CASTELNUOVO-MUMFORD REGULARITY OF SYMBOLIC POWERS OF TWO-DIMENSIONAL SQUARE-FREE MONOMIAL IDEALS

LE TUAN HOA AND TRAN NAM TRUNG

Abstract. Let I be a square-free monomial ideal of a polynomial ring R such that $\dim(R/I) = 2$. We give explicit formulas for computing the a_i-invariants $a_i(R/I^{(n)})$, $i = 1, 2$, and the Castelnuovo-Mumford regularity $\mathrm{reg}(R/I^{(n)})$ for all n. The values of these functions depend on the structure of an associated graph. It turns out that these functions are linear functions of n for all $n \geq 2$.

Introduction

Let I be a square-free monomial ideal of a polynomial ring $R = k[x_1, ..., x_r]$ over a field k. Then I can be considered as a Stanley-Reisner ideal associated to a simplicial complex. In recent years the study of powers I^n and symbolic powers $I^{(n)}$ attracts attention of many authors (see, e.g., [3, 5, 9, 12]). In the two-dimensional case, the associated simplicial complex is a graph G and we may write a two-dimensional square-free monomial ideal in the form:

$$I_G = \bigcap_{\{i,j\}\in E(G)} P_{ij} \bigcap_{i\in V_0(G)} P_i,$$

where $E(G)$ is the edge set of G, $V_0(G)$ the set of isolated vertices, $P_{ij} = (\{x_1, ..., x_r\} \setminus \{x_i, x_j\})$, and $P_i = (\{x_1, ..., x_r\} \setminus \{x_i\})$. Some algebraic properties of I_G^n and $I_G^{(n)}$ can be characterized in terms of G (see, e.g., [8, 7]). In this paper we are interested in computing the Castelnuovo-Mumford regularity. Let us recall this notion. Let J be a proper homogeneous ideal of R. Set

$$a_i(R/J) = \sup\{t \mid H^i_{\mathfrak{m}}(R/J)_t \neq 0\},$$

where $H^i_{\mathfrak{m}}(R/J)$ is the local cohomology module with the support $\mathfrak{m} = (x_1, \ldots, x_r)$. The Castelnuovo-Mumford regularity of R/J is defined by

$$\mathrm{reg}(R/J) = \max\{a_i(R/J) + i \mid 0 \leqslant i \leqslant \dim R/J\}.$$

Let $J^{(n)}$ be the n-th symbolic powers of J. It is well-known that $\mathrm{reg}(R/J^n)$ is a linear function of n for $n \gg 0$ (see [1, Theorem 1.1] or [6, Theorem 5]). Concerning $\mathrm{reg}(R/J^{(n)})$, it was shown in some cases that this function is bounded by a linear function of n (see [4, Section 2]). Moreover, when $J = I$ is a square-free monomial ideal, in [5, Theorem 4.1 and Theorem 4.9] we proved that $a_i(R/I^{(n)})$ and $\mathrm{reg}(R/I^{(n)})$

1991 *Mathematics Subject Classification.* 13D45.

Key words and phrases. Castelnuovo-Mumford regularity, symbolic power, square-free monomial ideals.

are quasi-linear functions of n for $n \gg 0$. But it is still not known whether they are linear functions of n for $n \gg 0$. Therefore, we start to investigate this problem when $\dim R/I = 2$, i.e. when $I = I_G$ for a graph G. The main purpose of this note is to give explicit formulas for computing $a_i(R/I_G^{(n)}),\ i = 1, 2$ and $\operatorname{reg}(R/I_G^{(n)})$ (see Theorem 2.3, Theorem 2.8 and Theorem 2.9). It turns out that all these functions are linear functions of n for $n \geq 2$. The proofs of these results are based on Takayama's formula for computing local cohomology modules of monomial ideals (see Lemma 1.1) and a formula to compute simplicial complexes associated to symbolic powers of square-free monomial ideals (see Lemma 1.3), which extends a result given in [8].

The paper is divided into two sections. In Section 1 we recall Takayama's formula - a generalized version of Hochster's formula - to compute local cohomology modules of monomial ideals, and then give some descriptions of associated simplicial complexes. In Section 2 we prove the main results.

1. Auxiliary results

A simplicial complex Δ on the finite set $[r] := \{1, \dots, r\}$ is a collection of subsets of $[r]$ such that $F \in \Delta$ whenever $F \subseteq F'$ for some $F' \in \Delta$. Notice that we do not impose the condition that $\{i\} \in \Delta$ for all $i \in [r]$. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. The Stanley-Reisner ideal of Δ is the following ideal of $R := k[x_1, ..., x_r]$:

$$I_\Delta := (x_{i_1} \cdots x_{i_s} \mid \{i_1, \dots, i_s\} \notin \Delta) = \bigcap_{F \in \mathcal{F}(\Delta)} P_F,$$

where P_F is the prime ideal of R generated by all variables x_i with $i \notin F$. It is a square-free monomial ideal. Conversely, if I is a square-free monomial ideal, then it is the Stanley-Reisner ideal associated to the following simplicial complex

$$\Delta(I) = \{\{i_1, ..., i_s\} \mid x_{i_1} \cdots x_{i_s} \notin I\}.$$

If I is an arbitrary monomial ideal we set $\Delta(I) = \Delta(\sqrt{I})$. For a subset F of $[r]$, let $R_F := R[x_i^{-1} \mid i \in F]$ and for $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r$, let $x^\alpha = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$. We define the co-support of α to be the set $CS_\alpha := \{i \mid \alpha_i < 0\}$. Set

$$\Delta_\alpha(I) = \{F \subseteq [r] \setminus CS_\alpha \mid x^\alpha \notin IR_{F \cup CS_\alpha}\}.$$

We set $\widetilde{H}_i(\emptyset; k) = 0$ for all i, $\widetilde{H}_i(\{\emptyset\}; k) = 0$ for all $i \neq -1$, and $\widetilde{H}_{-1}(\{\emptyset\}; k) = k$. Thanks to [2, Lemma 1.1] we may formulate Takayama's formula as follows.

Lemma 1.1. ([11, Theorem 2.2])

$$\dim_k H^i_{\mathfrak{m}}(R/I)_\alpha = \begin{cases} \dim_k \widetilde{H}_{i-|CS_\alpha|-1}(\Delta_\alpha(I); k) & \textit{if } CS_\alpha \in \Delta(I), \\ 0 & \textit{otherwise.} \end{cases}$$

It was then shown in [8, Lemma 1.3] that $\Delta_\alpha(I)$ is a subcomplex of $\Delta(I)$. For a face $F \in \Delta$, the link of F is defined by

$$\operatorname{lk}_\Delta(F) = \{G \subseteq [r] \setminus F \mid F \cup G \in \Delta\}.$$

The next lemma gives a more precise description of $\Delta_\alpha(I)$ and will be useful in its computation.

Lemma 1.2. *Assume that $CS_\alpha \in \Delta(I)$ for some $\alpha \in \mathbb{Z}^r$. Then*

$$\Delta_\alpha(I) = \{F \in \mathrm{lk}_{\Delta(I)}(CS_\alpha)|\ x^\alpha \notin IR_{F\cup CS_\alpha}\}.$$

Proof. Let $F \subseteq [r] \setminus CS_\alpha$. Note that if $F \cup CS_\alpha \notin \Delta(I)$, then $\sqrt{I}R_{F\cup CS_\alpha} = R_{F\cup CS_\alpha}$, which yields $IR_{F\cup CS_\alpha} = R_{F\cup CS_\alpha}$ and $F \notin \Delta_\alpha(I)$. So, if $F \in \Delta_\alpha(I)$ we must have $F \cup CS_\alpha \in \Delta(I)$, i.e. $F \in \mathrm{lk}_{\Delta(I)}(CS_\alpha)$. □

The n-th symbolic power $I_\Delta^{(n)}$ is defined by

$$I_\Delta^{(n)} = \bigcap_{F\in\mathcal{F}(\Delta)} P_F^n.$$

The following lemma extends [8, Lemma 2.1] and plays a crucial role in studying properties of $\Delta_\alpha(I_G^{(n)})$.

Lemma 1.3. *Assume that $CS_\alpha \in \Delta$ for some $\alpha \in \mathbb{Z}^r$. Then*

$$\mathcal{F}(\Delta_\alpha(I_\Delta^{(n)})) = \{F \in \mathcal{F}(\mathrm{lk}_\Delta(CS_\alpha)) \mid \sum_{i\notin F\cup CS_\alpha} \alpha_i \leqslant n-1\}.$$

Proof. By Lemma 1.2 it follows that a facet $F \in \Delta_\alpha(I_\Delta^{(n)})$ has the form $F = F' \setminus CS_\alpha$, where F' is a facet of Δ containing CS_α and $x^\alpha \notin I_\Delta^{(n)} R_{F'}$. Since $I_\Delta^{(n)} R_{F'} = (x_i|\ i \notin F')^n$, the last condition is equivalent to $x^{\alpha'} \notin (x_i|\ i \notin F')^n$, where $x^{\alpha'} = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_s}^{\alpha_{i_s}}$ if we set $[r] \setminus F' = \{i_1, ..., i_s\}$. Clearly, this condition is equivalent to $\sum_{i\notin F\cup CS_\alpha} \alpha_i \leq n-1$. □

Put $|\alpha| = \alpha_1 + \cdots + \alpha_r$.

Lemma 1.4. *If $\Delta_\alpha(I_\Delta^{(n)}) = \{\emptyset\}$, then $CS_\alpha \in \mathcal{F}(\Delta)$ and $|\alpha| \leq n-1-|CS_\alpha|$. Moreover, if $F \in \mathcal{F}(\Delta)$, then*

$$\max\{|\beta| \mid CS_\beta = F \text{ and } \Delta_\beta(I_\Delta^{(n)}) = \{\emptyset\}\} = n-1-|F|.$$

Proof. From Lemma 1.3 it immediately follows that $CS_\alpha \in \mathcal{F}(\Delta)$. Since $\alpha_i \leq -1$ for all $i \in CS_\alpha$, and $\emptyset \in \Delta_\alpha(I_\Delta^{(n)})$, again by Lemma 1.3 we have

$$|\alpha| = \sum_{i\in CS_\alpha} \alpha_i + \sum_{j\notin CS_\alpha} \alpha_j \leq -|CS_\alpha| + n - 1.$$

Now let $F \in \mathcal{F}(\Delta)$. Without loss of generality we may assume that $F = \{1, ..., s\}$. Let $\beta = (-1, ..., -1, n-1, 0, ..., 0)$ (s entries of -1). Then $CS_\beta = F$, $|\beta| = n-1-s$ and one can use Lemma 1.3 to verify that $\Delta_\beta(I_\Delta^{(n)}) = \{\emptyset\}$. Hence the second statement follows from the first one. □

A graph G is an undirected simple graph with the vertex set $V(G) \subseteq [r]$ having no loops. The set of isolated vertices is denoted by $V_0(G)$, which can be empty. The set of edges of G is denoted by $E(G)$ and is assumed to be not empty. We always consider G as the simplicial complex Δ of dimension one, such that $\mathcal{F}(\Delta) =$

$E(G) \cup \{\{i\} | i \in V_0(G)\}$. If there is no confusion we will use the same notation G to denote this simplicial complex. Recall that a connected graph without cycles is called a *tree*, and a disjoint union of trees is called a *forest*. The following result is probably known, but we could not find a reference. We provide a proof for the sake of completeness.

Lemma 1.5. *Let G be a graph considered as a simplicial complex of dimension one. Then $\widetilde{H}_1(G,k) = 0$ if and only if G is a forest.*

Proof. Let $G_1, \ldots, G_s$ be the connected components of G. Then the reduced Euler characteristic $\tilde{\chi}(G)$ of G can be computed in two ways (see, e.g. [10, Definition 3.2]):

$$\tilde{\chi}(G) = -1 + |V(G)| - |E(G)| = \dim_k \widetilde{H}_0(G;k) - \dim_k \widetilde{H}_1(G;k).$$

Since $\dim_k \widetilde{H}_0(G;k) = s - 1$, we deduce that

$$\dim_k \widetilde{H}_1(G;k) = |E(G)| + s - |V(G)| = \sum_{i=1}^{s} (|E(G_i)| + 1 - |V(G_i)|).$$

As each G_i is a connected graph, we have $|E(G_i)| + 1 \geq |V(G_i)|$ and the equality holds if and only if G_i is a tree. Thus, $\dim_k \widetilde{H}_1(G;k) = 0$ if and only if all $G_1, ..., G_s$ are trees, that means G is a forest, as required. □

2. Castelnuovo-Mumford regularity of symbolic powers

Since we are considering graphs with possibly isolated vertices, any square-free monomial ideal of dimension two can be seen as I_G for some graph G. Since $I_G^{(n)}$ has no $\mathfrak{m}$-primary component, $H^0_{\mathfrak{m}}(R/I_G^{(n)}) = 0$. Hence

$$\operatorname{reg}(R/I_G^{(n)}) = \max\{a_1(R/I_G^{(n)}) + 1,\ a_2(R/I_G^{(n)}) + 2\}.$$

So, in order to compute $\operatorname{reg}(R/I_G^{(n)})$ we have to compute $a_1(R/I_G^{(n)})$ and $a_2(R/I_G^{(n)})$. The computation of $a_1(R/I_G^{(n)})$ in the unmixed case was implicitly done in [8] and [7]. We formulate these results below. Since $\dim(R/I_G^{(n)}) = 2$, it follows that $a_1(R/I_G^{(n)}) = -\infty$ if and only if the ring $R/I_G^{(n)}$ is Cohen-Macaulay.

We recall some notions from graph theory. The distance between two vertices i and j is the minimal length of paths which connect them. The maximal distance between two vertices of G is called the diameter of G and denoted by $\operatorname{diam}(G)$. If G is not connected we set $\operatorname{diam}(G) = \infty$. A graph is called a *matroid* if any two of its disjoint edges are contained in a cycle of length 4.

Lemma 2.1. *The ring $R/I_G^{(n)}$ is a Cohen-Macaulay ring if and only if G is connected and one of the following conditions is satisfied:*

(i) *$n = 1$,*
(ii) *$\operatorname{diam}(G) = 2$ and either $n = 2$ or G is a matroid.*

Proof. It is well known that the Cohen-Macaulayness of $R/I_G^{(n)}$ implies the connectedness of G. This also immediately follows from Lemma 1.1 by setting $\alpha = (0, ..., 0)$ and $i = 1$. Hence we may assume from the beginning that G is connected. In particular, G has no isolated vertex, and the statement follows from [8, Theorem 2.3 and Theorem 2.4]. □

Lemma 2.2. *Assume that G has no isolated vertex and $R/I_G^{(n)}$ is not a Cohen-Macaulay ring. Then $a_1(R/I_G^{(n)}) = 2n-2$.*

Proof. By [7, Lemma 3.2(1)], $a_1(R/I_G^{(n)}) \leq 2n-2$. In order to show the reverse inequality we distinguish three cases.

If $n=1$, then by Lemma 2.1, G is not connected. Hence, by [7, Lemma 3.2(2)], $[H_{\mathfrak{m}}^1(R/I_G)]_0 \neq 0$.

If $n=2$, then by Lemma 2.1, $\operatorname{diam}(G) \geq 3$. Hence, by [7, Corollary 3.4], $[H_{\mathfrak{m}}^1(R/I_G^{(2)})]_2 \neq 0$.

Assume $n \geq 3$. By Lemma 2.1, G is not a matroid. Hence, by [7, Lemma 3.5], $[H_{\mathfrak{m}}^1(R/I_G^{(n)})]_{2n-2} \neq 0$.

Summing up, in all cases, $[H_{\mathfrak{m}}^1(R/I_G^{(n)})]_{2n-2} \neq 0$, which yields $a_1(R/I_G^{(n)}) \geq 2n-2$, as required. □

Theorem 2.3. *Assume that $R/I_G^{(n)}$ is not a Cohen-Macaulay ring. Then $a_1(R/I_G^{(n)}) = 2n-2$.*

Proof. By Lemma 2.2, it suffices to assume that G has an isolated vertex, say 1. Since $E(G) \neq \emptyset$, we may assume that $\{2,3\} \in E(G)$. Let $\beta = (n-1, n-1, 0, ..., 0)$. We have $CS_\beta = \emptyset$, and by Lemma 1.3, $\{1\}, \{2,3\} \in \Delta_\beta(I_G^{(n)})$. Hence, $\Delta_\beta(I_G^{(n)})$ is disconnected and by Lemma 1.1,

$$\dim_k [H_{\mathfrak{m}}^1(R/I_G^{(n)})]_\beta = \dim_k \widetilde{H}_0(\Delta_\beta(I_G^{(n)}); k) \neq 0,$$

which implies $a_1(R/I_G^{(n)}) \geq |\beta| = 2n-2$.

We now show that $a_1(R/I_G^{(n)}) \leq 2n-2$. Let $\alpha \in \mathbb{Z}^r$ such that $a_1(R/I_G^{(n)}) = |\alpha|$ and

$$\dim_k [H_{\mathfrak{m}}^1(R/I_G^{(n)})]_\alpha = \dim_k \widetilde{H}_{-|CS_\alpha|}(\Delta_\alpha(I_G^{(n)}); k) \neq 0. \tag{1}$$

Hence $|CS_\alpha| \leq 1$. If $|CS_\alpha| = 1$, the above inequality implies that $\Delta_\alpha(I_G^{(n)}) = \{\emptyset\}$. By Lemma 1.4, $|\alpha| \leq n-2$, a contradiction. Hence, $CS_\alpha = \emptyset$. In this case, by Lemma 1.2, $\Delta_\alpha(I_G^{(n)})$ is a subgraph of G and, by (1), it must be disconnected. We may assume that $\{1, i_1\}, \{2, i_2\}$ are facets of $\Delta_\alpha(I_G^{(n)})$ such that $i_1 \neq 2$, $i_2 \neq 1$ and $i_1 \neq i_2$ (but it may happen that $i_1 = 1$ and/or $i_2 = 2$). Then, by Lemma 1.3 $|\alpha| \leq \sum_{j \neq 1, i_1} \alpha_j + \sum_{j \neq 2, i_2} \alpha_j \leq 2n-2$, as required. □

We are now computing $a_2(R/I_G^{(n)})$. For that we need some preparation lemmas. Recall that the *girth* of G, denoted by $\operatorname{girth}(G)$, is the smallest length of cycles of G. If G contains no cycle (equivalently, G is a forest) we set $\operatorname{girth}(G) = \infty$. Thus, if $\operatorname{girth}(G)$ is finite, then $3 \leq \operatorname{girth}(G) \leq |V(G)|$.

From now on, let $\alpha \in \mathbb{Z}^r$ such that $[H_{\mathfrak{m}}^2(R/I_G^{(n)})]_\alpha \neq 0$. By Lemma 1.1,

$$\dim_k [H_{\mathfrak{m}}^2(R/I_G^{(n)})]_\alpha = \dim_k \widetilde{H}_{1-|CS_\alpha|}(\Delta_\alpha(I_G^{(n)}); k) \neq 0, \tag{2}$$

and CS_α is a face of the simplicial complex G. Hence, we must have $|CS_\alpha| \le 2$. We distinguish three cases.

Lemma 2.4. *Assume that* $|CS_\alpha| = 0$, *i.e.* $\alpha \in \mathbb{N}^r$. *Then* $3 \le s := \operatorname{girth}(G) \le r$, *and*

$$|\alpha| \le \left[\frac{s(n-1)}{s-2}\right].$$

Proof. Since $CS_\alpha = \emptyset$, by Lemma 1.2, $\Delta_\alpha(I_G^{(n)})$ is a subgraph of G. Since $\widetilde{H}_1(\Delta_\alpha(I_G^{(n)}); k) \neq 0$, by Lemma 1.5, $\Delta_\alpha(I_G^{(n)})$ must contain a cycle, say $C = (1, 2, ..., t)$, where $t \ge s = \operatorname{girth}(G)$. In particular, s is finite and $3 \le s \le r$. By Lemma 1.3, for all $l = 1, ..., t-1$ we have $\sum_{i \neq l,l+1} \alpha_i \le n-1$ and $\sum_{i\neq t,1} \alpha_i \le n-1$. Hence,

$$(t-2)|\alpha| \le \sum_{l=1}^{t-1} \sum_{i\neq l,l+1} \alpha_i + \sum_{i \neq t,1} \alpha_i \le t(n-1),$$

which yields $|\alpha| \le [\frac{t(n-1)}{t-2}] \le [\frac{s(n-1)}{s-2}]$. □

Lemma 2.5. *Assume that* $|CS_\alpha| = 1$. *Then* $|\alpha| \le 2n-3$.

Proof. We may assume that $CS_\alpha = \{r\}$. By Lemma 1.2, $\Delta_\alpha(I_G^{(n)}) \subseteq \operatorname{lk}_G(CS_\alpha)$, so it is $\emptyset$, or $\{\emptyset\}$, or a set of points. By (2), $\dim_k \widetilde{H}_0(\Delta_\alpha(I_G^{(n)}); k) \neq 0$. Therefore, $\Delta_\alpha(I_G^{(n)})$ must contain at least two points, say $1, 2$, and we must have $r \ge 3$. Since $\alpha_r \le -1$ and $\alpha \ge 0$ for $i \le r-1$, by Lemma 1.3 we get

$$|\alpha| \le \sum_{i\neq 1,r} \alpha_i + \sum_{i \neq 2,r} \alpha_i + \alpha_r \le 2(n-1) - 1 = 2n-3.$$

□

Lemma 2.6. *Assume that* $|CS_\alpha| = 2$. *Then* $|\alpha| \le n-3$.

Proof. Since $\operatorname{lk}_{CS_\alpha}(G) = \{\emptyset\}$, by Lemma 1.2, $\Delta_\alpha(I_G^{(n)})$ is either $\emptyset$ or equal to $\{\emptyset\}$. By (2) we must have $\widetilde{H}_{-1}(\Delta_\alpha(I_G^{(n)}); k) \neq 0$. Therefore, $\Delta_\alpha(I_G^{(n)}) = \{\emptyset\}$. By Lemma 1.4, $|\alpha| \le n-3$. □

Lemma 2.7. *Assume that* G *contains a vertex of degree at least 2. Then* $a_2(R/I_G^{(n)}) \ge 2n-3$.

Proof. We may assume that $\{1,2\}, \{1,3\} \in E(G)$. Let $\beta = (-1, n-1, n-1, 0, ..., 0)$. Then $CS_\beta = \{1\}$, $\operatorname{lk}_G(CS_\beta) \supseteq \{2,3\}$. By Lemma 1.3, one can check that $\Delta_\beta(I_G^{(n)}) = \{\emptyset, \{2\}, \{3\}\}$. Hence, by Lemma 1.1,

$$\dim_k[H^2_{\mathfrak{m}}(R/I_G^{(n)})]_\beta = \dim_k \widetilde{H}_0(\Delta_\beta(I_G^{(n)}); k) = 1,$$

which implies $a_2(R/I_G^{(n)}) \ge |\beta| = 2n-3$. □

We are now able to compute $a_2(R/I_G^{(n)})$:

Theorem 2.8. *For all* $n \geqslant 1$ *we have*

(1) *If* $\operatorname{girth}(G) = 3$, *then* $a_2(R/I_G^{(n)}) = 3n - 3$.
(2) *If* $\operatorname{girth}(G) = 4$, *then* $a_2(R/I_G^{(n)}) = 2n - 2$.
(3) *If* $\infty > \operatorname{girth}(G) \geqslant 5$, *then* $a_2(R/I_G) = 0$ *and* $a_2(R/I_G^{(n)}) = 2n - 3$ *for all* $n \geqslant 2$.
(4) *If* G *is a forest with some vertex of degree at least* 2, *then* $a_2(R/I_G^{(n)}) = 2n-3$.
(5) *If* G *consists of* $t \geq 1$ *disjoint edges and possibly isolated vertices, then*

$$a_2(R/I_G^{(n)}) = \begin{cases} -2 & \text{if } r = 2, \\ n - 3 & \text{if } r > 2, \end{cases}$$

where r *is the number of variables of* R.

Proof. Let $m := a_2(R/I_G^{(n)})$ at let α be chosen as in (2) such that $m = |\alpha|$. Let $s = \operatorname{girth}(G)$. In the case $s < \infty$, we may assume that $C = (1, 2, ..., s)$ is a cycle of G. We distinguish four cases.

Case 1: $s = 3$. By Lemmas 2.4, 2.5, 2.6, $m \leq 3n - 3$. Let $\beta = (n-1, n-1, n-1, 0, ..., 0)$. Using Lemma 1.3 one can immediately check that $\Delta_\beta(I_G^{(n)})$ is a subgraph of G and contains C. By Lemma 1.5, $\widetilde{H}_1(\Delta_\beta(I_G^{(n)}); k) \neq 0$. Then, by Lemma 1.1, $[H_{\mathfrak{m}}^2(R/I_G^{(n)})]_\beta \neq 0$, whence $m \geq |\beta| = 3n - 3$. Hence $m = 3n - 3$.

Case 2: $s = 4$. Again by Lemmas 2.4, 2.5, 2.6, $m \leq 2n-2$. Let $\beta = (n-1, 0, n-1, 0, ..., 0)$. With a similar argument as in Case 1, we get $m = 2n - 2$.

Case 3: $5 \leq s < \infty$. If $n = 1$, then again by Lemmas 2.4, 2.5, 2.6, $m \leq 0$. Using a similar argument as in Case 1 applied to $\beta = (0, ..., 0)$, we get $m = 0$. If $n \geq 2$, then $[s(n-1)/(s-2)] \leq 2n-3$. Again by Lemmas 2.4, 2.5, 2.6, $m \leq 2n-3$. Using Lemma 2.7 we then get $m = 2n - 3$.

Case 4: $s = \infty$, that means G is a forest. If G contains a vertex of degree at least 2, then combining Lemmas 2.5, 2.6, 2.7, we get $m = 2n - 3$. Otherwise, G consists of t disjoint edges, where $t \geq 1$, and possibly some isolated vertices. If $r = 2$, then $t = 1$ and $I_G^{(n)} = I_G = 0$. It is clear that $a_2(R/I_G^{(n)}) = -2$. Let $r \geq 3$. By Lemma 2.4 we must have $|CS_\alpha| = 1, 2$. Assume that $|CS_\alpha| = 1$. Since at most one vertex is joined to the vertex of CS_α, $\Delta_\alpha(I_G^{(n)})$ must be $\emptyset$ or $\{\emptyset\}$ or consists of exactly one point. In all cases, by Lemma 1.1

$$\dim_k[H_{\mathfrak{m}}^2(R/I_G^{(n)})]_\alpha = \dim_k \widetilde{H}_0(\Delta_\alpha(I_G^{(n)}); k) = 0,$$

a contradiction. Hence $|CS_\alpha| = 2$. By Lemma 2.6, $m = |\alpha| \leq n - 3$.

On the other hand, in this case we may assume that $\{1, 2\} \in E(G)$. Let $\beta = (-1, -1, n-1, 0, ..., 0)$. Then $CS_\beta = \{1, 2\}$, $\operatorname{lk}_G(CS_\beta) = \{\emptyset\}$. By Lemma 1.3, one can check that $\Delta_\beta(I_G^{(n)}) = \{\emptyset\}$. Hence, by Lemma 1.1,

$$\dim_k[H_{\mathfrak{m}}^2(R/I_G^{(n)})]_\beta = \dim_k \widetilde{H}_{-1}(\Delta_\beta(I_G^{(n)}); k) = 1,$$

which implies $m = a_2(R/I_G^{(n)}) \geq |\beta| = n - 3$, whence $m = n - 3$. □

Finally, we can state and prove the main result on the Castelnuovo-Mumford regularity. One can see that as $a_2(R/I_G^{(n)})$, the function $\operatorname{reg}(R/I_G^{(n)})$ mainly depends on the girth of G.

Theorem 2.9. *For all $n \geqslant 1$ we have*

(1) *If* $\operatorname{girth}(G) = 3$, *then* $\operatorname{reg}(R/I_G^{(n)}) = 3n - 1$.

(2) *If* $\operatorname{girth}(G) = 4$, *then* $\operatorname{reg}(R/I_G^{(n)}) = 2n$.

(3) *If* $\infty > \operatorname{girth}(G) \geqslant 5$, *then* $\operatorname{reg}(R/I_G) = 2$ *and* $\operatorname{reg}(R/I_G^{(n)}) = 2n - 1$ *for all* $n \geqslant 2$.

(4) *If G is a forest with at least two edges or at least one isolated vertex, then* $\operatorname{reg}(R/I_G^{(n)}) = 2n - 1$.

(5) *If G consists of exactly one edge, then* $\operatorname{reg}(R/I_G^{(n)}) = 0$ *if* $r = 2$ *and* $\operatorname{reg}(R/I_G^{(n)}) = n - 1$ *for all $r \geq 3$, where r is the number of variables of R.*

Proof. By Lemma 2.1 and Theorem 2.3, $a_1(R/I_G^{(n)}) + 1 \leq 2n - 1$. Since

$$\operatorname{reg}(R/I_G^{(n)}) = \max\{a_1(R/I_G^{(n)}) + 1,\ a_2(R/I_G^{(n)}) + 2\},$$

using Theorem 2.8 above one immediately get the statements in the first three cases and also in the case when G is a forest with a vertex of degree at least 2.

So, it is left to consider the case G being a forest and all its vertices have degree one or zero. In particular, all edges of G must be disjoint. Recall that G has at least one edge. If G is a forest consisting of at least two disjoint edges or at least one isolated vertex, then G is disconnected. By Lemma 2.1 and Theorem 2.3, $a_1(R/I_G^{(n)}) + 1 = 2n - 1$, while by Theorem 2.8(5), $a_2(R/I_G^{(n)}) + 2 = n - 1$. Hence $\operatorname{reg}(R/I_G^{(n)}) = 2n - 1$. In the last case, when G consists of exactly one edge, then $R/I_G^{(n)}$ is a Cohen-Macaulay ring. Therefore, $\operatorname{reg}(R/I_G^{(n)}) = a_2(R/I_G^{(n)}) + 2$ and the statement follows from Theorem 2.8(5). □

From Lemma 2.1 and Theorem 2.3, it is clear that $a_1(R/I_G^{(n)})$ is a linear function for all $n \geq 1$, and from Theorem 2.8 and Theorem 2.9, $a_2(R/I_G^{(n)})$ and $\operatorname{reg}(R/I_G^{(n)})$ are linear functions for all $n \geq 2$.

Acknowledgment

This work is partially supported by Vietnam NAFOSTED (Project 101.01-2011.48). The authors would like to thank the referee for his/her careful reading and comments.

References

[1] D. Cutkosky, J. Herzog, N. V. Trung, *Asymptotic behavior of the Castelnuovo-Mumford regularity*, Compositio Math. **118** (1999), 243–261.

[2] D. H. Giang and L. T. Hoa, *On local cohomology of a tetrahedral curve*, Acta Math. Vietnam., **35** (2010), 229-241.

[3] J. Herzog, T. Hibi and N. V. Trung, *Symbolic powers of monomial ideals and vertex cover algebra*, Adv. Math. **210** (2007), 304–322.

[4] J. Herzog, L. T. Hoa and N. V. Trung, *Asymptotic linear bounds for the Castelnuovo-Mumford regularity*, Trans. Amer. Math. Soc. **354** (2002), 1793-1809.

[5] L. T. Hoa and T. N. Trung, *Partial Castelnuovo-Mumford regularities of sums and intersections of powers of monomial ideals*, Math. Proc. Cambridge Philos Soc. **149** (2010), 1-18.

[6] V. Kodiyalam, *Asymptotic behaviour of Castelnuovo-Mumford regularity*, Proc. Amer. Math. Soc. **128**(2000), 407-411.

[7] N. C. Minh and Y. Nakamura, *The Buchsbaum property of symbolic powers of Stanley-Reisner ideals of demsion 1*, J. Pure and Appl. Algebra **215** (2011), 161 - 167.

[8] N. C. Minh and N. V. Trung, *Cohen-Macaulayness of powers of two-dimensional squarefree monomial ideals*, J. Algebra **322** (2009), 4219-4227.

[9] N. C. Minh and N. V. Trung, *Cohen-Macaulayness of monomial ideals and symbolic powers of Stanley-Reisner ideals* , Adv. Mathematics **226** (2011), 1285 - 1306.

[10] R. P. Stanley, Combinatorics and Commutative Algebra, Birkhauser 1996.

[11] Y. Takayama, *Combinatorial characterizations of generalized Cohen-Macaulay monomial ideals*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **48** (2005), 327-344.

[12] T. N. Trung, *Stability of depths of powers of edge ideals*, J. Algebra (to appear).

Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, 10307 Hanoi, Vietnam

E-mail address: `lthoa@math.ac.vn`

Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, 10307 Hanoi, Vietnam

E-mail address: `tntrung@math.ac.vn`