Julia set of a polynomial and its equilibrium measure

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Abstract. We compute the capacity of the Julia set of a polynomial and prove the invariance of the equilibrium measure (of the Julia set). We also prove a necessary condition for two polynomials of the same degree to have the same Julia set.

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1. INTRODUCTION

Let $P = az^d + \cdots$ denote a polynomial of degree $d > 1$. Denote by $P^n = \underbrace{P \circ P \circ \cdots \circ P}_{n}$ the $n-$th iterate of P. The Julia set $J = J(P)$ of P is compact and nonempty. The Hausdorff dimension of J is positive ($\geq \ln d / \ln K_0$, with $K_0 = \max\{|P'(x)| : x \in J\} > 1$) [2]. Let $\mathbb{C}_\infty$ denote the Riemann sphere and $A(\infty) = \left\{ z \in \mathbb{C}_\infty : \lim_{n\to\infty} P^n(z) = \infty \right\}$. Then $\partial A(\infty) = J$. Let μ denote the equilibrium measure of J. For example, if $P = z^d$ then $J(P)$ is the unit circle, $\text{cap}(J) = 1$ and $\mu = \frac{d\theta}{2\pi}$; if $P = T_d$ (Chebisev polynomial of the first kind) then $J(P) = [-1, 1]$, $\text{cap}(J) = 1/2$ and $\mu = \frac{dx}{\pi\sqrt{1-x^2}}$. Brolin [3] proved that for monic polynomial $P = z^d + \cdots$, the sequence of zero counting measures of $P^n - a$ converges weakly to the equilibrium measure of $J = J(P)$ for any fixed $a \in \mathbb{C}$. Moreover, the capacity of $J(P)$ is 1 and $J(P)$ is completely invariant under $T : z \to P(z)$, i.e., $T(J) = T^{-1}(J) = J$. On the other hand, as a transformation on $J = J(P)$ the transformation T is strongly mixing and measure preserving, $\mu(T^{-1}(E)) = \mu(E)$ for $E \subset J$ in this case. In this paper, we will prove that

$$\text{cap}(J) = \sqrt[d-1]{\frac{1}{|a|}}$$

for all $P = az^d + \cdots$ and the equilibrium measure of $J = J(P)$ is invariant under T. Recall that the logarithmic energy of a probability measure μ is

$$I(\mu) = \iint \ln \frac{1}{|z-w|} d\mu(z)\, d\mu(w).$$

The equilibrium measure is the only probability measure with minimal energy [4]. Moreover, for the equilibrium measure μ of $J(P)$ we have $I(\mu) = -\ln \text{cap}(J)$ and

$$\int \ln \frac{1}{|z-w|} d\mu(z) = I(\mu),$$

for q.e. $w \in J$.

2. Main Results

Theorem 1. Let $P = az^d + \cdots$ denote a polynomial of degree $d > 1$. Then the Julia set $J = J(P)$ of P is compact, the equilibrium measure of J is invariant under P and

$$\text{cap}(J) = \sqrt[d-1]{\frac{1}{|a|}}. \tag{1}$$

Proof. Let μ denote the equilibrium measure of J. Define the probability measure $P(\mu)$ by letting $P(\mu)\varphi = \mu(\varphi \circ P)$ for every continuous function $\varphi : J \to \mathbb{R}$. This is really probability measure by Riesz representation theorem. Moreover, the logarithmic energy of $P(\mu)$ is

$$\begin{aligned} I(P(\mu)) &= \iint \ln \frac{1}{|P(z) - P(w)|} d\mu(z)\, d\mu(w) \\ &= \ln \frac{1}{|a|} + dI(\mu) \geq I(\mu) \end{aligned}$$

since μ has minimal energy. Thus,

$$I(\mu) \geq \frac{1}{d-1} \ln |a|$$

and consequently,

$$\mathrm{cap}(J) \leq \sqrt[d-1]{\frac{1}{|a|}}.$$

The equality holds if and only $P(\mu) = \mu$ (μ is invariant under P). Let $E_0 \supset J$ denote a simply connected closed set such that $\mathbb{C}_\infty \backslash E_0 \subset A(\infty)$. Then $\mathbb{C}_\infty \setminus E_0$ is invariant. Put $E_n = \{z : P(z) \in E_{n-1}\}$ for $n = 1, 2, \cdots$. Then $E_0 \supseteq E_1 \supseteq \cdots \supseteq E_n \supseteq \cdots \supseteq J$, $\lim_{n\to\infty} \partial E_n = J$ and consequently, $\mathrm{cap}(J) = \lim\limits_{n\to\infty} \mathrm{cap}(E_n)$. Here, we assume without loss of generality that the boundary ∂E_0 of E_0 is a Jordan curve. Let $g_n(z, \infty)$ denote the Green function of $\mathbb{C}_\infty \backslash E_n$ singular at infinity. Then $g_{n+1}(z, \infty) = g_n(P(z), \infty)/d$ and $g_n(z, \infty) = \ln |z| - \ln \mathrm{cap}(E_n) + o(1)$ as $z \to \infty$. Hence,

$$\begin{aligned} g_n(P(z), \infty) &= \ln |P(z)| - \ln \mathrm{cap}(E_n) + o(1) \\ &= d \ln |z| + \ln |a| - \ln \mathrm{cap}(E_n) + o(1) \\ &= d g_{n+1}(z, \infty) = d \ln |z| - d \ln \mathrm{cap}(E_{n+1}) + o(1). \end{aligned}$$

Therefore, $\mathrm{cap}(E_n) = \ln |a| + d \ln \mathrm{cap}(E_{n+1})$. Let $n \to \infty$ we get $\mathrm{cap}(J) = \sqrt[d-1]{\frac{1}{|a|}}$. The proof is now complete.

Theorem 2. Let $\{\varphi_n = \kappa_n z^n + \cdots : \kappa_n > 0\}_{n=0}^\infty$ denote the orthonormal polynomial basis of $L^2(J, \mu)$. Here J is the Julia set of a polynomial $P(z) = az^d + bz^{d-1} + \cdots$ of degree $d > 1$ and μ is the equilibrium measure of J. Then

$$\varphi_n \circ P = \frac{a^n}{|a|^n} \varphi_{nd}, \qquad \kappa_{nd} = \kappa_n |a|^n, \qquad \kappa_{d^n} = \kappa_1 |a|^{d^{n-1} + \cdots + d + 1}. \tag{2}$$

Moreover,

$$\varphi_1(z) = \kappa_1\big(z + \frac{b}{ad}\big), \quad \varphi_{d^n} = \kappa_1\big(\underbrace{P \circ P \circ \cdots \circ P}_{n} + \frac{b}{ad}\big)\big(\frac{|a|}{a}\big)^{d^{n-1}+\cdots+d+1}.$$

Proof. Since μ is invariant under P, we get

$$\int_J \varphi_n \circ P\,(z)\,\overline{\varphi_m \circ P\,(z)} d\mu = \delta\,(n-m)$$

Moreover, $\sum\limits_{P(\alpha)=z} \alpha^m$ is independent of z for $m = 1, 2, \cdots, d-1$ and consequently,

$$\int_J z^m \overline{\varphi_1 \circ P(z)} d\mu = \frac{1}{d}\int_J \big(\sum_{P(\alpha)=z} \alpha^m\big)\overline{\varphi_1(z)} d\mu = \frac{1}{d}\big(\sum_{P(\alpha)=z} \alpha^m\big)\int_J \overline{\varphi_1(z)} d\mu = 0$$

for $m = 1, 2, \cdots, d-1$ (see Theorem 2 in [1]). Thus,

$$\varphi_1 \circ P = \frac{a}{|a|}\varphi_d$$

and by induction on n we can finish the proof of (2). To compute φ_1 note that

$$\int_J z d\mu = \frac{1}{d}\sum_{P(\alpha)=z} \alpha = -\frac{b}{da}.$$

The proof is now complete.

Remark. The mapping $P : \varphi \to \varphi \circ P$ is an unitary operator of the Hilbert space $L^2\,(J, \mu)$ and

$$\langle P^m \varphi_n, \varphi_n \rangle = 0$$

for all $m, n = 1, 2, \cdots$. For example, if $P = z^d$ then $J\,(P)$ is the unit circle, $\mu = P\,(\mu) = \frac{d\theta}{2\pi}$, and $\varphi_n = z^n$. If $P = T_d$ (Chebisev polynomial of the first kind) then $J\,(P) = [-1, 1]$ and $\mu = P\,(\mu) = \frac{dx}{\pi\sqrt{1-x^2}}$, $\varphi_n = \sqrt{2}T_n$. Both of mappings $z \to z^d$ and $z \to T_d(z)$ have exactly two invariant measures (the Dirac δ at $z = 1$ and the equilibrium measure).

Theorem 3. If two polynomials $P_1 = a_1 z^d + \cdots$ and $P_2 = a_2 z^d + \cdots$ have the same degree and Julia set then there is a point ξ in the unit circle such that $a_1 = \xi a_2$ and $P_1 = \xi P_2 + P_1(0) - \xi P_2(0)$. In this case the Julia set is invariant under the transformation $z \to \xi z + P_1(0) - \xi P_2(0)$. If $\xi = 1$ then $P_1 = P_2$.

Proof. Let J denote the common Julia set and μ the equilibrium measure of J. Then

$$\int_J z^m d\mu = \frac{1}{d} \sum_{P_1(\alpha)=0} \alpha^m = \frac{1}{d} \sum_{P_2(\alpha)=0} \alpha^m.$$

Thus, the non-constant coefficients of P_1/a_1 and P_2/a_2 are the same. On the other hand, it follows from Theorem 1 that $|a_1| = |a_2|$ so $P_1 = \xi P_2 + P_1(0) - \xi P_2(0)$. Moreover, if $z \in J$ then there is $\eta \in J$ such that $P_2(\eta) = z$. Therefore, $\xi z + P_1(0) - \xi P_2(0) = P_1(\eta) \in J$. The proof is now complete.

Theorem 4. If two polynomials $P_1 = a^{d-1}z^d + \cdots$ and $P_2 = a^{d^n-1}z^{d^n} + \cdots$ have the same Julia set then $P_2 = \underbrace{P_1 \circ P_1 \circ \cdots \circ P_1}_{n}$.

Proof. The Julia set of P_1 and $\underbrace{P_1 \circ P_1 \circ \cdots \circ P_1}_{n}$ are the same and P_2 and $\underbrace{P_1 \circ P_1 \circ \cdots \circ P_1}_{n}$ have the same degree and the highest coefficient. By Theorem 3 we get the theorem proved.

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