ON AN INJECTIVITY LEMMA IN THE PROOF OF TANNAKIAN DUALITY

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ABSTRACT. In this short work we give a very short and elementary proof of the injectivity lemma, which plays an important role in the Tannakian duality for Hopf algebras over a field. Based on this we provide some generalizations of this fact to the case of flat algebras over a noetherian domain.

INTRODUCTION

Let k be a field. The neutral Tannakian duality establishes a dictionary between k-linear tensor abelian categories, equipped with a fiber functor to the category of k-vector spaces, and affine group schemes over k. The duality was first obtained by Saavedra in [5], among other important results. In [2], Deligne and Milne gave a very readable self-contained account on the result.

The main part of the proof of Tannakian duality was to establish the duality between abelian category equipped with fiber functors to vect_k and k-coalgebras. Here, one first proves the claim for those categories which have a (pseudo-) generator. Such categories are in correspondence to finite dimensional coalgebras. The injectivity lemma claims that, under this correspondence, fully faithful exact functors, which preserves subobjects, correspond to injective homomorphisms of coalgebras (see Lemma 1.2 for the precise formulation). This lemma was implicitly used in the proof of Prop. 2.21 in [2]. In the original work of Saavedra this claim was obtained as a corollary of the duality, cf. [5, 2.6.3 (f)]. In his recent book Szamuely gave a more direct proof of the injectivity lemma, cf. [8, Prop. 6.4.4]. Szamuely's proof is nice but still quite involved. Similar treatment and some generalizations was also made in Hashimoto's book [3, Lem. 3.6.10].

In this short work we give a very short and elementary proof of the injectivity lemma. Based on this, we provide some generalizations of this fact to the case of flat coalgebras over a noetherian domain. We believe that these results will find applications in the Tannakian duality for Hopf algebras over noetherian domains.

Notations. Throughout this work we shall fix a commutative noetherian domain R. The quotient field of R will be denoted by K.

For an algebra (or more general a ring) A (commutative or not), $\mathrm{mod}(A)$ denotes the category of left A-modules and $\mathrm{mod}_f(A)$ denotes the subcategory finitely generated modules.

For a coalgebra C over R, $\mathrm{comod}(C)$ denotes the category of right C-comodules, and $\mathrm{comod}_f(C)$ denotes the subcategory of comodules which are *finite over* R.

1. A SIMPLE PROOF OF THE INJECTIVITY LEMMA

Let k be a field and $f : A \to B$ be a homomorphism of finite dimensional k-algebras. Then f induces a functor $\omega : \mathrm{mod}_f(B) \to \mathrm{mod}_f(A)$ between the categories of finite modules over A and B, which is identity on the underlying vector spaces. In particular, ω is a faithfully exact functor.

Lemma 1.1. *A homomorphism f given as above is surjective if and only if the induced functor ω is full and has the property: for any B-module X and any A-submodule Y of $\omega(X)$, there exists a B-submodule X' of X such that $\omega(X') = Y$. In other words f is surjective if and only if $\mathrm{mod}_f(B)$, by means of f, is a full (abelian) subcategory of $\mathrm{mod}_f(A)$, closed under taking submodules.*

Proof. If f is surjective then obviously ω has the claimed properties. We prove the converse statement. Thus for any B-module X and any submodule Y of X, considered as modules over A, we know that Y is also stable under the action of B (obtained by restricting the action of B on X). Assume the contrary that f is not surjective, i.e., $B_0 := \mathrm{im}(f)$ is a strict subalgebra of B. Then $B_0 \subset B$ is an inclusion of A-modules, B it self is a B-module, but B_0 is not stable under the action of B as it contains the unit of B. A contradiction.   □

By duality we have the following result for comodules.

Lemma 1.2. *Let $f : C \to D$ be a homomorphism of finite dimensional k-coalgebras. Then the category $\mathrm{comod}_f(C)$, considered by means of f as a subcategory of $\mathrm{comod}_f(D)$, is full and closed under taking subobjects if and only if f is injective.*

Remark 1.3. In the proof of Lemma 1.1, there is no need to assume that A is finite dimensional. Therefore, in Lemma 1.2 there is no need to assume that D is finite dimensional. On the other hand, it is known that each coalgebra is the union of its finite dimensional subcoalgebras. Therefore there is no need to impose the dimension condition on C either.

2. Generalizations

We give here several generalizations of the lemmas in Section 1 to the case when k is a noetherian domain.

2.1. **The case of algebras.** Let R be a noetherian domain. We consider R-algebras. Modules over such an algebra are automatically R-modules, we call such a module R-finite (resp. torsion-free, flat, projective, free) if it is finite (resp. torsion-free, flat, projective, free) over R. Let $R \to S$ be a homomorphism of commutative algebras. Then the base change $R \to S$ will be denoted by the subscript $()_S$. For instance $M_S := M \otimes_R S$, $f_S := f \otimes_R S$ for an R-linear map f.

Let $f : A \to B$ be a homomorphism of finite, torsion-free R-algebras. It induces a functor $\omega : \mathrm{mod}_f(B) \to \mathrm{mod}_f(A)$, which is identity functor on the underlying R-modules, therefore it is faithful and exact. The following lemma is a straightforward generalization of Lemma 1.1.

Lemma 2.1. *The map f as above is surjective if and only if $\mathrm{mod}_f(B)$ when considered by means of f as a subcategory of $\mathrm{mod}_f(A)$ is full and closed under taking subobjects.*

Proof. Only the "if" claim needs verification. Let $\mathfrak{m} \in R$ be a maximal ideal and let $k(\mathfrak{m}) := R/\mathfrak{m}$ be the residue field. Then the full subcategory of $\mathrm{mod}_f(A)$ annihilated by $\mathfrak{m}$ is equivalent to $\mathrm{mod}_f(A_{k(\mathfrak{m})})$. Note that this subcategory is also closed under taking subobjects.

Thus, by assumption, $\mathrm{mod}_f(B_{k(\mathfrak{m})})$ is a full subcategory of $\mathrm{mod}_f(A_{k(\mathfrak{m})})$, closed under taking subobjects. Therefore the map $f_{k(\mathfrak{m})} : A_{k(\mathfrak{m})} \to B_{k(\mathfrak{m})}$ is surjective, by means of Lemma 1.1. This holds for any maximal ideal $\mathfrak{m}$ of R, hence $(B/f(A))_{k(\mathfrak{m})} = 0$ for all maximal ideals $\mathfrak{m}$. According to [4, Thm 4.8], we conclude that $B/f(A)$ itself is zero. □

Lemma 2.1 can also be proved in the same way as Lemma 1.1. We prefer to present the above proof as its main idea (using base change) will be exploited further.

An A-submodule N of M is said to be *saturated* iff M/N is R-torsion-free. A homomorphism $f : A \to B$ is said to be *dominant* if $f_K : A_K \to B_K$ is surjective, or equivalently $B/f(A)$ is R-torsion. Here K denotes the quotient field of R.

Proposition 2.2. *Let $f : A \to B$ be a homomorphism of R-torsion free algebras and assume that B is R-finite. Let $\omega : \mathrm{mod}_f(B) \to \mathrm{mod}_f(A)$ be the induced functor. Then:*

(1) *The image of ω is closed under taking saturated subobjects of R-torsion-free objects iff f is dominant. In this case ω is also closed under taking saturated submodules of any modules and its restriction to the subcategory of R-torsion-free modules is full.*

(2) *The image of ω is closed under taking subobjects of R-torsion-free objects iff f is surjective. (In this case ω is also obviously full.)*

Proof. (1). Assume that ω has the required property. We show that f is dominant, i.e. f_K is surjective. It suffices to show that the functor $\omega_K : \mathrm{mod}(B_K) \to \mathrm{mod}(A_K)$ induced from f_K satisfies the condition of Lemma 1.1. Let X be a finite B_K-module and $Y \subset X$ an A_K-submodule. Consider X as a B-module, take a K-basis of X such that a part of it is a basis of Y and let M be the B-submodule generated by this basis, let $N := M \cap Y$. Then $N_K = Y$ (as it contains a K-basis of Y). By the diagram below M/N is R-torsion-free.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0 \\
& & \cap\downarrow & & \cap\downarrow & & \cap\downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & X/Y & \longrightarrow & 0.
\end{array}
$$

Thus N is a saturated submodule of M, hence, by assumption, N is stable under B, consequently $Y = N_K$ is stable under B_K.

Conversely, assume that f is dominant, i.e., f_K is surjective. Then ω_K is fully faithful and closed under taking submodules. Let M be an R-torsion-free B-module, $N \subset M$ be a saturated A-submodule. Then M/N is also R-torsion-free, hence $N = M \cap N_K$. Now N_K is stable under B_K and M is stable under B, showing that N is stable under B.

Let $\varphi : M \to N$ be an A-linear map, where M, N are both R-torsion-free then φ is determined by $\varphi_K : M_K \to N_K$. Since f_K is surjective, we know that φ_K is A_K-linear, hence also B_K-linear, implying that f is B-linear. Thus ω restricted to R-torsion free modules is full.

Let now M be a finite A-module, N be a finite B-module and $\varphi : M \to N$ be an injective A-linear map with $M/\varphi(N)$ being R-torsion free. Consider a finite free B-linear cover $\psi : N' \to N$. Let $\varphi' : M' \to N'$ be the pull-back of φ along ψ (as A-modules).

$$
\begin{array}{ccc}
M' & \overset{\varphi'}{\hookrightarrow} & N' \\
\psi'\downarrow & & \downarrow\psi \\
M & \underset{\varphi}{\hookrightarrow} & N.
\end{array}
$$

Then φ' is injective and $\psi' : M' \to M$ is surjective, moreover $N'/\varphi'(M') \cong N/\varphi(N)$ hence is R-torsion free. Consequently, M' is B-stable and, since $\varphi\psi' = \psi\varphi'$ is B-linear, there is a B-action on M making φ B-linear.

(2). According to the proof of Lemma 2.1, it suffices to show that for any maximal ideal $\mathfrak{m}$ of R, the image of $\omega_{k(\mathfrak{m})}$ is closed under taking submodules. Let V be a $B_{k(\mathfrak{m})}$-module and let $\varphi : U \to V$ be an inclusion of $A_{k(\mathfrak{m})}$-modules. Represent V as a quotient of some (free) B-modules M, then U will be a quotient

of some A-submodule N of M. We have N is R-torsion-free, hence, by assumption, N is stable under B, so that U must be $B_{k(m)}$-stable. $\qquad\square$

2.2. **The case of coalgebras.** In this subsection we consider R-flat coalgebras. For such coalgebras the comodule categories are abelian, see, e.g. [3, Lem. 3.3.3] or [1, Thm. 3.13]. Moreover, the kernel and image of a comodule homomorphism are the same as those of the underlying R-module homomorphism. A homomorphism of flat R-coalgebras $f : C \to D$ induces the restriction functor $\omega : \mathrm{comod}(C) \to \mathrm{comod}(D)$ which is the identity functor on the underlying R-module. Hence ω is faithful and exact.

We say that a homomorphism of flat R-coalgebras $f : C \to D$ is pure if it is a pure homomorphism of R-modules. This condition is the same as requiring $D/f(C)$ be R-flat. Note also that over a noetherian domain, finite flat modules are projective.

For the case C and D are R-projective and C is R-finite, the desired results can be deduced from the previous subsection by means of the following lemma.

Lemma 2.3. *Let C be an R-finite flat (hence projective) module and D be an R-projective module. Then:*

(1) f *is injective iff* $f^\vee : D^\vee \to C^\vee$ *is dominant;*
(2) f *is injective and pure iff* $f^\vee : D^\vee \to C^\vee$ *is surjective,*

where $C^\vee := \mathrm{Hom}_R(C, R)$.

Proof. Embedding D as a direct summand into a free module does not change the properties of f and $f^\vee$, hence we can assume that D is free. Since C is finite, there exists a finite direct summand of D which contains the image of f and we can replace D by this summand, that means we can assume that D is finite. The claims for finite projective modules are obvious. For (1), it involves only the generic fibers. For (2), $f : C \to D$ is pure iff $D/f(C)$ projective, and iff the sequence $0 \to C \to D \to D/f(C) \to 0$ is split exact, iff the sequence $0 \to (D/f(C))^\vee \to D^\vee \to C^\vee \to 0$ is split exact. $\qquad\square$

A subcomodule N of a C-comodule M said to be saturated if M/N is R-torsion free.

Proposition 2.4. *Let C, D be R-projective coalgebras. Let $f : C \to D$ be a homomorphism of R-coalgebras and $\omega : \mathrm{comod}_f(C) \to \mathrm{comod}_f(D)$ be the induced functor. Assume that C is R-finite. Then:*

(1) *The image of functor ω is closed under taking saturated subcomodules of R-torsion-free comodules iff f is injective. In this case ω is also closed under taking saturated subcomodules of any comodules and its restriction to the subcategory of R-torsion-free comodules is full.*

(2) *The image of functor ω is closed under taking subcomodules of R-torsion-free comodules iff f is injective and pure. (In this case ω is also obviously full.)*

Proof. Since D is projective, the natural functor $\mathrm{comod}(D) \to \mathrm{mod}(D^\vee)$ is fully faithful, exact with image closed under taking subobjects [3, 3.10]. Thus the functor, induced from ω, $\mathrm{mod}(D^\vee) \to \mathrm{mod}(C^\vee)$ is fully faithful, exact and has image closed under taking subobjects iff ω is. The claim follows from Proposition 2.2 and Lemma 2.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We say that a flat R-coalgebra C is locally finite if C is the union of its finite R-projective pure subcoalgebras C_α, $\alpha \in A$. This property is called IFP (ind-finite projective property) in [3]. As a corollary of Proposition 2.4, we have

Corollary 2.5. *Let* C, D *be projective* R-*coalgebras. Assume that* C *has IFP,* $C = \bigcup C_\alpha$. *Let* $f : C \to D$ *be a homomorphism of* R-*coalgebras. Then* f *is injective iff the induced functor* $\omega : \mathrm{comod}_f(C) \to \mathrm{comod}_f(D)$ *has image closed under taking subobjects. In particular,* $\mathrm{comod}_f(C)$ *is the union of its full subcategories* $\mathrm{comod}_f(C_\alpha)$, *which are closed under taking subobjects.*

Notice that there exist R-flat coalgebras which contains almost no finite pure subcoalgebras, as shown in the examples below.

Example 2.6 ([6]). (1) Assume that R is a Dedekind ring with characteristic equal to 2 and $\pi \neq 0$ is a non-unit in R. Consider the algebra $H := R[T]/(\pi T^2 + T)$ with the coalgebra structure given by

$$\Delta(T) = T \otimes 1 + 1 \otimes T.$$

Then H is torsion-free, hence flat over R. Let $M \subset H$ be a saturated R-finite subcomodule of H. Then M is free over R, more over its rank is at most 2, as H_K has dimension 2 over K. If M has rank 2 then H/M is R-torsion, contradiction, hence M has rank 1 over R. The coaction of H on M is thus given by a group-like element in H. On the other hand, the R-flatness of H implies $H \otimes H$ is a submodule of $H_K \otimes_K H_K$, hence the coproduct on H is the restriction of the coproduct on H_K:

$$
\begin{array}{ccc}
H & \xrightarrow{\ \Delta\ } & H \otimes H \\
\downarrow & & \downarrow \\
H_K & \xrightarrow[\ \Delta\ ]{} & H_K \otimes_K H_K.
\end{array}
$$

Consequently, a group-like element of H is a group-like element of H_K. But in H_K the unique group-like element is 1. Thus M is a trivial comodule of H.

(2) Similarly, assume that R is a Dedekind ring, in which 2 is invertible. Consider the algebra $H := R[T]/(\pi T^2 + 2T)$ where $\pi \neq 0$ is a non-unit. H is a Hopf algebra with the coaction given by

$$\Delta(T) = T \otimes 1 + 1 \otimes T + \pi T \otimes T.$$

Similar discussion shows that there are only two finite saturated subcomodules of H, the one generated by 1 and the other generated by the group-like element $1 + \pi T$.

To treat a general coalgebra homomorphism $f : C \to D$ we shall imitate the proof of Lemma 1.1. Our condition on ω will be some what stronger.

Proposition 2.7. *Let* C, D *be* R-*flat coalgebras and* $f : C \to D$ *be a homomorphism of* R-*coalgebras. Let* $\omega : \mathrm{comod}(C) \to \mathrm{comod}(D)$ *be the induced functor on comodule categories. Then:*

(1) *The image of functor* ω *is closed under taking saturated subcomodules of* R-*torsion-free comodules iff* f *is injective. In this case* ω *is also closed under taking saturated subcomodules of any comodules and its restriction to the subcategory of* R-*torsion-free comodules is full.*

(2) *The image of functor* ω *is closed under taking subcomodules of any comodules iff* f *is injective and pure. In this case* ω *is full.*

Proof. (1) Assume that ω has the required property. Let $C_0 := \ker(f)$. Then $C/C_0 \cong \mathrm{im}(f) \subset D$ is R-torsion-free, hence C_0 is a saturated subcomodule of C, considered as D-comodules (the big left square with curved vertical arrows below is commutative). By assumption, the coaction of D on C_0 lifts to a coaction of C. That is, there exists a coaction $C_0 \to C_0 \otimes C$ (the dotted arrow below) making the following diagram commutative:

In particular, the coaction of C on C_0 is the restriction of that on C (the upper-left square). That is, for any $c \in C_0$ we have a representation

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)},$$

with $c_{(1)} \in C_0$. On the other hand, as f is a coalgebra homomorphism, we have $\varepsilon \circ f = \varepsilon$. Consequently, $\varepsilon(C_0) = 0$. Applying $\varepsilon \otimes \mathrm{id}$ to the above equation we get

$$c = \sum_{(c)} \varepsilon(c_{(1)}) \otimes c_{(2)} = 0.$$

A contradiction. Thus $\ker(f) = 0$.

Conversely, assume that $f : C \to D$ is injective, then the map $f_K : C_K \to D_K$ is also injective, as the base change $R \to K$ is flat. Hence, according to 1.2, 1.3, ω_K is fully faithful and is closed under taking subcomodules. Thus ω is full when restricted to R-torsion free comodules.

Finally we show that the image of ω is closed under taking saturated subcomodules of any comodules. For an R-module M, let M^{tor} denote its torsion part, i.e. those elements of M killed by some non-zero element of R. Then we have

$$M^{tor} = \ker(M \to M \otimes K).$$

Therefore, for any flat R-module P we have

$$(M^{tor} \otimes P) \cong (M \otimes P)^{tor}.$$

Since any R-linear map preserves the torsion part, we conclude that, if M is a C-comodule then M^{tor} is a subcomodule. Let now $N \subset M$ be a subcomodule with respect to the action of D. If M/N is R-torsion free then $M^{tor} = N^{tor}$ and hence is stable under the coaction of C. Hence we can consider the saturated inclusion $N/N^{tor} \subset M/M^{tor}$, which by assumption shows that N/N^{tor} is stable under the coaction of C. As C is flat, we conclude that N itself is stable under C.

(2) Assume that ω is closed under taking subcomodules of any comodules. Then according to (1), f is injective. Assume f is not pure, then there exists an ideal I of R such that the induced map $R/I \otimes C \to R/I \otimes D$ is not injective. Let C_0 be the kernel of this map. Repeat the argument of the proof of (1) we conclude that C_0 is stable under the coaction of D but not under the coaction of C, a contradiction. Thus f has to be pure.

For the converse, assuming that $f : C \to D$ is injective and pure and $N \subset M$ be R-modules, then we have the equality of submodules of $M \otimes D$:

$$N \otimes D \cap M \otimes C = N \otimes C,$$

where C is considered as an R-submodule of D by means of f. Hence, if M is a C-comodule and N is a D-subcomodule of M, then, denote by δ the coaction, we have

$$\delta(N) \subset N \otimes D \cap M \otimes C = N \otimes C.$$

That is, N is stable under the coaction of C. $\hfill\square$

Remark 2.8. According to Serre [7, Prop. 2], any object in $\mathrm{comod}(C)$ is the union of its R-finite subcomodules (but generally not saturated). I don't know if one can prove Propsition 2.7 with $\mathrm{comod}(C)$, $\mathrm{comod}(D)$ replaced by $\mathrm{comod}_f(C)$, $\mathrm{comod}_f(D)$, respectively.

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