nabla f|(x) < +\infty \Rightarrow [f \not\leq \alpha] \neq \emptyset \Rightarrow [0 < g_\alpha] \neq \emptyset \Rightarrow \inf_{x \notin [g_\alpha \leq 0]} |\nabla g_\alpha| < +\infty.$$

Next, the relations $\Delta_{-K}(f(x') - f(x)) \geq \Delta_{-K}(f(x') - \alpha) - \Delta_{-K}(f(x) - \alpha)$ and

$$-\frac{\Delta_{-K}(f(x') - f(x))}{d(x'; x)} \leq \frac{\Delta_{-K}(f(x) - \alpha) - \Delta_{-K}(f(x') - \alpha)}{d(x'; x)} = \frac{g_\alpha(x) - g_\alpha(x')}{d(x'; x)}, \ \forall x, x' \in X$$

give

$$^{-}|\nabla f|(x) \leq |\nabla g_\alpha|(x).$$

Therefore, we have $0 < \inf_{x \notin [f \leq \alpha]} {}^{-}|\nabla f|(x) < \inf_{x \notin [g_\alpha \leq 0]} |\nabla g_\alpha|(x)$. Thus, we have $0 < \inf_{x \notin [g_\alpha \leq 0]} |\nabla g_\alpha|(x) < +\infty$. It follows from the result stated in [3, p.412] that $[g_\alpha \leq 0] \neq \emptyset$. This together with the equality $[f \leq \alpha] = [g_\alpha \leq 0]$ give $[f \leq \alpha] \neq \emptyset$, as it was to be shown. □

Next, we introduce a concept of error bound for the map f.

Definition 4.1. *Assume that the map f is K-level closed. For $\alpha \in Y$, we let $\sigma_\alpha(f)$ denote the supremum of the σ's in $[0, +\infty[$ such that*

$$\sigma d(x; [f \leq \alpha]) \leq d_{-K}(f(x) - \alpha) \ \text{ for every } x \notin [f \leq \alpha]$$

with the conventions

$$\sigma_\alpha(f) = \begin{cases} 0 & \text{if } [f \leq \alpha] = \emptyset, \\ +\infty & \text{if } [f \leq \alpha] = X \end{cases}$$

We say that f has a global error bound at the level α if $\sigma_\alpha > 0$.

It follows from the definition that

$$\sigma_\alpha(f) = \inf_{x \notin [f \leq \alpha]} \frac{\Delta_{-K}(f(x) - \alpha)}{d(x; [f \leq \alpha])}. \tag{16}$$

We note that a local version of Definition 4.1 (so called "error bound property at some point") has been recently studied by Bednarczuk and Kruger in [5].

When $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, the concept of global error bound for a vector-valued map in Definition 4.1 coincides with the one defined for a scalar-valued function presented in [3, Definition 2.2]. It is well-known that error bounds for scalar functions play a key role in variational analysis and numerous characterizations of the error bound property have been established in terms of various derivative-like objects either in the primal space (directional derivatives, slopes, etc.) or in the dual space (subdifferentials, normal cones). For the summary of the theory of error bounds and its various applications in optimality conditions, subdifferential calculus... as well as their characterizations, we refer the interested reader to the papers by Azé [1], Lewis and Pang [20], Pang [26] and Bednarczuk and Kruger [4].

The following result provides estimations of the global error bound in term of the lower/upper strong slopes.

Theorem 4.1. *Assume that the metric space X is complete, the map f is K-level closed and $\alpha \in Y$. Then*

$$\inf_{\gamma \not\leq \alpha} \inf_{x \notin [f \leq \gamma]} {}^{-}|\nabla f|(x) \leq \inf_{\gamma \not\leq \alpha} \sigma_\gamma \leq \inf_{x \notin [f \leq \alpha]} {}^{+}|\nabla f|(x). \tag{17}$$

Proof. Let us begin with the first inequality of (17). Let $\gamma \in Y$ such that $\gamma \not\leq \alpha$. Without loss of generality we may assume that $\sigma_\gamma < +\infty$ and $\inf_{x \notin [f \leq \gamma], \gamma \not\leq \alpha} {}^{-}|\nabla f|(x) > 0$. Proposition 4.1 then implies that $[f \not\leq \gamma] \neq \emptyset$ and $[f \leq \gamma] \neq \emptyset$. Let σ be a scalar such that $\sigma > \sigma_\gamma$. Then (16) implies

$$\sigma > \sigma_\gamma = \inf_{x \notin [f \leq \gamma]} \frac{\Delta_{-K}(f(x) - \gamma)}{d(x; [f \leq \gamma])}.$$

Choose $\bar{x} \in [f \not\leq \gamma]$ such that $\Delta_{-K}(f(\bar{x}) - \gamma) < \sigma d(\bar{x}; [f \leq \gamma])$. Define a function $\theta : X \to \mathbb{R}$ by $\theta(x) = [\Delta_{-K}(f(x) - \gamma)]_+$. Since the function $\Delta_{-K}(f(\cdot) - \gamma)$ is lsc by Proposition 2.3, so is the function θ. Further, since $\theta(x) \geq 0$, we have

$$\theta(\bar{x}) < \inf_{\mathbb{B}_r(\bar{x})} \theta(x) + \sigma r,$$

where $r := d(\bar{x}; [f \leq \gamma])$ and $\mathbb{B}_r(\bar{x})$ is the closed ball with the radius r centered at $\bar{x}$. Applying a slope version of the Ekeland variational principle in [2, Corollary 2.1], we find $\tilde{x} \in X$ such that $d(\tilde{x}; \bar{x}) < r$, $\theta(\tilde{x}) \leq \theta(\bar{x})$ and $|\nabla \theta|(\tilde{x}) < \sigma$. Since $d(\tilde{x}; \bar{x}) < r = d(\bar{x}; [f \leq \gamma])$, we get $\tilde{x} \notin [f \leq \gamma]$. Then $f(\tilde{x}) \not\leq \gamma$ and $\Delta_{-K}(f(\tilde{x}) - \gamma) > 0$. As the function $\Delta_{-K}(f(\cdot) - \gamma)$ is lsc, we get $\Delta_{-K}(f(x) - \gamma) > 0$ for x sufficiently close to $\tilde{x}$. Hence, $\theta(x) = [\Delta_{-K}(f(x) - \gamma)]_+ = \Delta_{-K}(f(x) - \gamma)$ for x sufficiently close to $\tilde{x}$. It follows that

$$\limsup_{x \to \tilde{x}, x \neq \tilde{x}} \frac{\Delta_{-K}(f(\tilde{x}) - \gamma) - \Delta_{-K}(f(x) - \gamma)}{d(\tilde{x}; x)} = |\nabla \theta|(\tilde{x}) < \sigma.$$

On the other hand, since

$$\Delta_{-K}(f(x) - f(\tilde{x})) = \Delta_{-K}((f(x) - \gamma) - (f(\tilde{x}) - \gamma)) \geq \Delta_{-K}((f(x) - \gamma) - \Delta_{-K}(f(\tilde{x}) - \gamma),$$

we obtain

$$-\Delta_{-K}(f(x) - f(\tilde{x})) \leq \Delta_{-K}((f(\tilde{x}) - \gamma) - \Delta_{-K}(f(x) - \gamma)$$

and therefore,

$$\limsup_{x \to \tilde{x}, x \neq \tilde{x}} -\frac{\Delta_{-K}(f(x) - f(\tilde{x}))}{d(\tilde{x}; x)} \leq \limsup_{x \to \tilde{x}, x \neq \tilde{x}} \frac{\Delta_{-K}((f(\tilde{x}) - \gamma) - \Delta_{-K}(f(x) - \gamma)}{d(\tilde{x}; x)}.$$

It follows that ${}^{-}|\nabla f|(\tilde{x}) \leq |\nabla \theta|(\tilde{x})$. Hence, ${}^{-}|\nabla f|(\tilde{x}) < \sigma$. Recall that $\tilde{x} \notin [f \leq \gamma]$, we obtain $\inf_{x \notin [f \leq \gamma]} {}^{-}|\nabla f|(x) < \sigma$. Since σ can be chosen as closer to σ_γ as possible, we get $\inf_{x \notin [f \leq \gamma]} {}^{-}|\nabla f|(x) \leq \sigma_\gamma$ and since $\gamma \in Y$ satisfying $\gamma \not< \alpha$ is arbitrarily chosen, we get

$$\inf_{\gamma \not< \alpha} \inf_{x \notin [f \leq \gamma]} {}^{-}|\nabla f|(x) \leq \inf_{\gamma \not< \alpha} \sigma_\gamma.$$

Next, we prove the second inequality of (17). By (16) we have to prove that

$$\inf_{\gamma \not< \alpha} \inf_{x \notin [f \leq \gamma]} \frac{\Delta_{-K}(f(x) - \gamma)}{d(x; [f \leq \gamma])} \leq \inf_{x \notin [f \leq \alpha]} {}^{+}|\nabla f|(x). \tag{18}$$

We may assume that $[f \not\leq \alpha] \neq \emptyset$ because otherwise we have $\inf_{x \notin [f \leq \alpha]} {}^{+}|\nabla f|(x) = +\infty$ and (18) holds automatically and that

$$\inf_{\gamma \not< \alpha} \inf_{x \notin [f \leq \gamma]} \frac{\Delta_{-K}(f(x) - \gamma)}{d(x; [f \leq \gamma])} > 0.$$

Let $\bar{x} \in X$ be such that $f(\bar{x}) \not\leq \alpha$. Let $\sigma > 0$ be such that

$$\inf_{x \notin [f \leq \gamma]} \frac{\Delta_{-K}(f(x) - \gamma)}{d(x; [f \leq \gamma])} > \sigma, \ \forall \gamma \not< \alpha. \tag{19}$$

Set $\gamma_n := f(\bar{x}) - k_0/n$. Since $f(\bar{x}) \not\leq \alpha$ or $f(\bar{x}) \in Y \setminus (\alpha - K)$ and K is a closed set, we deduce that $\gamma_n \in Y \setminus (\alpha - K)$ or $\gamma_n \not\leq \alpha$ for n sufficiently large. Further, as $f(x) = \gamma_n + k_0/n > \gamma_n$, by (19) we can find $x_n \in [f \leq \gamma_n]$ such that

$$\frac{\Delta_{-K}(f(\bar{x}) - \gamma_n)}{d(\bar{x}; x_n)} \geq \sigma.$$

Then, we have

$$0 < d(\bar{x}; x_n) \leq \frac{\Delta_{-K}(f(\bar{x}) - \gamma_n)}{\sigma} = \frac{\Delta_{-K}(k_0/n)}{\sigma} = \frac{\Delta_{-K}(k_0)}{\sigma n} = \frac{1}{\sigma n},$$

which means that $x_n \to \bar{x}$. On the other hand, since $f(x_n) \leq \gamma_n = f(\bar{x}) - k_0/n$, we get $f(x_n) < f(\bar{x})$. Thus, $\bar{x}$ is not a local Pareto minimum of f. Observe that the K-monotonicity of the function Δ_{-K} gives

$$\frac{\Delta_{-K}(f(\bar{x}) - f(x_n))}{d(\bar{x}; x_n)} \geq \frac{\Delta_{-K}(f(\bar{x}) - \gamma_n)}{d(\bar{x}; x_n)} \geq \sigma.$$

As $x_n \to \bar{x}$, we get ${}^{+}|\nabla f|(\bar{x}) \geq \sigma$. Hence, $\inf_{x \notin [f \leq \alpha]} {}^{+}|\nabla f|(x) \geq \sigma$. This together with (19) imply (18). $\square$

4.2. **A Hoffman-type error bound for a linear inequality system.** Let X be a Banach space, $a_i^* \in X^*$ and $c_i \in \mathbb{R}$ ($i \in I := \{1, ..., n\}$). Let S be the solution set of the linear inequality system

$$\langle a_i^*, x\rangle \leq c_i, i \in I, \tag{20}$$

i.e. $S := \{x \in X \mid \langle a_i^*, x\rangle \leq c_i, i \in I\}$. Assume that S is nonempty. The following fundamental result due to Hoffman [15] is well-known : when $X = \mathbb{R}^n$, there exists a constant $\tau > 0$ such that

$$d(x; S) \leq \tau\phi(x), \ \forall x \in X, \tag{21}$$

where $\phi(x) := [\max\{\langle a_i^*, x\rangle - c_i \mid i \in I\}]_+$ (ϕ is called a merit function for the set S, this means that $\phi(x) \geq 0$ on X and $[\phi = 0] = S$).

In the vector case (X is an infinite-dimensional space), Ng and Zheng [30] obtained, in particular, the following result: Let τ_* be the infimum of τ's satisfying (21). Then

$$\tau_* = \frac{1}{\bar{\sigma}}$$

where

$$\bar{\sigma} := \min_{D \in \mathcal{W}(I)} \min\{\| \sum_{i \in D} \|v_i a_i^*\| \mid v_i \geq 0, \ \sum_{i \in D} v_i = 1\}$$

and $\mathcal{W}(I)$ is a certain (finite) family of subsets D of I such that $\{a_i^* \mid i \in D\}$ is a maximal linearly independent subset of $\{a_i^* \mid i \in I\}$ and D has some property, see [30]. This formula allows to finitely compute the value τ_* because the subprogramme in computing σ is a convex optimization problem over a simplex which is a quadratic or linear optimization problem when X is a finite-dimensional space equipped with the Euclidean l_2-norm or l_1-, l_∞-norms. We refer the interested reader to the paper [30] for other results on this subject.

Intuitively, error bounds hold when a regularity condition is satisfied. In the linear case, such a condition is that the lower slope of a linear map under consideration is positive. Namely, we have the following result on a Hoffman-type error bound for the linear inequality system (20).

Proposition 4.2. *Assume that*

$$\sigma := \min\{\| \sum_{i=1}^{n} \|v_i a_i^*\| \mid (v_i) \in \mathbb{R}^n_+, \ \sum_{i=1}^{n} v_i = 1\} > 0. \tag{22}$$

Then

$$d(x; S) \leq \frac{1}{\sigma}\theta(x), \ \forall x \in X,$$

where $\theta(x) := d_{-\mathbb{R}^n_+}(g(x) - c)$, $g(x) := (\langle a_1^*, x\rangle, ..., \langle a_n^*, x\rangle)$ *and* $c := (c_i) \in \mathbb{R}^n$.

Note that $\theta(x) = 0$ when $x \in S$ and $\theta(x) = \sqrt{\sum_{i=1}^{n} [\langle a_i^*, x\rangle - c_i]_+^2}$ when $x \notin S$.

Proof. Proposition 3.8 implies ${}^{-}|\nabla g|(x) \equiv \sigma > 0$ for any $x \in X$. Clearly, $S = [g \leq c]$. The assertion then follows from Theorem 4.1 with $\sigma = \sigma_c$. □

Remark 4.1. *(i) If $0 \notin \mathrm{co}\{a_i^* \mid i = \overline{1,n}\}$ (for instance, if a_i^*, $i = \overline{1,n}$ are linearly independent), then $\sigma = d(0; \mathrm{co}\{a_i^* \mid i = \overline{1,n}\}) > 0$.*

(ii) The function θ is a merit function for the set S of solutions to the linear inequality system (20) and it is easy to see that $\phi(x) \leq \theta(x)$ for all $x \in X$, where ϕ is a function involved in Ng and Zheng's result. The regularity condition (22) immediately gives a Hoffman-type error bound.

5. Slopes and Vector Optimization

In this section, we use the notions of slopes to study several problems in vector optimization. Firstly, we formulate a version of the variational principle for the vector-valued map f. Next, we formulate sufficient conditions for a point to be a weak/strong Pareto minimum of f in the convexity setting. Further, we consider the concept of a weak sharp Pareto (strong Pareto) minimum and apply the results on error bounds established in the previous section to get the existence of such a minimum.

Theorem 5.1. *Let X be a complete metric space and Y be a Banach space with a closed pointed convex cone K (with nonempty interior). Suppose that $f : X \to Y$ is K-lsc and K-bounded. Then for any $\epsilon > 0$ there exists $\bar{x} \in X$, such that*

$$ {}^{-}|\nabla f|(\bar{x}) \leq \epsilon. $$

Recall that f is K-bounded if there exists a bounded nonempty set $M \subset Y$ such that $f(x) \in M + K$ for all $x \in \mathrm{dom} f$.

Proof. Let $k_0 \in \mathrm{int} K$ such that $\Delta_{-K}(k_0) = d_{-K}(k_0) = 1$. Since f is K-bounded, Proposition 3.1 in [13] yields the existence of $x_0 \in X$, such that $f(x) + \epsilon k_0 \not\leq f(x_0)$, $\forall x \in X$. A vector-valued version of the Ekeland variational principle established in [13] implies the existence of $\bar{x} \in X$ such that $f(x) + \epsilon d(x; \bar{x}) k_0 \not\leq f(\bar{x})$ for all $x \neq \bar{x}$. Then we have $f(x) + \epsilon d(x; \bar{x}) k_0 - f(\bar{x}) \notin -K$. It follows that $\Delta_{-K}(f(x) + \epsilon d(x; \bar{x}) k_0 - f(\bar{x})) > 0$. Taking into account the triangle inequality property of the function Δ_{-K}, we get

$$ \Delta_{-K}(f(x) - f(\bar{x})) + \Delta_{-K}(\epsilon d(x; \bar{x}) k_0) \geq \Delta_{-K}(f(x) + \epsilon d(x; \bar{x}) k_0 - f(\bar{x})) > 0. $$

Then we obtain $\Delta_{-K}(f(x) - f(\bar{x})) + \epsilon d(x; \bar{x}) \Delta_{-K}(k_0) \geq 0$ and, since $\Delta_{-K}(k_0) = 1$, we have

$$ -\frac{\Delta_{-K}(f(x) - f(\bar{x}))}{d(x; \bar{x})} \leq \epsilon $$

for $x \neq \bar{x}$. Thus, the inequality ${}^{-}|\nabla f|(\bar{x}) \leq \epsilon$ holds. □

The following result states that in the convex setting, the stationarity (slopes equal zero) at some point implies that this point is a global weak (strong) Pareto minimum of f.

Theorem 5.2. *Suppose that X is a normed space, f is K-convex and $\bar{x} \in \text{dom} f$.*

(i) *If $^{+}|\nabla f|(\bar{x}) = 0$, then $\bar{x}$ is a global strong Pareto minimum of f.*

(ii) *If $^{-}|\nabla f|(\bar{x}) = 0$, then $\bar{x}$ is a global weak Pareto minimum of f.*

Proof. (i) Suppose to the contrary that $\bar{x}$ is not a global strong Pareto minimum of f. Then one can find $u \in X$, $u \neq \bar{x}$ such that $f(\bar{x}) - f(u) \notin -K$. Since K is closed, we have $\Delta_{-K}(f(\bar{x}) - f(u)) > 0$. Let $t_i := 1/(i+1)$ for $i = 1, 2, ...$ and $x_i := \bar{x} + t_i(u - \bar{x}) = (1-t_i)\bar{x} + t_i u$. As f is K-convex, we have $f(x_i) \leq t_i f(u) + (1 - t_i)f(\bar{x})$ and

$$f(\bar{x}) - f(x_i) \geq t_i[f(\bar{x}) - f(u)]. \tag{23}$$

The K-monotonicity of the function Δ_{-K} yields

$$\Delta_{-K}(f(\bar{x}) - f(x_i)) \geq t_i \Delta_{-K}(f(\bar{x}) - f(u)) > 0.$$

Since $\|x_i - \bar{x}\| = t_i \|u - \bar{x}\|$ and $t_i \to 0$, we obtain $x_i \to \bar{x}$ and

$$^{+}|\nabla f|(\bar{x}) \geq \limsup_{i \to +\infty} \frac{\Delta_{-K}(f(\bar{x}) - f(x_i))}{\|\bar{x} - x_i\|} \geq \frac{\Delta_{-K}(f(\bar{x}) - f(u))}{\|\bar{x} - u\|} > 0,$$

which is a contradiction to the assumption that $^{+}|\nabla f|(\bar{x}) = 0$.

(ii) Suppose to the contrary that $\bar{x}$ is not a weak Pareto minimum of f. Then one can find $u \in X$, $u \neq \bar{x}$ such that $f(u) - f(\bar{x}) \in -\text{int} K$. Hence, $\Delta_{-K}(f(u) - f(\bar{x})) < 0$. Let t_i and x_i, $i = 1, 2, ...$ be as in the proof of (i) . It follows from (23) that $(1/t_i)(f(x_i) - f(\bar{x})) \leq f(u) - f(\bar{x})$. Again, the K-monotonicity of the function Δ_{-K} yields

$$(1/t_i)\Delta_{-K}(f(x_i) - f(\bar{x})) \leq \Delta_{-K}(f(u) - f(\bar{x})) < 0.$$

Similarly to the proof of (i), we get

$$^{-}|\nabla f|(\bar{x}) \geq \limsup_{i \to +\infty} -\frac{\Delta_{-K}(f(x_i) - f(\bar{x}))}{\|x_i - \bar{x}\|} \geq -\frac{\Delta_{-K}(f(u) - f(\bar{x}))}{\|u - \bar{x}\|} > 0,$$

which is a contradiction to the assumption that $^{-}|\nabla f|(\bar{x}) = 0$. □

Remark 5.1. *(i) Consider a function $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (x, 1)$. It is easy to see that $^{-}|\nabla f|(x) \equiv 0$ while $^{+}|\nabla f|(x) \equiv 1$ and that any point $x \in \mathbb{R}$ is a global weak Pareto minimum of f but f does not possess any local Pareto minimum.*

(ii) The assumption of K-convexity in the above theorem cannot be dropped. Indeed, let f be the function presented in Example 3.1 (ii). This function is not $\mathbb{R}^2_+$-convex and the equalities $^{-}|\nabla f|(0) =^{+} |\nabla f|(0) = 0$ hold but $\bar{x} = 0$ is not a weak Pareto minimum of f.

In the remainder of this section, we are concerned with the weak sharp minima in vector optimization. In scalar optimization, the notion of weak sharp minimum generalizes the notion of sharp minimum [12] to the case of non-isolated minima. We refer the interested reader to the works by Burke and Ferris [6], Burke and Deng [7], Studniarski [28], Ng, Liu and Zheng [21], Ng and Yang [24].

We will use the following notion of weak sharp global minima. For $\bar{x} \in \mathrm{dom}f$, denote

$$W := \{x \in X \mid f(x) = f(\bar{x})\}.$$

Definition 5.1. *Let $\bar{x} \in \mathrm{dom}f$. We say that $\bar{x}$ is a weak sharp Pareto minimum of f if there exist a scalar $\rho > 0$ such that*

$$\rho d(x; W) \leq d_{-K}(f(x) - f(\bar{x})), \ \forall x \in \mathrm{dom}f.$$

Remark 5.2. *(i) The concept of weak sharp Pareto minimum for the vector-valued map f presented here is a special (global) case of* [28, Definition 2] *and we refer the interested reader to this work for a general concept of weak φ-sharp local Pareto minimum.*

(ii) Assume that W is closed. If $\bar{x}$ is a weak sharp Pareto minimum of f, then $\bar{x}$ is a Pareto minimum of f. Indeed, suppose to the contrary that one can find $x \in \mathrm{dom}f$ such that $f(x) \leq f(\bar{x})$ and $f(x) \neq f(\bar{x})$, i.e. $x \notin W$. Then we have $\rho d(x; W) \leq d_{-K}(f(x) - f(\bar{x})) = 0$ and since W is closed, we get $x \in W$, a contradiction.

Next, we apply Proposition 4.1 and Theorem 4.1 to obtain the existence of weak sharp Pareto minimum.

Theorem 5.3. *Assume that the metric space X is complete and the map f is K-level closed. Let $\alpha \in Y$ be a lower bound of f in the sense that for any $x \in \mathrm{dom}f$ one has either $f(x) = \alpha$ or $f(x) \not\leq \alpha$. Assume further that*

$$0 < \chi := \inf_{\gamma \not< \alpha} \inf_{x \notin [f \leq \gamma]} {}^{-}|\nabla f|(x). \tag{24}$$

Then f possesses a weak sharp Pareto minimum $\bar{x}$ satisfying $f(\bar{x}) = \alpha$ and

$$\rho d(x; W) \leq d_{-K}(f(x) - f(\bar{x})), \ \forall x \in \mathrm{dom}f$$

holds for some constant $\rho \geq \chi$.

Proof. Since $\alpha \not< \alpha$, we have

$$\inf_{\gamma \not< \alpha} \inf_{x \notin [f \leq \gamma]} {}^{-}|\nabla f|(x) \leq \inf_{x \notin [f \leq \alpha]} {}^{-}|\nabla f|(x).$$

Inequality (24) and Proposition 4.1 imply that $[f \leq \alpha] \neq \emptyset$. Then we find $\bar{x} \in X$ such that $f(\bar{x}) \leq \alpha$. As α is a lower bound of f, we get $f(\bar{x}) = \alpha$ and $[f \leq \alpha] = [f \leq f(\bar{x})] = [f =$

$f(\bar{x})]$. Thus, $W = [f \leq f(\bar{x})]$. On the other hand, Theorem 4.1 gives

$$\chi = \inf_{\gamma \not< \alpha} \inf_{x \notin [f \leq \gamma]} {}^{-}|\nabla f|(x) \leq \inf_{\gamma \not< \alpha} \sigma_\gamma.$$

In particular, for $\gamma = \alpha$ we have $\sigma_\alpha \geq \chi$ and hence,

$$\sigma_\alpha d(x; W) = \sigma_\alpha d(x; [f \leq f(\bar{x})] \leq d_{-K}(f(x) - f(\bar{x})),\ \forall x \in \mathrm{dom} f$$

and $\bar{x}$ is a weak sharp Pareto minimum of f with constant $\rho \geq \chi$. □

Example 5.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a map $f(x, y) = (|x|, |y|)$ considered in Example 3.1 (iii). It has been established that ${}^{-}|\nabla f|(x, y) = \sqrt{2}/2$ for any $(x, y) \neq (0, 0)$. Theorem 5.3 yields that $(0, 0)$ is a weak sharp Pareto minimum of f with the constant $\tau \geq \chi = \sqrt{2}/2$.*

6. Slopes and Calmness

In this section, strong slopes will be used to characterize calmness of a vector-valued map. Recall that a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is calm at $\bar{x}$ from below with modulus $\kappa \in \mathbb{R}_+$ if $f(\bar{x})$ is finite and for some neighborhood V of $\bar{x}$ one has

$$f(x) \geq f(\bar{x}) - \kappa \|x - \bar{x}\|$$

when $x \in V$ [27, p.322]. The property of f being calm at $\bar{x}$ from above is defined similarly; it corresponds to $-f$ being calm at $\bar{x}$ from below.

In the case of vector valued maps, we define two types of calmness.

Definition 6.1. *Let $f : X \to Y \cup \{+\infty\}$ and $\bar{x} \in \mathrm{dom} f$. Let $k_0 \in \mathrm{int} K$ be fixed.*

(i) *f is calm at $\bar{x}$ from below with modulus $\kappa \in \mathbb{R}_+$ in the direction k_0 if on some neighborhood V of $\bar{x}$ one has*

$$f(x) \geq f(\bar{x}) - \kappa d(x; \bar{x}) k_0,\ \forall x \in V \cup \mathrm{dom} f.$$

(ii) *f is weakly calm at $\bar{x}$ from below with modulus $\kappa \in \mathbb{R}_+$ in the direction k_0 if on some neighborhood V of $\bar{x}$ one has*

$$f(x) \not\ll f(\bar{x}) - \kappa d(x; \bar{x}) k_0,\ \forall x \in V \cup \mathrm{dom} f.$$

The property of f being calm/weakly calm at $\bar{x}$ from above with modulus $\kappa \in \mathbb{R}_+$ in the direction k_0 is defined similarly; it corresponds to $-f$ being calm/weakly calm at $\bar{x}$ from below with modulus $\kappa \in \mathbb{R}_+$ in the direction k_0. In what follows we will fix k_0 and omit the expression "in the direction k_0" while concerning with calmness/weakly calmness. We will say that f is calm/weak calm if it is calm/weakly calm from below and from above.

It is clear that calmness implies weak calmness and that the above concepts of calmness reduce to the classical one of a function when $Y = \mathbb{R}$, $K = \mathbb{R}_+$ and $k_0 = 1$. In the case

$Y = \mathbb{R}^n$, $K = \mathbb{R}^n_+$, $f = (f_1, ..., f_n)$ with $f_i : X \to \mathbb{R}$ $(i = \overline{1,n})$ and $k_0 = \mathbf{e}$ $(k_0 = \mathbf{e_i})$, the calmness of f is equivalent to calmness of all functions f_i (of one function f_i, respectively).

The following result provides a characterization of calmness in terms of the norm.

Proposition 6.1. *Assume that K is normal. Then f is calm near $\bar{x} \in \mathrm{dom} f$ if and only if there exist a constant $l > 0$ and a neighborhood U of $\bar{x}$ such $\|f(x) - f(\bar{x})\| \leq l\|x - \bar{x}\|$ for $x \in U$.*

Proof. Since $k_0 \in \mathrm{int}K$, one can define a new norm $\|.\|_{k_0}$ in Y as follows: for any $y \in Y$

$$\|y\|_{k_0} := \inf\{t \geq 0| \; -tk_0 \leq y \leq tk_0\};$$

moreover, when K is normal, the norms $\|.\|$ and $\|.\|_{k_0}$ are equivalent in the sense that $t_1\|y\|_{k_0} \leq \|y\| \leq t_2\|y\|_{k_0}$ for some scalars $t_1, t_2 > 0$ and all $y \in Y$ [19]. Now, the definitions of the calmness and of the norm $\|.\|_{k_0}$ imply

$$\begin{aligned} f \text{ is calm at } \bar{x} \quad &\Longleftrightarrow \quad -\kappa d(x;\bar{x})k_0 \leq f(x) - f(\bar{x}) \leq \kappa d(x;\bar{x})k_0, \;\forall x \text{ near } \bar{x} \\ &\Longleftrightarrow \quad \|f(x) - f(\bar{x})\|_{k_0} \leq \kappa d(x;\bar{x}), \;\forall x \text{ near } \bar{x}. \end{aligned}$$

The assertion follows from the equivalence of the norms $\|.\|$ and $\|.\|_{k_0}$ stated above. □

Some authors use the term "f is lipschitz near $\bar{x}$" when the inequality $\|f(x) - f(\bar{x})\| \leq l\|x - \bar{x}\|$ holds for x near $\bar{x}$, see [17].

Necessary and/or sufficient conditions for the weak calmness and the calmness in terms of the slopes are formulated in the following theorem.

Theorem 6.1. *Let $\bar{x} \in \mathrm{dom} f$.*

(i) *A necessary condition for f to be weakly calm from below at $\bar{x}$ with modulus κ is*

$$^{-}|\nabla f|(\bar{x}) \leq \kappa d_{-K}(k_0)$$

and sufficient conditions for f to be weakly calm from below at $\bar{x}$ with modulus κ is

$$^{-}|\nabla f|(\bar{x}) < \kappa d_{Y\setminus K}(k_0) \text{ or } {}^{+}|\nabla f|(\bar{x}) < \kappa d_{-K}(k_0).$$

(ii) *A necessary condition for f to be calm from below at $\bar{x}$ with modulus κ is*

$$^{-}|\nabla f|(\bar{x}) \leq \kappa d_{Y\setminus K}(k_0) \text{ and } {}^{+}|\nabla f|(\bar{x}) \leq \kappa d_{-K}(k_0)$$

and a sufficient condition for f to be calm from below at $\bar{x}$ with modulus κ is

$$^{+}|\nabla f|(\bar{x}) < \kappa d_{Y\setminus K}(k_0).$$

Proof. (i) If f is weakly calm from below at $\bar{x}$ with modulus κ, then $\Delta_{-K}(f(x) - f(\bar{x}) + \kappa d(x;\bar{x})k_0) \geq 0$ for x near $\bar{x}$. The triangle inequality property of Δ_{-K} yields $\Delta_{-K}(f(x) - f(\bar{x})) + \Delta_{-K}(\kappa d(x;\bar{x})k_0)) \geq 0$ for x near $\bar{x}$. Therefore, $-\frac{\Delta_{-K}(f(x)-f(\bar{x}))}{d(x;\bar{x})} \leq \kappa d_{-K}(k_0)$, holds for x near $\bar{x}$, which gives ${}^{-}|\nabla f|(\bar{x}) \leq \kappa d_{-K}(k_0)$.

Further, if f is not weakly calm from below at $\bar{x}$ with modulus κ, then there exists a sequence $x_i \to \bar{x}$ such that $f(x_i) \ll f(\bar{x}) - \kappa d(x_i;\bar{x})k_0$ or $\Delta_{-K}(f(x_i) - f(\bar{x}) + \kappa d(x_i;\bar{x})k_0) < 0$ for all $i = 1, 2, ...$ Applying the triangle inequality property of the function Δ_{-K}, we get $\Delta_{-K}(f(x_i) - f(\bar{x})) - \Delta_{-K}(-\kappa d(x_i;\bar{x})k_0) < 0$ and $\Delta_{-K}(\kappa d(x_i;\bar{x})k_0) - \Delta_{-K}(f(\bar{x}) - f(x_i)) < 0$ for all $i = 1, 2, ...$ These inequalities imply ${}^{-}|\nabla f|(\bar{x}) \geq \kappa d_{Y\setminus K}(k_0)$ and ${}^{+}|\nabla f|(\bar{x}) \geq \kappa d_{-K}(k_0)$, which contradict the assumption.

(ii) The assertion can be proved similarly. □

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Institute of Mathematics, Vietnam Academy of Science and Technology, 18, Hoang Quoc Viet Road, Cau Giay District 10307, Hanoi, Vietnam

E-mail address: `txdha@math.ac.vn`