A REMARK ON THE LOWER SEMICONTINUITY ASSUMPTION IN THE EKELAND VARIATIONAL PRINCIPLE

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Abstract. What happens to the conclusion of the Ekeland variational principle (briefly, EVP) if a considered function $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous not on the whole metric space X but only on its domain? We provide a straightforward proof showing that it still holds but only for ϵ varying in some interval $]0, \beta - \inf_X f[$, where β is a quantity expressing quantitatively the violation of the lower semicontinuity of f outside its domain. The obtained result extends EVP to a larger class of functions.

1. Introduction

Let X be a complete metric space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ a function. The domain, the epigraph and the lower level set at $t \in \mathbb{R}$ of f are denoted by $\text{dom} f := \{x \in X \mid f(x) < +\infty\}$, $\text{epi} f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \le t\}$ and $[f \le t] := \{x \in X \mid f(x) \le t\}$, respectively. For any nonempty set $U \subset X$, $\text{int} U$ and $\text{bd} U$ denote the interior and the boundary of U.

Recall that f is lower semicontinuous (briefly, lsc) at $x \in X$ if the inequality

$$\liminf_{x' \to x} f(x') \ge f(x) \tag{1}$$

holds and f is lsc on X if it is lsc at every point $x \in X$ [1, Definition 2, p. 11].

The Ekeland variational principle (briefly, EVP), which has proved to be a potent and flexible tool in analysis and in optimization theory, is stated as follows.

Theorem 1.1 ([3, Theorem 1.1]). *Let f be lsc on X and bounded from below. Let $\epsilon > 0$. Then for any point $x_0 \in X$ satisfying $f(x_0) \le \inf_X f + \epsilon$ and any scalar $\lambda > 0$, there exists $\bar{x} \in \text{dom} f$ such that:*

(a) $f(\bar{x}) \le f(x_0)$;

(b) $d(x_0; \bar{x}) \le \lambda$;

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(c) $\bar{x}$ is a strict minimizer of the function $x \to f(x) + (\epsilon/\lambda)d(x;\bar{x})$, i.e.

$$f(\bar{x}) < f(x) + (\epsilon/\lambda)d(x;\bar{x}), \quad \text{for all } x \neq \bar{x}. \tag{2}$$

Recall that f is said to be Gâteaux differentiable at a point $x \in$int dom$f$ (X is assumed to be a Banach space) if there exists $f'(x) \in X^*$ such that for any $u \in X$

$$\lim_{h\to 0} \frac{f(x+hu) - f(x)}{h} = f'(x)(u).$$

Proposition 1.1 ([1, Proposition 5, p. 258])**.** *Assume that X is a Banach space and f is finite and Gâteaux differentiable at $\bar{x}$. Then condition (2) implies that*

$$\|f'(\bar{x})\|_* \leq \epsilon/\lambda,$$

where $\|f'(\bar{x})\|_*$ is the norm of $f'(\bar{x})$ in the dual space X^*.

The main assumption in EVP is that f is "lsc on X" although Ekeland did not clarify the meaning of the expression "lsc on X" in the paper [3]. His arguments, however, were essentially based on the fact that the epigraph of f is closed. It is well-known that f is lsc on X if and only if the epigraph and all lower level sets $[f \leq t]$ ($t \in \mathbb{R}$) of f are closed.

The assumption that f is lsc on the whole space X is in fact very important. As far as it is known to the author, there is no attention paid to the question what happens to the conclusion of EVP when f is only assumed to be lsc on its domain. Meanwhile, the expression "f is lsc on X" sometimes is understood as "f is lsc on its domain" [2, p. 3] [1].

In this note we study this question, namely:

(i) We present examples illustrating that the conclusion of EVP may not hold at all or it holds not for all positive ϵ when f is lsc only on its domain.

(ii) We provide a straightforward proof showing that under a natural assumption, the conclusion of EVP holds for ϵ varying in some interval $]0, \beta - \inf_X f[$, where β is a quantity expressing quantitatively the violation of inequality (1) outside the domain of f. This version of EVP collapses to the classical one when the function is lsc on the whole space.

2. Main results

First, we consider some examples showing that the conclusion of EVP may fail when f is lsc only on its domain.

[1]This has been recently corrected, see https://www.carma.newcastle.edu.au/jon/ToVA/errata.pdf

Example 2.1. *(i) Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a function defined by $f(x) = x$ if $x > 0$ and $f(x) = +\infty$ otherwise. This function is continuous on its domain but is not lsc at $x = 0$. The conclusion of EVP does not hold for any $\epsilon > 0$. Indeed, for $\lambda = \epsilon + 1$ if the assertion (b) holds for some $\bar{x} \in \mathrm{dom} f$, then Proposition 1.1 yields $|f'(\bar{x})| \leq \epsilon/(\epsilon + 1)$ while we have $f'(\bar{x}) = 1$.*

(ii) Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a function defined by

$$f(x) = \begin{cases} -x + 1 & \text{if } x < 0 \\ 5/x & \text{if } x > 0 \\ +\infty & \text{if } x = 0 \end{cases}$$

This function is continuous on its domain but is not lsc at $x = 0$ and $\inf_X f = 0$. We show that the conclusion of EVP does not hold for any $\epsilon > 1$. Indeed, let $x_0 = -\epsilon + \delta + 1$, where δ is a scalar satisfying $0 < \delta < \min\{\epsilon - 1, 1\}$. Then $x_0 < 0$ and we have $f(x_0) = -x_0 + 1 = \epsilon - \delta < \epsilon = \inf_X f + \epsilon$. Let λ be a scalar satisfying $\epsilon < \lambda < \epsilon + 1 - \delta$. Suppose to the contrary that we can find a point $\bar{x} \neq 0$ satisfying the assertions (a)-(b) of Theorem 1.1. Since $x_0 + \lambda < 2$, the inequality $d(\bar{x}; x_0) \leq \lambda$ implies $\bar{x} \in]-\infty, 0[\cup]0, 2[$. If $\bar{x} \in]-\infty, 0[$, then $f'(\bar{x}) = -1$ and if $\bar{x} \in]0, 2[$, then $f'(\bar{x}) = -5/\bar{x}^2 < -1$. In both cases, we have $|f'(\bar{x})| \geq 1$ while Proposition 1.1 yields $|f'(\bar{x})| \leq \frac{\epsilon}{\lambda} < 1$, a contradiction. We show later that the conclusion of EVP holds for any $\epsilon \in]0, 1]$.

To formulate a version of EVP for the case f is lsc on its domain, we need some characterization of how much the lower semicontinuity is violated outside the domain of f and how close f is to being "lsc on X". To this end, let us establish an auxiliary result.

Lemma 2.1. *Suppose that f is lsc on its domain. If $[f \leq t]$ is closed for some $t > \inf_X f$, then $[f \leq t']$ is also closed for any $t' \in]\inf_X f, t[$.*

Proof. Let $\{x_i\}$ be a sequence such that $x_i \in [f \leq t']$ for $i = 1, 2, ...$ and $x_i \to x$. Since $[f \leq t'] \subset [f \leq t]$ and $[f \leq t]$ is closed, we get $x \in [f \leq t]$, which means that $x \in \mathrm{dom} f$. As f is lsc on its domain, we have $f(x) \leq \liminf_{x' \to x} f(x') \leq \liminf_{i \to +\infty} f(x_i) \leq t'$. Thus, $x \in [f \leq t']$ and the lower level set $[f \leq t']$ is closed. □

Now, let us define two quantities, which play an important role in the mentioned above characterization. Set

$$\beta := \inf_{x \in \mathrm{bd}\ \mathrm{dom} f \setminus \mathrm{dom} f} \ \liminf_{x' \to x} f(x')$$

and (f is assumed to be bounded from below on X)

$$\gamma := \sup\{t > \inf_X f \mid [f \leq t] \text{ is closed}\}.$$

Here, we make the convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

Remark 2.1. *(i) Assuming that f is lsc on its domain, we consider the question at which points (outside the domain of f) the lower semicontinuity of f is violated. It is clear that f is lsc at any point $x \in \operatorname{int}(X \setminus \operatorname{dom} f)$, because in such a case one has $\liminf_{x' \to x} f(x') = f(x) = +\infty$. So any point x at which f is not lsc must lie on* bd dom$f$ *with $f(x) = +\infty$ and inequality (1) does not hold at x, i.e. we have*

$$\liminf_{x' \to x} f(x') < f(x) = +\infty.$$

The value $\liminf_{x' \to x} f(x')$ may be used as a tool measuring the violation of inequality (1) at x and the quantity β expresses quantitatively the largest violation of inequality (1) of f outside its domain. One can see that the larger the value β is, the smaller the violation of inequality (1) of f outside its domain is.

(ii) It is known that f is lsc on X if and only if all its lower level sets are closed and in such a case $\gamma = +\infty$. Clearly, when f is lsc only on its domain (dom$f \neq X$), we have $\gamma < +\infty$ and we could say that the larger the value γ is, the closer f is to being "lsc on X".

The following proposition shows that γ and β are in fact equal.

Proposition 2.1. *Assume that f is lsc on* dom$f$ *and bounded from below. If one of the two inequalities $\gamma > \inf_X f$ and $\beta > \inf_X f$ holds, then the other also holds and $\gamma = \beta$.*

Proof. First, suppose that $\gamma > \inf_X f$. We will prove that $\beta \geq \gamma$. Suppose to the contrary that $\beta < \gamma$. Then we can choose a scalar $\bar{t}$ such that $\beta < \bar{t} < \gamma$ and $\bar{t} > \inf_X f$. By the definition of β, one can find $u \in$ bd dom$f$ such that $f(u) = +\infty$ and $\liminf_{x' \to u} f(x') < \bar{t}$. Then we can find a sequence $\{x_i\}$ such that $x_i \to u$, $f(x_i) \leq \bar{t}$ and $\lim_{i \to \infty} f(x_i) = \liminf_{x' \to u} f(x') < \bar{t}$. Thus, $x_i \in \bar{S} := [f \leq \bar{t}]$ for $i = 1, 2, ...,$ $x_i \to u$ but $f(u) = +\infty$, i.e. $u \notin \bar{S}$ and $\bar{S}$ is not closed. On the other hand, since $\inf_X f < \bar{t} < \gamma$, the definition of γ implies that $\bar{S}$ is closed. The obtained contradiction shows that the inequality $\beta \geq \gamma$ holds and therefore, $\beta \geq \gamma > \inf_X f$.

Next, suppose that $\beta > \inf_X f$. We show that $\gamma \geq \beta$. It suffices to check that any lower level set $U := [f \leq t]$ with t satisfying $\inf_X f < t < \beta$ is closed. Observe that U is nonempty because $t > \inf_X f$ and that $U \subset$ dom$f$. Let $\{x_i\}$ be a sequence such that $x_i \in U$ for $i = 1, 2, ...$ and $x_i \to u$. We claim that $u \in$ dom$f$. Indeed, suppose to the contrary that $f(u) = +\infty$. Since $x_i \in U \subset$ dom$f$ and $x_i \to u$, we get $u \in$ bd dom$f$. Then we have

$$\beta = \inf_{x \in \text{bd dom} f, f(x) = +\infty} \liminf_{x' \to x} f(x') \leq \liminf_{x' \to u} f(x') \leq \liminf_{i \to \infty} f(x_i) \leq t < \beta,$$

which is a contradiction. Therefore, $u \in$ dom$f$. By the assumption, f is lsc at u. Hence, $f(u) \leq \liminf_{x' \to u} f(x') \leq \liminf_{i \to \infty} f(x_i) \leq t$, this means that $u \in U$ and U is closed. Thus, the inequality $\gamma \geq \beta$ holds and we obtain $\gamma \geq \beta > \inf_X f$.

Finally, it is clear from the used arguments that the assertion is true. ∎

A version of EVP for the case f is lsc on its domain reads as follows.

Theorem 2.1. *Let f be lsc on* $\mathrm{dom} f$ *and bounded from below. Suppose that*

$$\beta > \inf_X f. \tag{3}$$

Then the conclusion of EVP stated in Theorem 1.1 holds for any scalar ϵ satisfying

$$\epsilon \in]0, \beta - \inf_X f[. \tag{4}$$

Proof. Let ϵ be a scalar satisfying relation (4) and $x_0 \in \mathrm{dom} f$ satisfying $f(x_0) \leq \inf_X f + \epsilon$. Set $t := \inf_X f + \epsilon$. We will consider the lower level set S given by $S := [f \leq t]$. Since $\inf_X f < t$, the set S is nonempty. Further, since $t < \beta$ and $\beta = \gamma$ by Proposition 2.1, one can find a scalar t_0 such that $t < t_0 < \gamma$ and the set $[f \leq t_0]$ is closed. Lemma 2.1 implies that S is closed.

Define a function $\bar{f} : X \to \mathbb{R} \cup \{+\infty\}$ by

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in S \\ +\infty & \text{if } x \notin S. \end{cases}$$

We show that $\bar{f}$ satisfies conditions of Theorem 1.1. First, we show that this function is lsc on the whole space X. Let $x \in X$ be given. If $x \in \mathrm{dom} \bar{f} = S$, the lower semicontinuity of f at x and the equalities $\bar{f}(x) = f(x)$ and $\bar{f}(u) \geq f(u)$ for all $u \in X$ imply that $\bar{f}$ satisfies inequality (1) at x. If $x \in X \setminus S$, inequality (1) holds at x because the set $X \setminus S$ is open and, therefore, $\liminf_{x' \to x} \bar{f}(x') = \bar{f}(x) = +\infty$. Thus, $\bar{f}$ is lsc on X. Next, since $f(x_0) \leq \inf_X f + \epsilon = t$, we get $x_0 \in S = \mathrm{dom} \bar{f}$ and $\bar{f}(x_0) = f(x_0)$. We also have $\inf_X \bar{f} = \inf_X f > -\infty$. Therefore, $\bar{f}(x_0) \leq \inf_X \bar{f} + \epsilon$.

Applying Theorem 1.1 to the function $\bar{f}$, we obtain the existence of $\bar{x} \in \mathrm{dom} \bar{f}$ such that $\bar{f}(\bar{x}) \leq \bar{f}(x_0)$, $d(x_0; \bar{x}) \leq \lambda$, and $\bar{x}$ is a strict minimizer of the function $x \to \bar{f}(x) + (\epsilon/\lambda) d(x; \bar{x})$, i.e.

$$\bar{f}(\bar{x}) < \bar{f}(x) + (\epsilon/\lambda) d(x; \bar{x}), \ \forall x \in \mathrm{dom} \bar{f}, \ x \neq \bar{x}.$$

Further, as $\bar{x} \in \mathrm{dom} \bar{f} = S$, we have $\bar{f}(\bar{x}) = f(\bar{x})$. Recall that $\bar{f}(x_0) = f(x_0)$. Then it follows from $\bar{f}(\bar{x}) \leq \bar{f}(x_0)$ that $f(\bar{x}) \leq f(x_0)$. It remains to check that the assertion (c) stated in Theorem 1.1 holds. Let $x \in \mathrm{dom} f, x \neq \bar{x}$ be given. If $x \in S$, then $\bar{f}(x) = f(x)$ and we have $f(\bar{x}) = \bar{f}(\bar{x}) < \bar{f}(x) + (\epsilon/\lambda) d(x; \bar{x}) = f(x) + (\epsilon/\lambda) d(x; \bar{x})$. If $x \notin S$, then $f(x) > t \geq f(\bar{x})$ and hence, $f(\bar{x}) < f(x) < f(x) + (\epsilon/\lambda) d(x; \bar{x})$. Thus, the assertion (c) holds.

□

3. Some comments

(i) When f is lsc on X, we have

$$\sup\{t > \inf_X f \mid [f \leq t] \text{ is closed}\} = \inf_{x \in \text{bd } \text{dom} f \setminus \text{dom} f} \liminf_{x' \to x} f(x') = +\infty$$

and Theorem 2.1 reduces to Theorem 1.1.

(ii) If $\gamma = \min\{t > \inf_X f \mid [f \leq t] \text{ is closed}\}$, i.e. the set $[f \leq \gamma]$ is closed, the arguments used the proof of Theorem 2.1 show that relation (4) can be replaced by

$$\epsilon \in]0, \beta - \inf_X f].$$

(iii) Estimate of ϵ given in relation (4) cannot be improved in the sense that the conclusion of EVP may fail if we take $\epsilon > \beta - \inf_X f$. Let us consider the function f in Example 2.1 (ii). We have $\gamma = \beta = 1$, $\inf_X f = 0$ and the lower level set $[f \leq 1]$ is closed. The conclusion of EVP holds for any ϵ in $]0, 1]$ by Theorem 2.1 and comment (iii) but not for $\epsilon > 1$ as it has already been shown.

(iv) Theorem 2.1 extends Theorem 1.1 to a larger class of functions. For instance, if $\text{dom} f$ is a proper subset of X, which is not closed, and f is lsc and bounded from above on its domain, then f cannot be lsc on the whole space X. Indeed, if $x \in \text{bd } \text{dom} f \setminus \text{dom} f$, then $f(x) = +\infty$ while the boundedness from above of f on its domain implies $\liminf_{x' \to x} f(x') < +\infty$ and therefore, f is not lsc at x. Then only Theorem 2.1 is applicable to the function f while Theorem 1.1 is not. One can see that this also happens in Example 2.1(ii). Note that the domain of f being the solution set of some optimization problem (the case with a bilevel optimization problem) is often not closed.

(v) Theorem 2.1 may allow to get the same conclusion as in Theorem 1.1 but under the weaker assumption that f is lsc on its domain. For instance, one can derive from Theorem 2.1 results about the density of the range $f'(X)$ similar to the ones stated in [3, Corollaries 2.4 and 2.5] because only sufficiently small values of ϵ are needed for the proof. But in general, if the value ϵ is a priori given, one should check whether ϵ satisfies relation (4) or not before using the conclusion of EVP.

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