ON THE STRUCTURE OF AFFINE FLAT GROUP SCHEMES OVER DISCRETE VALUATION RINGS

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ABSTRACT. We study affine group schemes over a discrete valuation ring R using two techniques: Neron blowups and Tannakian categories. We employ the theory developed to define and study differential Galois groups of integrable connections on a scheme over a R. This throws light on how differential Galois groups of families degenerate.

1. INTRODUCTION

The thoughts the reader is about to see in this text come from our will to understand "models" of group schemes defined over a DVR and their Tannakian interpretation. In fact, we set out to identify if a more general theory would be able to accommodate the following two examples. (For unexplained notations, see page 4.)

Example 1.1 (SGA3, Exp. VI_B, 13.3)**.** Let R be a discrete valuation ring with uniformiser π and quotient field K. Consider the pro-system of affine group schemes

$$\ldots \longrightarrow \mathbf{G}_{a,R} \xrightarrow{\times\pi} \mathbf{G}_{a,R} \xrightarrow{\times\pi} \mathbf{G}_{a,R}, \tag{1}$$

which corresponds, on the level of rings, to the inductive system

$$R[x_0] \longrightarrow R[x_1] \longrightarrow \ldots, \qquad x_i \longmapsto \pi x_{i+1}. \tag{2}$$

The limit G of diagram (1) is a flat affine group scheme over R whose associated ring R[G] is the colimit of (2): $\{P \in K[T] \,|\, P(0) \in R\}$. Another meaningful way to express G is by writing it as a join:

$$G = \mathbf{G}_{a,K} \coprod_{e,\operatorname{Spec} K} \operatorname{Spec} R.$$

Note that the R-module R[G], being isomorphic to $R \oplus K \oplus K \oplus \cdots$ is *not* projective and that $G \otimes R_n$ is always the trivial group scheme over R_n.

Example 1.2 (cf. 3.2.1.5 of [And01])**.** Let R be a discrete valuation ring with uniformiser π and quotient field K. Let $\mathbf{G}_{m,R} = \operatorname{Spec} R[z, 1/z]$. If we set $x_0 = z - 1$ and $y_0 = 1/z - 1$, so that $R[z, 1/z] = R[x_0, y_0]/(x_0 + y_0 + x_0y_0)$, co-multiplication is given by

$$\begin{aligned} x_0 &\longmapsto x_0 \otimes 1 + 1 \otimes x_0 + x_0 \otimes x_0 \\ y_0 &\longmapsto y_0 \otimes 1 + 1 \otimes y_0 + y_0 \otimes y_0. \end{aligned} \tag{3}$$

Now we write

$$G_n = \operatorname{Spec} R[x_n, y_n]/(x_n + y_n + \pi^n x_n y_n),$$

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and define co-multiplication by

$$\begin{aligned} x_n &\longmapsto x_n \otimes 1 + 1 \otimes x_n + \pi^n x_n \otimes x_n, \\ y_n &\longmapsto y_n \otimes 1 + 1 \otimes y_n + \pi^n y_n \otimes y_n. \end{aligned} \tag{4}$$

It is immediately verified that

$$x_{n-1} \longmapsto \pi x_n$$
$$y_{n-1} \longmapsto \pi y_n$$

defines a morphism of R-algebras $R[G_{n-1}] \to R[G_n]$ which induces an isomorphism $K[G_{n-1}] \to K[G_n]$. This means that $R[G_n]$ is none other than the subring of $K[z, 1/z]$ generated by $\pi^{-n}x_0$ and $\pi^{-n}y_0$. It is also easily verified that the co-multiplication (4) on $R[G_n]$ is obtained from that in (3). We then arrive at the projective system of affine group schemes

$$\ldots \longrightarrow G_{n+1} \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \ldots$$

whose limit G is considered in [And01, 3.2.1.5]. Note that, as in Example 1.1,

$$R[G] = \{P \in K[z, 1/z] \, : \, \varepsilon P \in R\},$$

where ε is the co-identity $z \mapsto 1$. Again $G \otimes R_n = \operatorname{Spec} R_n$ for all n.

Furthermore, in [And01], the author shows that the group scheme G is a "differential Galois group"; a fact which kindred our interest in building a heftier theory.

Our method is to put together the theory of Tannakian categories over a DVR [DH14] and the theory of Neron blowups [WW80]. Once this is done, two basic principles appear:

- **P1** Tannakian theory over a DVR meets much more frequently group schemes of infinite type. More technically, "Galois groups" or "images" can be of infinite type.
- **P2** The theory of Neron blowups serves to render these inconvenients more tractable.

We then focus on two main tasks:

- **T1** Understand to what extent **P2** is completely responsible for the difficulty in **P1**.
- **T2** Detect in concrete cases the difficulty in **P1** is avoided.

Our findings concerning **T1**, respectively **T2**, is in Section 6, respectively Section 8. We now summarise the contents of the text in more detail.

In Section 2 we introduce and study some basic properties of the Neron blowup of a group scheme over a DVR along a closed subscheme of the special fibre. This is a central technique in studying group schemes over discrete valuation rings. The text relies heavily on the work of Waterhouse and Weisfeiler [WW80] and for most of the time (Sections 2.1–2.4) we simply present their findings in our perspective so that further explanations becomes more effective. In doing so, we explain how one result from [WW80] fits comfortably in general mathematical culture (Theorem 2.4), we elaborate on a hint appearing in [WW80] (Proposition 2.7), we give a more important role to the notion of standard sequence (Definition 2.12) by including it as part of the structure theorem on generic isomorphisms (Theorem 2.11) and we derive an expression for an arbitrary affine flat group scheme as a limit of flat *algebraic* ones (Theorem 2.17). On the other hand, Section 2.5 already starts the program mentioned in this introduction: we define automatic blowups of group schemes (Definition 2.18). This is an abstraction of Example 1.1 and Example 1.2, and fits into the theory of standard sequences. We then go on to nuance some basic properties of the automatic blowup.

In Section 3, we begin to put emphasis on the Tannakian approach and study how to produce faithful representations of Neron blowups. This is motivated by our will to study differential Galois groups. Indeed, differential Galois groups are the reflex in the

Tannakian mirror of certain tensor categories *generated by one object*. Performing blowups modifies the group on one side and our task is to understand how to modify the generating object on the other. We begin by finding faithful representations of Neron blowups of the identity (Proposition 3.1) and then proceed to treat the general case (Proposition 3.5). To repeat, the constructions involved in these last two propositions are taylored to serve a situation where the objects under the lenses is the category of representations, so that manipulations should take place in there. Once this principle is abandoned, we can offer the reader some simpler results (Proposition 3.8 and Proposition 3.9).

Section 4 studies possible candidates for images of group schemes. For a morphism $\rho : G \to H$ between affine group schemes over a field k, it is well-known that the schematic image $\mathrm{Im}(\rho)$ is a closed subgroup scheme of H and that the natural morphism $G \to \mathrm{Im}(\rho)$ is faithfully flat [Wa79, Ch. 14]. (This is a very pleasant feature of this theory.) Said differently, ρ can be written as the composition $\tau \circ \sigma$, with σ a closed embedding and τ faithfully flat. But, in the setting of group schemes over a DVR, the situation gets a bit more complicated since the aforementioned factorization ceases to exist in general. One is then led to consider factorizations into three morphisms (Definition 4.1), which gives two "images." This is not as simple as the case of affine group schemes over fields, but is not as complicated as the that of affine schemes. In possession of these definitions, we then move on to study the behavior of these images under reduction modulo π. The structure then becomes more sophisticated and a third "image" appears, see diagram (18). While these "images" are of course related to each other, they need not be identical (Example 4.8). We put in evidence some of these relations in Corollary 4.6 and Proposition 4.7. The section ends with a direct translation of some of the results found in terms of the accompanying tensor categories, which are properly defined in Definition 4.9 and recognized in Proposition 4.10.

In Section 5 we introduce the Neron blowup along a formal closed flat subgroup scheme (Definition 5.6). This is a generalisation of the notion of automatic blowup introduced in Section 2.5 and is the beginning of the march towards understanding how Neron blowups help in controlling "Galois groups" of infinite type. This section works out some fundamental properties of the new concept and special attention should be paid to Theorem 5.14, which says that the standard sequence of the blowup along a formal closed flat subgroup is "constant." This property is then isolated and studied from an abstract point of view in Section 6. There, the goal is simple: when is a standard sequence with "constant" centres the standard sequence of a blowup along a formal flat closed subgroup scheme? (The motivation for this question should be understood in the light of task **T1** mentioned earlier since standard sequences give rise to affine group schemes.) We give a partial answer, cf. Corollary 6.10.

In Section 7 we apply some of our previous results to differential Galois theory. Here we make fundamental use of Tannakian categories. The upshot is that for integrable connections on a scheme over R, one has two differential Galois groups over R in sight: the full and the restricted (see Definition 7.6 and Definition 7.7). Over k we arrive at three differential Galois groups. A simple example where all these groups differ is given (Example 7.9).

In Section 8 we present a result, Theorem 8.1, which moves in the direction of Bolibrukh's theorem on complete integrability and isomonodromic deformations [Sab07, Theorem 1.2]. It says that for inflated integrable connections (also called completely integrable) over proper and smooth ambient spaces, the many differential Galois groups coincide. This is of course an accomplishment of task **T2** mentioned above. Since the proof of Theorem 8.1 makes use of two other elaborate webs of ideas we include a technical section, Section 9, to support the proof of Theorem 8.1.

Notations, conventions and standard terminology.

(1) Throughout the text, R stands for a discrete valuation ring with quotient field K, residue field k, and uniformizer π. All the Hom-sets and tensor products, when not explicitly indicated, are understood to be taken over R.
(2) Given an object X over R (a scheme, a module, etc), we sometimes find useful to write X_k instead of $X \otimes_R k$, X_K instead of $X \otimes_R K$, etc.
(3) To avoid repetitions, by a **group scheme** over some ring A, we understand an **affine group scheme** over A.
(4) The category of group schemes over a ring A will be denoted by $(\mathbf{GSch}/A)$; the full subcategory whose objects are A-flat will be denoted by $(\mathbf{FGSch}/A)$.
(5) If V is a free R module of finite rank, we write $\mathbf{GL}(V)$ for the general linear group scheme representing $A \mapsto \mathrm{Aut}_A(V \otimes A)$. If $V = R^n$, then $\mathbf{GL}(V) = \mathbf{GL}_n$.
(6) If G is a group scheme over R, we let $\mathrm{Rep}_R(G)$ stands for the category of representations of G which are, as modules, *of finite type* over R. (We adopt Jantzen's and Waterhouse's definition of representation. See [J87, Part I, 2.7 and 2.8, 29ff] and [Wa79, 3.1-2, 21ff].) We also abuse notation and make no notational distinction between the homomorphism $G \to \mathbf{GL}(V)$ and the co-action $V \to V \otimes R[G]$.
(7) The full subcategory $\mathrm{Rep}_R(G)^o$ of $\mathrm{Rep}_R(G)$ has for objects those V whose underlying R-module is free.
(8) For an affine scheme X over R, we let $R[X]$ stands for the ring $\mathcal{O}(X)$. More generally, if A is any R-algebra, we write $A[X]$ to denote $\mathcal{O}(X \otimes_R A)$.
(9) If $M \subset N$ is an inclusion of R-modules, we write M^{sat} to stand for the saturation of M in N, i.e. $\cup_m (M : \pi^m)$.

2. INTRODUCING NERON BLOWUPS

2.1. **Definition and basic properties.** Let $G \in (\mathbf{FGSch}/R)$. In what follows we regard $R[G]$ as a subring of $K[G]$.

Let H_0 be a closed subgroup of G_k defined by an ideal J. Note that, in this case, $\pi \in J$. The Neron blowup G' of G at H_0 (cf. [WW80, p.551] and [Ana73, 2.1.2]) is the spectrum of the subring $R[G']$ of $K[G]$ generated by $R[G]$ and the elements of $\pi^{-1}J$. (It is obviously sufficient to adjoin elements $\pi^{-1}f$, where f runs over a system of generators of J.) The natural morphism $R[G] \to R[G']$ is a morphism of Hopf algebras over R as is readily verified. (See also [WW80, Proposition 1.1].) By construction, $G'_K \to G_K$ is an isomorphism and $G'_k \to G_k$ factors through H_0. In fact we have:

Lemma 2.1 (Universal property, see [WW80], p.551)**.** *Let $\mathcal{G} \to G$ be a morphism in* $\mathbf{FGSch}/R$ *such that $\mathcal{G}_k \to G_k$ factors through $H_0 \to G_k$. Then there exists a unique $\mathcal{G} \to G'$ rendering the diagram*

$$\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[dr] & G' \arrow[d] \\
& G
\end{tikzcd}$$

commutative. □

Remark 2.2. It should be noted that this very elementary definition of blowup in fact relates to the usual notion as follows. Let $(\pi, a_1, \dots, a_r) \subset R[G]$ be an ideal cutting out on G_k a closed subgroup scheme H_0. Write G' for its Neron blow-up. Then

$$\frac{R[G][\xi_1, \dots, \xi_r]}{(\pi\xi_1 - a_1, \dots \pi\xi_r - a_r)^{\mathrm{sat}}} \longrightarrow K[G], \quad \xi_i \longmapsto \frac{a_i}{\pi},$$

defines an isomorphism onto $R[G']$. As is well-known, the ring on the left above is the ring of an affine piece of the blowup [Liu02, Lemma 1.2, p.318].

2.2. **A result stemming directly from the theory of blowups.** We use this section to state a generalization of a result appearing in [WW80] and which is more transparently treated by using (nowadays) commonplace algebraic geometry. We begin by recalling the following important theorem, whose proof the reader can find in [Bour, Ch. X,§9,no.7].

Theorem 2.3. *Let* $\mathcal{I} \subset \mathcal{O}_X$ *be an ideal of the locally noetherian scheme* X. *Let* $\psi : \widetilde{X} \to X$ *stand for the blowup of* $\mathcal{I}$. *Let* $U = \operatorname{Spec} A$ *be an affine open of* X *where* $\mathcal{I}$ *is generated by a regular* A*-sequence* $x_1, \dots, x_c$. *Then the* U*-scheme* $\psi^{-1}(U)$ *is isomorphic to*

$$\operatorname{Proj} \frac{A[\xi_1, \dots, \xi_c]}{(\xi_i x_j - \xi_j x_i)}.$$

$\square$

From this and Theorem 1.5 of [WW80], we arrive at the following generalization of [WW80, Theorem 1.7]. (Recall from [EGA IV_4, 17.12.1, p.85] that a closed immersion of smooth schemes is always regular.)

Theorem 2.4. *Let* G *be a group scheme over* R *which is flat and of finite type. Let* $H_0 \subset G_k$ *be a regularly immersed closed sub-group scheme. Write* c *for the codimension of* H_0 *in* G_k. *If* $G' \to G$ *stands for the Neron blowup of* G *at* H_0, *then* $G' \otimes k \to H_0$ *is a smooth surjective morphism whose kernel is isomorphic to* $\mathbf{G}_{a,k}^c$. $\square$

Example 2.5. Let k be of positive characteristic p. Let $G = \mathbf{G}_{a,R}$ and denote the Neron blowup of $\alpha_p \subset G_k$ by $G' \to G$. Then $G'_k \to \alpha_p$ is a smooth morphism with kernel $\mathbf{G}_{a,k}$.

2.3. **The action of the centre of the Neron blowup on the kernel.** Let G be a group scheme over R which is flat and of finite type. Let $H_0 \subset G_k$ be a regularly immersed closed sub-group scheme. From Theorem 2.4 we know that $G'_k \to H_0$ is a faithfully flat morphism so that H_0 acts by group scheme automorphism on the *right of* the kernel F. This is explained in [DG70, Ch. III, §6, no.1, 431ff], but is really a triviality: "lift and conjugate." Since $F \simeq \mathbf{G}_{a,k}^c$, the one might ask if this action is linear and if it can be obtained from something simpler. The answer is Proposition 2.7 below, which is the theme of this section.

To grasp the meaning of Proposition 2.7, we need the notion of the *conormal* representation. Let $\mathcal{G}$ be an affine group scheme over k and let $\mathcal{H} \subset \mathcal{G}$ be a closed subgroup cut out by the ideal $I \subset k[\mathcal{G}]$. We let $\mathcal{H}$ act on $\mathcal{G}$ *on the right* by conjugation: $g \cdot h = h^{-1}gh$. In this way, we obtain an $\mathcal{H}$-module structure on $k[\mathcal{G}]$. If $\mathfrak{a} \subset k[\mathcal{G}]$ stands for the augmentation ideal, it is easily seen that $\mathfrak{a}$, I and $\mathfrak{a}I$ are all sub-$\mathcal{H}$-modules of $k[\mathcal{G}]$. Note that $I/\mathfrak{a}I$ is a finite dimensional k-space.

Definition 2.6. The $\mathcal{H}$-module $I/\mathfrak{a}I$ is called the co-normal representation. It will be denoted in this text by $\nu : \mathcal{H} \to \mathbf{GL}(I/\mathfrak{a}I)$.

Obviously, $I/\mathfrak{a}I$ is simply the fibre at e of the conormal sheaf of the immersion $\mathcal{H} \subset \mathcal{G}$ [EGA IV$_4$, p.5]. We invite the reader to explicate the relation between ν and the pertaining adjoint representations.

Proposition 2.7. *We maintain the setting of the first paragraph of this section. Then there exists an isomorphism of group schemes*

$$F \simeq \mathbf{G}_{a,k}^c \simeq \operatorname{Spec} k[\xi_1, \ldots, \xi_c]$$

such that the action of H_0 *on* F *is determined by*

$$\xi_j \longmapsto \sum_{i=1}^{c} \xi_i \otimes \nu_{ij},$$

where $[\nu_{ij}] \in \mathrm{GL}_c(k[H_0])$ *defines the conormal representation of* H_0.

Proof. In the sequel, we employ some useful notation:

- We write "$\times$" for the fibre product over R and note that if X and Y are k-schemes, then $X \times Y$ is canonically isomorphic to $X \times_{\operatorname{Spec} k} Y$ [EGA I, 3.2.4, p.105].
- If A and B are R-algebras and $\mathfrak{A}$ is an ideal of A, we write $\langle \mathfrak{A} \otimes 1\rangle$ for the ideal generated by the image of $\mathfrak{A} \otimes_R B \to A \otimes_R B$.
- If $\mathfrak{p}$ is a prime ideal of a ring A and $I \subset \mathfrak{p}$ is a sub-ideal, we write $I_\mathfrak{p}$ for $IA_\mathfrak{p}$.

Let $\gamma : G \times G \to G$ stand for the morphism defined by $(x, g) \mapsto g^{-1}xg$; it produces a right action of G on itself. Letting $\gamma' : G' \times G' \to G'$ stand for the analogous action of G' on itself, we arrive at a commutative diagram

$$\begin{tikzcd}
G' \times G' \arrow[r, "\gamma'"] \arrow[d] & G' \arrow[d] \\
G \times G \arrow[r, "\gamma"'] & G.
\end{tikzcd}$$

Let $V \subset G$ be *any* affine open whose intersection with H_0 is the source of a section to $G'_k \to H_0$:

$$\sigma : H_0 \cap V \longrightarrow G'_k.$$

By definition of the right action

$$\alpha : F \times H_0 \longrightarrow F$$

we have a commutative diagram

$$\begin{tikzcd}
G'_k \times G'_k \arrow[rr, "\gamma'_k"] & & G'_k \\
F \times G'_k \arrow[u, hook] \arrow[rr] & & F \arrow[u, hook] \\
F \times (H_0 \cap V). \arrow[u, "\mathrm{id} \times \sigma"] \arrow[urr, "\alpha"'] & &
\end{tikzcd} \tag{5}$$

Let $\mathfrak{n}$ be a closed point of $H_0 \cap V$ and complete diagram (5) as follows:

$$\begin{tikzcd}
& G'_k \times G'_k \arrow[r, "\gamma'_k"] & G'_k \\
& F \times G'_k \arrow[r] \arrow[u, hook] & F \arrow[u, hook] \\
F \times \operatorname{Spec} k[H_0]_{\mathfrak{n}} \arrow[r] & F \times (H_0 \cap V). \arrow[u, "\mathrm{id} \times \sigma"] \arrow[ur, "\alpha"'] &
\end{tikzcd} \tag{6}$$

We will show that the obvious composition

$$F \times \operatorname{Spec} k[H_0]_{\mathfrak{n}} \longrightarrow F \times (H_0 \cap V) \xrightarrow{\alpha} F$$

coincides with the composition

$$F \times \operatorname{Spec} k[H_0]_{\mathfrak{n}} \longrightarrow F \times (H_0 \cap V) \xrightarrow{\nu} F,$$

where ν is the conormal representation.

As usual, let $\varepsilon : R[G] \to G$ be the co-identity and write $\mathfrak{a}$ for its kernel. Write $\mathfrak{m}$ for the ideal $(\pi, \mathfrak{a})$ — it corresponds to the identity on the closed fibre G_k — and note that ε extends to $R[G]_{\mathfrak{m}}$. It goes without saying that the kernel of $\varepsilon : R[G]_{\mathfrak{m}} \to R$ is just $\mathfrak{a}R[G]_{\mathfrak{m}}$.

Lemma 2.8. *There exist functions* $x_{1,\mathfrak{m}}, \dots, x_{c,\mathfrak{m}} \in R[G]_{\mathfrak{m}}$ *such that*

(1) *for each* j, $\varepsilon(x_{j,\mathfrak{m}}) = 0$, *and*
(2) *the sequence* $\pi, x_{1,\mathfrak{m}}, \dots, x_{c,\mathfrak{m}}$ *is regular and generates the ideal of* H_0 *at* $R[G]_{\mathfrak{m}}$.

Proof. Let $\pi, x^*_{1,\mathfrak{m}}, \dots, x^*_{c,\mathfrak{m}}$ be a regular sequence generating the ideal of H_0 at $\mathfrak{m}$. We can always write $x^*_{j,\mathfrak{m}} = c_j + x_{j,\mathfrak{m}}$, with $c_j \in R$ and $x_{j,\mathfrak{m}} \in \mathfrak{a}R[G]_{\mathfrak{m}}$. Moreover, we know that $\varepsilon(x^*_{j,\mathfrak{m}}) \equiv 0 \mod \pi$, since $x^*_{j,\mathfrak{m}} + (\pi) \in k[G]$ belongs to the ideal of H_0. It then follows that $c_j \equiv 0 \mod \pi$. Write $c^*_j := \pi^{-1} c_j$. Then $x_{j,\mathfrak{m}} = x^*_{j,\mathfrak{m}} - \pi c^*_j$, which means that we have an equality of ideals

$$(\pi, x_{1,\mathfrak{m}}, \dots, x_{c,\mathfrak{m}}) = (\pi, x^*_{1,\mathfrak{m}}, \dots, x^*_{c,\mathfrak{m}}).$$

The proof is finished by EGA IV_4, 16.9.5, p.47. □

The morphism γ induces an arrow between local rings

$$\gamma^{\#} : R[G]_{\mathfrak{m}} \longrightarrow R[G]_{\mathfrak{m}} \otimes_R R[G]_{\mathfrak{n}}.$$

Since H_0 is a subgroup of G_k, if $J \subset R[G]$ stands for its ideal, it follows that

$$\gamma^{\#}(J_{\mathfrak{m}}) \subset \langle J_{\mathfrak{m}} \otimes 1 \rangle + \langle 1 \otimes J_{\mathfrak{n}} \rangle.$$

If $y_{1,\mathfrak{n}}, \dots, y_{c,\mathfrak{n}}$ denote elements of $R[G]_{\mathfrak{n}}$ such that $\pi, y_{1,\mathfrak{n}}, \dots, y_{c,\mathfrak{n}}$ forms a regular sequence generating $J_{\mathfrak{n}}$, we conclude that

$$\gamma^{\#}(x_{j,\mathfrak{m}}) = \sum_i a_{ij} x_{i,\mathfrak{m}} \otimes a'_{ij} + \sum_i b'_{ij} \otimes b_{ij} y_{i,\mathfrak{n}} + \pi s_j \tag{7}$$

for a_{ij}, b'_{ij} in $R[G]_{\mathfrak{m}}$, a'_{ij}, b_{ij} in $R[G]_{\mathfrak{n}}$ and s_j in $R[G]_{\mathfrak{m}} \otimes_R R[G]_{\mathfrak{n}}$.

Since $\varepsilon(x_{j,\mathfrak{m}}) = 0$ (Lemma 7), eq. (7) gives

$$0 = \sum_i \varepsilon(b'_{ij}) \cdot b_{ij} y_{i,\mathfrak{n}} + \pi \cdot \left(\varepsilon \otimes_R \mathrm{id}_{R[G]_{\mathfrak{n}}}\right)(s_j),$$

which proves that

$$\pi \cdot \left(\varepsilon \otimes_R \mathrm{id}_{R[G]_\mathfrak{n}}\right)(s_j) \in (\mathbf{y}_\mathfrak{n}).$$

Since the ideal $(\mathbf{y}_\mathfrak{n})$ has no π-torsion, because the sequence $\{y_{1,\mathfrak{n}}, \ldots, y_{c,\mathfrak{n}}, \pi\}$ is regular, it follows that

$$\left(\varepsilon \otimes_R \mathrm{id}_{R[G]_\mathfrak{n}}\right)(s_j) \in (\mathbf{y}_\mathfrak{n}).$$

Hence, abusing notation and confusing $\mathfrak{a}$ with $\mathfrak{a}R[G]_\mathfrak{m}$,

$$s_j \in \langle \mathfrak{a} \otimes_R 1\rangle + \langle 1 \otimes_R (\mathbf{y}_\mathfrak{n})\rangle. \tag{8}$$

Since the sequence $\{\pi, x_{1,\mathfrak{m}}, \ldots, x_{c,\mathfrak{m}}\}$ is regular, it follows that we have an isomorphism of $R[G]_\mathfrak{m}$-algebras:

$$R[G']_\mathfrak{m} = \frac{R[G]_\mathfrak{m}[\boldsymbol{\xi}]}{(\pi\boldsymbol{\xi} - \mathbf{x}_\mathfrak{m})}. \tag{9}$$

(Here $R[G']_\mathfrak{m}$ stands for the ring obtained by inverting the elements of $R[G']$ in the image of $R[G] \setminus \mathfrak{m}$.) Now, using (EGA IV$_1$, 15.2.4, p.21), there exists some affine neighbourhood V of $\mathfrak{n}$ in G and functions $\mathbf{y} \in R[V]$ inducing $\mathbf{y}_\mathfrak{n}$ such that $\{\pi, y_1, \ldots, y_c\}$ is $R[V]$-regular. Consequently, if $G'|_V$ stands for the inverse image of V in G', we have

$$R[G'|_V] \simeq \frac{R[V][\boldsymbol{\eta}]}{(\pi\boldsymbol{\eta} - \mathbf{y})} \tag{10}$$

as $R[V]$-algebras. In particular, $V \cap H$ is the source of a section $\sigma : H_0 \cap V \to G'_k$ defined on the level of rings by $\boldsymbol{\eta} \mapsto \mathbf{0}$.

Now, $\gamma'^{\#}$ fits into a diagram

$$\begin{tikzcd}
R[G']_\mathfrak{m} \arrow[r, "\gamma'^{\#}"] & R[G']_\mathfrak{m} \otimes_R R[G']_\mathfrak{n} \\
R[G]_\mathfrak{m} \arrow[u] \arrow[r, "\gamma^{\#}"'] & R[G]_\mathfrak{m} \otimes_R R[G]_\mathfrak{n}. \arrow[u]
\end{tikzcd}$$

Then, from eqs. (7) and (9), we have

$$\pi\gamma'^{\#}(\xi_j) = \pi \sum_i a_{ij}\xi_i \otimes a'_{ij} + \pi \sum_i b'_{ij} \otimes b_{ij}\eta_i + \pi s_j,$$

so that

$$\gamma'^{\#}(\xi_j) = \sum_i a_{ij}\xi_i \otimes a'_{ij} + \sum_i b'_{ij} \otimes b_{ij}\eta_i + s_j. \tag{11}$$

Consequently, eqs. (8) and (10) show that

$$\gamma'^{\#}(\xi_j) \equiv \sum_i \varepsilon(a_{ij})\xi_i \otimes a'_{ij} \mod (\pi) + \langle \mathfrak{a} \otimes_R 1\rangle + \langle 1 \otimes_R (\boldsymbol{\eta})\rangle. \tag{12}$$

Now we write diagram (6) on the level of rings and complete :

$$\begin{tikzcd}
& & & k[G']_m \arrow[dlll, dashed, bend right] \arrow[ddl, dashed, bend left] \\
k[G']_m \otimes k[G']_n \arrow[d, dashed] & k[G'] \otimes k[G'] \arrow[l] \arrow[d] & k[G'] \arrow[l, "\gamma'^{\#}_k"'] \arrow[d] \arrow[ur, dashed] & \\
k[F] \otimes k[G']_n \arrow[d, "\text{kill } 1 \otimes \eta"'] & k[F] \otimes k[G'] \arrow[l] \arrow[d, "\mathrm{id} \otimes \sigma^{\#}"'] & k[F] \arrow[l] \arrow[dl, "\alpha^{\#}"] & \\
k[F] \otimes k[H_0]_n & k[F] \otimes k[H_0 \cap V]. \arrow[l] & &
\end{tikzcd}$$

It then follows from eq. (12) that the image of ξ_j via

$$k[F] \xrightarrow{\alpha^{\#}} k[F] \otimes k[H_0 \cap V] \longrightarrow k[F] \otimes k[H_0]_n$$

is $\sum_i \varepsilon(a_{ij})\xi_i \otimes a'_{ij}$. By definition, this is the co-normal representation ν. □

Question 2.9. Is a regularly immersed closed affine algebraic subgroup scheme cut out by a regular sequence?

Remark 2.10. Proposition 2.7 is mentioned on p. 554 of [WW80].

2.4. The standard blowup sequence.

As remarked in [WW80], Neron blowups allow us to decompose *any* morphism of group schemes which is an isomorphism on generic fibres.

Theorem 2.11 (cf. Theorem 1.4 of [WW80]). *Let* $\rho : \mathcal{G} \to G$ *be an arrow of* **FGSch**/R *which is an isomorphism on generic fibres. Write* $\rho_0 = \rho$ *and* $G_0 = G$. *Let* $\rho_n : \mathcal{G} \to G_n$ *be constructed and define* G_{n+1}, *respectively* ρ_{n+1}, *as the blowup of* $\mathrm{Im}(\rho_n \otimes k)$, *respectively the natural morphism* $\mathcal{G} \to G_{n+1}$. *Then* $\varprojlim_n \rho_n : \mathcal{G} \to \varprojlim G_n$ *is an isomorphism.*

Proof. For the convenience of the reader and to avoid the hypothesis of finite generation present in [WW80], we summarize the proof.

We prove by induction that $\pi^{-n}R[G_0] \cap R[\mathcal{G}] \subset R[G_n]$. As the ideal of $R[G_0]$ cutting out $\mathrm{Im}(\rho_0 \otimes k)$ is $R[G_0] \cap \pi R[\mathcal{G}]$, the case $n = 1$ is readily proved. We now assume that $\pi^{-n}R[G_0] \cap R[\mathcal{G}] \subset R[G_n]$. Let $f_0 \in R[G_0]$ be such that $\pi^{-n-1}f_0$ lies in $R[\mathcal{G}]$. Then $\pi^{-n}f_0$ belongs to $\pi R[\mathcal{G}]$. On the other hand, the inclusion $\pi^{-n}R[G_0] \cap R[\mathcal{G}] \subset R[G_n]$ forces $\pi^{-n}f_0$ to be in $R[G_n]$, so that $\pi^{-n}f_0$ belongs to $R[G_n] \cap \pi R[\mathcal{G}]$. The latter ideal cuts out $\mathrm{Im}(\rho_n \otimes k)$, and therefore $\pi^{-1}\pi^{-n}f_0$ lies in $R[G_{n+1}]$.

Using the above inclusion, we arrive at $R[\mathcal{G}] \subset \cup R[G_n]$. Since each $R[G_n]$ is contained in $R[\mathcal{G}]$, the proof of finished. □

Definition 2.12. The sequence of morphism described in the statement of Theorem 2.11 is called the standard blowup sequence associated to $\mathcal{G} \to G$. If context prevents any misunderstanding, we simply speak of the standard sequence of $\mathcal{G} \to G$.

The concept of standard sequence also allows the following point of view [WW80, p.552, Remarks].

Definition 2.13. A diagram

$$\cdots \longrightarrow G_{n+1} \xrightarrow{\rho_n} \cdots \xrightarrow{\rho_0} G_0$$

in **FGSch**/R is a standard sequence if

(1) each arrow ρ_n is a blowup of some closed subgroup $B_n \subset G_n \otimes k$ and
(2) the induced morphism $B_{n+1} \to B_n$ is faithfully flat.

The overburdening of the term "standard sequence" caused by Definitions 2.12 and Definition 2.13 is ephemeral in view of:

Lemma 2.14. *Let*

$$\cdots \longrightarrow G_{n+1} \xrightarrow{\rho_n} \cdots \xrightarrow{\rho_0} G_0$$

be a standard sequence as in Definition 2.13. Then $\varprojlim_n G_n \to G_0$ *is an isomorphism on generic fibres and its standard blowup sequence is the above diagram.* □

We shall apply Theorem 2.11 to give a description of affine group schemes over R, see Theorem 2.17 below. First we give a definition.

Definition 2.15. Let H' be a flat Hopf algebra over R. A Hopf subalgebra H of H' is an R-submodule equipped with a Hopf algebra structure such that the inclusion $H \to H'$ is a homomorphism of Hopf algebras. We say that H is saturated in H' if H'/H is flat as an R-module.

Remarks 2.16. (1) The image of a Hopf algebra homomorphism is a Hopf subalgebra (of the target).

(2) One can always saturate a Hopf subalgebra. Adopting the notations of p. 4 and Definition 2.15, we put

$$\Delta_{H^{\mathrm{sat}}}(h) := \pi^{-m}\Delta_H(\pi^m h), \quad \text{if } \pi^m h \in H.$$

It can be extracted from the proof of Lemma 3.1.2 in [DH14] that this is a coproduct for H^{sat}.

(3) Some authors define Hopf subalgebras as our special Hopf subalgebras. But Neron blowups justify our definition, see also Theorem 2.17.

Theorem 2.17. *Let G be a flat group scheme over R. Then we can present it as the the limit of a pro-system of flat group schemes*

$$G := \varprojlim_i G_i,$$

in which all morphism are faithfully flat and the generic fiber of each G_i is of finite type over K. Further, each G_i can be obtained from a flat group scheme of finite type by a standard sequence.

Proof. Consider $R[G]$ as a (right) regular representation of G (i.e. as comodule on itself by means of the coproduct). According to Serre [Se68], $R[G]$ is the union of its subrepresentations, which are finite as R-modules. Let V be such a sub- representation (V is torsion free and finite over R, hence it is free over R). Then V induces a Hopf algebra homomorphism $R[GL(V)] \to R[G]$, the image of which is a Hopf subalgebra of $R[G]$, denoted by $R[G(V)]$. Notice that V is a subset of $R[G(V)]$. Thus $R[G]$ is a union of its Hopf subalgebras $R[G(V)]$. One can take the G_i in Theorem to be the saturation of the $R[G(V)]$ as V runs in the set of finite sub- representations of $R[G]$.

Now by means of Theorem 2.11, the saturation $R[G(V)]^{\mathrm{sat}}$ is obtained from $R[G(V)]$ by iterated Neron blowups. □

2.5. **Study of a particular case.** Examples 1.1 and 1.2 are a particular case of the following process. Let G be a flat group scheme over R.

Definition 2.18. The automatic blowup

$$N \longrightarrow G$$

of the identity is the limit of the diagram

$$\cdots \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_0$$

where $G_0 = G$ and $G_{n+1} \to G_n$ is the Neron blowup of $\{e\} \subset G_n \otimes k$.

Remark 2.19. The construction put forth in the above definition will be generalized in Section 5 below.

Proposition 2.20. *Write* $\mathfrak{a}$ *for the augmentation ideal of* $R[G]$. *Then, the algebra* $R[N]$ *of the automatic blowup of the identity is the sub-*$R[G]$*-algebra of* $K[G]$ *generated by* $\cup_n \pi^{-n}\mathfrak{a}$. *The latter union also generates the augmentation ideal of* $R[N]$.

Proof. We let G_n be as in Definition 2.18 and write $\mathfrak{a}_n$ for the augmentation ideal of $R[G_n]$. It is easily proved that $(\pi^{-1}\mathfrak{a}_n) = \mathfrak{a}_{n+1}$. Let us now show by induction that $R[G_0][\pi^{-n}\mathfrak{a}_0] = R[G_n]$ and $\mathfrak{a}_n = (\pi^{-n}\mathfrak{a}_0)$. As there is nothing to be checked for $n = 0$, we assume that $R[G_0][\pi^{-n}\mathfrak{a}_0] = R[G_n]$ and $\mathfrak{a}_n = (\pi^{-n}\mathfrak{a}_0)$. Now, if $A \subset K[G]$ is any R-algebra and S is any subset of A generating an ideal $\mathfrak{S}$, it is clear that $A[\pi^{-1}\mathfrak{S}] = A[\pi^{-1}S]$. Hence, $R[G_{n+1}] = R[G_n][\pi^{-1}\mathfrak{a}_n]$ is just $R[G_n][\pi^{-n-1}\mathfrak{a}_0]$, which equals $R[G_0][\pi^{-n-1}\mathfrak{a}_0]$. This proves the first claim. As $\mathfrak{a}_{n+1} = (\pi^{-1}\mathfrak{a}_n)$ and $\mathfrak{a}_n = (\pi^{-n}\mathfrak{a}_0)$, we conclude that $\mathfrak{a}_{n+1} = (\pi^{-n-1}\mathfrak{a}_0)$, and the second claim follows as well. □

Some noteworthy properties present in the examples remain valid. Others easily come forward.

Corollary 2.21. *If* $\varepsilon_K : K[G] \to K$ *stands for the co-identity, then the first projection*

$$K[G] \times_{\varepsilon_K, K} R \longrightarrow K[G]$$

induces an isomorphism onto $R[N]$. *Geometrically,*

$$N \simeq G_K \coprod_{e,\,\mathrm{Spec}\,K} \mathrm{Spec}\,R.$$

Proof. It is clear that the first projection is an isomorphism onto the subalgebra $A := \{f \in K[G] : \varepsilon_K(f) \in R\}$. Equally clear is the inclusion $R[N] \subset A$, so that we are left with the verification of $A \subset R[N]$. If $f \in A$, then $f = c + f'$ with $c \in R$ and $\varepsilon_K(f') = 0$; this means that $f' = \pi^{-m}f''$ with f'' in the augmentation ideal of $R[G]$. By Proposition 2.20, $f' \in R[N]$ and $A \subset R[N]$ is verified.

The proof of the final statement follows directly from the first and from the fact that the functor Spec sends the cartesian diagram of rings in sight into a co-cartesian diagrams of schemes [Fe03, Theorem 5.1, p. 568]. □

Corollary 2.22. *Let* $\mathcal{G} \to G$ *be a morphism of flat group schemes over* R *which induces an isomorphism on generic fibres. Then there exists a unique arrow* $N \to G$ *rendering commutative the diagram*

$$\begin{tikzcd}
N \arrow[r] \arrow[d] & \mathcal{G} \arrow[dl] \\
G. &
\end{tikzcd}$$

Said differently, $N \to G$ *is an initial object in the category of flat group schemes* $\mathcal{G} \to G$ *over* G *which induce an isomorphism on generic fibres.*

Proof. We need to prove that $R[\mathcal{G}] \subset K[G]$ is contained in $R[N]$. For that, let $\mathfrak{a}$ and $\mathfrak{a}_{\mathcal{G}}$ stand respectively for the augmentation ideals of $R[G]$ and $R[\mathcal{G}]$. Writing an $f \in \mathfrak{a}_{\mathcal{G}}$ as $\pi^{-m}f'$ with $f' \in \mathfrak{a}$, we conclude that $f \in \cup_n \pi^{-n}\mathfrak{a}$. But then $f \in R[N]$ (Proposition 2.20) and we are done since $R \cdot 1 \oplus \mathfrak{a}_{\mathcal{G}} = R[\mathcal{G}]$. ☐

Corollary 2.23. *Let* $N \to G$ *be the automatic blowup of the identity. Then, for each* $n \geqslant 0$, *the group scheme* $N \otimes R_n$ *over* R_n *is trivial, i.e. isomorphic to* $\operatorname{Spec} R_n$.

Conversely, let $\rho : \mathcal{G} \to G$ *be an arrow of* $\mathbf{FGSch}/R$ *which is an isomorphism on generic fibres. If* $\mathcal{G} \otimes R_n$ *is trivial for all* n, *then* ρ *is the automatic blowup of the identity.*

Proof. In the sequel, let $\mathfrak{a}$, $\mathfrak{a}_N$ and $\mathfrak{a}_{\mathcal{G}}$ stand respectively for the augmentation ideals of $R[G]$, $R[N]$ and $R[\mathcal{G}]$. We know that $R[N] = \cup_n R[G][\pi^{-n}\mathfrak{a}]$ and that $\mathfrak{a}_N = (\cup_n \pi^{-n}\mathfrak{a})$. Hence, $\mathfrak{a}_N \subset (\pi^{n+1})$ for any n; the first claim follows. On the other hand if $R_n[\mathcal{G}]$ is always trivial, then $\mathfrak{a}_{\mathcal{G}}$ is contained in (π^{n+1}). Hence, $\mathfrak{a} \cdot R[\mathcal{G}] \subset (\pi^{n+1})$, which shows that $R[\mathcal{G}] \supset \cup_n R[G][\pi^{-n}\mathfrak{a}] = R[N]$. In view of Corollary 2.22, the inclusion $R[N] \supset R[\mathcal{G}]$ always holds and the proof is finished. ☐

In fact, the second statement in the above corollary allows the following relevant amplification in case G is a closed subgroup scheme of $\mathbf{GL}_{n,R}$.

Corollary 2.24. *Let* $V \in \operatorname{Rep}_R(G)^{\mathrm{o}}$ *afford a faithful representation of* G. *Let* $\rho : \mathcal{G} \to G$ *be an arrow of* $\mathbf{FGSch}/R$ *which is an isomorphism on generic fibres. If* $V \otimes R_n$ *is the trivial representation of* $\mathcal{G} \otimes R_n$ *for all* n, *then* ρ *is the automatic blowup of the identity.*

Proof. Let $[a_{ij}] \in \mathrm{GL}_r(R[G])$ be the matrix associated to V on some unspecified basis. Write $b_{ij} = a_{ij} - \delta_{ij}$. Since $V \otimes R_n$ is the trivial representation of $\mathcal{G} \otimes R_n$, we conclude that $\pi^{n+1} | b_{ij}$ in $R[\mathcal{G}]$ for all n. Now, the augmentation ideal $\mathfrak{a}$ of $R[G]$ is generated by the functions b_{ij}, so that $\mathfrak{a} \cdot R[\mathcal{G}] \subset (\pi^{n+1})$. Hence, $R[\mathcal{G}] \supset \cup_n R[G][\pi^{-n}\mathfrak{a}] = R[N]$. Using Corollary 2.22, we have $R[N] = R[\mathcal{G}]$. ☐

3. FAITHFUL REPRESENTATIONS OF NERON BLOWUPS

Let G be a flat group scheme over R. We assume furthermore that G is of finite type over R, so that the Neron blowup (see Section 2) of any closed subgroup of G_k is again of finite type. As such a group scheme, it can be embedded into some $\mathbf{GL}_{r,R}$ (adapt the *proofs* in [Wa79, 3.3]) or, said differently, it possesses a faithful representation. In this section we describe a means to find faithful representations of Neron blowups. Of course, put this way, our task might seem pointless since it is sufficient to run a general argument. But our point of view is to produce faithful representations by performing linear algebraic constructions in Rep_R. This is most desirable when the category Rep_R is in fact one side of a Tannakian correspondence, see Example 7.9.

The principle is quite simple and we begin with a particular case:

$$G' \longrightarrow G$$

is the Neron blowup of the identity in the closed fibre. Let $V \in \operatorname{Rep}_R(G)^{\mathrm{o}}$ be a faithful representation of rank r; since $G'_k \to G_k$ is the *trivial* morphism, we know that $\mathbf{1}^r \otimes k \simeq$

$V \otimes k$ as representations of G'. We then obtain a diagram

$$\begin{tikzcd} & V \arrow[d] \\ \mathbf{1}^r \arrow[r, "\varphi"'] & V \otimes k \end{tikzcd}$$

in the category $\mathrm{Rep}_R(G')$. (Here φ is the obvious morphism.) Let V' stand for its push-out. In concrete terms,

$$V' = \{(\mathbf{v}, \mathbf{e}) \in V \oplus \mathbf{1}^r \,:\, \mathbf{v} \otimes 1 = \varphi(\mathbf{e})\}. \tag{13}$$

Proposition 3.1. *In the above setting, the representations V' of G' is faithful.*

The proof hinges on the following lemma, whose verification is omitted.

Lemma 3.2. *Let M and E be free R-modules. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a basis for M and $\{\mathbf{e}_1, \ldots, \mathbf{e}_s\}$ be one for E. Let $q \leqslant \min\{r, s\}$ and define $\varphi : E \to V \otimes k$ by*

$$\varphi(\mathbf{e}_j) = \begin{cases} \mathbf{v}_j \otimes 1, & 1 \leqslant j \leqslant q \\ \mathbf{0}, & \textit{otherwise.} \end{cases}$$

Then the R-module

$$\{(\mathbf{v}, \mathbf{e}) \in M \oplus E \,:\, \mathbf{v} \otimes 1 = \varphi(\mathbf{e})\}$$

is free on the basis

$$\begin{array}{lll} (\pi\mathbf{v}_1, \mathbf{0}), & \ldots, & (\pi\mathbf{v}_r, \mathbf{0}) \\ (\mathbf{v}_1, \mathbf{e}_1), & \ldots, & (\mathbf{v}_q, \mathbf{e}_q) \\ (\mathbf{0}, \mathbf{e}_{q+1}), & \ldots, & (\mathbf{0}, \mathbf{e}_s). \end{array}$$

□

Proof of Proposition 3.1. This is plain linear algebra. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a basis of V and let $a_{ij} \in R[G]$ stand for the matrix coefficients inducing the representation of G. Let $\varepsilon : R[G] \to R$ stand for the counit; by definition of a representation we have $\varepsilon(a_{ij}) = \delta_{ij}$, so that

$$a_{ij} = \delta_{ij} \cdot 1 + b_{ij}, \tag{14}$$

with $b_{ij} \in \operatorname{Ker} \varepsilon$. By definition of G', there exist $b'_{ij} \in R[G']$ such that

$$\pi b'_{ij} = b_{ij}. \tag{15}$$

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ be an ordered basis of $\mathbf{1}^r$ which is sent by φ to $\{\mathbf{v}_1 \otimes 1, \ldots, \mathbf{v}_r \otimes 1\}$. According to Lemma 3.2, V' is free on

$$\{\pi\mathbf{v}_1, \ldots, \pi\mathbf{v}_r, \mathbf{v}_1 + \mathbf{e}_1, \ldots, \mathbf{v}_r + \mathbf{e}_r\}.$$

Employing this basis and the equations (14) and (15), we conclude that the matrix defining the representation of G' on V' is

$$\left[\begin{array}{c|c} a_{ij} & b'_{ij} \\ \hline O & \delta_{ij} \end{array}\right].$$

Since the functions a_{ij} together with $1/\det[a_{ij}]$ generate the R-algebra $R[G]$, the functions b_{ij} generate the ideal $\operatorname{Ker} \varepsilon$; by definition the functions a_{ij}, $1/\det[a_{ij}]$ and $b'_{ij} = \pi^{-1} b_{ij}$ generate $R[G']$ and the proof is finished. □

Before moving to the search for a faithful representation of a general Neron blowup $G' \to G$, we record some valuable properties of V' and give an example.

Corollary 3.3. *We maintain the above notations.*

1. *The representation* V' *is an extension of* $\mathbf{1}^r$ *by* V. *In particular,* V *is a sub-representation of* V'.
2. *Let* ξ *be the evident extension class in* $\mathrm{Ext}_{G'}(V\otimes k, V)$. *Then the class of* V' *in* $\mathrm{Ext}_{G'}(\mathbf{1}^r, V)$ *is simply the image of* ξ *under the morphism induced by* $\mathbf{1}^r \to V\otimes k$.
3. *The cokernel of the injection* $V' \to V\oplus\mathbf{1}^r$ *is annihilated by* π.
4. *The natural arrow* $V'\otimes K \to (V\oplus\mathbf{1}^r)\otimes K$ *is an isomorphism of representations of* $G'\otimes K$.
5. *Let* $\{\mathbf{v}_1,\ldots,\mathbf{v}_r\}$ *and* $\{\mathbf{e}_1,\ldots,\mathbf{e}_r\}$ *be respectively ordered basis of* V *and* $\mathbf{1}^r$ *as constructed above. Write* $\rho_K : G_K \to \mathbf{GL}_{2r,K}$ *for the homomorphism associated to the representation* $(V\oplus\mathbf{1}^r)\otimes K$ *by means of these basis. Then, if*
$$\left[\begin{array}{ccc|c} \pi & & & \\ & \ddots & & \mathrm{Id}_r \\ & & \pi & \\ \hline & & & \\ & & & \mathrm{Id}_r \\ & & & \end{array}\right],$$
it follows that G' *is the closure of the image of* $\beta^{-1}\cdot\rho_K\cdot\beta$ *in* $\mathbf{GL}_{2r,R}$.
6. *As representations of* G', V' *is a sub-representation of* $V\oplus\mathbf{1}^r$, *and* V *is a sub-representation of* V'.

Proof. The only statements which needs comment are (2) and (5). Concerning (2), we only need to inform the reader that we follow [HS70, p.87] in constructing $\mathrm{Ext}(V\otimes k, V)\to \mathrm{Ext}(\mathbf{1}^r, V)$. Concerning (5), we produce the following justifications. Let $[a_{ij}]\in \mathrm{GL}_r(R[G])$ stand for the matrix associated to the representation V and the basis $\{\mathbf{v}_1,\ldots,\mathbf{v}_r\}$. We observe that

$$\beta^{-1}\cdot\rho_K\cdot\beta = \left[\begin{array}{c|c} a_{ij} & (a_{ij}-\delta_{ij})/\pi \\ \hline O & \delta_{ij} \end{array}\right].$$

Now the above matrix is also the matrix of a faithful representation $G'\to\mathbf{GL}_{2r,R}$, see the proof of Proposition 3.1. Since $G'\otimes K \xrightarrow{\sim} G\otimes K$, the closure of the image of $\beta^{-1}\cdot\rho_K\cdot\beta$ is the same as the closure of the image of $G'\otimes K$ by $\beta^{-1}\cdot\rho_K\cdot\beta$. This is G', since two flat and closed subschemes having the same generic fibre must be equal. □

Example 3.4. Let G' be the Neron blowup of $\mathbf{G}_{m,R} = \operatorname{Spec} R[u, 1/u]$ at the origin in $\mathbf{G}_{m,k}$. If V is the obvious representation of $\mathbf{G}_{m,R}$ (corresponding to id), then V' is the representation of G' corresponding to the matrix

$$\begin{bmatrix}\pi & 1\\ 0 & 1\end{bmatrix}^{-1}\cdot\begin{bmatrix}u & \\ & 1\end{bmatrix}\cdot\begin{bmatrix}\pi & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}u & (u-1)/\pi\\ 0 & 1\end{bmatrix}.$$

It follows that G' is

$$\left\{\begin{bmatrix}u & v\\ 0 & 1\end{bmatrix} : \pi v + 1 = u\right\}.$$

Let us now assume that

$$G' \longrightarrow G$$

is the Neron blowup of a closed subgroup scheme $H_0\subset G\otimes k$. The idea to construct a faithful representation of G' by means of a faithful representation of G is to express H_0 as the stabilizer of some line. This means that the problem is divided into two:

Step 1; express H_0 as a stabilizer, and

Step 2; construct a faithful representation of the Neron blow of a stabilizer in the special fibre.

We start by discussing **Step 2** and presenting our findings as Proposition 3.5. **Step 1** follows from standard material in the theory of affine group schemes and is explained in Lemma 3.7 below.

Let $V \in \mathrm{Rep}_R(G)^o$ be faithful and let $\mathbf{v} \otimes 1 \in V \otimes k$ be a nonzero vector. Let $H_0 \subset G_k$ stand for the stabilizer of the *line*

$$\ell := \mathbf{v} \otimes k.$$

Clearly, it is possible to find an ordered basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ of V such that $\mathbf{v}_1 = \mathbf{v}$. Let $[a_{ij}] \in \mathrm{GL}_r(R[G])$ be the matrix of coefficients associated to $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$. It then follows that the ideal cutting out H_0 in $R[G]$ is simply

$$(\pi, a_{21}, \ldots, a_{r1}).$$

For future use, we write

$$a_{21} = \pi a'_{21}, \ldots, a_{r1} = \pi a'_{r1} \tag{16}$$

for functions $a'_{21}, \ldots, a'_{r1} \in R[G']$.

The line ℓ is now fixed by G'_k and we obtain a character $G'_k \to \mathbf{G}_{m,k}$ (defined by the group-like element $a_{11} + (\pi)$ in $k[G']$). In order to follow the method developed to treat the particular case, we should find a representation $L \in \mathrm{Rep}_R(G')^o$ lifting ℓ. So here we need to modify the argument since there is no reason for L to exist. Let $E \in \mathrm{Rep}_R(G')^o$ be the source of a surjection

$$\varphi : E \longrightarrow \ell$$

in $\mathrm{Rep}_R(G')$. (That E exists is proved in [Se68, Proposition 3, p.41]). We fix an element $\mathbf{e}_1 \in E$ above $\mathbf{v} \otimes 1 \in \ell$. A bit of common sense shows that there exists elements $\mathbf{e}_2, \ldots, \mathbf{e}_s \in \mathrm{Ker}\,\varphi$ which together with $\mathbf{e}_1$ for a basis of E. Let $[b_{ij}] \in \mathrm{GL}_s(R[G'])$ be the matrix associated to the representation E and the ordered basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_s\}$. Note that in this case we have

$$\begin{aligned} b_{11} &= a_{11} + \pi c_{11} \\ b_{12} &= \pi c_{12} \\ \cdots &= \cdots \\ b_{1s} &= \pi c_{1s} \end{aligned} \tag{17}$$

since $\varphi(\mathbf{e}_1) = \mathbf{v}_1 \otimes 1$ and the subspace of E_k generated by $\mathbf{e}_2 \otimes 1, \ldots, \mathbf{e}_s \otimes 1$ is stable under G'_k. We then consider the pull-back diagram

$$\begin{tikzcd} V' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ E \arrow[r, "\varphi"'] & V_k, \end{tikzcd}$$

so that V' is naturally a sub-G'-module of $V \oplus E$. From Lemma 3.2 we know that

$$\begin{array}{lll} (\pi\mathbf{v}_1, \mathbf{0}), & \ldots, & (\pi\mathbf{v}_r, \mathbf{0}) \\ (\mathbf{v}_1, \mathbf{e}_1), & & \\ (\mathbf{0}, \mathbf{e}_2), & \ldots, & (\mathbf{0}, \mathbf{e}_s) \end{array}$$

is a basis of V'. A simple computation using equations (16) and (17) shows that the matrix of the representation of G' on V' associated to this basis is

$$\left[\begin{array}{ccc|c|ccc} a_{11} & \cdots & a_{1r} & -c_{11} & -c_{12} & \cdots & -c_{1s} \\ \vdots & \ddots & \vdots & a'_{21} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a'_{r1} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1s} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{s1} & b_{s2} & \cdots & b_{ss} \end{array}\right].$$

Since the a_{ij} together with $1/\det[a_{ij}]$ generate $R[G]$, and $a'_{21},\dots,a'_{r1}$ generate $R[G']$ over $R[G]$, we conclude that V' is a faithful representation of G'. We have then proved the following.

Proposition 3.5. *Let $V \in \mathrm{Rep}_R(G)$ be faithful and let $\ell \subset V_k$ be a line with whose stabilizer is denoted by $H_0 \subset G_k$. Let G' be the Neron blowup of G at H_0 and let $\varphi : E \to \ell$ be a surjective morphism in $\mathrm{Rep}_R(G')$ such that the R-module E is free. Then $V' := V \times_{V_k} E$ is a faithful representation of G'.* ☐

Corollary 3.6. *Let $V \in \mathrm{Rep}_R(G)^{\mathrm{o}}$ be faithful and let $\mathbf{v}\otimes 1 \in V_k\backslash 0$ be a vector whose stabilizer is denoted by $H_0 \subset G_k$. (Note that H_0 is smaller than the stabilizer of the line!) Let G' be the Neron blowup of G at H_0 and note that there is an obvious G'-equivariant morphism $\mathbf{1} \to V_k$ whose image is the line $\mathbf{v}\otimes k$. Then $V' := V \times_{V_k} \mathbf{1}$ is a faithful representation of G'. Moreover, if $[a_{ij}] \in \mathbf{GL}_r(R[G])$ is the matrix associated to some basis $\{\mathbf{v}_1,\dots,\mathbf{v}_r\}$ with $\mathbf{v}_1 = \mathbf{v}$, then the matrix of the representation V' of G' is*

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1r} & \alpha_{11} \\ \vdots & \ddots & \vdots & a'_{21} \\ \vdots & & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a'_{r1} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right],$$

where $\pi\alpha_{11} = a_{11} - 1$ and $\pi a'_{i1} = a_{i1}$ for each $i \geqslant 2$. ☐

To address **Step 1** we only need the following.

Lemma 3.7. *Let $H_0 \subset G_k$ be a closed subgroup scheme of G. Then there exists a faithful representation V of G and a line $\ell \subset V_k$ whose stabilizer is exactly H_0.*

Proof. By [Wa79, 16.1, Corollary] there exists $\rho : G_k \to \mathbf{GL}_{r,k}$ and a line $\ell \subset k^r$ whose stabilizer is H_0. Using [Se68, Proposition 3, p.41] we can find $W \in \mathrm{Rep}_R(G)^{\mathrm{o}}$ together with a G_k-equivariant injection $k^r \to W_k$. Obviously, H_0 is still the stabilizer of the line $\ell \subset W_k$. Replacing W by $V = W \oplus F$, with F faithful, we are done. ☐

Let us now criticize Proposition 3.5. Finding explicitly the "covering" representation $\varphi : E \to \ell$ is by no means a simple task. (But in many important cases, the character of $G'_k \to \mathbf{G}_{m,k}$ associated to ℓ will be a reduction of a character $G' \to \mathbf{G}_{m,R}$.) Also, if we abandon the need to construct the faithful representation V' by means of linear algebra in the abelian category $\mathrm{Rep}_R(G')$, a more efficient path is:

Proposition 3.8. *Let* V *be a free* R*-module on the basis* $\mathbf{v}_1, \ldots, \mathbf{v}_r$ *affording a faithful representation of* G. *Let* $H_0 \subset G_k$ *be the stabilizer of the line spanned by* $\mathbf{v}_1 \otimes 1$ *in* V_k *and denote by* $G' \to G$ *the Neron blowup of* H_0. *Write* $[a_{ij}] \in \mathrm{GL}_r(R[G])$ *for the matrix associated to the representation of* G *on* V, *and let* $a'_{21}, \ldots, a'_{r1}$ *be functions of* $R[G']$ *which, when multiplied by* π, *become respectively* $a_{21}, \ldots, a_{r1}$. *Then the following claims are true.*

(1) *The sub-*R*-module of* V_K *freely generated by* $\mathbf{v}_1 \otimes \pi^{-1}, \mathbf{v}_2 \otimes 1, \ldots, \mathbf{v}_r \otimes 1$ *has the structure of an* $R[G']$*-comodule. Its associated matrix is*

$$\left[\begin{array}{c|ccc} a_{11} & \pi a_{12} & \cdots & \pi a_{1r} \\ \hline a'_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{r1} & a_{r2} & \cdots & a_{rr} \end{array}\right].$$

(2) *If* V' *stands for the representation of* G' *considered in the previous item, then* $V \oplus V'$ *is a faithful representation of* G'.

Proof. Once one knows what to look for, the proof is a triviality. □

We now explain how the point of view of [SS90] provides another means to construct faithful representations of Neron blowups centred at *normal subgroups*.

Proposition 3.9. *Let* $H \subset G$ *be an* R*-flat, normal subgroup scheme and write* A *for the quotient group scheme. Denote by* G' *and* A' *the Neron blowups of* G *at* H_k *and of* A *at* $\{e\} \subset A_k$, *respectively. Let* $G' \to A'$ *be the morphism obtained by the "universal property" (Lemma 2.1). If* $\rho : G \to \mathbf{GL}_r$ *and* $\sigma : A' \to \mathbf{GL}_s$ *are faithful representations, then* $\rho \oplus \sigma$ *is a faithful representation of* G'.

Proof. Here are the properties of the quotient which we use in the sequel: $R[A]$ is a subring of $R[G]$ and if $\mathfrak{a}_A$ stands for the augmentation ideal of $R[A]$, then $\mathfrak{a}_A \cdot R[G]$ cuts out H. Hence, if $a_1, \ldots, a_m$ are generators of $\mathfrak{a}_A$, it follows that

$$R[G'] = R[G][\pi^{-1}a_1, \ldots, \pi^{-1}a_m].$$

We now verify that

$$\rho^* \otimes \sigma^* : R[\mathbf{GL}_r] \otimes R[\mathbf{GL}_s] \longrightarrow R[G']$$

is a surjection. By definition of ρ, $R[G] \subset \mathrm{Im}(\rho^* \otimes \sigma^*)$. By definition of σ, the elements $\pi^{-1}a_i$ belong to $\mathrm{Im}(\sigma^*)$. This finishes the proof.

□

4. IMAGES OF MORPHISMS BETWEEN FLAT GROUP SCHEMES

Let $\rho : \Pi \to G$ be a morphism in **FGSch**/R. There are two natural ways of defining "images" of ρ.

Definition 4.1 (The diptych). Define Ψ_ρ as the group scheme whose Hopf algebra is the image of $R[G]$ in $R[\Pi]$. Define $R[\Psi'_\rho]$ as the saturation of the latter inside $R[\Pi]$. The obvious commutative diagram

$$\begin{tikzcd} \Psi'_\rho \arrow[r] & \Psi_\rho \arrow[d] \\ \Pi \arrow[u] \arrow[r, "\rho"'] & G \end{tikzcd}$$

is called the diptych of ρ.

Implicit in the above definition is the fact that $R[\Psi'_\rho]$ is a Hopf algebra. This can be extracted from the proof of Lemma 3.1.2 in [DH14] (see Remarks 2.16). Another relevant fact, whose proof the reader can find in [DH14, Theorem 4.1.1], is the following.

Theorem 4.2. *The morphism* $\Pi \to \Psi'_\rho$ *is faithfully flat.* ☐

A basic property of Ψ_ρ is as follows.

Lemma 4.3. *The group scheme* Ψ_ρ *is the closure of the image of* $\Pi_K \to G_K$ *in* G. ☐

Other useful facts having simple proofs are collected in the next lemma.

Lemma 4.4. *Let* $\rho : \Pi \to G$ *be as before and consider the following factorization of* ρ *in the category* **FGSch**$/R$*:*

$$\Pi \longrightarrow G' \longrightarrow G.$$

1. *If* $\Pi \to G'$ *faithfully flat, then there exists a unique dotted arrow rendering the diagram*
$$\begin{tikzcd} & \Psi'_\rho \arrow[rd] & \\ \Pi \arrow[ru] \arrow[r] & G' \arrow[u, dashed] \arrow[r] & G \end{tikzcd}$$
commutative.
2. *If* $G' \to G$ *is a closed embedding, then there exists a unique dotted arrow*
$$\begin{tikzcd} & \Psi_\rho \arrow[rd] \arrow[d, dashed] & \\ \Pi \arrow[ru] \arrow[r] & G' \arrow[r] & G \end{tikzcd}$$
rendering this diagram commutative.
3. *If* $G' \to G$ *is a closed embedding, and the arrow* $\Psi_\rho \to G'$ *envisaged in (2) induces an isomorphism on generic fibres, then* $\Psi_\rho \to G'$ *is an isomorphism.* ☐

Lemma 4.5. *If* $\Psi'_{\rho,k} \to \Psi_{\rho,k}$ *is faithfully flat, then* $\Psi'_\rho \to \Psi_\rho$ *is an isomorphism.*

Proof. By construction, $R[\Psi_\rho] \to R[\Psi'_\rho]$ is injective. If $k[\Psi_\rho] \to k[\Psi'_\rho]$ is also injective, which follows from the assumption, then $R[\Psi_\rho] \subset R[\Psi'_\rho]$ is saturated. The equality $K[\Psi_\rho] = K[\Psi'_\rho]$ then finishes the proof. ☐

Over the residue field k, there is yet another interesting group scheme in sight: the image of ρ_k. We then have the *triptych* of ρ, which is the commutative diagram

(18)
$$\begin{tikzcd} \Psi'_{\rho,k} \arrow[rr] \arrow[rd, two heads] & & \Psi_{\rho,k} \arrow[dd, hook] \\ & \mathrm{Im}(\rho_k) \arrow[ru, hook] \arrow[rd, hook] & \\ \Pi_k \arrow[uu, two heads] \arrow[ru, two heads] \arrow[rr, "\rho_k"'] & & G_k. \end{tikzcd}$$

$\hookrightarrow$ = closed immersion
$\twoheadrightarrow$ = faithfully flat.

Together with Lemma 4.5, diagram (18) proves the following:

Corollary 4.6. *The following claims are true.*

i) *If* $\mathrm{Im}(\rho_k) \to \Psi_{\rho,k}$ *is an isomorphism, then* $\Psi'_\rho \to \Psi_\rho$ *is an isomorphism.*
ii) *If* $\Psi'_\rho \to \Psi_\rho$ *is an isomorphism, then* $\mathrm{Im}(\rho_k) \to \Psi_{\rho,k}$ *is an isomorphism.*
iii) *The image of* $\Psi'_{\rho,k}$ *in* $\Psi_{\rho,k}$ *is none other than* $\mathrm{Im}(\rho_k)$.

□

Proposition 4.7. *The kernel fo* $\Psi'_{\rho,k} \to \mathrm{Im}(\rho_k)$ *is unipotent.*

Proof. Obviously, the kernel in question is also $\mathrm{Ker}\,\Psi'_{\rho,k} \to \Psi_{\rho,k}$ as we learn from diagram 18. Using Theorem 2.11, we are able to write Ψ'_ρ as $\varprojlim_i \Psi'_i$, where each $\Psi'_{i+1} \to \Psi'_i$ is a Neron blowup and $\Psi'_0 = \Psi_\rho$. The reader can immediately verify that

$$\mathrm{Ker}\left(\Psi'_{\rho,k} \to \Psi_{\rho,k}\right) \quad = \quad \varprojlim_i \mathrm{Ker}\left(\Psi'_{i,k} \to \Psi_{\rho,k}\right).$$

Consequently, using [DG70, Proposition 2.3 on p. 485], we only need to show that each kernel in the above limit is unipotent. This, in turn follows directly from [WW80, Theorem 1.5] and two fundamental properties of unipotent group schemes: a closed subgroup scheme of an unipotent group scheme is unipotent, and the extension of an unipotent group scheme by a unipotent group scheme is still unipotent [DG70, Proposition 2.3 on p. 485]. □

We show in the following example that it is possible that neither $\Psi'_{\rho,k} \to \mathrm{Im}(\rho_k)$ nor $\mathrm{Im}(\rho_k) \to \Psi_{\rho,k}$ be an isomorphism.

Example 4.8. Let $\rho : G' \to G$ be the Neron blowup of the origin in the closed fibre of $G = \mathbf{G}_{m,R}$. The diptych of ρ is

$$\begin{tikzcd}
G' \arrow[r, "\rho"] & G \\
G' \arrow[u, "\mathrm{id}"] \arrow[r, "\rho"'] & G. \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Then $\Psi'_{\rho,k} = \mathbf{G}_{a,k}$, $\Psi_{\rho,k} = \mathbf{G}_{m,k}$ and $\mathrm{Im}(\rho_k) = \{e\}$.

Further ahead, in Example 7.9, we will show how this example fits in a more empiric situation.

Let V be a free R-module of finite rank and assume that our G above equals $\mathbf{GL}(V)$. We now interpret Ψ_ρ and Ψ'_ρ in terms of their representations categories. This amounts simply to finding proper references in the literature.

Definition 4.9. Let $\Pi \in \mathbf{FGSch}/R$, as before.

(1) Let V be an object of $\mathrm{Rep}_R(\Pi)^o$. Write $\mathbf{T}^{a,b}V$ for the representation $V^{\otimes a} \otimes V^{\vee \otimes b}$ and denote by $\langle V \rangle_\otimes$ the full subcategory of $\mathrm{Rep}_R(\Pi)$ having as objects subquotients of direct sums $\mathbf{T}^{a_1,b_1}V \oplus \cdots \oplus \mathbf{T}^{a_m,b_m}V$ for varying a_i, b_i.
(2) Let $\alpha : V' \to V$ be a monomorphism in $\mathrm{Rep}_R(\Pi)$ with both V and V' free as R-modules. If $\mathrm{Coker}(\alpha)$ is also free as an R-module, we say, following [dS09, Definitions 10 and 23], that α is a special monomorphism. Call an object $V'' \in \mathrm{Rep}_R(\Pi)^o$ a special sub-quotient of V if there exists a special monomorphism $V' \to V$ and an epimorphism $V' \to V''$. The category of all special sub-quotients of various $\mathbf{T}^{a_1,b_1}V \oplus \cdots \oplus \mathbf{T}^{a_m,b_m}V$ is denoted by $\langle V \rangle_\otimes^s$.

Proposition 4.10. *Let* V *be an object of* $\mathrm{Rep}_R(\Pi)^o$ *and* ρ *be the natural homomorphism* $\Pi \to G := \mathbf{GL}(V)$.

(1) *The obvious functor* $\mathrm{Rep}_R(\Psi_\rho) \to \mathrm{Rep}_R(\Pi)$ *defines an equivalence of categories between* $\mathrm{Rep}_R(\Psi_\rho)^o$ *and* $\langle V\rangle_\otimes^s$.
(2) *The obvious functor* $\mathrm{Rep}_R(\Psi'_\rho) \to \mathrm{Rep}_R(\Pi)$ *defines an equivalence between* $\mathrm{Rep}_R(\Psi'_\rho)$ *and* $\langle V\rangle_\otimes$.

Proof. The first claim is a result of [dS09, Proposition 12]. The second claim follows from the definition of Ψ'_ρ and [DH14, Theorem 4.2]. □

Note that, in general, $\mathrm{Rep}_R(\Psi_\rho)$ is not a full subcategory of $\mathrm{Rep}_R(\Pi)$. This means that we have the following interpretation of the diptych (Definition 4.1) in terms of representation categories:

$$\begin{tikzcd}
\langle V\rangle_\otimes \arrow[d] & \langle V\rangle_\otimes^s \arrow[l] \arrow[d] \\
\mathrm{Rep}_R(\Pi) & \mathrm{Rep}_R(\mathbf{GL}(V))^o. \arrow[l]
\end{tikzcd}$$

We now deal with the representation theoretic interpretation of the triptych (diagram (18)) of ρ. For that, given an R-linear category $\mathcal{C}$, write $\mathcal{C}_{(k)}$ to denote the full subcategory whose objects W are annihilated by π, i.e. $\pi \cdot \mathrm{id}_W = 0$ in $\mathrm{Hom}_{\mathcal{C}}(W, W)$. We then have a commutative diagram of *solid* arrows between k-linear abelian categories:

$$\begin{tikzcd}
\mathrm{Rep}_R(\Psi'_\rho)_{(k)} \arrow[dd] & & \mathrm{Rep}_R(\Psi_\rho)_{(k)} \arrow[ll] \arrow[dl, dashed] \\
& \langle V\otimes k\rangle_\otimes \arrow[ul] \arrow[dl] & \\
\mathrm{Rep}_R(\Pi)_{(k)}. & &
\end{tikzcd} \tag{19}$$

From [J87, Part I, 10.1, 162] the categories $\mathrm{Rep}_R(-)_{(k)}$ are simply the corresponding representations categories of the group schemes obtained by base change $R \to k$. Since V is a faithful representation of Ψ_ρ (recall that $\Psi_\rho \to \mathbf{GL}(V)$ is a closed immersion by construction), $V \otimes k$ is a faithful representation of $\Psi_\rho \otimes k$, so that each object of $\mathrm{Rep}_R(\Psi_\rho)_{(k)}$ is a sub-quotient of some $\bigoplus \mathbf{T}^{a_i,b_i}(V\otimes k)$. This means that the upper horizontal arrow in diagram (19) factors through $\langle V\otimes k\rangle_\otimes$, i.e. the dotted arrow exists and still produces a commutative diagram. We conclude that diagram (19) captures the essence of diagram (18) as the former can easily be completed by introducing the representation category of the general linear group on the lower right corner.

5. NERON BLOWUPS OF FORMAL SUBGROUP SCHEMES

Let G be group scheme over R which is flat and of finite type. Let $H_n \subset G \otimes R_n$ be a closed sub-group scheme cut out by the ideal $I_n \subset R[G]$. We note that, in this case, $\pi^{n+1} \in I_n$.

Definition 5.1. The subring of $K[G]$ obtained by adjoining to $R[G]$ all elements of the form $\pi^{-n-1}a$ with $a \in I_n$ will be denoted by E^n.

Lemma 5.2. *Let* Δ, ε, *and* S *denote respectively the co-multiplication, the co-identity and the antipode of* $K[G]$. *Then* Δ, ε *and* S *send* E^n *into* $E^n \otimes E^n$, R *and* E^n *respectively.*

Proof. Let $a \in I_n$. Since $(\operatorname{Ker}\varepsilon, \pi^{n+1}) \supset (I_n, \pi^{n+1})$, there exists some $c \in \operatorname{Ker}\varepsilon$ and some $f \in R[G]$ such that $a = c + \pi^{n+1}f$. It then follows that $\varepsilon(a) = \pi^{n+1}\varepsilon(f)$, so that $\varepsilon(a\pi^{-n-1}) \in R$. Also, there are $a_i, a_i' \in I_n$ together with $\tau \in R[G] \otimes R[G]$ such that

$$\Delta(a) = \sum a_i \otimes x_i + x_i' \otimes a_i' + \pi^{n+1}\tau.$$

Then,

$$\Delta\left(\pi^{-n-1}a\right) = \sum \pi^{-n-1}a_i \otimes x_i + x_i' \otimes \pi^{-n-1}a_i' + \tau.$$

The verification of the statement concerning the antipode is equally trivial and is omitted. □

Definition 5.3. The group scheme

$$\operatorname{Spec} E^n$$

is called the Neron blowup of H_n.

Obviously, there is a morphism of group schemes

$$\operatorname{Spec} E^n \longrightarrow G$$

which, when tensored with K, becomes an isomorphism.

We now wish to concentrate on the case "$n = \infty$." We assume that R is *complete*. Let $\widehat{G}$ be the completion of G along its closed fibre (it is automatically flat over R) and let $\mathfrak{H} \subset \widehat{G}$ be a closed formal subgroup scheme which is moreover flat over R. We follow traditional notation in the theory of adic algebras and write $R\langle G\rangle$, respectively $R\langle\mathfrak{H}\rangle$, to denote the algebra associated to the formal schemes $\widehat{G}$, respectively $\mathfrak{H}$. This being so, $R\langle\mathfrak{H}\rangle$ is a quotient of $R\langle G\rangle$ by some ideal I. In order to carry on proofs, we shall find useful to let $I_n \subset R[G]$ be the ideal of the closed subgroup H_n of $G \otimes R_n$ induced by $\mathfrak{H}$. Note that $\pi^{n+1} \in I_n$ and $I_n R\langle G\rangle = (\pi^{n+1}, I)$.

Lemma 5.4. *Each element of* E^n *is also an element of* E^{n+1}.

Proof. We start by observing that $I_n = (I_{n+1}, \pi^{n+1})$ since H_n is obtained from H_{n+1} by base change $R_{n+1} \twoheadrightarrow R_n$. Any $b \in I_n$ can be written as $\pi^{n+1}b' + b''$ where $b' \in R[G]$ and $b'' \in I_{n+1}$. Then $\pi^{-n-1}b = b' + \pi^{-n-1}b''$, which finishes the proof since $\pi^{-n-1}b'' = \pi \cdot \pi^{-n-2}b''$. □

Definition 5.5. We denote by E^∞ the algebra $\cup_n E^n$.

Of course, replacing "n" by "∞" in the statement of Lemma 5.2 still produces a true claim so that we have the following.

Definition 5.6. The group scheme associated to the R-algebra E^∞ of Definition 5.5 will be denoted by $\mathcal{N}_{\mathfrak{H}}^\infty$ or $\mathcal{N}_{\mathfrak{H}}^\infty(G)$ if the need presents. The group scheme $\mathcal{N}_{\mathfrak{H}}^\infty$ will be called the blowup of G along $\mathfrak{H}$.

The group scheme $\operatorname{Spec} E^n$ associated to $H_n = \mathfrak{H} \otimes R_n$ (see Definition 5.3) will be denoted by $\mathcal{N}_{\mathfrak{H}}^n$ or $\mathcal{N}_{\mathfrak{H}}^n(G)$ if the need presents. The group scheme $\mathcal{N}_{\mathfrak{H}}^n$ will be called the partial blowup of G of level n along $\mathfrak{H}$.

If $\mathfrak{H}$ is the completion of a closed subgroup H of G, we put $\mathcal{N}_H^* = \mathcal{N}_{\mathfrak{H}}^*$.

Example 5.7. The automatic blowup of the identity introduced in Definition 2.18 is, due to Proposition 2.20, $\mathcal{N}_{\{e\}}^\infty(G)$.

Note that the obvious arrow of groups $\mathcal{N}_{\mathfrak{H}}^\infty \to G$ becomes an isomorphism when tensored with K. The following is a trivial observation which will prove useful further ahead.

Lemma 5.8. *Let* $\varphi : G' \to G$ *be a morphism in* $\mathbf{FGSch}/R$*. Denote by* $\hat{\varphi} : \widehat{G}' \to \widehat{G}$ *the morphism induced between* π*-adic completions. If* $\mathfrak{H}' \subset \widehat{G}'$ *is a closed and* R*-flat formal subgroup which is taken by* $\hat{\varphi}$ *to* $\mathfrak{H}$*, then there exists a unique arrow of affine group schemes* $\mathcal{N}^{\infty}_{\mathfrak{H}'}(G') \to \mathcal{N}^{\infty}_{\mathfrak{H}}(G)$ *rendering the diagram*

$$\begin{tikzcd}
\mathcal{N}^{\infty}_{\mathfrak{H}'}(G') \arrow[r] \arrow[d] & G' \arrow[d, "\varphi"] \\
\mathcal{N}^{\infty}_{\mathfrak{H}}(G) \arrow[r] & G
\end{tikzcd}$$

commutative. ☐

In the following lines we wish to present some elementary features of the blowup along a formal subgroup. We maintain the above notations and introduce

$$K\langle G\rangle = K \otimes R\langle G\rangle \quad \text{and} \quad K\langle \mathfrak{H}\rangle = K \otimes R\langle \mathfrak{H}\rangle.$$

For the sequel, the following commutative diagram can be helpful.

$$\begin{tikzcd}
R[G] \arrow[r] \arrow[d, hook] & R\langle G\rangle \arrow[r, two heads] \arrow[d, hook] & R\langle \mathfrak{H}\rangle \arrow[d, hook] \\
K[G] \arrow[r] & K\langle G\rangle \arrow[r, two heads] & K\langle \mathfrak{H}\rangle.
\end{tikzcd}$$

Lemma 5.9. *An element* $b \in K[G]$ *belongs to* E^{∞} *if and only if its image in* $K\langle \mathfrak{H}\rangle$ *belongs to the image of* $R\langle \mathfrak{H}\rangle$*.*

Proof. Let $b \in R[G]$ and $m \geqslant 1$. We assume that the image of $\pi^{-m}b$ in $K\langle \mathfrak{H}\rangle$ coincides with the image of an element $a^* \in R\langle G\rangle$. Then $\pi^m a^*$ and b have the same image in $K\langle \mathfrak{H}\rangle$. This implies that $\pi^m a^* \equiv b \mod I$. As $IR_n[G] = I_n R_n[G]$, we have $0 \equiv b \mod I_{m-1}R_{m-1}[G]$. Hence $\pi^{-m}b \in E^{m-1}(G)$.

Let $a \in I_n$. Since $(I, \pi^{n+1}) = I_n R\langle G\rangle$, we can write $a = \pi^{n+1}a^* + a^{**}$, where $a^* \in R\langle G\rangle$ and $a^{**} \in I$. Then, $\pi^{-n-1}a = a^* + \pi^{-n-1}a^{**}$ in $K\langle G\rangle$ with $a^* \in R\langle G\rangle$ and $\pi^{-n-1}a^{**} \in I \cdot K\langle G\rangle$. The image of $\pi^{-n-1}a$ in $K\langle \mathfrak{H}\rangle$ is then the image of a^*. ☐

Corollary 5.10. *The first projection*

$$K[G] \times_{K\langle \mathfrak{H}\rangle} R\langle \mathfrak{H}\rangle \longrightarrow K[G]$$

induces an isomorphism between $K[G] \times_{K\langle \mathfrak{H}\rangle} R\langle \mathfrak{H}\rangle$ *and* E^{∞}*.* ☐

Corollary 5.11. *For each* $n \in \mathbb{N}$*, the obvious morphism of* R_n*-groups* $\mathcal{N}^{\infty} \otimes R_n \to G \otimes R_n$ *induces an isomorphism of* $\mathcal{N}^{\infty} \otimes R_n$ *with* $H_n = \mathfrak{H} \otimes R_n$*.*

Proof. In view of Corollary 5.10, it is enough to show that the obvious morphism

$$\theta : R[G] \longrightarrow K[G] \times_{K\langle \mathfrak{H}\rangle} R\langle \mathfrak{H}\rangle,$$

when tensored with R_n, gives the ensuing factorization:

$$\begin{tikzcd}
R_n[G] \arrow[r] \arrow[d] & R_n \otimes \left(K[G] \times_{K\langle \mathfrak{H}\rangle} R\langle \mathfrak{H}\rangle\right) \\
R_n \otimes R\langle \mathfrak{H}\rangle \arrow[ur, "\sim"'] &
\end{tikzcd}$$

We consider the commutative diagram of R-algebras

$$\begin{tikzcd}
K[G] \times_{K\langle\mathfrak{H}\rangle} R\langle\mathfrak{H}\rangle \arrow[r, "\mathrm{pr}_2"] & R\langle\mathfrak{H}\rangle \\
R[G] \arrow[u, "\theta"] \arrow[r] & R\langle G\rangle, \arrow[u, two heads]
\end{tikzcd}$$

and claim that $\mathrm{pr}_2 \otimes R_n$ is an isomorphism. Surjectivity is easily checked and injectivity is justified as follows: If $(f, \varphi) \in K[G] \times_{K\langle\mathfrak{H}\rangle} R\langle\mathfrak{H}\rangle$ is such that $\varphi = \pi^{n+1}\varphi'$, then $(f, \varphi) = \pi^{n+1}(\pi^{-n-1}f, \varphi')$, so that (f, φ) belongs to (π^{n+1}). The proof then yields by tensoring the above commutative square with R_n. $\square$

We now analyze the standard sequence (see Definition 2.12) associated to $\mathcal{N}^{\infty}_{\mathfrak{H}} \to G$. We will see that this sequence can be produced by a spontaneous process. For that we need the following concept.

Definition 5.12 (Strict transform)**.** Let $\mathfrak{H} \subset \widehat{G}$ be as before. Write $I \subset R\langle G\rangle$ for the ideal of $\mathfrak{H}$ and let $\rho : G' \to G$ be the Neron blowup of some closed subgroup of $G \otimes k$. Then the strict transform of $\mathfrak{H}$, denoted $\rho^{\#}(\mathfrak{H})$, is the closed formal subgroup of $\widehat{G}'$ cut out by the saturation of the ideal $I \cdot R\langle G'\rangle$, namely $\cup_n (I \cdot R\langle G'\rangle : \pi^n)$. If $H \subset G$ is a closed R-flat subgroup, we can equally define the strict transform $\rho^{\#}(H)$, which now is a closed, subgroup of G'.

Remarks 5.13. (1) By construction, all strict transforms are flat over R.

(2) The strict transform of closed subgroups $H \subset G$ is used in [SS90, Section 1].

(3) If $\rho : G' \to G$ and $H \subset G$ are as in Definition 5.12, then the strict transform $\rho^{\#}(H)$ is just the closure in G of the closed subgroup $H \otimes K$ in $G' \otimes K$. The same reasoning can be applied in the case of a formal subgroup as $\mathfrak{H} \subset \widehat{G}$ if we use the concept of schematic closure in the setting of rigid geometry, see [Be91, p.17].

We are now ready to state:

Theorem 5.14. *Let $\mathfrak{H} \subset \widehat{G}$ be as above.*

(1) *Let $G_0 := G$, $\mathfrak{H}_0 := \mathfrak{H}$ and denote by $\rho_0 : G_1 \to G_0$ the Neron blowup of $\mathfrak{H}_0 \otimes k$. If $\mathfrak{H}_n$ is defined, write $\rho_n : G_{n+1} \to G_n$ for the Neron blowup of $\mathfrak{H}_n \otimes k$ and put $\mathfrak{H}_{n+1} := \rho_n^{\#}(\mathfrak{H}_n)$. Then*

$$\cdots \xrightarrow{\rho_1} G_1 \xrightarrow{\rho_0} G_0 = G$$

is the standard sequence (see Definition 2.12) of $\mathcal{N}^{\infty}_{\mathfrak{H}}(G) \to G$

(2) *The image of the canonical morphism $\mathcal{N}^{\infty}_{\mathfrak{H}}(G) \otimes k \to G_{n+1} \otimes k$ is taken isomorphically by $\rho_n \otimes k$ to the image of $\mathcal{N}^{\infty}_{\mathfrak{H}}(G) \otimes k \to G_n \otimes k$.*

The proof will need the following preparation. From Corollary 5.11 we know that the image of $\rho_0 \otimes k$ is just $\mathfrak{H}_0 \otimes k$. Let $a_1^*, \dots, a_r^*$ generate the ideal I of $\mathfrak{H}$ and let $a_i \in R[G_0]$ be such that $a_i^* \equiv a_i \mod \pi$. Obviously, the ideal of $\mathfrak{H}_0 \otimes k$ is $(\pi, a_1, \dots, a_r)$. The Neron blowup $\rho_0 : G_1 \to G_0$ of $\mathfrak{H}_0 \otimes k$ is given by the inclusion

$$R[G_0] \subset E = R[G_0]\left[\frac{a_1}{\pi}, \dots, \frac{a_r}{\pi}\right].$$

In the sequel, we write $A\langle\xi_1, \dots, \xi_r\rangle$ for the π-adic completion of $A[\xi_1, \dots, \xi_r]$.

Lemma 5.15. *Let* $\widehat{E}$ *stand for the* π*-adic completion of* E*. Then*

$$\widehat{E} \simeq \frac{R\langle G_0\rangle\langle \zeta_1,\ldots,\zeta_r\rangle}{(\pi\zeta_1 - a_1^*,\ldots,\pi\zeta_r - a_r^*)^{\mathrm{sat}}}.$$

Proof. We use bold letters to stand for r-tuples. It is clear that, as $R\langle G_0\rangle$-algebras,

$$\frac{R\langle G_0\rangle\langle \boldsymbol{\xi}\rangle}{(\pi\boldsymbol{\xi} - \boldsymbol{a})} \simeq \frac{R\langle G_0\rangle\langle \boldsymbol{\zeta}\rangle}{(\pi\boldsymbol{\zeta} - \boldsymbol{a}^*)}$$

by means of $\xi_i \mapsto \zeta_i - \pi^{-1}(a_i^* - a_i)$. We then recall that for a noetherian R-algebra A, its completion $\widehat{A}$ is A-flat [Mat, Theorem 8.8, p.60]. From this, we conclude that for any noetherian R-algebra A and for any ideal J of A, the equality $J^{\mathrm{sat}}\widehat{A} = (J\widehat{A})^{\mathrm{sat}}$ holds. Now

$$\begin{aligned}
\widehat{E} &= \left(\frac{R[G_0][\boldsymbol{\xi}]}{(\pi\boldsymbol{\xi} - \boldsymbol{a})^{\mathrm{sat}}}\right)^{\wedge} && \text{(by Remark 2.2)}\\
&= \frac{R\langle G_0\rangle\langle \boldsymbol{\xi}\rangle}{(\pi\boldsymbol{\xi} - \boldsymbol{a})^{\mathrm{sat}} R\langle G_0\rangle\langle \boldsymbol{\xi}\rangle} && \text{(by [Mat, Theorem 8.7,p.60])}\\
&= \frac{R\langle G_0\rangle\langle \boldsymbol{\xi}\rangle}{(\pi\boldsymbol{\xi} - \boldsymbol{a})^{\mathrm{sat}}} && \text{(by the argument above).}
\end{aligned}$$

□

The algebra $\widehat{E}$ in the statement of the lemma is quite close to a fundamental object in rigid analytic Geometry. Following the terminology of [FvdP04, 4.1], the analytic space $\mathrm{Sp}\,K \otimes \widehat{E}$ is just the *rational domain* $|a_i^*| \leqslant |\pi|$ in the rigid analytic space $\mathrm{Sp}\,K\langle G_0\rangle$.

Proposition 5.16. *The morphism of affine formal schemes* $\widehat{G}_1 \to \widehat{G}_0$ *induces an isomorphism between* $\rho_0^{\#}(\mathfrak{H}_0)$ *and* $\mathfrak{H}_0$*. The same statement also holds if, instead of a formal subgroup* $\mathfrak{H}_0 \subset \widehat{G}_0$*, we choose a closed subgroupo* $H_0 \subset G_0$ *which is flat over* R.

Remark 5.17. In the usual theory of blowups, this is known as the statement that the strict transform *is* the blowup of the intersection [EiHa00, Proposition IV-21] and that blowing up a "hypersurface" operates no change. In the present setting, a separate consideration has to be made due to the fact that the generic fibre of $\widehat{G}_1$ is not the generic fibre of $\widehat{G}_0$.

Proof. We will omit the verification of the statement concerning a closed subgroup $H_0 \subset G_0$ and concentrate on the formal case.

Write $I^{\#}$ for the ideal of $\rho_0^{\#}(\mathfrak{H}_0)$. We must show that the obvious arrow $\phi : R\langle G_0\rangle/I \to R\langle G_1\rangle/I^{\#}$ is an isomorphism. By the description of $\widehat{E} = R\langle G_1\rangle$ offered in Lemma 5.15, the equality $I = (a_1^*,\ldots,a_r^*)$ and the definition of $I^{\#}$ as the saturation of $I \cdot R\langle G_1\rangle$, it is evident that ϕ is surjective. Injectivity will follow from injectivity of $\phi\otimes K$, which for those accustomed to rigid analytic geometry is a triviality. Indeed, there exists a dotted arrow rendering the diagram

$$\begin{tikzcd}
K\langle G_1\rangle \arrow[dr, dashed] & \\
K\langle G_0\rangle \arrow[u] \arrow[r] & K\langle G_0\rangle/I\cdot K\langle G_0\rangle
\end{tikzcd}$$

commutative, see [FvdP04, 4.1.2, p.71]. The injectivity of $\phi \otimes K$ is then obvious as $I \cdot K\langle G_1\rangle = I^{\#} \cdot K\langle G_1\rangle$. □

Proof of Theorem 5.14. We have now obtained an affine and flat group scheme G_1 together with a closed formal subgroup scheme $\rho_0^\#(\mathfrak{H}_0) = \mathfrak{H}_1 \subset \widehat{G}_1$. By functoriality (Lemma 5.8), we arrive at a commutative diagram

$$\begin{tikzcd}
\mathcal{N}^\infty_{\mathfrak{H}_1}(G_1) \arrow[r] \arrow[d] & G_1 \arrow[d] \\
\mathcal{N}^\infty_{\mathfrak{H}_0}(G_0) \arrow[r] & G_0.
\end{tikzcd}$$

Furthermore, we can affirm that inserting the obvious morphism $\mathcal{N}^\infty_{\mathfrak{H}_0}(G_0) \to G_1$ in the above diagram still produces a commutative one. (Recall that G_1 is the Neron blowup of $\mathfrak{H}_0 \otimes k$.) Using Corollary 5.11 and Corollary 5.16 we conclude that

$$\mathcal{N}^\infty_{\mathfrak{H}_1}(G_1) \otimes k \xrightarrow{\sim} \mathcal{N}^\infty_{\mathfrak{H}_0}(G_0) \otimes k.$$

As

$$\mathcal{N}^\infty_{\mathfrak{H}_1}(G_1) \otimes K \xrightarrow{\sim} \mathcal{N}^\infty_{\mathfrak{H}_0}(G_0) \otimes K,$$

we deduce that $\mathcal{N}^\infty_{\mathfrak{H}_0}(G_0) \to \mathcal{N}^\infty_{\mathfrak{H}_1}(G_1)$ is an isomorphism and the image of $\mathcal{N}^\infty_{\mathfrak{H}_0}(G_0) \otimes k \to G_1 \otimes k$ is the image of $\mathcal{N}^\infty_{\mathfrak{H}_1}(G_1) \otimes k \to G_1 \otimes k$. Theorem 5.14 can now be proved by induction. □

To end this section, we use Theorem 5.14 to study the standard sequence of the partial blowup of level m of $\mathfrak{H} \subset \widehat{G}$ (cf. Definition 5.3). One probable candidate for this sequence is the truncation of the standard sequence of $\mathcal{N}^\infty_{\mathfrak{H}}(G)$. The situation here seems to be a bit more subtle and we only propose an easy consequence of Theorem 5.14.

Corollary 5.18. *We maintain the notations of Theorem 5.14. Let* $n \geqslant 0$ *be given. Then there exists* $\mu \geqslant 0$ *depending on* n *and* $\mathfrak{H}$ *such that the standard sequence of* $\mathcal{N}^m_{\mathfrak{H}} \to G$*, for all* $m \geqslant \mu$ *starts as*

$$G_{n+1} \xrightarrow{\rho_n} \cdots \xrightarrow{\rho_0} G_0 = G.$$

Proof. We write $J_\nu \subset R[G_\nu]$ for the ideal of $\mathfrak{H}_\nu \otimes k \subset \widehat{G}_\nu \otimes k = G_\nu \otimes k$. From Theorem 5.14, we know that

(1) each $R[G_\nu]$ is contained in $E^\infty = \cup_m E^m$ and
(2) the ideal $\pi E^\infty \cap R[G_\nu]$, which cuts out the image of $\mathcal{N}^\infty_{\mathfrak{H}}(G) \otimes k$ in $G_\nu \otimes k$, is just J_ν.

Since $G = G_0$ is assumed to be of finite type, there exists some $\mu_0 \geqslant 0$ such that $R[G_{n+1}] \subset E^m$ for all $m \geqslant \mu_0$. We now take $\mu \geqslant \mu_0$ to be such that the generators of each one of the ideals $J_0, \ldots, J_n$ belong to πE^μ. This μ is the one predicted in the statement as in this case, for $m \geqslant \mu$, the image of $\mathcal{N}^m_{\mathfrak{H}}(G) \otimes k \to G_\nu \otimes k$ is cut out by J_ν for each $\nu \leqslant n$. □

Question 5.19. Under which conditions the standard sequence of $\mathcal{N}^n_{\mathfrak{H}}(G) \to G$ beings as $G_{n+1} \to \cdots \to G_0$, where the G_i are as in Theorem 5.14?

6. STUDY OF A PARTICULAR CLASS OF STANDARD SEQUENCES

In this section, we assume that R is complete. Let G be group scheme over R which is flat and of finite type. Theorem 5.14 guarantees that the centres appearing in the standard sequence of

$$\mathcal{N}^\infty_{\mathfrak{H}}(G) \longrightarrow G$$

are all isomorphic. In this section we wish to understand a possible converse for this: Corollary 6.10.

As the next paragraph argues, standard sequences with "constant" centres may easily appear; the utility of this study is therefore not restricted to proving a converse to Theorem 5.14.

Assume that k is of characteristic zero and let

$$\cdots \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0$$

be a standard sequence (cf. Definition 2.13). As $n \mapsto \dim B_n$ is nondecreasing and

$$\dim B_n \leqslant \dim G_{n,k} = \dim G_{0,K}$$

(the equality follows from [SGA3, VI_B, Corollary 4.3]) there exists $n_0 \in \mathbf{N}$ such that $\dim B_{n_0} = \dim B_n$ for all $n \geqslant n_0$. Since the kernel of $B_{n+1} \to B_n$ is either trivial or positive dimensional (it is a subgroup scheme of some $\mathbf{G}_{a,k}^r$ due to Theorem 2.4) we conclude that for $n \geqslant n_0$, the arrows $B_{n+1} \to B_n$ are all *isomorphisms*.

It is quite possible that a result more general than Corollary 6.10 holds, but for the moment, our best effort needs higher control on the relation between centres (i.e., the conclusion of Proposition 6.1 hold) so that more hypothesis were introduced.

Proposition 6.1. *Let*

$$\cdots \longrightarrow G_{n+1} \xrightarrow{\rho_n} \cdots \xrightarrow{\rho_0} G_0 = G$$

be a standard sequence (Definition 2.13) where the centre of ρ_n *is* $B_n \subset G_n \otimes k$. *Assume the following particularities.*

i) *The group* G_0 *is smooth over* R, *and* B_0 *is smooth over* k.
ii) *For every* $n \geqslant 0$, *the restriction* $\rho_n \otimes k : B_{n+1} \to B_n$ *is an isomorphism.*
iii) *There exists an* R*-flat closed subgroup* $L_0 \subset G_0$ *with* $L_0 \otimes k = B_0$.
iv) *For each representation* V *of* B_0, *the first cohomology group* $H^1(B_0; V)$ *vanishes (linear reductivity).*

Then, for every $n \geqslant 1$, *there exist an internal automorphism* $a_n : G_n \to G_n$ *together with a closed and* R*-flat subgroup* $L_n \subset G_n$ *enjoying the following properties.*

(1) *For any* $n \geqslant 1$, *the closed subgroup* L_n *is* $a_n(\rho_{n-1}^{\#} L_{n-1})$. *(See Definition 5.12 for notation.)*
(2) *The special fibre of* L_n *is* B_n.

Proof. We construct a_1 and L_1. From assumption (ii) and [Wa79, Theorem 14.1], we know that $G_1 \otimes k \to B_0$ is faithfully flat; we arrive at an exact sequence

$$1 \longrightarrow U_1 \longrightarrow G_1 \otimes k \longrightarrow B_0 \longrightarrow 1 \tag{e}$$

which, again by property (ii), assures that $G_1 \otimes k$ is a semidirect product $U_1 \rtimes B_1$. In $G_1 \otimes k$, besides B_1, we have another closed subgroup isomorphic to B_0, viz. the special fibre of $\Lambda_1 := \rho_0^{\#}(L_0)$ (see Definition 5.12 and Proposition 5.16). As U_1 is isomorphic to $\mathbf{G}_{a,k}^r$, due to (i) and Theorem 2.4, the action of B_0 on U_1 by conjugation defines a linear action of B_0 on U_0. (See Section 2.3 for details.) Assumption (iv) together with the exercise proposed in I.5.1 of [Se97] shows that there exists an internal automorphism $a_1 : G_1 \otimes k \to G_1 \otimes k$ taking $\Lambda_1 \otimes k$ to B_1.

Now G_1 is smooth over R due to the "fibre-by-fibre" smoothness criterion [EGA IV_4, 17.8.2, p. 79], the isomorphism $G_1 \otimes K \simeq G_0 \otimes K$ and smoothness of $G_1 \otimes k \to B_0$ (Theorem 2.4). The infinitesimal lifting criterion allows us to find an internal automorphism of G_1, parsimoniously called a_1, such that $a_1(\Lambda_1 \otimes k) = B_1$; define $L_1 := a_1(\Lambda_1)$. This deals with the case $n = 1$. The inductive step "$n \Rightarrow n+1$" yields in the exactly same fashion. □

It is perhaps worth expressing the conclusion of Proposition 6.1 in a pictorial form by means of the following *stair*:

(20) higher level

$$\begin{tikzcd}
 & & \vdots \\
 & G_2 \arrow[r, "a_2"'] \arrow[d, "\rho_1"] & G_2 \\
G_1 \arrow[r, "a_1"] \arrow[d, "\rho_0"] & G_1 & \\
G_0 & &
\end{tikzcd}$$

where

$$\begin{aligned}
\rho_0 &= \text{blowup of } L_0 \otimes k, \\
a_1 &= \text{internal automorphism of } G_1, \\
L_1 &= a_1\left[\rho_0^{\#}(L_0)\right], \\
\rho_1 &= \text{blowup of } L_1 \otimes k, \\
a_2 &= \text{internal automorphism of } G_2 \\
L_2 &= a_2\left[\rho_1^{\#}(L_1)\right], \\
\cdots &= \cdots .
\end{aligned} \tag{21}$$

For the sake of discussion, we make some definitions.

Definition 6.2 (Spontaneous sequences)**.** A standard sequence as depicted in the above stair is called an almost spontaneous sequence associated to $L_0 \subset G_0$. An almost spontaneous standard sequence is spontaneous if all the internal automorphisms appearing in it equal the identity.

Note that another way of stating Theorem 5.14(1) is then:

Theorem 6.3. *Let* $H \subset G$ *be a closed immersion in* **FGSch**/R. *Then the standard sequence of* $\mathcal{N}_H^\infty(G) \to G$ *is the spontaneous sequence of* $H \subset G$. □

We now wish to prove Theorem 6.8 below which sheds light on the nature of *almost spontaneous sequences*. We fix, in addition to G, a closed immersion $L_0 \subset G_0 = G$ in **FGSch**/R and a stair as in (20)-(21), which is the almost spontaneous sequence associated to L_0 and the internal automorphisms a_i.

Proposition 6.4. *We maintain the above notations. For each* $n \geqslant 1$, *define internal automorphisms* $a_{0,n}, \ldots, a_{n,n}$ *by decreeing that* $a_{n,n} = a_n$ *and that*

$$\begin{tikzcd}
G_m \arrow[r, "a_{m,n}"] \arrow[d, "\rho_{m-1}"'] & G_m \arrow[d, "\rho_{m-1}"] \\
G_{m-1} \arrow[r, "a_{m-1,n}"'] & G_{m-1}
\end{tikzcd}$$

commutes. In other words, we complete the stair (20) *as suggested by:*

(case $n = 2$)

$$\begin{tikzcd}
 & & G_3 \arrow[d, "\rho_2"] \\
 & G_2 \arrow[r, "a_2"] \arrow[d, "\rho_1"'] & G_2 \arrow[d, dotted, "\rho_1"] \\
G_1 \arrow[r, "a_1"] \arrow[d, "\rho_0"'] & G_1 \arrow[r, dotted, "a_{1,2}"] \arrow[d, dotted, "\rho_0"] & G_1 \arrow[d, dotted, "\rho_0"] \\
G_0 \arrow[r, dotted, "a_{0,1}"'] & G_0 \arrow[r, dotted, "a_{0,2}"'] & G_0
\end{tikzcd}$$

Then

$$G_{n+1} \xrightarrow{\rho_n} \cdots \xrightarrow{\rho_0} G_0 = G$$

is the spontaneous sequence associated to

$$a_{0,n} \cdots a_{0,2} a_{0,1}(L_0)$$

and truncated at level $n + 1$.

Proof. We proceed by induction and begin with $n = 1$. In this case, we have a commutative diagram

$$\begin{tikzcd}
 & G_2 \arrow[d, "\rho_1"] \\
G_1 \arrow[r, "a_1"] \arrow[d, "\rho_0"'] & G_1 \arrow[d, dotted, "\rho_0"] \\
G_0 \arrow[r, dotted, "a_{0,1}"'] & G_0.
\end{tikzcd}$$

We remark that $a_{0,1} : G_0 \otimes k \to G_0 \otimes k$ is conjugation by an element of $L_0 \otimes k$. Hence, $L_0 \otimes k = a_{0,1}(L_0) \otimes k$ and commutativity of the above diagram guarantees that $\rho_0^{\#}\left[a_{0,1}(L_0)\right] = a_1\left[\rho_0^{\#}(L_0)\right]$. The desired result then follows.

Let $n > 1$ and assume that the result is valid for all almost spontaneous sequences associated to $L_1 = a_1\left[\rho_0^{\#}(L_0)\right] \subset G_1$ and truncated at level n. Since

$$\begin{tikzcd}
G_1 \arrow[r, "a_{1,1}"] \arrow[d, "\rho_0"'] & G_1 \arrow[r, "a_{1,2}"] \arrow[d, "\rho_0"'] & \cdots \arrow[r, "a_{1,n}"] & G_1 \arrow[d, "\rho_0"] \\
G_0 \arrow[r, "a_{0,1}"'] & G_0 \arrow[r, "a_{0,2}"'] & \cdots \arrow[r, "a_{0,n}"'] & G_0
\end{tikzcd}$$

is commutative, functoriality of the strict transform and the definition of L_1 give us

$$\begin{aligned}
\rho_0^{\#}\left[a_{0,n} \cdots a_{0,1}(L_0)\right] &= a_{1,n} \cdots a_{1,1}(\rho_0^{\#}(L_0)) \\
&= a_{1,n} \cdots a_{1,2}(L_1).
\end{aligned} \tag{22}$$

(Recall that by construction $a_{1,1} = a_1$.) Since $a_{0,j}$ is conjugation by an element of $G_0(R)$ belonging to the image of $G_1(R) \to G_0(R)$, we can affirm that $a_{0,j}$ leaves $L_0 \otimes k$ invariant.

Consequently, L_0 and $a_{0,n}\cdots a_{0,1}(L_0)$ have the same closed fibre, from which we conclude that ρ_0 is also the blowup of

$$(a_{0,n}\cdots a_{0,1}(L_0))\otimes k.$$

Since

$$G_{n+1}\xrightarrow{\rho_n}\cdots\xrightarrow{\rho_1}G_1$$

is, by induction hypotehsis, the spontaneous sequence of $a_{1,n}\cdots a_{1,2}(L_1)\subset G_1$ truncated at level n and, as (see eqs. (22))

$$a_{1,n}\cdots a_{1,2}(L_1)=\rho_0^{\#}\left[a_{0,n}\cdots a_{0,1}(L_0)\right],$$

we obtain that

$$G_{n+1}\xrightarrow{\rho_n}\cdots\xrightarrow{\rho_1}G_1\xrightarrow{\rho_0}G_0$$

is the spontaneous sequence associated to $a_{0,n}\cdots a_{0,1}(L_0)$ truncated at level $n+1$. □

We maintain the above notations and write

$$C_n:=a_{0,n}\cdots a_{0,1}(L_0).$$

Proposition 6.5. *For each $n\geqslant m$, we have $C_n\otimes R_m=C_m\otimes R_m$ as closed subschemes of $G\otimes R_m$. The limit $\mathfrak{C}:=\varinjlim_n C_n\otimes R_n$ defines a closed formal subgroup of $\widehat{G}_0=\varinjlim_n G_0\otimes R_n$ which is furthermore flat over R.*

The proof relies on the following result.

Lemma 6.6. *Let $\mathcal{L}\subset\mathcal{G}$ be a closed immersion in* **FGSch**/R. *Let*

$$\cdots\xrightarrow{\sigma_0}\mathcal{G}_0=\mathcal{G}$$

stand for the spontaneous sequence associated to $\mathcal{L}$. Let $g_{n+1}\in\mathcal{G}_{n+1}(R)$. Then its image by the obvious arrow

$$\mathcal{G}_{n+1}(R)\longrightarrow\mathcal{G}(R)\longrightarrow\mathcal{G}(R_n)$$

lies in $\mathcal{L}(R_n)$.

Proof. To help with the verification, we employ the following observation.

Lemma 6.7. *Let $\mathcal{L}\subset\mathcal{G}$ be a closed immersion in* **FGSch**/R. *If J stands for the ideal of $\mathcal{L}$ and $\sigma:\mathcal{G}'\to\mathcal{G}$ for the blowup of $\mathcal{L}\otimes k$, then the ideal of the strict transform $\sigma^{\#}(\mathcal{L})$ of $\mathcal{L}$ contains $\pi^{-1}J$.*

Proof. Evident since the ideal of the strict transform is the saturation $\cup_\nu(JR[\mathcal{G}']:\pi^\nu)$. □

We carry on the verification of the lemma. Write $\mathcal{L}_0=\mathcal{L}$ and define $\mathcal{L}_{n+1}$ as the strict transform of $\mathcal{L}_n$ in $\mathcal{G}_{n+1}$. Let J_n stand for the ideal of $\mathcal{L}_n$ in $R[\mathcal{G}_n]$. From Lemma 6.7, we know that for any $f_0\in J_0$, there exists an element $f_{n+1}\in R[\mathcal{G}_{n+1}]$ (belonging to J_{n+1}) such that $\pi^{n+1}f_{n+1}=f_0$. Consequently, if $g_{n+1}:R[\mathcal{G}_{n+1}]\to R$ is a morphism of R-algebras, we conclude that $g_{n+1}(f_0)=\pi^{n+1}g_{n+1}(f_0)$, so that the morphism $g_0:R[\mathcal{G}_0]\to R$ associated to it satisfies $g_0(J_0)\subset(\pi^{n+1})$. Hence, $g_0\otimes R_n:R_n[\mathcal{G}_0]\to R_n$ annihilates J_0, i.e. the corresponding point belongs to $\mathcal{L}_0(R_n)$. □

Proof of Proposition 6.5. We wish first to verify

$$C_{n+1}\otimes_R R_n=C_n\otimes_R R_n. \tag{23}$$

Proposition 6.4 assures us that for each fixed n,

$$G_{n+1}\xrightarrow{\rho_n}\cdots\xrightarrow{\rho_0}G_0=G$$

is the spontaneous sequence of $C_n \subset G_0$ truncated at level $n+1$. We know that the internal automorphism $a_{0,n+1} : G_0 \to G_0$, which sends C_n isomorphically to C_{n+1}, is conjugation by an element of the form

$$\rho_n \circ \cdots \circ \rho_0(g_{n+1}), \qquad g_{n+1} \in G_{n+1}(R).$$

Hence, due to Lemma 6.6, it is true that

$$a_{0,n+1}(C_n \otimes_R R_n) = C_n \otimes_R R_n.$$

Therefore,

$$\begin{aligned} C_{n+1} \otimes_R R_n &= a_{0,n+1}(C_n) \otimes_R R_n \\ &= a_{0,n+1}(C_n \otimes R_n) \\ &= C_n \otimes R_n. \end{aligned}$$

Another way of expressing equality (23) is by saying that, if $\mathfrak{k}_n$ stands for the kernel of $R_n[G_0] \to R_n[C_n]$, then $R_{n+1}[G_0] \to R_n[G_0]$ takes $\mathfrak{k}_{n+1}$ onto $\mathfrak{k}_n$. Consequently, $\varprojlim_n R_n[G_0] \to \varprojlim_n R_n[C_n]$ is surjective and $\mathfrak{C}$ is a closed formal subscheme of $\widehat{G}_0$. We omit the verification that $\mathfrak{C}$ is a subgroup of $\widehat{G}_0$ and refer the reader to [Mat, 22.3, p.174] for the statement about flatness. □

Everything is now in place for the proof of

Theorem 6.8. *Let*

$$\cdots \longrightarrow G_{n+1} \xrightarrow{\rho_n} \cdots \xrightarrow{\rho_0} G_0 = G$$

be the almost spontaneous sequence (see Definition 6.2) associated to $L_0 \subset G_0$ *as defined on page 27. If* $\mathfrak{C} = \varinjlim C_n \otimes R_n$ *stands for the formal closed subgroup of* $\widehat{G}_0$ *obtained by means of Proposition 6.4 and Proposition 6.5, then the standard sequence above is the the standard sequence of* $\mathcal{N}^\infty_{\mathfrak{C}}(G) \to G$.

In order to provide a clear proof of Theorem 6.8, we need the following lemma.

Lemma 6.9. *Let* $G' \to G$ *and* $H \to G$ *be respectively an isomorphism and a closed immersion of* **FGSch**/R. *Denote by* $H' \to G'$ *the closed immersion corresponding to* $H \to G$ *and let* n *be an integer.*

Assume that for some $\mu \in \mathbf{N}$, *the standard sequence of* $\mathcal{N}^\mu_H(G) \to G$ *coincides with the spontaneous sequence associated to* H *when both are truncated at level* $n+1$. *Then the same property holds for* $\mathcal{N}^\mu_{H'}(G') \to G'$. □

Proof of Theorem 6.8. We fix some $n \geqslant 0$ and show that the spontaneous sequence of $\mathfrak{C} \subset \widehat{G}$ truncated at level $n+1$ is

$$G_{n+1} \xrightarrow{\rho_n} \cdots \xrightarrow{\rho_0} G_0 = G.$$

The goal is to apply Proposition 6.4. To avoid repetitions we let "σ_{n+1}" abbreviate "standard sequence truncated at level $n+1$." We also omit reference to G when possible.

Let μ be a positive integer satisfying the following properties.

- **P1.** The σ_{n+1} of $\mathcal{N}^\infty_{\mathfrak{C}}$ coincides with the σ_{n+1} of $\mathcal{N}^\mu_{\mathfrak{C}}$.
- **P2.** The spontaneous sequence of $L_0 \subset G_0$ truncated at level $n+1$ coincides with the σ_{n+1} of $\mathcal{N}^\mu_{L_0}$.
- **P3.** $\mu \geqslant n$.

That μ exists is a consequence of Corollary 5.18 and Theorem 6.3. From **P1**, the σ_{n+1} of $\mathcal{N}_{\mathfrak{C}}^{\infty}$ is the σ_{n+1} of $\mathcal{N}_{\mathfrak{C}}^{\mu} = \mathcal{N}_{C_\mu}^{\mu}$. From **P2** and Lemma 6.9, the σ_{n+1} of $\mathcal{N}_{C_\mu}^{\mu}$ is the truncation of the spontaneous sequence of $C_\mu \subset G_0$ at level $n+1$. Due to Proposition 6.4 and **P3**, the spontaneous sequence associated to $C_\mu \subset G_0$, truncated at level $n+1$, is just what we started with:

$$G_{n+1} \xrightarrow{\rho_n} \cdots \xrightarrow{\rho_0} G_0 = G.$$

□

We then obtain our main converse of Theorem 5.14 as a consequence of Proposition 6.1, Definition 6.2 and Theorem 6.8.

Corollary 6.10. *Let*

$$\cdots \longrightarrow G_{n+1} \xrightarrow{\rho_n} \cdots \xrightarrow{\rho_0} G_0 = G$$

be a standard sequence (Definition 2.12). Denote the center of the Neron blowup ρ_n *by* $B_n \subset G_n \otimes k$*. Assume that the four hypothesis in Proposition 6.1 hold:*

i) *The group* G_0 *is smooth over* R*, and* B_0 *is smooth over* k*.*
ii) *For every* $n \geqslant 0$*, the restriction* $\rho_n \otimes k : B_{n+1} \to B_n$ *is an isomorphism.*
iii) *There exists an* R*-flat closed subgroup* $L_0 \subset G_0$ *with* $L_0 \otimes k = B_0$*.*
iv) *For each representation* V *of* B_0*, the first cohomology group* $H^1(B_0; V)$ *vanishes (linear reductivity).*

Then the above standard sequence is the standard sequence of $\mathcal{N}_{\mathfrak{C}}^{\infty}(G) \to G$*, where* $\mathfrak{C}$ *is a formal,* R*-flat, closed subgroup scheme of* $\widehat{G}$*.* □

7. GROUP SCHEMES OVER R IN DIFFERENTIAL GALOIS THEORY

We now wish to apply the theory so far developed to study differential Galois theory. We shall restrict ourselves to the case of equicharacteristic zero, since our main result, Theorem 8.1, needs this assumption. It should be noted that some definitions and results can be carried over to greater generality once we replace "modules with integrable connections" by "$\mathcal{D}$-modules."

Let $f : X \to S$ be a morphism locally of finite type between locally noetherian schemes. We let **MIC**(X/S) stand for the category of relative integrable connections on coherent $\mathcal{O}_X$-modules as in [K70, Section 1]. (But note that our **MIC** is more restrictive than Katz'.) Objects of **MIC**(X/S) are couples

$$(\mathcal{M}, \nabla)$$

where $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module and

$$\nabla : \mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$$

is an integrable connection. To render certain parts of the text more readable, we frequently suppress the symbol ∇ from the notation. Arrows in **MIC**(X/S) are just morphisms of $\mathcal{O}_X$-modules which are "compatible" with connections. It is obvious that **MIC**(X/S) is an abelian category and that the evident functor from **MIC**(X/S) to coherent modules is exact and faithful. Furthermore, the tensor product and the "dualization" of coherent modules induces a tensor product (denoted by $\otimes$) and a "dualization" (denoted by the superscript $(-)^{\vee}$) in **MIC**(X/S). Details are in [K70, Section 1.1].

Remark 7.1. In order to render referencing more effective, we inform the reader that **MIC**(X/S) coincides with the category of stratified sheaves **str**(X/S) (see [DH14], [dS09],

[BO78]) once S is a $\mathbf{Q}$-scheme and f is a smooth morphism. A proof is given in [BO78, 2.15].

We now fix some assumptions which will be in force for the rest of this section: we assume that k is algebraically closed of characteristic 0, that R contains a copy of k, and that $f : X \to \operatorname{Spec} R$ is a geometrically connected smooth morphism admitting a section $\xi : \operatorname{Spec} R \to X$.

Enforcing the obvious notations, we are then led to consider three categories of modules with connections:

$$\mathbf{MIC}(X/k), \quad \mathbf{MIC}(X/R) \quad \text{and} \quad \mathbf{MIC}(X_k/k).$$

Objects of $\mathbf{MIC}(X/R)$ are here called *relative connections*, while those on $\mathbf{MIC}(X/k)$ are called *absolute connections*. (We omit the adjective integrable.) An absolute connection $(\mathcal{M}, \nabla)$ can be "inflated" to a relative connection $\mathbf{Inf}(\mathcal{M})$ by means of the composition

$$\mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes \Omega^1_{X/k} \xrightarrow{\mathrm{id}\otimes c} \mathcal{M} \otimes \Omega^1_{X/R},$$

where $c : \Omega^1_{X/k} \to \Omega^1_{X/R}$ is the canonic morphism. It is easily verified that $\mathbf{Inf} : \mathbf{MIC}(X/k) \to \mathbf{MIC}(X/R)$ is a tensor functor.

Objects of $\mathbf{MIC}(X/R)$ which underlie locally free $\mathcal{O}_X$-modules are the objects of a full subcategory $\mathbf{MIC}(X/R)^{\mathrm{o}}$. A basic result in the theory is the following. (For proofs, the reader is directed to [K70, Proposition 8.9] for the first statement and to [K90, p.40], [dS09, p.82] or [DH14, 5.1.1] for the second.)

Proposition 7.2. *Assume that f is smooth. Any $M \in \mathbf{MIC}(X_k/k)$ is locally free as an $\mathcal{O}_{X_k}$-module. Any $\mathcal{M} \in \mathbf{MIC}(X/R)$ which is free of π-torsion belongs to $\mathbf{MIC}(X/R)^{\mathrm{o}}$ (and conversely).* ☐

Another cornerstone, which is explained in [DH14, Proposition 5.1.1], ensues. (The reader should also bear in mind Remark 7.1.)

Proposition 7.3. *Let $\xi : \operatorname{Spec} R \to X$ be an R-point of X. Then the pull-back functor $\xi^* : \mathbf{MIC}(X/R) \to \mathbf{mod}(R)$ is exact and faithful.* ☐

In possession of these facts, we can put forward the main definitions of this section.

Definition 7.4.
1. Let $\mathcal{M}$ be an object of $\mathbf{MIC}(X/R)^{\mathrm{o}}$. Write $\mathbf{T}^{a,b}\mathcal{M}$ for the connection $\mathcal{M}^{\otimes a} \otimes \mathcal{M}^{\vee \otimes b}$ and denote by $\langle \mathcal{M} \rangle_\otimes$ the full subcategory of $\mathbf{MIC}(X/R)$ having as objects sub-quotients of direct sums $\mathbf{T}^{a_1,b_1}\mathcal{M} \oplus \cdots \oplus \mathbf{T}^{a_m,b_m}\mathcal{M}$ for varying a_i, b_i.
2. Let $\alpha : \mathcal{M}' \to \mathcal{M}$ be a monomorphism in $\mathbf{MIC}(X/R)$ with both $\mathcal{M}$ and $\mathcal{M}'$ locally free as $\mathcal{O}_X$-modules. If $\operatorname{Coker}(\alpha)$ is also locally free, we say, following [dS09, Definitions 10 and 23], that α is a special monomorphism. Call an object $\mathcal{M}'' \in \mathbf{MIC}(X/R)^{\mathrm{o}}$ a special sub-quotient of $\mathcal{M}$ if there exists a special monomorphism $\mathcal{M}' \to \mathcal{M}$ and an epimorphism $\mathcal{M}' \to \mathcal{M}''$. The category of all special sub-quotients of various $\mathbf{T}^{a_1,b_1}\mathcal{M} \oplus \cdots \oplus \mathbf{T}^{a_m,b_m}\mathcal{M}$ is denoted by $\langle \mathcal{M} \rangle^{\mathrm{s}}_\otimes$.

The structure result which will enable us to see the theory of modules with connections through group theoretical lenses is the following. (The proof, which is in Saavedra's seminal work [S72], is explained concisely in [DH14].)

Theorem 7.5. *Let $\mathcal{M} \in \mathbf{MIC}(X/R)$. The R-point ξ induces an equivalence of abelian tensor categories*

$$\xi^* : \langle \mathcal{M} \rangle_\otimes \longrightarrow \mathrm{Rep}_R(\mathrm{Gal}'(\mathcal{M})),$$

where $\mathrm{Gal}'(\mathcal{M})$ is a flat group scheme over R. ☐

Definition 7.6. The group scheme $\mathrm{Gal}'(\mathcal{M})$ is called the full differential Galois group of $\mathcal{M}$.

Definition 7.7. Let $\mathcal{M} \in \mathbf{MIC}(X/R)^{\mathrm{o}}$ and write

$$\rho : \mathrm{Gal}'(\mathcal{M}) \longrightarrow \mathbf{GL}(\xi^*\mathcal{M})$$

for the associated representation of the full differential Galois group. The restricted differential Galois group of $\mathcal{M}$, denoted $\mathrm{Gal}(\mathcal{M})$, is the group scheme Ψ_ρ of Definition 4.1.

Using Proposition 4.10, we then arrive at the diptych of $\mathcal{M}$ as in Definition 4.1:

$$\begin{tikzcd}
\Psi'_\rho \arrow[r] & \mathrm{Gal}(\mathcal{M}) \arrow[d, hook] \\
\mathrm{Gal}'(\mathcal{M}) \arrow[u, "\mathrm{id}"] \arrow[r, "\rho"'] & \mathbf{GL}(\xi^*\mathcal{M}).
\end{tikzcd}$$

Of course, the presence of the identity as the leftmost arrow above might seem an improper way of presenting things; the reader who wishes to complete the diagram by including the fundamental group scheme $\Pi(X/S, \xi)$ (see [DH14, Definition 5.1.4]) in the lower left corner is invited to do so at the expense of having to understand that $\mathrm{Rep}_R(\Pi)$ is not what one might naively think it is. Since we focus on differential Galois groups and not fundamental group schemes, we leave $\Pi(X/S, \xi)$ as inspirational.

Then, as in Section 4 we have:

Proposition 7.8. *The following claims are true.*

1. *The arrow* $\mathrm{Gal}'(\mathcal{M}) \to \mathrm{Gal}(\mathcal{M})$ *induces an isomorphism on generic fibres and is a possibly infinite iteration of Neron blowups.*
2. *The functor* ξ^* *induces an equivalence*

$$\langle \mathcal{M} \rangle^{s}_{\otimes} \stackrel{\sim}{\longrightarrow} \mathrm{Rep}_R(\mathrm{Gal}(\mathcal{M}))^{\mathrm{o}}.$$

□

We are then ready to present the "empiric" version of Example 4.8.

Example 7.9. Let $X = \operatorname{Spec} A$, where $A = R[x, 1/x]$. Let $\mathcal{L} = \mathcal{O}_X\mathbf{e} \in \mathbf{MIC}(X/R)^{\mathrm{o}}$ be defined by

$$\partial \mathbf{e} = -\frac{\pi}{x}\mathbf{e};$$

here $\partial = d/dx$. We wish to compute $\mathrm{Gal}(\mathcal{L})$ and $\mathrm{Gal}'(\mathcal{L})$ (at some unspecified R-point).

In $\langle \mathcal{L} \rangle_\otimes$ we have the connection $\mathcal{V}$ given by

$$\begin{aligned}
\partial \mathbf{v}_1 &= \frac{1}{x}(\mathbf{v}_2 - \pi \mathbf{v}_1) \\
\partial \mathbf{v}_2 &= 0;
\end{aligned}$$

it is the sub-object of $\mathbf{1} \oplus \mathcal{L}$ generated by $\mathbf{v}_1 = (1, \mathbf{e})$ and $\mathbf{v}_2 = (\pi, 0)$. (Needless to say, the construction of $\mathcal{V}$ parallels that in Proposition 3.1.) Since $\pi(\mathbf{1} \oplus \mathcal{L}) \subset \mathcal{V}$, we conclude that $\langle \mathcal{V} \rangle_\otimes = \langle \mathcal{L} \rangle_\otimes$. We denote by Gal' the full differential Galois group of $\mathcal{L}$, which is the same as that of $\mathcal{V}$, at some R-point, so that we have, associated to $\mathcal{L}$, the representation

$\rho : \mathrm{Gal}' \to \mathbf{G}_{m,R}$. Let $\mathbf{G}'_{m,R}$ stand for the Neron blowup of $\mathbf{G}_{m,R}$ at the origin of the closed fibre. Since ρ_k is the trivial representation (it corresponds to $\mathcal{L}_k$), we derive a factorization

$$\begin{tikzcd}
\mathrm{Gal}' \arrow[r, "\rho'"] \arrow[dr, "\rho"'] & \mathbf{G}'_{m,R} \arrow[d, "\gamma"] \\
& \mathbf{G}_{m,R}
\end{tikzcd}$$

of ρ. Under this factorization, the faithful representation of $\mathbf{G}'_{m,R}$ constructed in Proposition 3.1 is taken to $\mathcal{V}$. In particular, we have a faithful representation of $\mathbf{G}'_{m,R} \otimes k$ which is taken to $\mathcal{V}_k \in \langle \mathcal{L} \rangle_\otimes$. Now we note that $\mathcal{V}_k$ is the module of the logarithm, so that $\langle \mathcal{V}_k \rangle_\otimes \xrightarrow{\sim} \mathrm{Rep}_k(\mathbf{G}_{a,k})$. We arrive at a commutative diagram

$$\begin{tikzcd}
\mathrm{Gal}'_k \arrow[r, "\rho'_k"] \arrow[d] & \mathbf{G}'_{m,R} \otimes k \\
\mathbf{G}_{a,k}. \arrow[ur]
\end{tikzcd}$$

Since the arrow

$$\mathrm{Rep}_k(\mathbf{G}'_{m,R} \otimes k) \longrightarrow \mathrm{Rep}_k(\mathbf{G}_{a,k})$$

takes a faithful representation of $\mathbf{G}'_{m,R} \otimes k$ to a faithful representation of $\mathbf{G}_{a,k}$, we conclude that $\mathbf{G}_{a,k} \to \mathbf{G}'_{m,R} \otimes k$ is a closed embedding. As $\mathbf{G}'_{m,R} \otimes k = \mathbf{G}_{a,k}$ and k is of characteristic zero, $\mathbf{G}_{a,k} \to \mathbf{G}'_{m,R} \otimes k$ is an isomorphism. This guarantees that ρ' is is an isomorphism [WW80, Lemma 1.3, p.551] so that the diptych of ρ (see Section 4) is

$$\begin{tikzcd}
\mathbf{G}'_{m,R} \arrow[r, "\gamma"] & \mathbf{G}_{m,R} \arrow[d, "\mathrm{id}"] \\
\mathrm{Gal}' \arrow[u, "\rho'"] \arrow[r] & \mathbf{G}_{m,R}.
\end{tikzcd}$$

To summarise, $\mathrm{Gal}' = \mathbf{G}'_{m,R}$ and $\mathrm{Gal} = \mathbf{G}_{m,R}$. When reduced modulo π, this gives

$$\begin{tikzcd}
\mathbf{G}_{a,k} \arrow[r] & \mathbf{G}_{m,k} \arrow[d, "\mathrm{id}"] \\
\mathrm{Gal}'_k \arrow[u, "\sim"] \arrow[r, "\rho_k"'] & \mathbf{G}_{m,k},
\end{tikzcd}$$

in which the image of ρ_k is the trivial group [Wa79, 8.3, Corollary, p.65].

Remark 7.10. Note that this example shows that [And01, Theorem 3.3.1.1, p. 729] cannot be true. In fact, what in [And01] is defined as $\mathrm{Gal}(\mathcal{L})$, respectively $\mathrm{Gal}(\mathcal{L}_k)$, is what we here call Gal', respectively $\mathrm{Im}(\rho_k)$, and we have showed that $\mathrm{Gal}'_k \not\simeq \mathrm{Im}(\rho_k)$. It seems that the inconsistency in the proof lies in the definition of the arrow u in [And01, 3.2.3.4, p.727].

8. COINCIDENCE OF GALOIS GROUPS IN THE CASE OF INFLATED CONNECTIONS

We maintain the notations of Section 7. In particular, we assume that k is algebraically closed of characteristic 0, that R contains a copy of k, and that $f : X \to \operatorname{Spec} R$ is a geometrically connected smooth morphism admitting a section $\xi : \operatorname{Spec} R \to X$.

Theorem 8.1. *We assume in addition that* $f : X \to \operatorname{Spec} R$ *is projective. Let* $\mathcal{M} \in \mathbf{MIC}(X/k)$ *be an absolute connection and let* $\mathbf{Inf}(\mathcal{M})$ *be its inflated relative connection. Then the morphism of group schemes over* R

$$\mathrm{Gal}'(\mathbf{Inf}(\mathcal{M})) \longrightarrow \mathrm{Gal}(\mathbf{Inf}(\mathcal{M}))$$

is an isomorphism.

Example 8.2. Here is an example showing that the Theorem 8.1 does not hold if X is simply affine. Let R be the localization of $\mathbf{C}[\pi]$ at the maximal ideal (π), $X = \mathbf{A}^1_R$ and $\mathcal{L} = (\mathcal{O}\mathbf{e}, \nabla)$ be the object of $\mathbf{MIC}(X/\mathbf{C})$ defined by

$$\nabla(\mathbf{e}) = -\mathbf{e} \otimes (\pi\, dx + x\, d\pi).$$

Its inflation is determined by

$$\nabla_{\mathbf{Inf}}(\mathbf{e}) = -\mathbf{e} \otimes \pi\, dx,$$

which is the differential equation for the function $\exp(\pi x)$. This being so, we have

$$\nabla_{\mathbf{Inf}} \sum_{\nu=0}^{n} \frac{(\pi x)^\nu}{\nu!}\mathbf{e} = -\mathbf{e}\pi^{n+1}\frac{x^n}{n!} \otimes dx,$$

which gives rise to an isomorphism

$$\mathbf{Inf}(\mathcal{L}) \otimes R_n \simeq \mathbf{1} \otimes R_n.$$

Using Corollary 2.24, we conclude that $\mathrm{Gal}'(\mathbf{Inf}(\mathcal{L}))$ is the automatic blowup of $\mathrm{Gal}(\mathbf{Inf}(\mathcal{L})) = \mathbf{G}_{m,R}$ at $\{e\}$.

Our proof of Theorem 8.1 depends on two known results (found in [EH06], [dS12], [Zha14]). Since some work has to be done in order to adapt the substance of these theorems to our setting, we prefer to put explain them in Section 9 and give a proof of Theorem 8.1 now.

Proof of Theorem 8.1. To lighten notation, we write M, G and G' in place of $\mathbf{Inf}(\mathcal{M}) \otimes k$, $\mathrm{Gal}(\mathbf{Inf}(\mathcal{M}))$ and $\mathrm{Gal}'(\mathbf{Inf}(\mathcal{M}))$, respectively. It is enough to prove that the functor

$$\mathrm{Rep}_k(G_k) \longrightarrow \langle M \rangle_\otimes \subset \langle \mathbf{Inf}(\mathcal{M}) \rangle_{\otimes,(k)} = \mathrm{Rep}_k(G'_k)$$

is an equivalence (cf. Corollary 4.6(i) and diagram (19)). The proof is a consequence of the following claims.

Claim 1. For each couple of objects $\mathcal{N}, \mathcal{N}'$ of $\mathbf{MIC}(X/k)$ and each arrow

$$\theta : \mathcal{N}|_{X_k} \longrightarrow \mathcal{N}'|_{X_k}$$

of $\mathbf{MIC}(X/k)$, there exists an arrow of $\mathbf{MIC}(X/R)$,

$$\widetilde{\theta} : \mathbf{Inf}(\mathcal{N}) \longrightarrow \mathbf{Inf}(\mathcal{N}'),$$

inducing θ modulo π.

Verification. We view θ as a global horizontal section of

$$(\mathcal{N}|_{X_k})^\vee \otimes \mathcal{N}'|_{X_k};$$

as such, it must belong to the maximal trivial sub-object T of $(\mathcal{N}|_{X_k})^\vee \otimes \mathcal{N}'|_{X_k}$. According to Theorem 9.1(1), there exists $\mathcal{T} \in \mathbf{MIC}(R/k)$, a monomorphism

$$\mu : f^*\mathcal{T} \longrightarrow \mathcal{N}^\vee \otimes \mathcal{N}'$$

which restricts to the monomorphism $T \to (\mathcal{N}|_{X_k})^\vee \otimes \mathcal{N}'|_{X_k}$ mentioned above. Hence, there exists a global basis $\{e_1, \ldots, e_m\}$ of $f^*\mathcal{T}$ such that the connection is determined by

$$e_j \longmapsto \sum_{i=1}^{m} a_{ij} e_i \otimes d\pi, \qquad a_{ij} \in R.$$

In particular, the section θ of T can be lifted to a global section $\widetilde{\theta}$ of $f^*\mathcal{T}$ which is horizontal for the connection $\mathbf{Inf}(f^*\mathcal{T})$. Therefore, we have obtained an arrow

$$\mathbf{Inf}(\mathcal{N}) \longrightarrow \mathbf{Inf}(\mathcal{N}')$$

which, when reduced modulo π, gives back θ.

Claim 2. For each $V \in \langle M \rangle_\otimes$, there exist $\mathcal{E}, \mathcal{E}' \in \langle \mathbf{Inf}(\mathcal{M}) \rangle_\otimes^s$ and an arrow in $\mathbf{MIC}(X/R)$

$$\widetilde{\theta} : \mathcal{E} \longrightarrow \mathcal{E}'$$

such that

$$V \simeq \operatorname{Coker}\left(\widetilde{\theta}_k : \mathcal{E}_k \longrightarrow \mathcal{E}'_k\right).$$

Verification. According to Theorem 9.1(2), the connection V on the special fibre X_k can be "almost lifted" to a connection on the whole X. Precisely, we can find

$$\mathcal{N}, \mathcal{N}' \in \langle \mathcal{M} \rangle_\otimes$$

(recall that $\langle \mathcal{M} \rangle_\otimes \subset \mathbf{MIC}(X/k)$) and an exact sequence in $\mathbf{MIC}(X_k/k)$:

$$\mathcal{N}|_{X_k} \xrightarrow{\theta} \mathcal{N}'|_{X_k} \longrightarrow V \longrightarrow 0.$$

Using the above Claim, the arrow θ is the reduction of and arrow $\widetilde{\theta}$ from $\mathbf{Inf}(\mathcal{N})$ to $\mathbf{Inf}(\mathcal{N}')$. We then take $\mathcal{E} = \mathbf{Inf}(\mathcal{N})$ and $\mathcal{E}' = \mathbf{Inf}(\mathcal{N}')$.

Claim 3. Denote by η the composition of functors

$$\mathrm{Rep}_R(G) \longrightarrow \mathrm{Rep}_R(G') \xrightarrow{\sim} \langle \mathbf{Inf}(\mathcal{M}) \rangle_\otimes.$$

For each $V \in \mathrm{Rep}_k(G_k)$, there exists $N \in \mathrm{Rep}_R(G)^o$ such that

(a) V is a quotient of N_k and

(b) $\eta(N)$ is of the form $\mathbf{Inf}(\mathcal{N})$ for some $\mathcal{N} \in \langle \mathcal{M} \rangle_\otimes$.

Verification. According to [Se68, Proposition 3, p.41] we can "almost lift" V. Precisely, there exists $E \in \mathrm{Rep}_R(G)^o$ and a surjection $E_k \to V$. By means of the equivalence

$$\eta : \mathrm{Rep}_R(G)^o \xrightarrow{\sim} \langle \mathbf{Inf}(\mathcal{M}) \rangle_\otimes^s$$

of Proposition 7.8, we can find a diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r] \arrow[d] & \mathcal{T} \\
\eta(E), &
\end{tikzcd}$$

where $\mathcal{T}$ is some tensor power of $\mathbf{Inf}(\mathcal{M})$ and the vertical arrow is special (see Definition 7.4). Theorem 9.2 says that any special sub-object of an inflated connection must "almost" come from an inflation itself. Precisely, there exists $\mathcal{N} \in \langle \mathcal{M} \rangle_{\otimes}$ and an epimorphism

$$\mathbf{Inf}(\mathcal{N}) \longrightarrow \mathcal{F}.$$

A moment's thought using the fact that the $\mathcal{O}_X$-modules underlying objects of $\mathbf{MIC}(X/k)$ shows that $\mathbf{Inf}(\mathcal{N})$ must belong to $\langle \mathbf{Inf}(\mathcal{M}) \rangle_{\otimes}^{s}$; the above equivalence then produces the desired $N \in \mathrm{Rep}_R(G)^o$.

Claim 4. The functor

$$\eta_k : \mathrm{Rep}_k(G) = \mathrm{Rep}_R(G)_{(k)} \longrightarrow \langle \mathbf{Inf}(\mathcal{M}) \rangle_{\otimes,(k)}$$

is full.

Verification. Let $\varphi : \eta_k(V) \to \eta_k(V')$ be a morphism in $\mathbf{MIC}(X_k/k)$. It then fits into a commutative diagram

$$\begin{tikzcd}
\eta(N) \otimes k \arrow[r, "\theta"] \arrow[d, two heads] & \eta(N') \otimes k \\
\eta_k(V) \arrow[r, "\varphi"'] & \eta_k(V'), \arrow[u, hook]
\end{tikzcd}$$

where $N = \mathbf{Inf}(\mathcal{N})$ and $N' = \mathbf{Inf}(\mathcal{N}')$ are as in Claim 3. Claim 1 gives us a lift

$$\widetilde{\theta} : \eta(N) \longrightarrow \eta(N')$$

of θ. As in the proof of Claim 3, both $\mathbf{Inf}(\mathcal{N})$ and $\mathbf{Inf}(\mathcal{N}')$ lie in $\langle \mathbf{Inf}(\mathcal{M}) \rangle_{\otimes}^{s}$. Since η is an equivalence between $\mathrm{Rep}_R(G)^o$ and $\langle \mathbf{Inf}(\mathcal{M}) \rangle_{\otimes}^{s}$, there exists $\sigma : N \to N'$ such that $\eta(\sigma) = \widetilde{\theta}$. Since the vertical arrows in the above diagram also belong to the image of η_k, the proof of the claim is finished. □

9. ADAPTING TWO KNOWN RESULTS TO OUR SETTING

Our goal in this section is to adapts two known results to the setting of Theorem 8.1.

Theorem 9.1. *Let X be a proper and smooth R-scheme with geometrically connected fibres. Let $\mathcal{M}$ belong to $\mathbf{MIC}(X/k)$.*

1. *The maximal trivial subobject of $\mathcal{M}|_{X_k}$ (this belongs to $\mathbf{MIC}(X_k/k)$) is the restriction of a sub-object $\mathcal{T} \subset \mathcal{M}$. Moreover, $\mathcal{T}$ is the pull-back to $\mathbf{MIC}(X/k)$ of an object of $\mathbf{MIC}(R/k)$.*
2. *If N belongs to $\langle \mathcal{M}|_{X_k} \rangle_{\otimes}$, then there exists $\mathcal{N}$ in $\langle \mathcal{M} \rangle_{\otimes}$ and a monic $N \to \mathcal{N}|_{X_k}$.*

Proof. There exists a smooth and projective morphism of smooth irreducible k-schemes $\mathbf{f} : \mathbf{X} \to \mathbf{S}$ together with a morphism $\mathrm{Spec}\, R \to \mathbf{S}$ giving back f. This is explained in EGA IV$_3$, 8.8.2 and 8.10.5, and in EGA IV$_4$, 17.7.8. Moreover, since the set of points $s \in \mathbf{S}$ where $\mathbf{f}^{-1}(s)$ is geometrically integral is open [EGA IV$_3$, 12.2.4], we can assume that $\mathbf{f}$ has geometrically integral fibres. We can also assume that $\mathcal{M}$ comes from $\mathbf{MIC}(\mathbf{X}/k)$. Now, if $\Pi(-)$ stands for the affine group scheme associated to the category $\mathbf{MIC}(-/k)$, then the sequence of fundamental group schemes

$$\Pi(\mathbf{X}_{s_0}) \longrightarrow \Pi(\mathbf{X}) \longrightarrow \Pi(\mathbf{S}) \longrightarrow 1$$

is exact (cf. [Zha14, Theorem 1.1] or [dS12, Theorem 1]). Using the characterisation of exactness presented in [EHS07, Appendix], we immediately arrive at the desired conclusion. Moreover, the proof in loc.cit. shows that $\mathcal{N}$ can be chosen in $\langle \mathcal{M} \rangle_{\otimes}$. □

Theorem 9.2 (Cf. [EH06], Theorem 5.10)**.** *Let* X *be a smooth and geometrically connected* R*-scheme. Let* $\mathcal{M} \in \mathbf{MIC}(X/k)$ *be an absolute connection and*

$$\mathcal{V} \longrightarrow \mathbf{Inf}(\mathcal{M})$$

a special monic of $\mathbf{MIC}(X/R)$*. Then there exists an absolute connection* $\widetilde{\mathcal{V}} \in \mathbf{MIC}(X/k)$ *together with an epimorphism*

$$\mathbf{Inf}(\widetilde{\mathcal{V}}) \longrightarrow \mathcal{V}$$

of relative connections. Furthermore, $\widetilde{\mathcal{V}}$ *may be chosen in* $\langle\mathcal{M}\rangle_\otimes$*.*

Proof. The idea is to apply Lemma 9.3 to the socle series obtained from Lemma 9.4. We begin by assuming that $\mathcal{V}$ is of rank one as an $\mathcal{O}_X$-module.

Employing the notation of Lemma 9.4, define $\mathrm{Soc}_1(\mathcal{V}) := \mathrm{Soc}(\mathcal{V})$. If Soc_i is defined, put

$$\mathrm{Soc}_{i+1}(\mathcal{V}) = \begin{array}{l}\text{inverse image in } \mathbf{Inf}(\mathcal{M}) \text{ of} \\ \mathrm{Soc}(\mathcal{V}) \subset \mathbf{Inf}(\mathcal{M})/\mathrm{Soc}_i(\mathcal{V}),\end{array}$$

so that

$$\mathrm{Soc}_{i+1}(\mathcal{V})/\mathrm{Soc}_i(\mathcal{V}) \simeq \mathrm{Soc}(\mathcal{V}).$$

Using that the socle is a special sub-object (Lemma 9.4) we see that $\mathrm{Soc}_i(\mathcal{V}) \subset \mathbf{Inf}(\mathcal{M})$ is special so that for some $r \in \mathbf{N}$,

$$\cdots \subset \mathrm{Soc}_i(\mathcal{V}) \subset \mathrm{Soc}_{i+1}(\mathcal{V}) \subset \cdots$$

becomes stationary. Due to the definition of $\mathrm{Soc}(\mathcal{V})$, we conclude that there are no submodules of

$$\mathbf{Inf}(\mathcal{M})/\mathrm{Soc}_r(\mathcal{V})$$

isomorphic to $\mathcal{V}$. The assumption on the rank of $\mathcal{V}$ and Proposition 7.2 then force all arrows

$$\mathcal{V} \longrightarrow \mathbf{Inf}(\mathcal{M})/\mathrm{Soc}_r(\mathcal{V})$$

in $\mathbf{MIC}(X/R)$ to be null. Using that

$$\mathrm{Soc}_1(\mathcal{V}), \quad \frac{\mathrm{Soc}_2(\mathcal{V})}{\mathrm{Soc}_1(\mathcal{V})}, \quad \ldots$$

are all isomorphic to direct sums of $\mathcal{V}$, we see that any arrow

$$\mathrm{Soc}_r(\mathcal{V}) \longrightarrow \mathbf{Inf}(\mathcal{M})/\mathrm{Soc}_r(\mathcal{V})$$

is null. By Lemma 9.3, the relative connection $\mathrm{Soc}_r(\mathcal{V})$ is an inflation, i.e. there exists an absolute connection $\widetilde{\mathcal{V}}$ together with a monomorphism $\widetilde{\mathcal{V}} \to \mathcal{M}$ in $\mathbf{MIC}(X/k)$ such that $\mathbf{Inf}(\widetilde{\mathcal{V}}) \to \mathbf{Inf}(\mathcal{M})$ is our monic $\mathrm{Soc}_r(\mathcal{V}) \to \mathbf{Inf}(\mathcal{M})$. Since $\mathcal{V}$ is a quotient of $\mathrm{Soc}_r(\mathcal{V}) = \mathbf{Inf}(\widetilde{\mathcal{V}})$ and since $\widetilde{\mathcal{V}}$ is a sub-object of $\mathcal{M}$, we are done.

The general case follows from the fact that if $m = \mathrm{rank}(\mathcal{V})$, then

$$\left\{\wedge^{m-1}\mathcal{V}\right\}^\vee \otimes \det(\mathcal{V}) \simeq \mathcal{V},$$

which is a quotient of $\mathbf{Inf}(\mathcal{M})^\vee \otimes \det(\mathcal{V})^{\widetilde{}}$. □

Lemma 9.3. *Let* X *be a smooth* R*-scheme. Let* $\mathcal{E} \to \mathbf{Inf}(\mathcal{M})$ *be a monomorphism of* $\mathbf{MIC}(X/R)$*. If*

$$\mathrm{Hom}_{\mathbf{MIC}(X/R)}(\mathcal{E}, \mathbf{Inf}(\mathcal{M})/\mathcal{E}) = 0,$$

then the relative connection on $\mathcal{E}$ *extends to an absolute connection such that the arrow* $\mathcal{E} \to \mathcal{M}$ *is a morphism of* $\mathbf{MIC}(X/k)$*.*

Proof. Let us assume that X is $\operatorname{Spec} A$ and that $\Omega^1_{X/R}$ is free on $dx_1, \ldots, dx_n$. From the exact sequence [EGA IV$_4$, 17.2.3]

$$0 \longrightarrow A \otimes_R \Omega^1_{R/k} \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/R} \longrightarrow 0$$

we conclude that $\Omega^1_{X/k}$ is free on $d\pi$ and $dx_1, \ldots, dx_n$. Write ∂_i for the R-linear derivation of A induced by dx_i and ∂_π for the derivation of A induced by $d\pi$, that is, $\partial_\pi(dx_i) = 0$, $\partial_\pi(d\pi) = 1$, etc. Note that

$$[\partial_i, \partial_j] = [\partial_i, \partial_\pi] = 0. \tag{24}$$

Now, if $\mathcal{E} \to \mathcal{M}$ is a monomorphism of **MIC**(X/R), we define

$$\varphi : \mathcal{E} \longrightarrow \mathcal{M}/\mathcal{E},$$

$$e \longmapsto \partial_\pi(e) \mod \mathcal{E}.$$

The Leibniz rule shows that φ is A-linear and the equations (24) prove that φ is also an arrow of **MIC**(X/R). We note that if $x'_1, \ldots, x'_n$ are elements of A whose differentials $dx'_1, \ldots, dx'_n$ generate $\Omega^1_{X/R}$, then the arrow $\varphi' : \mathcal{E} \to \mathcal{M}/\mathcal{E}$ obtained by means of the derivations ∂'_i and ∂'_π is identical with φ. Therefore, the assumption $X = \operatorname{Spec} A$ can be abandoned and we obtain an arrow in **MIC**(X/R) from $\mathcal{E}$ to $\mathcal{M}/\mathcal{E}$.

By hypothesis, $\varphi = 0$ and we conclude that for any open set $\operatorname{Spec} A$ and any ∂_π as above, we have $\partial_\pi(\mathcal{E}) \subset \mathcal{E}$, which proves that the relative connection on $\mathcal{E}$ extends to an absolute connection such that $\mathcal{E} \to \mathcal{M}$ is horizontal. □

Lemma 9.4. *Let X be a smooth R-scheme having connected fibres. Let $\mathcal{L}$ and $\mathcal{E}$ be relative connections. Assume that $\mathcal{E}$, respectively $\mathcal{L}$, is locally free, respectively locally free of rank one, as an $\mathcal{O}_X$-module. Define*

$$\operatorname{Soc}(\mathcal{L}) \subset \mathcal{E}$$

as the sum of all sub-objects of $\mathcal{E}$ which are isomorphic to $\mathcal{L}$. The following properties are true.

1. *The sub-object* $\operatorname{Soc}(\mathcal{L})$ *of $\mathcal{E}$ is special.*
2. *In* **MIC**(X/R) *we have* $\operatorname{Soc}(\mathcal{L}) \simeq \mathcal{L}^{\oplus r}$ *for some* r.

Proof. Assume that $\mathcal{L}$ is the *trivial* relative connection. Let $V \subset X$ be an affine open meeting the special fibre. Let $s \in \mathcal{E}(V)$ be a horizontal section having the form πs_1 for some $s_1 \in \mathcal{E}(V)$. Obviously, s_1 is then horizontal. From Lemma 9.5 below, there exists a *global* horizontal section $\widetilde{s}_1$ such that $\widetilde{s}_1|_V = s_1$. This means that $\widetilde{s}_1$ is a section of $\operatorname{Soc}(\mathcal{L})$, and hence s_1 is a section of $\operatorname{Soc}(\mathcal{L})$. This proves (1) in our particular case. Let us now address (2), still assuming $\mathcal{L}$ to be trivial. From what we proved before, it is clear that each monic $s : \mathcal{L} \to \mathcal{E}$ in **MIC**(X/R) must be of the form $\pi^m s_1 : \mathcal{L} \to \mathcal{E}$ for some special monic $s_1 \in \operatorname{Hom}_{\mathbf{MIC}(X/R)}(\mathcal{L}, \mathcal{E})$. Consequently, $\operatorname{Soc}(\mathcal{L})$ is the sum of all special sub-objects of $\mathcal{E}$ isomorphic to $\mathcal{L}$. Now let $\mathcal{L}_1 \subset \mathcal{E}$ and $\mathcal{L}_2 \subset \mathcal{E}$ be special sub-objects isomorphic to $\mathcal{L}$. Then $\mathcal{L}_1 \cap \mathcal{L}_2$ is a special sub-object of $\mathcal{L}_1$ and of $\mathcal{L}_2$, so that Proposition 7.2 forces $\mathcal{L}_1$ to equal $\mathcal{L}_2$ or $\mathcal{L}_1 \cap \mathcal{L}_2 = (0)$. We conclude that there are only finitely many special distinct sub-objects $\mathcal{L}_1, \ldots, \mathcal{L}_r$ which are isomorphic to $\mathcal{L}$. It then follows that $\operatorname{Soc}(\mathcal{L}) \simeq \mathcal{L}^{\oplus r}$.

The general case is treated by employing the relative connection $\mathcal{L}^\vee \otimes \mathcal{E}$. □

Lemma 9.5. *Let X be a smooth R-scheme having connected fibres. Let $\mathcal{E}$ be a relative connection whose underlying $\mathcal{O}_X$-module is locally free. Let s be a horizontal section over some* *non-empty open* V *of* X *which meets the special fibre* X_k*. Then there exists a unique global horizontal section* $\widetilde{s}$ *inducing* s*. Said differently, the restriction arrow*

$$\mathrm{Hom}_{\mathbf{MIC}(X/R)}(\mathbf{1},\mathcal{E})\longrightarrow \mathrm{Hom}_{\mathbf{MIC}(V/R)}(\mathbf{1}|_V,\mathcal{E}|_V)$$

is bijective.

Proof. This is essentially in the literature, but since we are unable to find a reference with a similar set up, we prefer to give a proof. The reason lies in the fact that the closed subscheme of singularities of s will be "invariant" under the action of vector fields.

We begin by noting that, due to the uniqueness statement, it is sufficient to assume $X = \mathrm{Spec}\,A$. We then define $I = \{\varphi \in A \,:\, \varphi|_V \cdot s \text{ extends to } X\}$. It now follows that for each $\partial \in \mathrm{Der}_R(A)$, we have $\partial I \subset I$. Since X_K is connected, we must have $IA_K = A_K$, which means that $\sqrt{I} \supset \sqrt{(\pi)}$. Therefore, there exists some global section $\widetilde{t}$ of $\mathcal{E}$ whose restriction to V is $\pi^m s$.

We claim that in this case $\widetilde{t} = \pi^m\widetilde{s}$ for some global section $\widetilde{s}$. With this goal, we observe that if $U \subset X$ is an affine open and $\mathfrak{p} \in U \cap V \cap X_k$, then $\pi^m\mathcal{O}(U)_{\mathfrak{p}} \cap \mathcal{O}(U) = \pi^m\mathcal{O}(U)$ because the ideal $\pi\mathcal{O}(U)$ is prime and π is not a zero divisor. Hence, if U is an open affine meeting X_k on which $\mathcal{E}$ is free, it follows that $\widetilde{t}|_U = \pi^m\widetilde{s}_U$ for some $\widetilde{s}_U \in \mathcal{E}(U)$. Moreover, $\widetilde{s}_U$ is unique for such a property. In this way we arrive at the desired section $\widetilde{s}$. Moreover, local freeness of $\mathcal{E}$ again guarantees that $\widetilde{s}$ is horizontal. □

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