Recovering a polynomial of a given degree from its Julia set

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Abstract. We prove several interesting conditions of polynomials with real Julia set. We also extend excellent results in [1, 2] about monic polynomials of real coefficients to arbitrary polynomials.

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1. INTRODUCTION

Let $P = az^d + bz^{d-1} + \cdots$ denote a polynomial of degree $d > 1$. Denote by $P^n = \underbrace{P \circ P \circ \cdots \circ P}_{n}$ the $n-$th iterate of P. The Julia set $J = J(P)$ of P is compact and nonempty. More exactly, J is the boundary of the set [4]

$$A(\infty) = \left\{ z \in \mathbb{C}_\infty : \lim_{n\to\infty} P^n(z) = \infty \right\}.$$

Here, $\mathbb{C}_\infty$ denotes the Riemann sphere. The Hausdorff dimension of J is positive ($\geq \ln d/\ln K_0$, with $K_0 = \max\{|P'(x)| : x \in J\} > 1$) [3]. Recently, we proved that the logarithmic capacity of J is $\sqrt[d-1]{\frac{1}{|a|}}$ and μ (the equilibrium measure) of J is invariant under P [5]. We refer [6] for readers interested in potential theory (capacity, equilibrium measure of a compact set). The following theorem is also proved in [5]

Theorem 0. If P_1 and P_2 are two polynomials of the same degree having the same Julia set then there exist $\xi \in \partial\mathbb{D}$ (the unit circle) and $c \in \mathbb{C}$ such that $P_1 = \xi P_2 + c$. In this case, the common Julia set is invariant under the transformation $z \to \xi z + c$. Hence, if $\xi = 1$ then $c = 0$. If the common Julia set is real then $\xi = \pm 1$.

Now we can recover the polynomial of a given degree $d > 1$ from its Julia set as follows. Let $\{\varphi_n = \kappa_n z^n + \cdots : \kappa_n > 0\}_{n=0}^{\infty}$ denote the orthonormal polynomial basis of $L^2(J,\mu)$. We also prove in [5] that

$$\varphi_1 = \kappa_1\big(z + \frac{b}{da}\big), \qquad \varphi_d(z) = \kappa_1(P(z) + \frac{b}{da})\frac{|a|}{a}$$

so

$$P(z) = \frac{a}{|a|\,\kappa_1}\varphi_d(z) - \frac{b}{da}.$$

Moreover,

$$\varphi_n \circ P = \frac{a^n}{|a|^n}\varphi_{nd}, \qquad \kappa_{nd} = \kappa_n|a|^n \qquad \kappa_{d^n} = \kappa_1|a|^{d^{n-1}+\cdots+d+1}$$

$$\varphi_{d^n} = \kappa_1\left(\underbrace{P \circ P \circ \cdots \circ P}_{n} + \frac{b}{da}\right)\big(|a|/a\big)^{d^{n-1}+\cdots+d+1}.$$

Example 1. Let $J = R\partial\mathbb{D}$ (the circle of radius R). Then $\operatorname{cap}(J) = R$, $\mu = \frac{d\theta}{2\pi}$ and $\varphi_n = z^n$. Thus, $|a| = 1/R^{d-1}$, $b = 0$ and $P(z) = \xi z^d/R^{d-1}$, where ξ is fixed on the unit circle.

2. Main Result.

In this paper we consider the case if J is real. We will extend excellent results in [1, 2]. First of all, we get a recurrence formula for real orthogonal polynomials $\{\varphi_n\}_{n=0}^{\infty}$:

$$\alpha_{n+1}\varphi_{n+1} + \beta_n\varphi_n + \alpha_{n-1}\varphi_{n-1} = x\varphi_n.$$

Here,

$$\alpha_n = \int_J x\varphi_n(x)\varphi_{n-1}(x)d\mu(x) = \frac{\kappa_{n-1}}{\kappa_n}$$

and

$$\beta_n = \int_J x\varphi_n^2(x)d\mu(x).$$

But $\kappa_{nd} = \kappa_n|a|^n$ so

$$\alpha_n = \frac{|a|\kappa_{nd-d}}{\kappa_{nd}} = |a|\alpha_{nd}\alpha_{nd-1}\cdots\alpha_{nd-d+1}$$

It is also prove in [5] that

$$\int_J x^m d\mu(x) = \frac{1}{d}\sum_{P(\lambda)=0}\lambda^m$$

for $m = 1, 2, \cdots, d-1$. Therefore, $\frac{P-P(0)}{a}$ is a monic polynomial with real coefficients. Hence, we have

Theorem 1. If the Julia set of a polynomial P is real then $P = aQ + P(0)$, where Q is a monic polynomial with real coefficients.

Example 2. Let $P = T_d$ (the $d-$th Chebisev polynomial of the first kind). Then

$$A(\infty) = \mathbb{C}_\infty \setminus [-1, 1]$$

and $J = [-1, 1]$. The equilibrium measure of J is

$$\mu = \frac{1}{\pi\sqrt{1-x^2}}.$$

and $\mathrm{cap}(J) = 1/2$. Moreover, $\varphi_n = \sqrt{2}T_n$ for $n = 1, 2, \cdots$.

Theorem 2. If the Julia set J of a polynomial $P = az^d + bz^{d-1} + \cdots$ of degree $d > 1$ is a compact interval $[t_1, t_2]$ then $a = \pm\left(\frac{4}{t_2-t_1}\right)^{d-1}$, $b = -da(t_1+t_2)/2$ and

$$P(x) = \pm\frac{t_2-t_1}{2}\cdot T_d\left(\frac{2x-(t_1+t_2)}{t_2-t_1}\right) + \frac{t_1+t_2}{2}.$$

Moreover, we can compute t_1 and t_2 from the coefficients (a,b) of P as follows

$$t_1 = -\left(\frac{b}{ad} + 2\cdot \sqrt[d-1]{\frac{1}{|a|}}\right), \qquad t_2 = 2\cdot \sqrt[d-1]{\frac{1}{|a|}} - \frac{b}{ad}.$$

Proof. It is well known that the equilibrium measure of J is

$$\mu = \frac{dx}{\pi\sqrt{(x-t_1)(t_2-x)}} \qquad \text{and} \qquad \operatorname{cap}(J) = \frac{|t_1-t_2|}{4} = \sqrt[d-1]{\frac{1}{|a|}}.$$

Hence, the orthonormal polynomials are

$$\varphi_n(x) = \sqrt{2}T_n\left(\frac{2x-(t_1+t_2)}{t_2-t_1}\right)$$

for $n = 1, 2, \cdots$. On the other hand,

$$\varphi_1(x) = \kappa_1\big(x + \frac{b}{da}\big) = \sqrt{2}\cdot\frac{2x-(t_1+t_2)}{t_2-t_1}$$

and

$$\varphi_d\left(x\right) = \kappa_1(P\left(x\right) + \frac{b}{da})\frac{|a|}{a} = \sqrt{2}T_d\left(\frac{2x-(t_1+t_2)}{t_2-t_1}\right),$$

so we get

$$\kappa_1 = \frac{2\sqrt{2}}{t_2-t_1}, \quad b = -\frac{da(t_1+t_2)}{2}$$

and

$$P\left(x\right) = \frac{t_2-t_1}{2}\frac{a}{|a|}T_d\left(\frac{2x-(t_1+t_2)}{t_2-t_1}\right) + \frac{t_1+t_2}{2}.$$

On the other hand,

$$P\left(x\right) = \frac{t_2-t_1}{2}T_d\left(\frac{2x-(t_1+t_2)}{t_2-t_1}\right) + \frac{t_1+t_2}{2}$$

is the polynomial of degree d having Julia set $J = [t_1, t_2]$ which is invariant under $z \to -z + t_1 + t_2$. Hence, it follows from Theorem 0 that another polynomial of degree d having $J = [t_1, t_2]$ as its the Julia set is the only $-P + t_1 + t_2$. The last assertion follows directly from the fact that $b = -da(t_1+t_2)/2$. We fiinish the proof here.

Theorem 3. Let $J = [-t_2, -t_1] \cup [t_1, t_2]$. Here, $0 < t_1 < t_2$. The polynomial of degree $2d$ having J as its Julia set is

$$\pm\sqrt{t_2^2 + t_1^2} \cdot T_d\left(\frac{2x^2 - (t_1^2 + t_2^2)}{t_2^2 - t_1^2}\right).$$

Proof. Let μ denote the equilibrium measure of J. Then

$$\mu = \frac{1}{\pi} \cdot \frac{|x|dx}{\sqrt{(x^2 - t_1^2)(t_2^2 - x^2)}} \qquad \text{and} \qquad \operatorname{cap}(J) = \frac{\sqrt{t_2^2 - t_1^2}}{2} = \sqrt[d-1]{\frac{1}{|a|}}.$$

The orthonormal polynomial basis φ_n of $L^2(J, \mu)$ can be computed as follows:

$$\varphi_1(x) = \kappa_1 x, \qquad \kappa_1 = \sqrt{\frac{2}{t_1^2 + t_2^2}}$$

$$\varphi_{2d}(x) = \pm\kappa_1 P(x) = \sqrt{2} T_d\left(\frac{2x^2 - (t_1^2 + t_2^2)}{t_2^2 - t_1^2}\right).$$

The proof is now complete.

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